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Sensitivity analysis of hyperbolic optimal control problems

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Abstract The aim of this paper is to perform sensitivity analysis of optimal control problems defined for the wave equation. The small parameter describes the size of an imperfection in the form of a small hole or cavity in the geometrical domain of integration. The initial state equation in the singularly perturbed domain is replaced by the equation in a smooth domain. The imperfection is replaced by its approximation defined by a suitable Steklov's type differential operator. For approximate optimal control problems the well-posedness is shown. One term asymptotics of optimal control are derived and justified for the approximate model. The key role in the arguments is played by the so called "hidden regularity" of boundary traces generated by hyperbolic solutions.

Keywords Sensitivity analysis · Optimal control problems · Hyperbolic boundary value problems · Linear partial differential operators · Steklov-Poincaré operator · Kondratiev weighted spaces

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1 Introduction

1.1 Modelling of imperfections

If a defect is included in the domain of integration of elliptic PDE, for example a crack, the domain becomes nonsmooth, i.e., looses the property of being Lipschitz. In such case the theory of boundary value problems defined on nonsmooth domains should be applied in order to show that the boundary value problem under consideration is well-posed in the scale of Kondratiev weighted spaces. If the size of the defect can be considered as a small parameter, the analysis can be performed on a suitable smooth domain, but the asymptotics [26–36] are derived according to the rules for singularly perturbed geometrical domains. The other possibility is to use the regular perturbations in line with homogenization techniques in optimal design: the real material is the *strong* material, but instead of the holes the *weak* material is introduced. The contrast parameter which stands for the properties of the weak material can be considered as a tool to obtain the holes by a limit passage, if necessary. This approach is useful for stationary problems, however it fails for evolution problems. This issue is particularly pronounced in low regularity hyperbolic models such as wave equations. The reason is simple, the asymptotic analysis performed for the stationary problems [42–47] gives useful information for low frequencies only, one can see this phenomenon when dealing with the spectral problems.

On the other hand, the models which are useful for applications should be simple and easy for computations. Therefore, we propose in this paper to conduct the analysis of the influence of imperfections for a simple model, just by taking only one term asymptotics of the energy functional obtained for a singular domain perturbations with nucleation of a hole. The question is whether even in such a case the presence of imperfections described in an approximate fashion destabilizes the control problems? We will show that the answer is quite complicated and a suitable regularization of the model is needed. The latter involves insertion of an additional small parameter in the boundary conditions. This parameter will force Lopatinski condition to hold for a Neumann problem which then will result in the so called "hidden regularity" [16] on the boundary. The idea of "hidden regularity" regularization has been used in the past successfully for boundary control problems—particularly in the context of numerical approximations [5, 6, 13, 20]. Regularizing parameter allows to obtain smooth on the boundary approximations, which can be then taken to appropriate limits. We refer to [1–4, 8, 9, 17, 22–25, 37, 39–41, 48] for the related results on modeling and optimization of distributed parameter systems.

1.2 Optimal control problems for the wave equation

We consider an optimal control problem for the wave equation. The control in $L_2(\Gamma \times (0, T))$ is applied on a portion of lateral boundary of the cylinder $\Omega_\rho \times (0, T)$. We assume that in the domain Ω_ρ a small defect is present, in the form of a void, its size is measured by small parameter $\rho \to 0$. We want to find how the defect influences the optimal control. It seems that such analysis for the class of problems can be important for applications, with respect, for instance, to nucleation of small cracks. The exact analysis of the asymptotic behavior of optimal controls with respect to

$\rho \to 0$ is out of reach for engineers and also quite involved. In particular, the required asymptotic analysis of hyperbolic problems for high frequencies in singularly perturbed domains is a mathematical topic still in its early stages of development. Therefore, we perform only some approximate sensitivity analysis based on asymptotic analysis of elliptic operators with respect to the parameter ρ. Unfortunately, such an analysis is far from being precise, more precise analysis would be based on the so-called self-adjoint extensions of elliptic operators [36], which is the topic of further studies (cf. [11]). However, even in the case of simplified analysis, the result seems to be useful and simple since we can replace the singular domain perturbations in Ω_ρ by the regular perturbations in boundary conditions on the truncated domain Ω_R. Here, we use the idea which can be very useful in the domain decomposition technique for the numerical solution of hyperbolic equations. We have also a precise result of sensitivity analysis, interesting on its own i.e., one term asymptotic expansion of the optimal control is obtained with respect to the parameter ρ for the control problem with constraints. The solutions considered are of "finite energy" controlled by physically significant $L_2(\Sigma)$ boundary inputs.

From the PDE point of view, the main difficulty of the problem is due to intrinsic low regularity of solutions to hyperbolic problems driven by L_2 Neumann boundary data. Of particular relevance is the regularity due to non-homogeneous boundary data which undergo infinitesimal perturbations. Standard hyperbolic regularity, is of no use in such analysis. What is essential instead, is the so-called "hidden regularity" property displayed by hyperbolic flows which satisfy the Lopatinski condition [7, 16, 18, 19, 38]. However, the model under consideration is equipped with the Neumann type of boundary conditions where hidden regularity does not hold [18, 19] unless the dimension of the domain is equal to one. Thus, the additional technical difficulty is related to the Neumann control in $L_2(\Gamma \times (0, T))$ where Lopatinski condition fails. This has implications on regularity theory which leads to the loss of $1/3$ derivative when analyzing the control-input map [18]. In order to deal with this difficulty, we shall impose absorbing boundary conditions (typical boundary friction) which can also be considered as a feedback stabilizer for the wave equation [12, 13, 20]. These boundary conditions, while producing long time stabilizing effect allow also to prove a weak version of "hidden regularity" for finite energy solutions [5, 6, 13, 16]. This latter property turns out critical for the analysis of sensitivity conducted in the present work. We shall show that the resulting control problem is well-posed, with the unique optimal control, and the first order perturbation of the optimal control with respect to the parameter ρ is uniquely determined by the solution of the control problem in the unperturbed domain. In other words, for a small defect in the domain of integration, its influence on the optimal control is determined by solving an auxiliary optimal control problem in unperturbed domain. Such an information could be useful for practical purposes, since the cost of numerical solution in singularly perturbed geometrical domain could be substantially higher, due to the singularities, compared to the cost of numerics performed on smooth unperturbed domains. It should be noted that the idea of "hidden regularity" regularization, in the context of wave equation, has been explored in the past. For instance, [5, 6] appplies the same regularization to approximation of Riccati operators arising in boundary control of wave equation with Neumann boundary conditions. In fact, the entire theory of con-

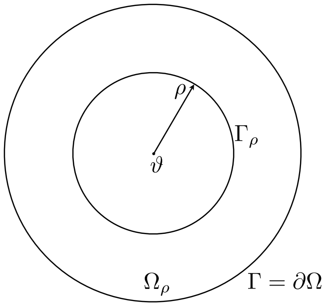


Fig. 1 The domain Ω_ρ in two spatial dimensions

vergence of FEM approximations of Riccati solutions rests on a suitable prior "hidden" regularization of the problem. The passage with the limit on the parameter of regularization leads to the ultimate convergence result for the original wave equation. The method of "hidden" regularization has been also used in domain decomposition procedures introduced and described in [13].

2 Geometry of $\boldsymbol{\Omega_\rho}$

To fix the ideas, we consider the following model problem. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary Γ and $\mathbb{B}_\rho$ be a *defect* included in Ω, in the form of a void. The case of a small crack can also be considered in our framework. The domain with the defect is denoted by Ω_ρ (Fig. 1). Usually, if the asymptotic analysis in singularly perturbed domains is applied for the construction of an approximate problem of simpler nature, some attributes of the defect like [26, 37] mass matrix, polarization matrix etc. are necessary in order to replace the domain Ω_ρ, which is singularly perturbed, by a punctured domain which *remembers* the presence of the defect by means of a singular potential located at the center of the defect. Here we avoid this type of approximation, we apply only the non-local Steklov-Poincaré operator which results from the asymptotic energy expansion for the Laplacian. This operator depends on the small parameter $\rho \to 0$, and we can use its expansion with respect to ρ in order to obtain the constructive formulae from our sensitivity analysis. For simplicity we fix the spatial dimension $n = 2$.

We denote by

$$\begin{aligned} \Omega_\rho \;\; &= \Omega \setminus \overline{\mathbb{B}}_\rho \subset \mathbb{R}^2, \\ \partial\Omega_\rho &= \Gamma \cup \Gamma_\rho, \end{aligned} \tag{1}$$

where: Ω is a domain on the plane $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ and

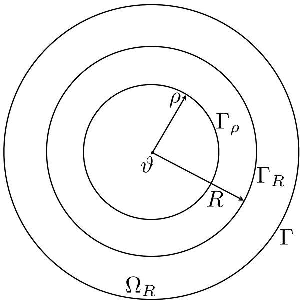
$$\mathbb{B}_\rho = \{x : |x - \vartheta| < \rho\} \tag{2}$$

with a smooth boundary Γ_ρ.

3 Domain decomposition $\boldsymbol{\Omega = \mathbb{B}_R \cup \Gamma_R \cup \Omega_R}$

Another useful geometrical construction for our problem is based on the domain decomposition technique. The idea is simple, we want to perform the asymptotic

Fig. 2 The domain Ω_R



analysis with respect to singular perturbations of domains in subdomain $\mathbb{B}_R$ with $R > \rho$ fixed once forever. The goal is then to study the influence of the small parameter $\rho \to 0$ on the optimal control in disjoint subdomain Ω_R. This decomposition allows us to obtain a simple problem in Ω_R, with regular perturbations of the boundary conditions imposed on the interface Γ_R between two subdomains. In this way we introduce the new hyperbolic problem to be considered, defined in the cylinder $\Omega_R \times (0, T)$, and avoid in fact any interaction with the boundary layer created in $\mathbb{B}_R$ by the presence of the defect. However, we need the preliminary analysis of the defect on the Steklov-Poincaré operator which lives on $\Gamma_R = \partial\mathbb{B}_R$. This analysis is performed only for the elliptic operator, and we use the result to construct the asymptotics for the elliptic Steklov-Poincaré operator. We consider the geometry from the figure below. Let us surround Γ_ρ by the circle Γ_R such that $R > \rho > 0$ (Fig. 2). Consequently, we denote

$$\Omega_R = \Omega \setminus \overline{\mathbb{B}}_R, \tag{3}$$

where:

$$\mathbb{B}_R = \{x : |x - \vartheta| < R\} \tag{4}$$

and we assume that the centre $\vartheta := \mathcal{O}$ is just the origin.

For further purposes we set the non-local Neumann boundary condition on Γ_R:

$$-\frac{\partial y}{\partial n} = A_\rho(y) \quad \text{on } \Gamma_R, \tag{5}$$

where: A_ρ is a Steklov-Poincaré operator defined below in the domain $C(R, \rho) = \mathbb{B}_R \setminus \overline{\mathbb{B}}_\rho$. The operator A_ρ is a mapping of $H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$. We recall the definition of the operator.
For given element $v \in H^{1/2}(\Gamma_R)$ we solve the boundary value problem

$$-\Delta w = 0 \quad \text{in } C(R, \rho), \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_\rho, \qquad w = v \quad \text{on } \Gamma_R, \tag{6}$$

and set

$$A_\rho v = \frac{\partial w}{\partial \nu} \quad \text{on } \Gamma_R, \tag{7}$$

where ν is the unit exterior normal vector on $\partial C(R, \rho)$, note that the unit exterior normal vector n on $\Gamma_R \subset \partial\Omega_R$ is $n = -\nu$.

Notation The following notation will be adopted.

We shall denote by $|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)}$ Sobolev's norm of order s defined on a domain Ω. Similarly, the inner products $(u,v)_\Omega = \int_\Omega uvdx$ denote L_2 products integrated over Ω. $(u,v)_\Gamma$ denotes $L_2(\Gamma)$ inner product. The same notation will be used for Γ_R.

Trace operators or restrictions to the boundary are denoted by $\gamma w \equiv w|_\Gamma$, and $\gamma_R w \equiv w|_{\Gamma_R}$.

Time-space function spaces are denoted by $C(X) \equiv C([0,T];X)$, $L_p(X) \equiv L_p(0,T;X)$ where X is a given Banach space.

Projection operators $P_i : \mathbb{R}^2 \mapsto \mathbb{R}^1$ are given by $P_i(u_1,u_2) = u_i$, $i = 1.2$.

The Constant C or c denote generic constants that do not depend on solution or the parameter ρ.

We begin with the following preliminary Lemma which is known to the specialists.

Lemma 1 *For all $\rho \geq 0$ the operator $A_\rho : L_2(\Gamma_R) \mapsto L_2(\Gamma_R)$ is self-adjoint and it is continuous $H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$. Moreover, for all $u \in H^{1/2}(\Gamma_R)$ the following bounds are uniform in $\rho \in [0,1]$.*

$$(A_\rho u, u)_{L_2(\Gamma_R)} = |A_\rho^{1/2} u|^2_{\Gamma_R} \sim |u|^2_{H^{1/2}(\Gamma_R)} = |u|^2_{1/2,\Gamma_R},$$
$$c|u|^2_{H^{1/2}(\Gamma_R)} \leq |A_\rho^{1/2} u|^2_{\Gamma_R} \leq C|u|^2_{H^{1/2}(\Gamma_R)}.$$

Proof We denote by $D : L_2(\Gamma_R) \mapsto L_2(\Omega_R)$ the Dirichlet map defined as

$$Dv \equiv w, \ w \quad \text{given by (6)}.$$

By using Green's formula one finds that for all $v, z \in H^{1/2}(\Gamma_R)$ one has

$$\begin{aligned}(A_\rho v, z)_{\Omega_R} &= \left(\frac{\partial}{\partial \nu} Dv, Dz\right)_{\Gamma_R} = \left(\frac{\partial}{\partial \nu} Dv, Dz\right)_{\Gamma_R \cup \Gamma_\rho} \\ &= (\Delta Dv, Dv)_{\Omega_R} + (\nabla Dv, \nabla Dz)_{\Omega_R} = (\nabla Dv, \nabla Dz)_{\Omega_R}\end{aligned}$$

which shows self-adjointness. In addition

$$(A_\rho v, z)_{\Omega_R} \leq |Dv|_{1,\Omega_R} |Dz|_{1,\Omega_R} \leq C|v|_{1/2,\Gamma_R} |z|_{1/2,\Gamma_R}$$

and

$$(A_\rho v, v)_{\Omega_R} = |\nabla Dv|^2_{\Omega_R} \sim |v|^2_{1/2,\Gamma_R}.$$

Since $\nabla Dv \in L_2(\Omega)$ implies that $Dv \in H^1(\Omega)$ and $Dv|_{\Gamma_R} \in H^{1/2}(\Gamma_R)$, we infer that $v \in H^{1/2}(\Gamma_R)$. The above concludes, via Closed Graph Theorem, the conclusion stated in Lemma 1. □

The domain decomposition technique allows for replacing the singular perturbation in the form of the small hole $\mathbb{B}_\rho$ in the domain Ω_ρ by a regular perturbation in the truncated domain Ω_R on the boundary Γ_R. However, for this purpose elliptic theory is used only. This leads to a consideration of the asymptotic approximation which is of the first order with respect to a small parameter ρ. More precisely, the energy in the ring $\mathbb{B}_R \setminus \mathbb{B}_\rho$ for the Laplacian with non-homogeneous Dirichlet condition on Γ_R and the homogeneous Neumann condition on Γ_ρ is considered as a function of $\rho \to 0$. The asymptotic approximation of the energy is equivalent to the asymptotic approximation of the Steklov-Poincaré operator on Γ_R. In addition, we determine below the first term of order ρ^2 which turns out to be a bounded non-local operator on $L_2(\Gamma_R)$. The boundary condition imposed on Γ_R reflects the presence of the defect in the ring $\mathbb{B}_R \setminus \mathbb{B}_\rho$. In this way, the presence of a singular perturbation in Ω_ρ is modeled by a regular perturbation of non-local boundary conditions on the boundary of truncated domain. The precision of such approximation for the hyperbolic evolution problems is still to be evaluated for some numerical examples.

Using the results of the Appendix we obtain the following expansion for the elliptic Steklov-Poincaré operator A_ρ in the norm $\mathcal{L}(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R))$:

$$A_\rho = A_0 + \rho^2 B + O(\rho^4) = A_0 + \rho^2 B + R_\rho. \tag{8}$$

More specifically, $A_0 \in \mathcal{L}(H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R))$, the remainder $O(\rho^4)$ is uniformly bounded on bounded sets in the space $H^{1/2}(\Gamma_R)$ with values in $H^{-1/2}(\Gamma_R)$ and also can be considered as bounded from $H^1(\Gamma_R) \mapsto L_2(\Gamma_R)$. The operator B is self-adjoint, from $H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$ and can be shown to be bounded operator $H^{1/2}(\Gamma_R) \mapsto L_2(\Gamma_R)$. The bounds are uniform in ρ.

The first term A_0 in the expansion of the operator A_ρ is the Dirichlet to Neumann operator or Steklov-Poincaré operator of the ball, hence it is simply given by the standard Green formula for the ball $\mathbb{B}_R$. In other words, for a given function w_0 harmonic in $\mathbb{B}_R$, with the Dirichlet trace $v \in H^{1/2}(\Gamma_R)$, the value of the operator $A_0(v) \in H^{-1/2}(\Gamma_R)$ is just the Neumann trace of the harmonic function cf. (5).

The second term of the expansion in ρ^2 can be represented in two spatial dimensions in the equivalent form of the product of the line integrals over the circle $\Gamma_R = \{x : |x - \mathcal{O}| = R\}$ with the centre at the origin $\mathcal{O}$. $\langle Bu, u\rangle$ is just the sum of squares of the line integrals, the trace on Γ_R is integrated with polynomials of degree one in both space variables. The operator B is self-adjoint since the bilinear form is symmetric

$$\langle Bu, u\rangle = b(\Gamma_R; u, u) = -\frac{1}{2\pi R^6}\left[\left(\int_{\Gamma_R} u x_1\, ds\right)^2 + \left(\int_{\Gamma_R} u x_2\, ds\right)^2\right]. \tag{9}$$

From the above representation, since the line integrals on Γ_R are well defined for functions in $L_2(\Gamma_R)$, or even in $L_1(\Gamma_R)$, it follows that the operator B can be extended to the bounded operator on $L_2(\Gamma_R)$,

$$B \in \mathcal{L}(L_2(\Gamma_R) \mapsto L_2(\Gamma_R)) \tag{10}$$

since the symmetric bilinear form of the operator, given by the equality

$$\begin{aligned}\langle Bu, v\rangle &= b(\Gamma_R; u, v)\\ &= -\frac{1}{2\pi R^6}\left[\left(\int_{\Gamma_R} u x_1\, ds\right)\left(\int_{\Gamma_R} v x_1\, ds\right) + \left(\int_{\Gamma_R} u x_2\, ds\right)\left(\int_{\Gamma_R} v x_2\, ds\right)\right]\end{aligned}$$

is continuous for all $u, v \in L_2(\Gamma_R)$. In fact, the bilinear form

$$L_2(\Gamma_R) \times L_2(\Gamma_R) \ni (u, v) \mapsto b(\Gamma_R; u, v) \in \mathbb{R}$$

is continuous with respect to the weak convergence since it has a simple structure

$$b(\Gamma_R; u, v) = l_1(u)l_1(v) + l_2(u)l_2(v), \quad u, v \in L_1(\Gamma_R)$$

with two linear forms $v \mapsto l_i(v), i = 1, 2,$ given by the line integrals on Γ_R. This gives us the additional regularity when replacing the singular perturbation of geometrical domain by the regular non-local perturbation B of the non-local boundary operator A_ρ. The numerical results are required, however, to confirm if this regular approximation of the hole is robust, and efficient e.g., for the low frequencies, which seems to be the case on the strength of theoretical considerations. The presence of "hidden regularity" regularization is known to produce strong stability properties for approximations of boundary conditions [5, 13].

3.1 Approximate model in Ω_R

For the domain Ω_ρ with defect in the form of a hole $\mathbb{B}_\rho$, the wave equation should be considered in the singularly perturbed domain with a small hole. Our aim is to consider a model with the domain without any hole, but with some influence of the defect modeled by means of the asymptotic analysis, cf. Appendix for the case of the energy functional and of the asymptotic analysis for the Steklov-Poincaré operator. The domain decomposition method consists in using the truncated domain Ω_R which contains no defect, however, the defect is modeled by a regular perturbation of the boundary conditions by the non-local Steklov-Poincaré operator. By the asymptotic analysis, the exact Steklov-Poincaré operator is approximated by its one term asymptotic approximation. Therefore, the approximate model in $\Omega_R \times (0, T)$ leads to the following hyperbolic equation with absorbing boundary conditions (where $\varepsilon > 0$ is a fixed parameter corresponding to the regularization)

$$\left.\begin{array}{ll}
\dfrac{\partial^2 y}{\partial t^2} - \Delta y = f & \text{in } \Omega_R \times (0, T), \\
\dfrac{\partial y}{\partial \eta} + \varepsilon y_t = v & \text{on } \Gamma \times (0, T) = \Sigma, \\
\dfrac{\partial y}{\partial \eta} + \varepsilon y_t + A_\rho(y) = 0 & \text{on } \Gamma_R \times (0, T) = \Sigma_R, \\
y(x, 0) = y_0(x) & \text{in } \Omega_R, \\
\dfrac{\partial y(x, 0)}{\partial t} = y_1(x) & \text{in } \Omega_R.
\end{array}\right\} \tag{11}$$

We are interested in optimizing finite energy solutions $(y, y_t) \in C(H^1(\Omega_R) \times L_2(\Omega_R))$ by means of boundary control $v \in L_2(\Sigma)$. We shall show that for the associated optimal control problem, the solutions are stable with respect to the small parameter $\rho \to 0$. In order to make the meaning of "finite energy" solutions to (11) precise, we shall define weak solutions via standard variational equality

$$\begin{aligned}&\frac{d}{dt}(y_t, \phi)_{\Omega_R} + (\nabla y, \nabla \phi)_{\Omega_R} + (\varepsilon y_t - v, \phi)_\Gamma \\ &\quad + (\varepsilon y_t, \phi)_{\Gamma_R} + (A_\rho^{1/2} y, A_\rho^{1/2})_{\Gamma_R} = 0, \quad \forall \phi \in H^1(\Omega_R)\end{aligned} \tag{12}$$

with the initial conditions $Y(0) = (y_0, y_1) \in H^1(\Omega_R) \times L_2(\Omega_R)$.

Remark 1 The presence of the parameter $\varepsilon > 0$ in both boundary conditions provides for "hidden regularity" effect on boundary traces of solutions. It will be shown that finite energy solutions satisfy $y_t|_{\Gamma \cup \Gamma_R} \in L_2(\Sigma \cup \Sigma_R)$. Thus, the boundary terms involving time derivatives restricted to the boundary are well defined in the definition of weak solution given in (12).

As already mentioned before, this type of "hidden regularization" regularization has been used in the context of FEM approximations of Riccati solutions [5, 20] and in the context of domain decomposition [13].

4 Neumann control problem in $U = L_2(\Gamma \times (0, T))$. Main results

We consider the following optimal boundary control problem defined in domain Ω_R. Let $U = L_2(\Gamma \times (0, T))$ be the space of controls. The time horizon T is fixed and the parameter of regularization $\varepsilon > 0$.

With $Y \equiv [y, y_t]$, a solution to (11), and

$$H \equiv H^1(\Omega) \times L_2(\Omega), \quad \mathcal{R} \in \mathcal{L}(H)$$

the functional cost is given by

$$I(v) = \frac{1}{2}|\mathcal{R}(Y(T; v) - Y_d)|_H^2 + \frac{\alpha}{2}\int_0^T \int_\Gamma |v|^2 ds dt. \tag{13}$$

The following constraints are imposed on the controls $v \in U_{ad}$:

$$U_{ad} = \{v \in L_2(\Gamma \times (0, T)), 0 \le v(x, t) \le 1\}. \tag{14}$$

Our first result pertains to existence and regularity of optimal pair (v_ρ^0, Y_ρ^0) corresponding to optimal control problem consisting of minimizing the functional (13) subject to (14) and (11).

Theorem 1 *For all initial data $Y(0) \in H$, terminal data $Y_d \in H$, right-hand side $f \in L_1(0, T; L_2(\Omega_R))$, and all $\rho > 0$ there exists a unique optimal control $v_\rho^0 \in L_2(\Sigma) \cap U_{ad}$ and such that optimal state $Y_\rho^0 \in C([0, T]; H)$. In addition, for $\varepsilon > 0$, the following boundary regularity holds: $\frac{d}{dt} y_\rho^0|_{\partial\Omega_R} \in L_2(\Sigma \cup \Sigma_R)$.*

Remark 2 We note that the boundary regularity of the velocity does not follow from interior regularity. This is an independent trace regularity result which turns out to be essential in characterizing optimal control.

When $\varepsilon = 0$ one can still deduce existence of optimal control and optimal trajectory, however the boundary regularity of solutions fails. Since this regularity is critical for further sensitivity analysis, we shall henceforth be assuming that $\varepsilon > 0$.

Our *GOAL*: is the asymptotic analysis of the optimal control when $\rho \to 0$. In reference to problem (11), the following result on the stability of optimal controls for $\rho \to 0$ holds.

Theorem 2 *Let us consider the minimization of the cost functional* (13) *evaluated for the state equation* (11) *with* $f = 0$ *and initial-terminal data in* $H = H^1(\Omega_R) \times L_2(\Omega_R)$, *subject to the control constraints* (14). *Then the unique optimal control* u_ρ^0 *admits for* $\rho \to 0$ *the one term asymptotics*

$$u_\rho = u + \rho^2 q + o(\rho^2) \quad \text{in } L_2(\Gamma \times (0, T)),$$

where q *is given by a unique solution of the auxiliary optimal control problem* (*cf. Lemma* 8) *with the state equation* (63), *the cost functional* (57), *and the set of admissible controls* (61).

The remaining part of the paper is devoted to the proof of the main theorem.

5 Existence and regularity theory

In this section we study forward regularity properties of solutions to the initial-boundary value problem given in (11).

Theorem 3 Regularity theorem
Let

- $f \in L_1(0, T; L_2(\Omega_R))$, $v \in L_2(0, T; L_2(\Gamma))$
- $y_0 \in H^1(\Omega_R)$, $y_1 \in L_2(\Omega_R)$.

Then, there exists a unique solution of the state equation in the truncated domain

$$y \in C(0, T; H^1(\Omega_R)) \cap C^1(0, T; L_2(\Omega_R))$$

and such that the following hidden regularity holds:

$$y|_{\Gamma_R} \in H^1(\Sigma_R) \cap C(0, T; H^{1/2}(\Gamma_R)).$$

$$y|_\Gamma \in H^1(\Sigma).$$

In addition, the following bound is available for all $0 \le t \le T$:

$$|y(t)|^2_{1,\Omega_R} + |y_t(t)|^2_{0,\Omega_R} + 2\varepsilon \int_0^t |y_t|^2_\Gamma ds + 2\varepsilon \int_0^t [|y_t|^2_{\Gamma_R} + |y|^2_{H^1(\Gamma_R \cup \Gamma)}] ds$$
$$\le C[|y(0)|^2_{1,\Omega_R} + |y_t(0)|^2_{0,\Omega_R}] + \frac{c}{\varepsilon}\|v\|^2_{L_2(\Sigma)},$$

where the constants are uniform in $\rho \ge 0$.

Proof We shall approach the problem of existence of solutions to (30)—or equivalently to its variational form defined in (12)—by using semigroup theory. In fact, it is enough to show that the semigroup solutions generated by differential equation (30) have desired boundary regularity. This allows to obtain variational formulation by taking strong limits of strong semigroup solutions defined on H. To this aim we define several operators. Let A_N denotes the Laplacian with zero Neumann boundary data on $\partial\Omega_R$. This is to say $A_N : D(A_N) \subset L_2(\Omega_R) \mapsto L_2(\Omega_R)$ is defined by

$$A_N u = -\Delta u, \quad u \in D(A_N) \equiv \left\{ u \in H^2(\Omega_R) : \frac{\partial}{\partial \nu} u = 0 \text{ on } \Gamma \cup \Gamma_R \right\}$$

Let N (resp. N_R) denote the Neumann harmonic extension from Γ (resp. Γ_R) into Ω_R. This is to say $N : L_2(\Gamma) \mapsto L_2(\Omega_R)$ is defined by $w \equiv Nv$ iff $\Delta w = 0$ in Ω_R and $\frac{\partial}{\partial \nu} w = v$ on Γ, $\frac{\partial}{\partial \nu} w = 0$ on Γ_R.

Similarly, $N_R : L_2(\Gamma_R) \mapsto L_2(\Omega_R)$ is defined by $w \equiv Nv$ iff $\Delta w = 0$ in Ω_R and $\frac{\partial}{\partial \nu} w = 0$ on Γ, $\frac{\partial}{\partial \nu} = v$ on Γ_R.

In defining Neumann harmonic extensions, without loss of generality we may assume that zero eigenvalue is mode out. This has no effect on further analysis, since eventually

$$|A_N^{1/2} u|^2_{\Omega_R} + |A_\rho^{1/2} u|^2_{\Gamma_R} \sim |u|^2_{H^1(\Omega_R)},$$

so the static elliptic operator controls full H^1 norm.

The abstract second order form of the problem under consideration (see [15, 20]) is the following:

$$\begin{aligned} &y_{tt} + A_N y + \varepsilon A_N N N^* A_N y_t + \varepsilon A_N N_R N_R^* A_N y_t + A_N N_R A_\rho(y) \\ &\quad = f + A_N N v \end{aligned} \tag{15}$$

with the initial conditions $y_0 \in H^1(\Omega_R)$, $y_1 \in L_2(\Omega_R)$.

The above representation uses the following identifications:

$$\begin{aligned} N^* A_N y &= y|_\Gamma, \quad y \in H^1(\Omega_R), \\ N_R^* A_N y &= y|_{\Gamma_R}, \quad y \in H^1(\Omega_R), \end{aligned} \tag{16}$$

which follow from the application of Green's formula [15].

STEP 1. We shall prove that (11) with $v = 0$, $f = 0$ generates a strongly continuous semigroup on $H^1(\Omega_R) \times L_2(\Omega_R)$. In order to accomplish this, we find it convenient to topologize H with equivalent norm given by

$$|y|_H^2 \equiv |A_N^{1/2} y_1|_{\Omega_R}^2 + |A_0^{1/2}(y_1)|_{\Gamma_R}^2 + |y_2|_{\Omega_R}^2,$$

$$|y|_{H_\rho}^2 \equiv |A_N^{1/2} y_1|_{\Omega_R}^2 + |A_\rho^{1/2}(y_1)|_{\Gamma_R}^2 + |y_2|_{\Omega_R}^2.$$

In view of Lemma 1 these are equivalent norms to the standard $H^1(\Omega_R) \times L_2(\Omega_R)$ norms. The inner product generated by H is the following

$$(y, w)_{H_\rho} \equiv (A_N^{1/2} y_1, A_N^{1/2} w_1)_{\Omega_R} + (A_\rho^{1/2} y_1, A_\rho^{1/2} w_1)_{\Gamma_R} + (y_2, w_2)_{\Omega_R}.$$

We introduce the operator $\mathcal{A}_\rho : H \mapsto H$ whose action is defined by

$$\mathcal{A}_\rho(y_1, y_2) \equiv \begin{pmatrix} 0 & I \\ -A_N - A_N N_R A_\rho N_R^* A_N & -\varepsilon A_N N_R N_R A_N^* - \varepsilon A_N N N^* A_N \end{pmatrix}.$$

It follows that for $y = (y_1, y_2) \in D(\mathcal{A}_\rho)$, where $D(\mathcal{A}_\rho)$ is the maximal domain, we obtain:

$$\begin{aligned}
&(\mathcal{A}_\rho y, y)_{H_\rho} \\
&\quad = -\varepsilon |N^* A_N y_2|_\Gamma^2 - \varepsilon |N_R^* A_N y_2|_{\Gamma_R}^2 - (A_N N A_\rho N_R^* A_N y_1, y_2)_{\Omega_R} \\
&\qquad - (A_N y_1, y_2) + (A_\rho^{1/2} N_R^* A_N y_1, A_\rho^{1/2} N_R^* A_N y_2)_{\Omega_R} + (A_N^{1/2} y_1, A_N^{1/2} y_2)_{\Omega_R} \\
&\quad = -\varepsilon |N^* A_N y_2|_\Gamma^2 - \frac{1}{2}\varepsilon |N_R^* A_N y_2|_{\Gamma_R}^2 + C_\varepsilon |y|_H^2, \qquad (17)
\end{aligned}$$

where we have used inner product defined on H_ρ. This gives that $\mathcal{A}_\rho$ is dissipative.

In order to prove the generation of the semigroup we need to establish maximal dissipativity. This is done as follows:

Maximal dissipativity, by Minty's Theorem, is equivalent to the range conditions, i.e. solvability of

$$\mathcal{A}_\rho y - y = f \in H$$

for every $f \in H$. Writing in the coordinates

$$\begin{aligned}
&y_2 - y_1 = f_1 \in H^1(\Omega_R), \\
&A_N y_1 + A_N N_R A_\rho N_R^* A_N y_1 + y_2 + \varepsilon A_N N_R N_R^* A_N y_2 + \varepsilon A_N N N^* A_N y_2 \qquad (18) \\
&\quad = f_2 \in L_2(\Omega_R),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&A_N y_1 + A_N N_R A_\rho N_R^* A_N y_1 + y_1 + \varepsilon A_N N_R N_R^* A_N y_1 + \varepsilon A_N N N^* A_N y_1 \\
&\quad = -f_1 - f_2 - \varepsilon A_N N_R N_R^* A_N f_1 - \varepsilon A_N N N^* A_N f_1. \qquad (19)
\end{aligned}$$

The operator

$$\mathcal{B} \equiv A_N + A_N N_R A_\rho N_R^* A_N + I + \varepsilon A_N N_R N_R^* A_N + \varepsilon A_N N N^* A_N$$

is a Lax-Milgram operator on the space $V \equiv H^1(\Omega_R)$. Indeed,

$$(\mathcal{B}u, u)_{L_2(\Omega_R)} = |A_N^{1/2} u|^2_{\Omega_R} + |A_\rho^{1/2} N_R^* A_N u|^2_{\Gamma_R} + |u|^2_{\Omega_R} \\ + \varepsilon |N^* A_N y|^2_\Gamma + \varepsilon |N_R^* A_N u|^2_{\Gamma_R} \geq |u|^2_{1,\Omega_R}.$$

The continuity of the associated form follows by Lemma 1.

Since with $f \in H^1(\Omega_R)$, and $R(N) \subset D(A_N^\alpha)$ for $0 \leq \alpha < 3/4$, we have

$$A_N N_R N_R^* A_N f_1 \in [D(A_N^{1/2})]' \subset V'$$

and

$$A_N N N^* A_N f_1 \in [D(A_N^{1/2})]' \subset V'.$$

The above leads to the solvability of (19). Thus, the generation of a strongly continuous semigroup is deduced via monotone operator theory.

Remark 3 We note that maximal dissipativity property of the operator $\mathcal{A}_\rho$ does not depend on strict positivity of the parameter ε. Thus, the conclusion on the existence of semigroup solution is independent on the regularization. However, in order to prove the energy inequality, as stated in the Theorem 3, the presence of $\varepsilon > 0$ is critical. This provides for additional boundary regularity.

In order to prove the energy inequality stated in Theorem 3, we first apply the energy-multipliers method. Taking first $v = 0$ and multiplying (11) by y_t gives the following energy equality:

$$|Y(t)|^2_{H_\rho} + \varepsilon \int_0^t |N_R^* A_N y_t|^2_{\Gamma_R} + \varepsilon \int_0^t |N^* A_N y_t|^2_\Gamma = |Y(0)|^2_{H_\rho}.$$

Since $y|_{\Gamma_R \cup \Gamma} \in H^1(0, T; L_2(\Gamma_R))$, $Y \in C(0, T; H)$ and $\frac{\partial}{\partial \nu} y \in L_2(\Sigma)$, hidden regularity applies [16, 21] and implies the L_2 regularity of tangential derivatives on the boundary Γ. The same applies to Γ_R after taking into considerations the regularity for "small frequencies" exhibited by elliptic problem resulting from microlocalization of the wave operator to "small" time dual variables (frequencies). This leads to consideration of elliptic problem (microlocally) which are driven by L_2 internal force and L_2 boundary data (see [18]). This is to say that microlocal solutions satisfy

$$\Delta y = f \in L_2(\Omega), \qquad \frac{\partial}{\partial \nu} y = h \quad \text{on } \Gamma, \qquad \frac{\partial}{\partial \nu} y + A_\rho y = g \quad \text{on } \Gamma_R$$

display the regularity:

$$|y|_{1+s,\Omega_R} + |y|_{H^{1/2+s}(\Gamma \cup \Gamma_R)} \leq C(|f|_{L_2(\Omega_R)} + |h|_{-1/2+s,\Gamma} + |g|_{-1/2+s,\Gamma_R}) \tag{20}$$

for all $s \in [0, 1/2]$ uniformly in the parameter $\rho > 0$. Applying the above inequality with $s = 1/2$ and accounting for the fact that we already know that $y_t \in L_2(\Sigma_R \cup \Sigma)$

gives:

$$\int_0^T |\nabla_{\Gamma_R} y|^2_{\Gamma_R} + |\nabla|_\Gamma y|^2_\Gamma ds \le c\varepsilon^{-1}|Y(0)|^2_{H_\rho}.$$

By Lemma 1 we have that $|Y|_{H_\rho} \sim |Y|_H$, uniformly in $\rho \in [0,1]$, so the norms H_ρ can be replaced (with appropriate change of the constants) by norms in H. This provides the desired bound with $v = 0$.

STEP 2: We go back to the main equation (15) which is the boundary perturbation of the ω dissipative semigroup. As such, it can be written as $Y \equiv [y, y_t]$

$$\frac{d}{dt}Y = \mathcal{A}_\rho Y + \begin{pmatrix} 0 \\ A_N N_R v + f \end{pmatrix}.$$

It is at this point where absorbing damping on the boundary Γ is critical. (Absorbing damping on Γ_R will be needed for sensitivity analysis).

Indeed, multiplying (15) by y_t and integrating by parts (this procedure is formally applied to smooth approximations of the problem and followed by limit process [20]) yields:

$$\begin{aligned} &|Y(t)|^2_{H_\rho} + 2\varepsilon \int_0^t |N^* A_N y_t|^2_\Gamma ds + 2\varepsilon \int_0^t |N_R^* A_N y_t|^2_{\Gamma_R} ds \\ &\quad \le |Y(0)|^2_{H_\rho} + \int_0^t |f|_{\Omega_R} |y_t|_{\Omega_R} ds + \int_0^t (A_N N v, y_t)_{\Omega_R} ds. \end{aligned} \tag{21}$$

The above implies:

$$\begin{aligned} &|A_N^{1/2} y(t)|^2_{\Omega_R} + |y_t(t)|^2_{\Omega_R} + 2\varepsilon \int_0^t |N^* A_N y_t|^2_\Gamma ds + 2\varepsilon \int_0^t |N_R^* A_N y_t|^2_{\Gamma_R} ds \\ &\quad \le C|A_N^{1/2} y(0)|^2_{\Omega_R} + C|y_t(0)|^2_{\Omega_R} + 1/2 \sup_{s\in[0,t]} |y_t(s)|^2_{\Omega_R} \\ &\qquad + C\left(\int_0^t |f|_{\Omega_R} ds\right)^2 + \varepsilon \int_0^t |N^* A_N y_t|^2_\Gamma ds + \frac{2}{\varepsilon}\int_0^t \int_\Gamma |v|^2 dx ds. \end{aligned} \tag{22}$$

Hence

$$\begin{aligned} &|Y(t)|^2_H + \varepsilon \int_0^t |y_t|^2_{\Gamma_R} + \varepsilon \int_0^t |y_t|^2_\Gamma \\ &\quad \le C_t\left[|Y(0)|^2_H + C\left(\int_0^t |f|_{\Omega_R} ds\right)^2 + \frac{c}{\varepsilon}\int_0^t \int_\Gamma |v|^2 dx ds\right]. \end{aligned} \tag{23}$$

Since $y_t|_{\Gamma_R} \in L_2(\Sigma_R)$ and solutions are of finite energy, hidden regularity [16] along with elliptic estimate (20) gives the control of all tangential-time and space derivatives on the boundary. This completes the proof of Theorem 3. □

It is convenient to introduce the following notation:

– $S_{\rho,t}: H \mapsto H$ is the semigroup generated by (11) with $f = 0, v = 0$.
– $(L_\rho u)(t) = \int_0^t S_{\rho,t-s} Du(s)ds$ where $Du \equiv [0, A_N N u]$, or equivalently solution to (11) with $f = 0$, $y_0, y_1 = 0$, $v = u$. The bounds are independent on ρ.

Theorem 3 implies for all $t \in [0, T]$.

Corollary 1

1. $|S_{\rho,t} Y|_H \le C_t |Y|_H$
2. $\varepsilon \int_0^T [|\gamma S_{\rho,t} Y|^2_{L_2(\Gamma)} + |\gamma_R S_{\rho,t} Y|^2_{L_2(\Gamma_R)}] \le C|Y|^2_H$
3. $|L_\rho(u)|_{C(0,T,H)} \le \frac{C}{\sqrt{\varepsilon}} |u|_{L_2(\Sigma)}$
4. $|\gamma P_2 L_\rho u|_{L_2(\Sigma)} + |\gamma_R P_1 L_\rho u|_{H^1(\Sigma_R \cup \Sigma)} \le \frac{C}{\varepsilon} |u|_{L_2(\Sigma)}$

where the bounds are uniform in $\rho \ge 0$ and also $t \in [0, T]$.

6 Optimal control problem

We shall analyze next the optimal boundary control problem (11)–(13) in the domain Ω_R with the fixed parameter $\rho > 0$. The solution of the formulated optimal control problem is equivalent to seeking a $v^0_\rho \in U_{ad}$ such that $I(v^0_\rho) \le I(v)$ for all $v \in U_{ad}$. Standard arguments in calculus of variations lead to the following results for $\alpha > 0$ a unique optimal control v^0_ρ is characterized by the following condition

$$I'(v^0_\rho)(v - v^0_\rho) \ge 0 \quad \forall v \in U_{ad}. \tag{24}$$

Using the form of the performance functional (13) we can express (24) in the following form:

$$\begin{aligned} &(\mathcal{R}^*\mathcal{R}(Y(x,T;v^0_\rho) - Y_d), (Y(x,T;v) - Y(x,T;v^0_\rho))_H \\ &\quad + \frac{\alpha}{2} \int_0^T \int_\Gamma v^0_\rho (v - v^0_\rho) dx dt \ge 0 \quad \forall v \in U_{ad}. \end{aligned} \tag{25}$$

After denoting $L_{\rho,T} u = L_\rho u(T)$ and observing that $Y(T, v) = S_{\rho,T} Y_0 + L_{\rho,T} v$ (25) gives:

$$\begin{aligned} &\int_0^T (L^*_{\rho,T} \mathcal{R}^*\mathcal{R}(L_{\rho,T} v^0_\rho + S_{\rho,T} Y_0 - Y_D), v - v^0_\rho)_\Gamma dt \\ &\quad + \frac{\alpha}{2} \int_0^T (v^0_\rho, v - v^0_\rho)_\Gamma dt \ge 0 \quad \forall v \in U_{ad}, \end{aligned} \tag{26}$$

which provides the following characterization of the optimal control.

$$v^0_\rho \equiv P_{U_{ad}} \left[\frac{-1}{\alpha} L^*_{\rho,T} \mathcal{R}^*\mathcal{R}(L_{\rho,T} v^0_\rho + S_{\rho,T} Y_0 - Y_D) \right] = P_{U_{ad}}[-C_\rho v^0_\rho + F_\rho], \tag{27}$$

where

$$C_\rho \equiv \frac{1}{\alpha} L^*_{\rho,T} \mathcal{R}^*\mathcal{R} L_{\rho,T}, \qquad F_\rho \equiv \frac{-1}{\alpha} L^*_{\rho,T} \mathcal{R}^*\mathcal{R}(S_{\rho,T} Y_0 - Y_D). \tag{28}$$

Here we use the topology of H (independent on the parameter ρ) for the computations of the adjoints.

Direct calculation of the adjoint operator $L^*_{\rho,T}$ yields:

$$L^*_{\rho,T}\Phi \equiv \gamma\Psi_2(t), \qquad \Psi(t) \equiv S^*_\rho(T-t)\Phi.$$

In order to calculate the adjoint explicitly we find useful to introduce the notation

$$\tilde{A}_N \equiv A_N + A_N N_R A_0 \gamma_R = A_N + A_N N_R A_0 N^*_R A_N.$$

This is self-adjoint, positive operator acting on $L_2(\Omega_R)$ and $|\tilde{A}_N^{1/2} u|_{\Omega_R} \sim |u|_{1,\Omega_R}$. We also note that

$$|Y|^2_H = |\tilde{A}_N^{1/2} y_1|^2_{\Omega_R} + |y_2|^2_{\Omega_R}.$$

With the above notation we have

$$\mathcal{A}^*_\rho(y_1,y_2) = [-y_2 - \tilde{A}_N^{-1} A_N N_R R^*_\rho \gamma_R y_2,\ \tilde{A}_N y_1 - \varepsilon A_N N_R \gamma_R y_2 - \varepsilon A_N N \gamma y_2]$$

where R_ρ is introduced in (8).

To simplify (25), we introduce the adjoint equation. For every $v \in U_{ad}$, we define the adjoint variable

$$P_\rho(t) = [p_{\rho,1}(t), p_{\rho,2}(t)] \equiv S^*_{\rho,T-t}\mathcal{R}^*\mathcal{R}(Y^0(T) - Y_d).$$

We verify that the vector P_ρ is the solution of the following system of equations:

$$\begin{aligned} &\frac{d}{dt} p_{\rho,1} = p_{\rho,2} + \tilde{A_N}^{-1} A_N N_R R^*_\rho \gamma_R p_{\rho,2}, \\ &\frac{d}{dt} p_{\rho,2} = -\tilde{A}_N p_{\rho,1} + \varepsilon A_N N_R \gamma_R p_{\rho,2} + \varepsilon A_N N \gamma p_{\rho,2}. \end{aligned} \tag{29}$$

The above system can be rewritten as the following PDE-non-local system

$$\left.\begin{aligned} &\frac{d}{dt} p_{\rho,1} = p_{\rho,2} + \tilde{A_N}^{-1} A_N N_R R^*_\rho \gamma_R p_{\rho,2}, \qquad \frac{d}{dt} p_{\rho,2} = \Delta p_{\rho,1} && \text{in } \Omega_R \times (0,T), \\ &\frac{\partial p_{\rho,1}}{\partial \eta} - \varepsilon p_{\rho,2} = 0 && \text{on } \Gamma \times (0,T), \\ &\frac{\partial p_{\rho,1}}{\partial \eta} - \varepsilon p_{\rho,2} + A_0(p_{\rho,1}) = 0 && \text{on } \Gamma_R \times (0,T), \\ &P_\rho(x,T;v) = \mathcal{R}^*\mathcal{R}(Y^0_\rho(T) - Y_d), && \text{in } \Omega_R, \end{aligned}\right\} \tag{30}$$

where Y^0_ρ is the optimal trajectory corresponding to ρ problem.

Remark 4 We note that for $\rho = 0$, we have $R_\rho = 0$ and the adjoint equation for the variable $P^0(t) = [p^0(t), p^0_t(t)]$ can be written as:

$$\begin{aligned} p^0_{tt} &= \Delta p^0, \\ \frac{\partial}{\partial \nu} p^0 - \varepsilon p^0_t &= 0 \quad \text{on } \Gamma, \\ \frac{\partial}{\partial \nu} p^0 - \varepsilon p^0_t + A_0(p^0) &= 0 \quad \text{on } \Gamma_R, \\ P^0(T) &= \mathcal{R}^*\mathcal{R}(Y^0(T) - Y_d). \end{aligned} \tag{31}$$

Lemma 2 *The following estimate holds for solutions to* (30) *for all* $t \in [0, T]$.

$$\begin{aligned} &|p_{\rho,1}(t)|^2_{1,\Omega_R} + |p_{\rho,2}(t)|^2_{0,\Omega_R} + \varepsilon \int_t^T |p_{\rho,2}|^2_{\Gamma} \\ &\quad + \varepsilon \int_t^T |p_{\rho,2}|^2_{\Gamma_R} \le C_t(|p_{\rho,1}(T)|^2_{1,\Omega_R} + |p_{\rho,2}(T)|^2_{0,\Omega_R}). \end{aligned} \tag{32}$$

Proof This result follows from the same arguments as used in the proof of Theorem 3. In order to obtain the energy estimate we multiply the first equation in (29) by $\tilde{A}_N p_1$, the second equation by p_2 and use duality pairings. This gives:

$$|P_\rho(t)|^2_H + \varepsilon \int_t^T |p_{2,\rho}|^2_{\Gamma \cup \Gamma_R} ds = |P_\rho(T)|^2_H + \int_t^T (R^*_\rho \gamma_R p_{\rho,2}, \gamma_R p_{1,\rho})_{\Gamma_R} ds. \tag{33}$$

Since $R^*_\rho : L_2(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$ is bounded uniformly in ρ, we obtain

$$\begin{aligned} &(R^*_\rho \gamma_R p_{\rho,2}, \gamma_R p_{1,\rho})_{\Omega_R} \\ &\quad \le C|p_1|_{1,\Omega_R} |\gamma_R p_2|_{0,\Gamma_R} \le C_\varepsilon |p_1|^2_{1,\Omega_R} + 1/4\varepsilon |\gamma_R p_2|^2_{0,\Gamma_R}. \end{aligned}$$

The above estimate leads, via Gronwall's inequality, to the final conclusion in Lemma 2. □

Remark 5 One could prove additional tangential regularity of $p_{\rho,1}|_{\Gamma_R} \in H^1(\Gamma_R)$ for ρ sufficiently small, which would allow to relax regularity of $R^*_\rho : L_2 \mapsto H^{-1}$. This step, however, would require analysis similar to that given in Regularity Theorem 3 but applied to $p_{\rho,1}$ equation and followed by perturbation argument in order to incorporate non-local operator on the right side of the first equation in (30). It is for this point where smallness of ρ will be needed. Since this point is not essential, we shall not insist on the additional technicalities.

Theorem 4 *Let the hypothesis of Theorem 1 be satisfied. Then for given* $Y_d, Y_0 \in H$, $v^0_\rho \in U_{ad}$, *there exists a unique solution to* (30)

$$P_\rho(v_\rho) = [p_\rho(v^0_\rho), p_{\rho,t}(v^0_\rho)] \in C(H)$$

and such that $p^0_{\rho,2}|_{\partial\Omega_R} \in L_2(\Sigma \cup \Sigma_R)$.

We simplify (25) using the adjoint equation (30). This leads to:

$$\int_0^T \int_\Gamma (p_{\rho,2} + \alpha v_\rho^0)(v - v_\rho^0)dxdt \geq 0 \quad \forall v \in U_{ad}. \tag{34}$$

Theorem 5 (Optimality Theorem) *For the problem* (11) *with the performance functional* (13) *with* $Y_d \in H$ *and* $\alpha > 0$, *and with constraints on the control* (14), *there exists a unique optimal control* v_ρ^0 *which satisfies the maximum condition* (34). *Moreover,* $v_\rho^0 = P_{U_{ad}}(-\frac{1}{\alpha} p_{\rho,2})$ *where* $P_{U_{ad}}$ *is a projection operator on* U_{ad} *with respect to* L_2 *topology.*

Remark 6 Note, that the boundary regularity of the adjoint variable $p_{\rho,2}|_{\Gamma \cup \Gamma_R}$ represents hidden regularity of the solutions to the adjoint equation. This is critical in characterizing the optimal solution.

Remark 7 By using inner product induced by $A_N + A_N N_R A_\rho \gamma_R$ the adjoint equation becomes just the wave equation (second order in time), rather then the system of two equations of first order.

7 Sensitivity of optimal controls in $U = L_2(\Gamma \times (0, T))$

By using variational definition of the projector operator $P_{U_{ad}}$ we infer the following characterization of the optimal control.

$$(v_\rho^0 + C_\rho v_\rho^0 + F_\rho, u - v_\rho^0)_\Sigma \geq 0 \quad \forall u \in U_{ad}. \tag{35}$$

The above characterization leads to the following error inequality satisfied by the difference of two optimal controls corresponding to $\rho > 0$ and $\rho = 0$ and denoted respectively by v_ρ and v^0.

$$\begin{aligned} &|v^0 - v_\rho^0|_\Sigma^2 - (C_\rho(v^0 - v_\rho^0), v^0 - v_\rho^0)_\Sigma \\ &\quad \leq ((C_0 - C_\rho)v^0, v^0 - v_\rho^0)_\Sigma + (F_0 - F_\rho), v^0 - v_\rho^0)_\Sigma. \end{aligned} \tag{36}$$

Since $-C_\rho$ is nonnegative, we obtain:

$$|v^0 - v_\rho^0|_\Sigma^2 \leq ((C_0 - C_\rho)v^0, v^0 - v_\rho^0)_\Sigma + (F_0 - F_\rho), v^0 - v_\rho^0)_\Sigma. \tag{37}$$

Therefore, sensitivity analysis is reduced to sensitivity analysis of operators C_ρ and F_ρ respectively.

The first step toward sensitivity analysis of optimal control is sensitivity analysis of state operator due to specified control input.

We define the control-state operator L_ρ; $L_2(\Sigma) \mapsto C(H)$ as:

$$L_\rho u \equiv \left[y_\rho, \frac{d}{dt} y_\rho \right] = Y_\rho,$$

where y_ρ satisfies (11) with $f = 0$, $y(0) = 0$, $y_t(0) = 0$.

We recall trace operator denoted by $\gamma y \equiv y|_\Gamma$. From Theorem 3 we have

$$\begin{aligned} L_\rho &\in \mathcal{L}(L_2(\Sigma); C(H)) \\ \gamma P_1 L_\rho &\in \mathcal{L}(L_2(\Sigma) \mapsto H^1(\Sigma)), \\ \gamma_R P_1 L_\rho &\in \mathcal{L}(L_2(\Sigma) \mapsto H^1(\Sigma_R)) \end{aligned} \tag{38}$$

We shall also introduce the following notation:
$K(v) \equiv [z, z_t] = Z$, where z satisfies

$$\begin{aligned} z_{tt} &= \Delta z, \\ \frac{\partial}{\partial \nu} z + \varepsilon z_t &= 0 \quad \text{on } \Sigma, \\ \frac{\partial}{\partial \nu} z + \varepsilon z_t + A_0(z) &= v \quad \text{on } \Sigma_R, \\ z(0) = z_t(0) &= 0, \end{aligned} \tag{39}$$

$$(K(v))(t) = \int_0^t S_{0,t-s} \begin{pmatrix} 0 \\ A_N N_R v(s) \end{pmatrix} ds. \tag{40}$$

We already know from Theorem 3

$$\begin{aligned} &K : L_2(\Sigma_R) \mapsto C(H), \quad \text{is bounded}, \\ &P_1 K v|_{\partial\Omega_R} \in H^1(\Sigma \cup \Sigma_R) \quad \forall v \in L_2(\Sigma_R). \end{aligned} \tag{41}$$

Lemma 3 *Let $u \in L_2(\Sigma)$. Then*

$$L_\rho u - L_0 u = \rho^2 L' u + r_1(\rho),$$

where

$$L' u = K(B(\gamma_R y_0)) = K(B(\gamma_R P_1 L_0 u)) \in C(H), \qquad L' \in \mathcal{L}(L_2(\Sigma) \mapsto C(H))$$

and

$$\frac{|r_1(\rho)|_{C(H)}}{\rho^2} \mapsto 0 \quad \forall u \in L_2(\Sigma).$$

Proof Denote $\hat{Y} \equiv L_\rho u - L_0 u$. Then $\hat{Y} = [\hat{y}, \hat{y}_t]$ satisfies

$$\begin{aligned} \hat{y}_{tt} &= \Delta \hat{y} \quad \text{in } Q_R, \\ \frac{\partial}{\partial \nu} \hat{y} + \varepsilon \hat{y}_t &= 0 \quad \text{on } \Sigma, \\ \frac{\partial}{\partial \nu} \hat{y} + \varepsilon \hat{y}_t + A_0(\hat{y}) + B(\gamma_R y_0)\rho^2 &= -B(\gamma_R \hat{y})\rho^2 + O_{\rho^4}(\gamma_R P_1 L_\rho u) \quad \text{on } \Sigma_R, \\ \hat{y}(0) = \hat{y}_t(0) &= 0, \end{aligned} \tag{42}$$

where for $u \in L_2(\Sigma)$ we have

$$\frac{|O_{\rho^4}(\gamma_R y_\rho)|_{L_2(\Sigma_R)}}{\rho^4} \leq c|\gamma_R y_\rho(t)|_{L_2(H^1(\Gamma_R))} \leq C|u|_{L_2(\Sigma)}. \tag{43}$$

This last conclusion follows from the fact that for $Y_\rho \in C(H)$ we have by Theorem 3 $\gamma_R y_\rho \in L_2(H^1(\Gamma_R))$ and the higher order term $\mathcal{O}(\rho^4)$ satisfies

$$|\mathcal{O}(\rho^4)(z)|_{L_2(\Gamma)} \leq \rho^4 |z|_{H^1(\Gamma)}.$$

From (42) and regularity Theorem 3 we obtain for all $u \in L_2(\Sigma)$

$$L'u = -K(B(\gamma_R y_0)) = -K(B(\gamma_R P_1 L_0 u)) \in C(H), \tag{44}$$

with $P_1 : \mathbb{R}^2 \mapsto \mathbb{R}^1$, $P^1(x, y) \equiv x$, so that $P_1^T u = (0, u)$. From (42)

$$\hat{Y}\rho^{-2} - L'u = K(B(\gamma_R \hat{y})) + K(\rho^{-2} O_{\rho^4}(\gamma_R P_1 L_\rho u)).$$

Using the fact that the operator B is bounded from $H^{1/2}(\Gamma) \mapsto L_2(\Gamma)$ and $K : L_2(\Sigma) \mapsto C(H)$ is bounded we obtain the estimate

$$\begin{aligned} \left|\frac{\hat{Y}}{\rho^2} - L'u\right|_{C(H)} &\leq \rho^2 \left|\frac{\hat{Y}}{\rho^2}\right|_{C(H)} + \rho^{-2}|O_{\rho^4}(\gamma_R P_1 L_\rho u)|_{L_2(\Sigma)} \\ &\leq \rho^2 \left|\frac{\hat{Y}}{\rho^2}\right|_{C(H)} + \rho^2 |u|_{L_2(\Sigma)}, \end{aligned}$$

where in the last step we have used regularity theorems and the bound in (43). Thus, taking ρ sufficiently small gives first via (44)

$$\left|\frac{\hat{Y}}{\rho^2}\right|_{C(H)} \leq C, \quad \forall u \in L_2(\Sigma).$$

Hence

$$|B(\gamma_R \hat{y})|_{C(L_2(\Gamma)} \leq C|\hat{y}|_{C(H^1(\Omega_R))} \leq C|\hat{Y}|_{C(H)}$$

and

$$|B(\gamma_R \hat{y})|_{L_2(\Sigma)} \mapsto 0, \quad \text{when } \rho \to 0,$$

consequently by (41)

$$|K(B(\gamma_R \tilde{y})|_{C(H)} \to 0, \quad \text{as } \rho \to 0.$$

This leads to

$$r_1(\rho) \equiv K(B(\gamma_R \hat{y})\rho^2 + K(O_{\rho^4}(L_\rho(u))), \tag{45}$$

where after recalling $L_\rho u|_\Gamma \in L_2(H^1(\Gamma)$ (hidden tangential-space regularity in (38)) we obtain

$$|r_1(\rho)|_{C(H)}\rho^{-2} \to 0,$$

when $\rho \to 0$ for all $u \in U_{ad}$, as desired. $\square$

PDE interpretation of the derivative $(L_T)'$ is given below.
$L'_T u = Z(T) = [z(t), z_t(t)](t = T)$ where $z(t)$ satisfies:

$$\begin{aligned} z_{tt} &= \Delta z \quad \text{in } Q_R, \\ \frac{\partial}{\partial \nu} z + \varepsilon z_t &= 0 \quad \text{on } \Sigma, \\ \frac{\partial}{\partial \nu} z + \varepsilon z_t + A_0 z &= B\gamma_R P_1 L_0 u \quad \text{on } \Sigma_R, \\ Z(0) &= 0 \quad \text{in } \Omega_R. \end{aligned} \tag{46}$$

The analysis of the adjoint operator follows from duality. We recall that the duality is always considered with respect to the norm in H topologized by $\tilde{A}_N^{1/2}$ for the first coordinate. We recall that this norm accounts for the effect of Steklov's operator.

Lemma 4 *Let $W \in H$. Then*

$$L^*_{\rho,T} W - L^*_{0,T} W = \rho^2 (L^*_T)' W + r_2(\rho),$$

where

$$[L^*_T]' = (L'_T)^* \in \mathcal{L}(H \mapsto L_2(\Sigma))$$

and

$$\frac{|r_2(\rho)|_{L_2(\Sigma)}}{\rho^2} \to 0 \quad \forall W \in H.$$

Proof The proof follows by duality. Let $W \in H$ and $u \in L_2(\Sigma)$. By Lemma 3

$$\begin{aligned} &(L^*_{\rho,T} W - L^*_{0,T} W, u)_\Sigma \\ &\quad = (W, L_{\rho,T} u - L_{0,T} u)_H = \rho^2 (W, L'_T u)_H + (W, r_1(\rho)(T))_H \\ &\quad = \rho^2 ((L'_T)^* W, u)_\Sigma + (r_2(\rho), u)_\Sigma \end{aligned}$$

where $(r_2(\rho), u)_\Sigma = (W, r_1(\rho)(T))_H \le |W|_H |r_1(\rho)(T)|_H$. Thus

$$[L^*_T]' = (L'_T)^* \in \mathcal{L}(H \mapsto L_2(\Sigma)),$$

and by using the structure of $r_1(\rho)(T)$

$$r_2(\rho) \equiv [K_T(B(\gamma_R P_1([L_\rho - L_0])]^* W + [K_T(O_{\rho^4}(\gamma_R P_1 L_\rho)]^* W,$$

which, by duality and Lemma 3, exhibits the prescribed rate of convergence. $\square$

We recall projector operators $P_i : \mathbb{R}^2 \mapsto \mathbb{R}^1, i = 1, 2$ given by

$$P_1(u, v) = u, \qquad P_2(u.v) = v.$$

Considering $P_1 : H \mapsto L_2(\Omega_R)$ we introduce the adjoint $P_1^* : L_2(\Omega) \mapsto H$ given by

$$P_1^* = P_1^T \tilde{A}_N^{-1}, \tag{47}$$

where

$$P_1^T \phi = (\phi, 0), \qquad P_2^T \phi = (0, \phi).$$

Note that with the above notation: $L'_T(u) = K_T(B\gamma_R P_1 L_0 u)$ and recalling $\gamma_R^* = A_N N_R$ we obtain

$$[K_T(B(\gamma_R P_1 L_0)]^* = L_0^* P_1^* A_N N_R B^* K_T^*.$$

Since $K_T(v) = (Kv)(T)$

$$K_T(v) = \int_0^T S_{0,T-t} \begin{pmatrix} 0 \\ A_N N_R v(t) \end{pmatrix} dt$$

we have

$$[K_T]^* W(t) = \gamma_R P_2 S^*_{0,T-t} W$$

and by (47)

$$(L'_T)^* W = L_0^* P_1^T \tilde{A}_N^{-1} A_N N_R B^* \gamma_R P_2 S^*_{0,T-\cdot} W. \tag{48}$$

The above can be interpreted as follows:
Let $\Psi(t) \equiv S^*_{0,T-t} W = [\psi_1(t), \psi_2(t)]$. Then $\gamma_R P_2 S^*_{0,T-t} W = \gamma_R \psi_2(t)$.
Since $(L_0^* F)(t) = N^* A_N P_2 \int_t^T S^*_{0,s-t} F(s) ds$ we obtain the following lemma:

Lemma 5

$$v(t) \equiv (L'_T)^* W(t) = (L_0^* P_1^T \tilde{A}_N^{-1} A_N N B^* \gamma_R \psi_t(\cdot))(t)$$
$$= \gamma P_2 \int_t^T S^*_{0,s-t} \begin{pmatrix} \tilde{A}_N^{-1} A_N N B^* \gamma_R \psi_t(s) \\ 0 \end{pmatrix} ds$$

has (after some calculations) the following representation

$$v(t) = \gamma \phi(t),$$

where

$$\phi_{tt} = \Delta \phi$$

$$\frac{\partial}{\partial \nu} \phi - \varepsilon \phi_t = 0 \quad \text{on } \Gamma,$$

$$
\begin{aligned}
\frac{\partial}{\partial \nu}\phi - \varepsilon\phi_t + A_0\phi &= B^*\gamma_R\psi_t \quad \text{on } \Gamma_R,\\
\phi(T) = \phi_t(T) &= 0,\\
\psi_{tt} &= \Delta\psi,\\
\frac{\partial}{\partial \nu}\psi - \varepsilon\psi_t &= 0 \quad \text{on } \Gamma,\\
\frac{\partial}{\partial \nu}\psi - \varepsilon\psi_t + A_0\psi &= 0 \quad \text{on } \Gamma_R,\\
\Psi(T) &= W
\end{aligned}
$$

and Ψ *satisfies the following energy inequality*

$$
|\Psi(t)|_H^2 + \varepsilon\int_0^t |\psi_t(s)|^2_{\Gamma\cup\Gamma_R} + \varepsilon|\psi|^2_{H^1(\Gamma\cup\Gamma_R)}ds \le C|W|_H^2. \tag{49}
$$

We note that the regularity of the system above is not obvious, when considering PDE representation (since $(B^*\gamma_R\psi_t \in L_2(0,T;H^{-1/2}(\Gamma_R)))$. However, this result can be easily deduced from semigroup argument applied to integral representation of $v(t)$ (as given in the Lemma) along with hidden regularity given in Theorem 3.

The above results lead to the sensitivity analysis of operators C_ρ and F_ρ introduced in (28)

Lemma 6 *Let* $u \in L_2(\Sigma)$, $Y_0, Y_D \in H$. *The following expansion holds*

$$
\begin{aligned}
C_\rho u - C_0 u &= \rho^2 C' u + o(\rho^2),\\
C' &= \frac{1}{\alpha}L_T^*\mathcal{R}^*\mathcal{R}L_T' + \frac{1}{\alpha}(L_T^*)'\mathcal{R}^*\mathcal{R}L_T \in \mathcal{L}(L_2(\Sigma)),\\
F_\rho - F_0 &= F'\rho^2 + o(\rho^2),\\
F' &= \frac{1}{\alpha}L_T^*\mathcal{R}^*\mathcal{R}S_{0,T}'Y_0 + \frac{1}{\alpha}(L_T^*)'\mathcal{R}^*\mathcal{R}[S_{0,T}Y_0 - Y_D] \in L_2(\Sigma).
\end{aligned}
$$

We establish the directional differentiability of the optimal controls with respect to the parameter $\rho = 0^+$.

Theorem 6 *We have the following expansion of the optimal control in* $L_2(\Gamma \times (0,T))$, *with respect to the small parameter,*

$$
v_\rho^0 = v^0 + \rho^2 q + o(\rho^2) \tag{50}
$$

for $\rho > 0$.

Proof From (36) and formulas in Lemma 6 we obtain

$$
|v_\rho^0 - v^0|_\Sigma \le |(C_0 - C_\rho)v^0|_\Sigma + |F_0 - F_\rho|_\Sigma \le C\rho^2 \tag{51}
$$

for $Y(0) \in H$. Therefore, there exists $q \in L_2(\Sigma)$ such that

$$v_\rho = v_0 + \rho^2 q + o(\rho^2). \tag{52}$$

In order to find the representation for the Gateau differential, we will be using representation of optimal controls v_ρ^0 given in (27). We write

$$\begin{aligned} v_\rho^0 - v^0 &= P_{U_{ad}}[-C_\rho v_\rho^0 + F_\rho] - P_{U_{ad}}[-C_0 v^0 + F_0] \\ &= P'_{U_{ad}}(-C_0 v^0 + F_0)[(-C_\rho + C_0)v^0 - C_\rho(v_\rho^0 - v^0) + F_\rho - F_0] + o(\rho^2) \\ &= [P_{U_{ad}}]'(-C_0 v^0 + F_0)[(-C_0 q - C_0' v^0 + F'] + o(\rho^2), \end{aligned} \tag{53}$$

where we have been using the fact that P_{ad} is Lipschitz on $L_2(\Sigma)$. Comparing leading terms in (52) and the last equality we obtain

$$q = [P_{U_{ad}}]'(-C_0 v^0 + F_0)[-C_0 q - C_0' v^0 + F']. \qquad \square$$

Moreover, we assume that ρ is a sufficiently small. By exploiting explicit representations of C_ρ operators the function q can be written as

$$q = [P_{U_{ad}}]'(-C_0 v^0 + F_0)[-L_T^* \mathcal{R}^* \mathcal{R} W(T) - (L_T^*)' \mathcal{R}^* \mathcal{R}(Y^0(T) - Y_D)], \tag{54}$$

where $Y^0(t)$ is the optimal trajectory and $W(t) = [w(t), w_t(t)]$ satisfies the state equation

$$\left.\begin{array}{ll} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = 0 & \text{in } \Omega_R \times (0,T), \\ \dfrac{\partial w}{\partial \eta} + \varepsilon w_t = q & \text{on } \Gamma \times (0,T), \\ \dfrac{\partial w}{\partial \eta} + \varepsilon w_t + A_0(w) + B(\gamma_R y^0) = 0 & \text{on } \Gamma_R \times (0,T), \\ w(x,0) = 0 & \text{in } \Omega_R, \\ \dfrac{\partial w}{\partial t}(x,0) = 0 & \text{in } \Omega_R. \end{array}\right\} \tag{55}$$

By using Regularity Theorem 3 one easily obtains:

Lemma 7 *Solution w satisfies*:

$$|w(t)|^2_{1,\Omega_R} + |w_t|^2_{0,\Omega_R} + \int_0^t |w_t|^2_{0,\Gamma} \le \frac{c}{\varepsilon} \int_0^t [|y^0|^2_{1/2,\Gamma_R} + |q|^2_\Gamma]. \tag{56}$$

We shall also introduce the performance functional

$$I(u) = \frac{1}{2} |\mathcal{R} W(T,x)|^2_H dx + \frac{\alpha}{2} \int_0^T \int_\Gamma |u|^2 dx dt, \tag{57}$$

and the adjoint equation

$$\left.\begin{aligned} &\frac{\partial^2 z}{\partial t^2} + \Delta z = 0 && \text{in } \Omega_R \times (0, T), \\ &\frac{\partial z}{\partial \eta} - \varepsilon z_t = 0 && \text{on } \Gamma \times (0, T), \\ &\frac{\partial z}{\partial \eta} - \varepsilon z_t + A_0(z) + B^*(\gamma_R p^0) = 0 && \text{on } \Gamma_R \times (0, T), \\ &Z(T) = W(T) && \text{in } \Omega_R \end{aligned}\right\} \tag{58}$$

and $P^0(t) = \Psi(t)$ satisfies the adjoint equation in Lemma 5 with $\Psi(T) \equiv \mathcal{R}^*\mathcal{R}(Y^0(T) - Y_d)$ (see Remark 4).

Lemma 8 *Regularity for* z.

$$\begin{aligned} &|z(t)|^2_{1,\Omega_R} + |z_t(t)|^2_{0,\Omega_R} + \varepsilon \int_0^T |z_t|^2_{\Gamma\cup\Gamma_R} ds \\ &\quad \le |w(T)|^2_{1,\Omega_R} + |w_t(T)|^2_{0,\Omega_R} + \frac{c}{\varepsilon}\int_0^T |p^0|^2_{1/2,\Gamma_R} \end{aligned} \tag{59}$$

With the above notation, the formula (54) can be written as

$$q = [P_{U_{ad}}]'(-C_0 v_0 + F_0)[-\gamma z_t].$$

Then, the optimal control q is characterized by

$$\begin{aligned} &(\mathcal{R}W(x, T; q), \mathcal{R}(W(x, T; u) - W(x, T; q)) + \int_0^T \int_\Gamma q(u - q) dx dt \ge 0 \\ &\forall u \in S_{ad}, \end{aligned} \tag{60}$$

where: S_{ad} is a set of admissible controls such that

$$\begin{aligned} S_{ad} = \Big\{ u \in L_2(\Gamma \times (0, T)) \Big| \;& u(x, t) \ge 0 \text{ on the set } E_0 = \{(x, t)|v^0(x, t) = 0\}, \\ & u(x, t) < 0 \text{ on the set } E_1 = \{(x, t)|v^0(x, t) = 1\}, \\ & \int_0^T \int_\Gamma (p_t^0 + \alpha v^0) u dx dt = 0 \Big\}, \end{aligned} \tag{61}$$

where: p_t^0 is a adjoint state for $\rho = 0$, v^0 is a optimal solution for $\rho = 0$ such that $0 \le v^0(x, t) \le 1$.
We simplify (60) using the adjoint equation (58). After transformations we obtain the following maximum condition

$$\int_0^T \int_\Gamma (z_t + \alpha q)(u - q) dx dt \ge 0 \quad \forall u \in S_{ad}. \tag{62}$$

Theorem 7 *For the hyperbolic problem*

$$\left.\begin{array}{ll}
\dfrac{\partial^2 w}{\partial t^2} - \Delta w = 0 & \text{in } \Omega_R \times (0,T), \\
\dfrac{\partial w}{\partial \eta} + \varepsilon w_t = u & \text{on } \Gamma \times (0,T), \\
\dfrac{\partial w}{\partial \eta} + \varepsilon w_t + A_0(w) + B(y^0) = 0 & \text{on } \Gamma_R \times (0,T), \\
w(x,0) = 0 & \text{in } \Omega_R, \\
\dfrac{\partial w}{\partial t}(x,0) = 0 & \text{in } \Omega_R,
\end{array}\right\} \tag{63}$$

with the performance functional (57) *with* $w(T) \in L_2(\Omega_R)$ *and* $\alpha > 0$, *and with constraints on the control* (61), *there exists a unique optimal control* q *which satisfies the maximum condition* (62).

8 Conclusions

The approximation used in this paper is obtained from the asymptotic analysis of the energy functional for the stationary problem. The energy functional is written for the Laplacian. The approximation is governed by a small parameter which describes singular perturbations of the domain. Such perturbations can be considered as a defect in a real world. The results presented are obtained for a defect in the form of a circular hole. For applications in the structural mechanics, the Laplace operator is replaced by a system of linear elasticity, and the defects can be some cracks, cavities or some other singularities with geometrical boundaries. Our method applies for such situations as well. The difference encountered is that instead of a scalar wave equation one should consider dynamic system of elasticity. There are several works [3, 10, 35, 47] which furnish the same kind of approximation for the energy functional, with the explicit expressions for the first order term, which can be used in our framework. The only difficulty is that instead of scalar problem, vectorial system of elasticity should be considered. This is possible, since all the ingredients are in place, including the hidden regularity [12, 13, 15].
In this paper the mixed initial-boundary value problems of hyperbolic type is considered. One could also consider similar optimal control problems defined for time delay hyperbolic systems. The ideas mentioned above will be developed in forthcoming papers.

Appendix: Asymptotic analysis with Steklov-Poincaré operator

For the convenience of the reader we provide the asymptotic analysis used for the elliptic problem in a singularly perturbed geometrical domain.

9.1 Steklov-Poincaré operator in the domain $C(R, \rho) = \mathbb{B}_R \setminus \overline{\mathbb{B}}_\rho$

The main result, we obtain is based on the expansion (66) of the Steklov-Poincaré operator with respect to the parameter ρ. The expansion is established in Sect. 9.1 by an application of elementary Fourier analysis.

We consider the mapping $A_\rho : H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$ defined by the boundary value problem

$$\begin{aligned} -\Delta w_\rho &= 0 \quad \text{in } C(R, \rho), \\ w_\rho &= v \quad \text{on } \Gamma_R = \partial\mathbb{B}_R, \\ \partial_n w_\rho &= 0 \quad \text{on } \Gamma_\rho, \end{aligned}$$

and we set

$$\partial_n w_\rho = A_\rho(v) \quad \text{on } \Gamma_R.$$

By an elementary evaluation of the associated energy functional, we refer the reader to Sect. 9.2 for details, taking into account the relation which follows by integration by parts, we find that

$$\langle A_\rho(v), v\rangle_{\Gamma_R} = \int_{C(R,\rho)} |\nabla w_\rho(v; x)|^2 dx,$$

and for $\rho > 0$, ρ small enough,

$$\int_{C(R,\rho)} |\nabla w_\rho(v; x)|^2 dx = \int_{\mathbb{B}_R} |\nabla w_0(v; x)|^2 dx + \rho^2 b(\Gamma_R; v, v) + O(\rho^4), \tag{64}$$

where w_0 denotes the solution in the intact domain without any hole, and the remainder $O(\rho^4)$ is uniformly bounded on bounded sets in the space $H^{1/2}(\Gamma_R)$.

By the properties of harmonic functions the second term can be represented in two spatial dimensions in the equivalent form of a line integral over the circle $\Gamma_R = \{x : |x - \mathcal{O}| = R\}$ with the centre at the origin $\mathcal{O}$

$$b(\Gamma_R; u, u) = -\frac{1}{2\pi R^6}\left[\left(\int_{\Gamma_R} ux_1\, ds\right)^2 + \left(\int_{\Gamma_R} ux_2\, ds\right)^2\right]. \tag{65}$$

Therefore, we obtain the expansion

$$A_\rho = A_0 + \rho^2 B + O(\rho^4), \tag{66}$$

in the operator norm $\mathcal{L}(H^{1/2}(\Gamma_R); H^{-1/2}(\Gamma_R))$.

9.2 Compactness of asymptotic energy expansion

In this section we provide a simple proof for (66) which is equivalent [42, 46] to (64).

Remark 8 We refer e.g., to [42, 46] for the derivation of topological derivatives of the energy functionals for a class of elliptic boundary value problems including the linear elasticity. However, in our applications the form of the first term of the energy expansion should have some specific properties, therefore, we use the equivalent form of the topological derivative of the energy, which is bounded in the Sobolev energy space. The same property is required in the topological sensitivity analysis of the contact problems [46, 47].

Let $0 \in \Omega$ and $\mathbb{B}_R$ be a ball around 0, while $C(\rho, R)$ is a ring $C(\rho, R) = \{\mathbf{x} \mid \rho < \|\mathbf{x}\| < R\}$ with inner boundary Γ_ρ and outer boundary Γ_R. Additionally we use notation $\Omega_R = \Omega \setminus \overline{\mathbb{B}}_R$, We consider functions $u \in H^1(\Omega_R)$ with traces (still denoted by u) on Γ_R belonging to $H^{1/2}(\Gamma_R)$. The following implication is true

$$\|u\|_{H^1(\Omega_R)} \leq \Lambda_0 \quad \Longrightarrow \quad \|u\|_{H^{1/2}(\Gamma_R)} \leq \Lambda(R),$$

and since R is fixed, we shall omit it, writing Λ instead of $\Lambda(R)$ (by Λ we shall denote generic constant depending only on Λ_0). Finally, we denote by (r, ϕ) spherical coordinates around 0.

From the fact that $u \in H^{1/2}(\Gamma_R)$ follows the existence of the Fourier series expansion in terms of ϕ:

$$u = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \sin k\phi + b_k \cos k\phi)$$

with coefficients satisfying

$$\sum_{k=1}^{\infty}\sqrt{1+k^2}(a_k^2 + b_k^2) \leq \Lambda.$$

This implies two important for us properties:

$$\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \Lambda, \qquad \sum_{k=1}^{\infty}k(a_k^2 + b_k^2) \leq \Lambda. \tag{67}$$

Now we shall consider in $\mathbb{B}_R$ the solution of the Laplace equation with Dirichlet boundary condition on Γ_R coinciding with u, denoted by w, and the solution of the same equation in $C(\rho, R)$, with the same condition on Γ_R and homogeneous Neumann condition on Γ_ρ, denoted by w_ρ. We define energies

$$E(u) = \int_{\mathbb{B}_R}\|\nabla w\|^2\, dS, \qquad E_\rho(u) = \int_{C(\rho,R)}\|\nabla w_\rho\|^2\, dS \tag{68}$$

which depend on u via boundary conditions. Our goal is to prove that E_ρ has an expansion in which the remainder is uniformly bounded. More precisely this can be expressed as follows.

Lemma 9 *The energy $E_\rho(u)$ admits the expansion, for $\rho > 0$, $\rho > 0$ small enough,*

$$E_\rho(u) = E(u) + \rho^2 b(\Gamma_R; u, u) + \mathcal{R}(u),$$

where

$$|\mathcal{R}(u)| \leq \Lambda \rho^4$$

uniformly on any fixed compact set in $H^1(\Omega_R)$, i.e. Λ depends on this set only.

Proof Since any compact set may be covered by finite number of balls, it is enough to prove the Lemma for a fixed ball in $H^1(\Omega_R)$. We may therefore assume that (67) holds. The proof will consist in obtaining explicit formulas for w and w_ρ as series, using the well known methods, similarly as in [42]. Then the energies may be computed exactly and the desired property of the remainder $\mathcal{R}(u)$ proven.

Constructing w from the Fourier series of its boundary condition we get

$$w = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k (a_k \sin k\phi + b_k \cos k\phi). \tag{69}$$

Similarly, for w_ρ in $C(\rho, R)$ holds

$$w_\rho = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} v_k(\rho)(a_k \sin k\phi + b_k \cos k\phi),$$

where

$$v_k(\rho) = A_k r^k + B_k \frac{1}{r^k} \ ,$$

and

$$A_k R^k + B_k \frac{1}{R^k} = 1,$$

$$k A_k \rho^{k-1} - B_k \frac{1}{\rho^{k+1}} = 0.$$

Hence

$$A_k = \frac{R^k}{R^{2k} + \rho^{2k}}, \qquad B_k = A_k \rho^{2k}$$

and finally

$$v_k(\rho) = \frac{r^k}{R^k} + \frac{\rho^{2k}}{R^{2k} + \rho^{2k}} \left(\frac{R^k}{r^k} - \frac{r^k}{R^k}\right).$$

Substituting this into the expansion for w_ρ gives

$$w_\rho = w + \sum_{k=1}^{\infty} \frac{\rho^{2k}}{R^{2k} + \rho^{2k}} \left(\frac{R^k}{r^k} - \frac{r^k}{R^k}\right)(a_k \sin k\phi + b_k \cos k\phi) := w + z_\rho \tag{70}$$

with

$$z_\rho = \sum_{k=1}^{\infty} \frac{\rho^{2k}}{R^{2k}+\rho^{2k}} \left(\frac{R^k}{r^k} - \frac{r^k}{R^k} \right) (a_k \sin k\phi + b_k \cos k\phi)\,. \tag{71}$$

For any function f we denote by $f_{/r}$, $f_{/\phi}$ the partial derivatives with respect to the polar coordinates, thus the norm of the gradient with respect to the cartesian coordinates takes the form

$$\|\nabla f\|^2 = f_{/r}^2 + \frac{1}{r^2} f_{/\phi}^2$$

and therefore

$$\begin{aligned}
E_\rho(u) &= \int_{C(\rho,R)} \|\nabla w + \nabla z_\rho\|^2 \, dS \\
&= E(u) + \int_{C(\rho,R)} [(z_{\rho/r})^2 + \frac{1}{r^2}(z_{\rho/\phi})^2] \, dS \\
&\quad + 2\int_{C(\rho,R)} [w_{/r} z_{\rho/r} + \frac{1}{r^2} w_{/\phi} z_{\rho/\phi}] \, dS \\
&\quad - \int_{\mathbb{B}_\rho} \|\nabla w\|^2 \, dS \\
&:= E(u) + I_1 + I_2 + I_3.
\end{aligned} \tag{72}$$

Now we have

$$z_{\rho/r} = -\sum_{k=1}^{\infty} \frac{\rho^{2k}}{R^{2k}+\rho^{2k}} k \frac{1}{r} \left(\frac{R^k}{r^k} - \frac{r^k}{R^k} \right) (a_k \sin k\phi + b_k \cos k\phi),$$

$$\frac{1}{r} z_{\rho/\phi} = \sum_{k=1}^{\infty} \frac{\rho^{2k}}{R^{2k}+\rho^{2k}} k \frac{1}{r} \left(\frac{R^k}{r^k} - \frac{r^k}{R^k} \right) (a_k \cos k\phi - b_k \sin k\phi).$$

After taking into account the orthogonality of trigonometric functions on $[0, 2\pi]$ and integrating with respect to ϕ one gets

$$I_1 = \frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\rho^{2k}}{R^{2k}+\rho^{2k}} \right)^2 k^2 (a_k^2 + b_k^2) \cdot I_{\rho k},$$

where

$$I_{\rho k} = \int_\rho^R \left[\left(\frac{R^k}{r^{k+1}} + \frac{r^{k-1}}{R^k} \right)^2 + \left(\frac{R^k}{r^{k+1}} - \frac{r^{k-1}}{R^k} \right)^2 \right] r dr.$$

Since, after integration

$$I_{\rho k} = \frac{1}{k} \left[\frac{R^{2k}}{\rho^{2k}} - \frac{\rho^{2k}}{R^{2k}} \right]$$

we obtain

$$I_1 = \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{\rho^{2k}}{R^{2k}+\rho^{2k}}\right)^2 k(a_k^2+b_k^2)\left[\frac{R^{2k}}{\rho^{2k}}-\frac{\rho^{2k}}{R^{2k}}\right]. \tag{73}$$

In order to compute I_2 we observe that

$$w_{/r} = \sum_{k=1}^{\infty} k\frac{r^{k-1}}{R^k}(a_k \sin k\phi + b_k \cos k\phi),$$

$$w_{/\phi} = \sum_{k=1}^{\infty} k\frac{r^{k}}{R^k}\,(a_k \cos k\phi - b_k \sin k\phi)$$

and after easy computations

$$I_2 = 0. \tag{74}$$

There remains I_3. It has the form

$$I_3 = -\int_{\mathbb{B}_\rho}\|\nabla w\|^2\,dS = -\int_{\mathbb{B}_\rho}\left(w_{/r}^2 + \frac{1}{r^2}w_{/\phi}^2\right)dS,$$

and in view of the written above expressions for $w_{/r}$, $w_{/\phi}$ and orthogonality

$$I_3 = -\frac{1}{\pi}\sum_{k=1}^{\infty}k^2(a_k^2+b_k^2)\int_0^\rho \frac{r^{2k-2}}{R^{2k}}\,r\,dr.$$

Finally

$$I_3 = -\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\rho^{2k}}{R^{2k}}k(a_k^2+b_k^2). \tag{75}$$

There remains to observe that, for $\rho \le \frac{1}{2}R$,

$$\frac{\rho^{2k}}{R^{2k}+\rho^{2k}} = \frac{\rho^{2k}}{R^{2k}}\left[1-\frac{\rho^{2k}}{R^{2k}}+\left(\frac{\rho^{2k}}{R^{2k}}\right)^2+\dots\right].$$

Collecting the formulas (73), (74), (75) we may single out the first terms containing ρ^2 and the rest, which in view of the regularity of boundary conditions and implied by this inequalities (67) is uniformly bounded by $\Lambda\rho^4$. $\square$

It is worth noticing that as a byproduct of this proof we have once again derived the formula for energy correction $b(\Gamma_R; u, u)$.

References

1. Ammari, H.: An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume. SIAM J. Control Optim. **41**, 1194–1211 (2002)

2. Berezin, F.A., Faddeev, L.D.: Remark on the Schrödinger equation with singular potential. Dokl. Akad. Nauk SSSR **137**, 1011–1014 (1961). (Engl transl. in Soviet Math. Dokl. **2**, 372–375 (1961))
3. Cardone, G., Nazarov, S.A., Sokołowski, J.: Asymptotic analysis, polarization matrices and topological derivatives for piezoelectric materials with small voids. SIAM J. Control Optim. **48**, 3925–3961 (2010)
4. Frémiot, G., Horn, W., Laurain, A., Rao, M., Sokołowski, J.: On the analysis of boundary value problems in nonsmooth domains. Dissertat. Math. **462** (2009)
5. Hendrickson, E., Lasiecka, I.: Numerical approximations and regularizations of Riccati equations arising in hyperbolic dynamics with unbounded control operators. Comput. Optim. Appl. **2**, 343–390 (1993)
6. Hendrickson, E., Lasiecka, I.: Finite dimensional approximations of boundary control problems arising in partially observed hyperbolic systems. Dyn. Contin. Discrete Impuls. Syst. **1**, 101–142 (1995)
7. Hormander, L.: The Analysis of Linear Partial Differential Operators, vol. III. Springer, Berlin (1985)
8. Jackowska, L., Sokołowski, J., Żochowski, A., Henrot, A.: On numerical solution of shape inverse problems. Comput. Optim. Appl. **23**, 231–255 (2002)
9. Karpeshina, Yu.E., Pavlov, B.S.: Interaction of the zero radius for the biharmonic and the polyharmonic equation. Mat. Zametki **40**, 49–59 (1986) (in Russian)
10. Khludnev, A., Novotny, A., Sokołowski, J., Żochowski, A.: Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions. J. Mech. Phys. Solids **57**, 1718–1732 (2009)
11. Kurasov, P., Posilicano, A.: Finite speed of propagation and local boundary conditions for wave equations with point interactions. Proc. Am. Math. Soc. **133**, 3071–3078 (2005)
12. Lagnese, J.: Stabilization of Thin Plates. SIAM, Philadelphia (1989)
13. Lagnese, J., Leugering, G.: Domain Decomposition Methods in Optimal Control of Partial Differential Equations. Birkhäuser, Basel (2004)
14. Landkof, N.S.: Fundamentals of Modern Potential Theory. Nauka, Moscow (1966) (in Russian)
15. Lasiecka, I.: Control Theory for Coupled PDE's. CBMS-SIAM-NSF Lecture Notes. SIAM, Philadelphia (2002)
16. Lasiecka, I., Lions, J.L., Triggiani, R.: Non-homogeneous boundary value problems for second order hyperbolic operators. J. Math. Pures Appl. **65**, 149–192 (1986)
17. Lasiecka, I., Sokołowski, J.: Sensitivity analysis of constrained optimal control problem for wave equation. SIAM J. Control Optim. **29**, 1128–1149 (1991)
18. Lasiecka, I., Triggiani, R.: Sharp regularity results for second order hyperbolic equations of Neumann type. Ann Mat. Pura Appl. **CLVII**, 1128–1149 (1990)
19. Lasiecka, I., Triggiani, R.: Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions. J. Differ. Equ. **94**, 112–164 (1991)
20. Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations. Cambridge University Press, Cambridge (2000)
21. Lasiecka, I., Triggiani, R., Zhang, X.: In: Differential Geometric Methods in the Control of Partial Differential Equations. Contemporary Math., pp. 227–327. AMS, Providence (2000)
22. Lions, J.L.: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin (1971)
23. Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications, vols. 1 and 2. Springer, Berlin (1972)
24. Malanowski, K.: Stability and sensitivity analysis for optimal control problems with control-state constraints. Dissertat. Math. **394** (2001)
25. Malanowski, K., Sokołowski, J.: Sensitivity of solutions to convex, control constrained optimal control problems for distributed parameter systems. J. Math. Anal. Appl. **120**, 240–263 (1986)
26. Maz'ya, V., Nazarov, S., Plamenevskij, B.: Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, vol. 1. Birkhäuser, Basel (2000)
27. Nazarov, S.A.: Self-adjoint extensions of the Dirichlet problem operator in weighted function spaces. Mat. Sb. **137**, 224–241 (1988) (English transl.: Math. USSR Sbornik **65**, 229–247 (1990))
28. Nazarov, S.A.: Asymptotic conditions at a point, self-adjoint extensions of operators and the method of matched asymptotic expansions. Am. Math. Soc. Transl. **198**, 77–125 (1999)
29. Nazarov, S.A., Slutskij, A.S., Sokołowski, J.: Topological derivative of the energy functional due to formation of a thin ligament on a spatial body. Acta Univ. Lodz., Folia Math. **12**, 39–72 (2005)
30. Nazarov, S.A., Sokołowski, J.: Asymptotic analysis of shape functionals. J. Math. Pures Appl. **82**, 125–196 (2003)

31. Nazarov, S.A., Sokołowski, J.: Self-adjoint extensions of differential operators in application to shape optimization. C. R., Méc. **331**, 667–672 (2003)
32. Nazarov, S.A., Sokołowski, J.: The topological derivative of the Dirichlet integral due to formation of a thin ligament. Sib. Math. J. **45**, 341–355 (2004)
33. Nazarov, S.A., Sokołowski, J.: Self-adjoint extensions for elasticity system in application to shape optimization. Bull. Pol. Acad. Sci., Math. **52**, 237–248 (2004)
34. Nazarov, S.A., Sokołowski, J.: Self-adjoint extensions for the Neumann Laplacian and applications. Acta Math. Sin. **22**, 879–906 (2006)
35. Nazarov, S.A., Specovius-Neugebauer, M., Sokołowski, J.: Polarization matrices in anisotropic elasticity. Asymptot. Anal. **68**, 189–221 (2010)
36. Pavlov, B.S.: The theory of extension and explicitly soluble models. Usp. Mat. Nauk **42**, 99–131 (1987) (Engl. transl. Soviet Math. Surveys **42**, 127–168 (1987))
37. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Princeton University Press, Princeton (1951)
38. Sakamoto, R.: Hyperbolic Boundary Value Problems. Cambridge University Press, Cambridge (1982)
39. Sokołowski, J.: Differential stability of solutions to constrained optimization problems. Appl. Math. Optim. **13**, 97–115 (1985)
40. Sokołowski, J.: Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. SIAM J. Control Optim. **25**, 1542–1556 (1987)
41. Sokołowski, J.: Shape sensitivity analysis of boundary optimal control problems for parabolic systems. SIAM J. Control Optim. **26**, 763–787 (1988)
42. Sokołowski, J., Żochowski, A.: On topological derivative in shape optimization. SIAM J. Control Optim. **37**, 1251–1272 (1999)
43. Sokołowski, J., Żochowski, A.: Topological derivative for optimal control problems. Control Cybern. **28**, 611–626 (1999)
44. Sokołowski, J., Żochowski, A.: Topological derivatives for elliptic problems. Inverse Probl. **15**, 123–134 (1999)
45. Sokołowski, J., Żochowski, A.: Optimality conditions for simultaneous topology and shape optimization. SIAM J. Control Optim. **42**, 1198–1221 (2003)
46. Sokołowski, J., Żochowski, A.: Modelling of topological derivatives for contact problems. Numer. Math. **102**, 145–179 (2005)
47. Sokołowski, J., Żochowski, A.: Asymptotic analysis and topological derivatives for shape and topology optimization of elasticity problems in two spatial dimensions. Eng. Anal. Bound. Elem. **32**, 533–544 (2008)
48. Sokołowski, J., Zolesio, J.-P.: Introduction to Shape Optimization. Shape Sensitivity Analysis. Springer, Berlin (1992)