# AN ALTERNATING DIRECTION METHOD WITH INCREASING PENALTY FOR STABLE PRINCIPAL COMPONENT PURSUIT 

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#### Abstract

The stable principal component pursuit (SPCP) is a non-smooth convex optimization problem, the solution of which enables one to reliably recover the low rank and sparse components of a data matrix which is corrupted by a dense noise matrix, even when only a fraction of data entries are observable. In this paper, we propose a new algorithm for solving SPCP. The proposed algorithm is a modification of the alternating direction method of multipliers (ADMM) where we use an increasing sequence of penalty parameters instead of a fixed penalty. The algorithm is based on partial variable splitting and works directly with the non-smooth objective function. We show that both primal and dual iterate sequences converge under mild conditions on the sequence of penalty parameters. To the best of our knowledge, this is the first convergence result for a variable penalty ADMM when penalties are not bounded, the objective function is non-smooth and its sub-differential is not uniformly bounded. Using partial variable splitting and adopting an increasing sequence of penalty multipliers, together, significantly reduce the number of iterations required to achieve feasibility in practice. Our preliminary computational tests show that the proposed algorithm works very well in practice, and outperforms ASALM, a state of the art ADMM algorithm for the SPCP problem with a constant penalty parameter.


1. Introduction. Suppose a matrix $D \in \mathbb{R}^{m \times n}$ is of the form $D=L^{0}+S^{0}$, where $L^{0}$ is a low-rank matrix, i.e. $\operatorname{rank}\left(L^{0}\right) \ll \min \{m, n\}$, and $S^{0}$ is a sparse matrix. The matrix $S^{0}$ is interpreted as gross errors in the measurement of the low rank matrix $L^{0}$. Wright et al. 31, Candés et al. 88 and Chandrasekaran et al. 9 proposed recovering the low-rank $L^{0}$ and sparse $S^{0}$ by solving the principal component pursuit (PCP) problem

$$
\begin{equation*}
\min _{L \in \mathbb{R}^{m \times n}}\|L\|_{*}+\xi\|D-L\|_{1} \tag{1.1}
\end{equation*}
$$

where $\xi=\frac{1}{\sqrt{\max \{m, n\}}}$. Here the nuclear norm $\|L\|_{*}:=\sum_{i=1}^{r} \sigma_{i}(L)$, where $\left\{\sigma_{i}(L)\right\}_{i=1}^{r}$ denotes the singular values of $L \in \mathbb{R}^{m \times n}$, and the $\ell_{1}$-norm $\|L\|_{1}:=\sum_{i=1}^{m} \sum_{j=1}^{n}\left|L_{i j}\right|$.

Theorem 1.1. [8] Suppose $D=L^{0}+S^{0} \in \mathbb{R}^{m \times n}$. Let $r=\operatorname{rank}\left(L^{0}\right)$ and $L^{0}=U \Sigma V^{T}=\sum_{i=1}^{r} \sigma_{i} u_{i} v_{i}^{T}$ denote the singular value decomposition (SVD) of $L^{0}$. Suppose there exists $\mu>0$ such that

$$
\begin{equation*}
\max _{i}\left\|U^{T} e_{i}\right\|_{2}^{2} \leq \frac{\mu r}{m}, \quad \max _{i}\left\|V^{T} e_{i}\right\|_{2}^{2} \leq \frac{\mu r}{n}, \quad\left\|U V^{T}\right\|_{\infty} \leq \sqrt{\frac{\mu r}{m n}} \tag{1.2}
\end{equation*}
$$

where $e_{i}$ denotes the $i$-th unit vector, and the non-zero components of the sparse matrix $S^{0}$ are chosen uniformly at random. Then there exist constants $c, \rho_{r}$, and $\rho_{s}$, such that the solution of the PCP problem (1.1) exactly recovers $L^{0}$ and $S^{0}$ with probability of at least $1-c n^{-10}$, provided

$$
\begin{equation*}
\operatorname{rank}\left(L^{0}\right) \leq \rho_{r} m \mu^{-1}(\log (n))^{-2} \quad \text { and } \quad\left\|S^{0}\right\|_{0} \leq \rho_{s} m n \tag{1.3}
\end{equation*}
$$

where the $\ell_{0}$-norm $\left\|S^{0}\right\|_{0}$ denotes the number of non-zero components of the matrix $S^{0}$.
Now, suppose the data matrix $D$ is of the form $D=L^{0}+S^{0}+N^{0}$ such that $L^{0}$ is a low-rank matrix, $S^{0}$ is a sparse gross "error" matrix, $N^{0}$ is a dense noise matrix with $\left\|N^{0}\right\|_{F} \leq \delta$, where the Frobenius norm $\|Z\|_{F}:=\sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} Z_{i j}^{2}}$. In [32], it was shown that it was still possible to recover the low-rank and sparse components $\left(L^{0}, S^{0}\right)$ of $D$ by solving the stable principal component pursuit (SPCP) problem

$$
\begin{equation*}
\min _{L, S \in \mathbb{R}^{m \times n}}\left\{\|L\|_{*}+\xi\|S\|_{1}:\|L+S-D\|_{F} \leq \delta\right\} \tag{1.4}
\end{equation*}
$$

Theorem 1.2. 32/ Suppose $D=L^{0}+S^{0}+N^{0}$, where $L^{0} \in \mathbb{R}^{m \times n}$ with $m<n$ satisfies (1.2) for some $\mu>0$, and the non-zero components of the sparse matrix $S^{0}$ are chosen uniformly at random. Suppose $L^{0}$ and $S^{0}$ satisfy (1.3). Then for any $N^{0}$ such that $\left\|N^{0}\right\|_{F} \leq \delta$, the solution $\left(L^{*}, S^{*}\right)$ to the SPCP problem (1.4) satisfies $\left\|L^{*}-L^{0}\right\|_{F}^{2}+\left\|S^{*}-S^{0}\right\|_{F}^{2} \leq C m n \delta^{2}$ for some constant $C$ with high probability.

[^0]In many applications, some of the entries of $D$ in (1.4) may not be available. Let $\Omega \subset\{i: 1 \leq i \leq$ $m\} \times\{j: 1 \leq j \leq n\}$ be the index set of the observable entries of $D$. Define the projection operator $\pi_{\Omega}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ as follows

$$
\left(\pi_{\Omega}(L)\right)_{i j}= \begin{cases}L_{i j}, & (i, j) \in \Omega  \tag{1.5}\\ 0, & \text { otherwise }\end{cases}
$$

Note that the adjoint operator $\pi_{\Omega}^{*}=\pi_{\Omega}$. For applications with missing observations, Tao and Yuan [29] proposed recovering the low rank and sparse components of $D$ by solving

$$
\begin{equation*}
\min _{L, S \in \mathbb{R}^{m \times n}}\left\{\|L\|_{*}+\xi\|S\|_{1}:\left\|\pi_{\Omega}(L+S-D)\right\|_{F} \leq \delta\right\} \tag{1.6}
\end{equation*}
$$

PCP and SPCP both have numerous applications in diverse fields such as video surveillance and face recognition in image processing [8], and clustering in machine learning [3] to name a few. (1.1), (1.4) and (1.6) can be reformulated as semidefinite programming (SDP) problems, and therefore, in theory they can be solved in polynomial time using interior point algorithms; however, these algorithms require very large amount of memory, and are, therefore, impractical for solving large instances. Recently, a number of first-order algorithms have been proposed to solve PCP and SPCP. For existing approaches to solve PCP and SPCP problems see [1, 2, 8, 15, 22, 23, 29, 31, 32, and references therein.

Our contribution. We propose a new alternating direction method of multipliers (ADMM) with an increasing penalty sequence called ADMIP to solve the SPCP problem (1.6). The ADMIP algorithm, detailed in Figure 1.1, uses partial variable splitting on (1.6), and works directly with the non-smooth objective function. In the context of method of multipliers, where the primal iterates are computed by minimizing the augmented Lagrangian function, under assumptions related to strong second-order conditions for optimality, it was shown in [27, 28] that the primal and dual iterates converge to an optimal pair superlinearly when the penalty parameters $\rho_{k} \nearrow \infty$, while the rate is only linear when $\sup _{k} \rho_{k}<\infty$. However, this result has not been extended to ADMM. In a recent survey, Boyd et al. 6] (see Section 3.4.1) remark that it is difficult to prove the convergence of ADMM when penalty multipliers change in every iteration. We show that both primal and dual ADMIP iterates converge to an optimal primal-dual solution for (1.6) under mild conditions on the penalty multiplier sequence. To the best of our knowledge, this is the first convergence result for a variable penalty ADMM when penalties are not bounded, the objective function is non-smooth and its subdifferential is not uniformly bounded.

The work of He et al. [16, 17, 18, on variable penalty ADMM algorithms implicitly assumes that both terms in the objective function are differentiable; therefore, these results do not extend to non-smooth optimization problem in (2.1), i.e. to the ADMM formulation of (1.6). The variable penalty ADMM algorithms in [16, 17, 18] are proposed to solve variational inequalities (VI) of the form:

$$
\left(x-x^{*}\right)^{\top} F\left(x^{*}\right)+\left(y-y^{*}\right)^{\top} G\left(y^{*}\right) \geq 0, \quad \forall(x, y) \in \Omega:=\{(x, y): x \in \mathcal{X}, y \in \mathcal{Y}, A x+B y=b\}
$$

where $A \in \mathbb{R}^{m \times n_{1}}, B \in \mathbb{R}^{m \times n_{2}}$, and $b \in \mathbb{R}^{m}$. The convergence proofs in [16, 17, 18] require that both $F: \mathcal{X} \rightarrow \mathbb{R}^{n_{1}}$ and $G: \mathcal{Y} \rightarrow \mathbb{R}^{n_{2}}$ are continuous point-to-point maps that are monotone with respect to the non-empty closed convex sets $\mathcal{X} \subset \mathbb{R}^{n_{1}}$ and $\mathcal{Y} \subset \mathbb{R}^{n_{2}}$, respectively. When these variable penalty ADMM methods for VI are applied to the VI reformulation of convex optimization problems of the form $\min \{f(x)+g(y): \quad(x, y) \in \Omega\}$, the requirement that $F$ and $G$ be continuous point-to-point maps implies that $F(x)=\nabla f(x)$, and $G(y)=\nabla g(y)$. On the other hand, if $f(x)$ and $g(x)$ are non-smooth convex functions, then both $F$ and $G$ should be point-to-set maps, i.e., multi-functions; therefore, the convergence proofs for variable penalty ADMM algorithms in [16, [17, 18 ] do not extend to our problem which is a nonsmooth convex optimization problem - see Assumption A and the following discussion on page 107 in [18]. The ADMM algorithm in [19] can solve $\min \{f(x)+g(y):(x, y) \in \Omega\}$ when both $f$ and $g$ are non-smooth convex functions; however, the convergence proof requires that the penalty sequence $\left\{\rho_{k}\right\}$ increases only finitely many times; i.e., $\left\{\rho_{k}\right\}$ is bounded above (17, 18, also assume bounded $\left\{\rho_{k}\right\}$ ). Recently, Lin et al. [22] have proposed an ADMM algorithm for solving PCP problem in (1.1), i.e. (1.6) with $\delta=0$, and show that the algorithm converges for a nondecreasing $\left\{\rho_{k}\right\}$ such that $\sum_{k=1}^{\infty} \rho_{k}^{-1}=\infty$. The analysis in [22] relies on the fact that the subdifferentials of any norm are uniformly bounded. When $\delta>0$ in (1.6), the results in [22] do not hold because the subdifferentials of the objective function in the ADMM formulation (2.1) are no longer uniformly bounded because of the indicator function used to model the constraint.

[^1]In ADMM algorithms [6, 11, 12], the penalty parameter is typically held constant, i.e. $\rho_{k}=\rho>0$, for all $k \geq 1$. Although convergence is guaranteed for all $\rho>0$, the empirical performance of ADMM algorithms is critically dependent on the choice of penalty parameter $\rho$ - it deteriorates very rapidly if the penalty is set too large or too small [13, 14, 19]. Moreover, it is discussed in 24 that there exists $\rho^{*}$ which optimizes the convergence rate for the constant penalty ADMM scheme; however, estimating $\rho^{*}$ is difficult in practice [17].

The main advantages of adopting an increasing sequence of penalties are as follows:
(i) The algorithm is robust in the sense that there is no need to search for an optimal $\rho^{*}$.
(ii) The algorithm is likely to achieve primal feasibility faster. ADMM algorithms can be viewed as inexact variant of augmented Lagrangian algorithms where one updates the dual iterate after all primal iterates are updated by taking a single block-coordinate descent step in each block. The primal infeasibility in augmented Lagrangian methods can be approximated by $\mathcal{O}\left(\rho_{k}^{-1}\left\|Y_{k}-Y^{*}\right\|\right)$, where $Y_{k}$ is an estimate of optimal dual $Y^{*}$ at the $k$-th iteration (see, e.g. Section 17.3 in [25]). Consequently, a suitably chosen increasing sequence of penalties can improve the convergence rate.
(iii) The complexity of initial (transient) iterations can be controlled through controlling the growth in $\left\{\rho_{k}\right\}$. The main computational bottleneck in ADMIP (see Figure 1.1) is Step 4 that requires an SVD computation (see 4.1). Since the optimal $L^{*}$ is of low-rank, and $L_{k} \rightarrow L^{*}$, eventually the SVD computations are likely to be very efficient. However, since the initial iterates may have large rank, the complexity of the SVD in the initial iterations can be quite large. From (4.1) it follows that one does not need to compute singular values smaller than $1 / \rho_{k}$; hence, starting ADMIP with a small $\rho_{0}>0$ will significantly decrease the complexity of initial iterations.
In this paper, we propose an algorithm that uses an increasing sequence of penalties. This may appear as a regressive step that ignores the accumulated numerical experience with penalty and augmented Lagrangian algorithms. However, we argue that this experience does not immediately carry over to ADMM-type algorithms, and hence, one should re-examine the role of increasing penalty parameters. The reluctance to use increasing penalty sequence goes back and is associated with the experience of solving convex optimization problems of the form $P \equiv \min _{x}\{f(x): A x=b\}$ using quadratic penalty methods (QPM). These methods solve $P$ by inexactly solving a sequence of subproblems $P_{k} \equiv \min _{x}\left\{f(x)+\rho_{k}\left\|A x-b_{k}\right\|_{2}^{2}\right\}$ with $b_{k}=b$ for all $k \geq 1$. Let $x_{k}$ denote an inexact minimizer of $P_{k}$ such that the violation in the optimality conditions is within a specified tolerance. Then the infeasibility $\left\|A x_{k}-b\right\|_{2}$ is $\mathcal{O}\left(\frac{1}{\rho_{k}}\right)$; therefore, the penalty parameter $\rho_{k}$ must be increased to infinity in order to ensure feasibility. Traditionally, each inexact solution $x_{k}$ is computed using a second-order method where the Hessian is of the form $\nabla^{2} f(x)+2 \rho_{k} A^{T} A$. It is important to note that since the condition number is an increasing function of $\rho_{k}$, one encounters numerical instabilities while solving $P_{k}$ for large $k$ values. On the other hand, in augmented Lagrangian methods (ALM), i.e. method of multipliers, one computes an inexact solution $x_{k}$ to the subproblem $P_{k}$ with $b_{k}=b+y_{k}$, and then updates $y_{k+1}=\frac{\rho_{k}}{\rho_{k+1}}\left(b_{k}-A x_{k}\right)$, for all $k \geq 1$. In contrast to QPM, ALM guarantees primal convergence for a constant penalty sequence, i.e. $\rho_{k}=\rho$ for all $k \geq 1$; hence, obviating the need to choose an increasing penalty sequence, and avoiding the numerical instability encountered while solving $P_{k}$ for large $k$. In this context, proposing an algorithm, ADMIP, that uses an increasing sequence of penalties would appear to be contradictory, ignoring the accumulated numerical experience with penalty and augmented Lagrangian algorithms. However, this experience does not immediately carry over to ADMM-type algorithms; there are significant differences between ADMIP and the quadratic penalty methods, that suggest that the numerical issues observed in penalty methods are not likely to arise in ADMIP, and therefore, an increasing sequence of penalties is worth revisiting. Indeed, ADMIP is a first-order algorithm that only employs shrinkage 10 type operations in each iteration (see Step 4 and Step 5 of ADMIP displayed in Figure 1.1). Moreover, unlike quadratic penalty methods that solve the subproblems $P_{k}$ to an accuracy that increases with $k$, ADMIP takes only one step for each $P_{k}$; more importantly, each step can be computed in closed form and is not prone to numerical instability; thus, avoiding the numerical problems associated with quadratic penalty methods due to use of an increasing penalty sequence. Furthermore, the results of our numerical experiments reported in Section 4 clearly indicate that using an increasing sequence of penalty multipliers results in faster convergence in practice; in fact, the performance of ADMIP dominates the performance of ADMM-type algorithms for any fixed penalty term. The numerical experiments also confirm that ADMIP is significantly more robust to changes in problem parameters.

Organization. We propose ADMIP in Section 2 and prove its convergence in Section 3 In Section 4 we report the results of our numerical experiments where we compare the performance of ADMIP with ASALM on a set of synthetic randomly generated problems and on a large-scale problem involving foreground extraction from a noisy surveillance video.

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Algorithm \(\operatorname{ADMIP}\left(Z_{0}, Y_{0},\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}\right)\)
    input: \(Z_{0} \in \mathbb{R}^{m \times n}, Y_{0} \in \mathbb{R}^{m \times n},\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}} \subset \mathbb{R}_{++}\)such that \(\rho_{k+1} \geq \rho_{k}, \rho_{k} \rightarrow \infty\)
    \(k \leftarrow 0\)
    while \(k \geq 0\) do
        \(L_{k+1} \leftarrow \operatorname{argmin}_{L}\left\{\|L\|_{*}+\left\langle Y_{k}, L-Z_{k}\right\rangle+\frac{\rho_{k}}{2}\left\|L-Z_{k}\right\|_{F}^{2}\right\}\)
        \(\left(Z_{k+1}, S_{k+1}\right) \leftarrow \operatorname{argmin}_{\left\{(Z, S):\left\|\pi_{\Omega}(Z+S-D)\right\|_{F} \leq \delta\right\}}\left\{\xi\|S\|_{1}+\left\langle-Y_{k}, Z-L_{k+1}\right\rangle+\frac{\rho_{k}}{2}\left\|Z-L_{k+1}\right\|_{F}^{2}\right\}\)
        \(Y_{k+1} \leftarrow Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k+1}\right)\)
        \(k \leftarrow k+1\)
    end while
```

Fig. 1.1. ADMIP: Alternating Direction Method with Increasing Penalty
2. An ADMM algorithm with partial variable splitting and increasing penalty sequence. Let

$$
\chi:=\left\{(Z, S) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}:\left\|\pi_{\Omega}(Z+S-D)\right\|_{F} \leq \delta\right\}
$$

denote the feasible set in (1.6) and let $\mathbf{1}_{\chi}(\cdot, \cdot)$ denote the indicator function of the closed convex set $\chi \subset$ $\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$, i.e. if $(Z, S) \in \chi$, then $\mathbf{1}_{\chi}(Z, S)=0$; otherwise, $\mathbf{1}_{\chi}(Z, S)=\infty$. We use partial variable splitting, i.e. we only split the $L$ variables in (1.4), to arrive at the following equivalent problem

$$
\begin{equation*}
\min _{L, Z, S \in \mathbb{R}^{m \times n}}\left\{\|L\|_{*}+\xi\|S\|_{1}+\mathbf{1}_{\chi}(Z, S): L=Z\right\} . \tag{2.1}
\end{equation*}
$$

The augmented Lagrangian function of (2.1) is defined as follows:

$$
\begin{equation*}
\mathcal{L}_{\rho}(L, Z, S ; Y)=\|L\|_{*}+\xi\|S\|_{1}+\mathbf{1}_{\chi}(Z, S)+\langle Y, L-Z\rangle+\frac{\rho}{2}\|L-Z\|_{F}^{2} . \tag{2.2}
\end{equation*}
$$

In each iteration of ADMIP in Figure 1.1, the next iterate $L_{k+1}$ is computed by minimizing (2.2) over $L \in \mathbb{R}^{m \times n}$ by setting $\rho=\rho_{k}$ and $(Y, Z, S)=\left(Y_{k}, Z_{k}, S_{k}\right)$; the next iterate $\left(Z_{k+1}, S_{k+1}\right)$ is computed by minimizing (2.2) over $(Z, S) \in \chi$, by setting $\rho=\rho_{k}$ and $(Y, L)=\left(Y_{k}, L_{k+1}\right)$; finally we set the next dual variable $Y_{k+1}=Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k+1}\right)$.

The computational complexity of each iteration of ADMIP is determined by the subproblems solved in Step 4 and Step 5. The subproblem in Step 4 is a matrix shrinkage problem and can be solved efficiently by computing an SVD of an $m \times n$ matrix. The explicit solution of the matrix shrinkage problem is given in (4.1). The subproblem in Step 5 has the following generic form:

$$
\begin{equation*}
\left(P_{n s}\right): \min \left\{\xi\|S\|_{1}+\langle Q, Z-\tilde{Z}\rangle+\frac{\rho}{2}\|Z-\tilde{Z}\|_{F}^{2}:(Z, S) \in \chi\right\} \tag{2.3}
\end{equation*}
$$

where $\rho>0, Q, \tilde{Z} \in \mathbb{R}^{m \times n}$ are given problem parameters.
Lemma 2.1. The optimal solution $\left(Z^{*}, S^{*}\right)$ to problem $\left(P_{n s}\right)$ can be written in closed form.
(i) Suppose $\delta>0$. Then

$$
\begin{align*}
& S^{*}=\operatorname{sgn}\left(\pi_{\Omega}(D-q(\tilde{Z}))\right) \odot \max \left\{\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|-\xi \frac{\left(\rho+\theta^{*}\right)}{\rho \theta^{*}} E, \mathbf{0}\right\},  \tag{2.4}\\
& Z^{*}=\pi_{\Omega}\left(\frac{\theta^{*}}{\rho+\theta^{*}}\left(D-S^{*}\right)+\frac{\rho}{\rho+\theta^{*}} q(\tilde{Z})\right)+\pi_{\Omega^{c}}(q(\tilde{Z})) \tag{2.5}
\end{align*}
$$

where $q(\tilde{Z}):=\tilde{Z}-\rho^{-1} Q ; E$ and $\mathbf{0} \in \mathbb{R}^{m \times n}$ are matrices with all components equal to ones and zeros, respectively; $\odot$ denotes the component-wise multiplication operator. When $\left\|\pi_{\Omega}(D-q(\tilde{Z}))\right\|_{F} \leq \delta$, the multiplier $\theta^{*}=0$; otherwise, $\theta^{*}$ is the unique positive solution of the nonlinear equation $\phi(\theta)=\delta$, where

$$
\begin{equation*}
\phi(\theta):=\left\|\min \left\{\frac{\xi}{\theta} E, \frac{\rho}{\rho+\theta}\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|\right\}\right\|_{F} \tag{2.6}
\end{equation*}
$$

The multiplier $\theta^{*}$ can be efficiently computed in $\mathcal{O}(|\Omega| \log (|\Omega|))$ time.
(ii) Suppose $\delta=0$. Then

$$
\begin{equation*}
S^{*}=\operatorname{sgn}\left(\pi_{\Omega}(D-q(\tilde{Z}))\right) \odot \max \left\{\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|-\xi \rho^{-1} E, \quad \mathbf{0}\right\}, \tag{2.7}
\end{equation*}
$$

and $Z^{*}=\pi_{\Omega}\left(D-S^{*}\right)+\pi_{\Omega^{c}}(q(\tilde{Z}))$.
Proof. Proof is almost the same with that of Lemma 6.1 in [1]. For the sake of completeness, we included the proof in Appendix A.1

Note that Lemma 2.1 also gives the worst case computational complexity of proximal gradient type firstorder methods such as FISTA [4] and Algorithm 2 in 30 applied to the "smoothed" version of the SPCP problem $\min _{L, S \in \mathbb{R}^{m \times n}}\left\{f_{\mu}(L)+\xi\|S\|_{1}:(L, S) \in \chi\right\}$, where $f_{\mu}(L)=\max _{U \in \mathbb{R}^{m \times n}:\|U\|_{2} \leq 1}\langle L, U\rangle-\frac{\mu}{2}\|U\|_{F}^{2}$. For $\mu=\Theta(\epsilon)$, Lemma 2.1 implies that FISTA computes an $\epsilon$-optimal solution of problem (1.6) in $\mathcal{O}(1 / \epsilon)$ iterations.

The following lemma will be used later in Section 3. However, we state it here since it is related to problem $\left(P_{n s}\right)$.

Lemma 2.2. Suppose that $\delta>0$. Let $\left(Z^{*}, S^{*}\right)$ be an optimal solution to problem $\left(P_{n s}\right)$ and $\theta^{*}$ be an optimal Lagrangian multiplier such that $\left(Z^{*}, S^{*}\right)$ and $\theta^{*}$ together satisfy the Karush-Kuhn-Tucker (KKT) conditions. Then $\left(W^{*}, W^{*}\right) \in \partial \mathbf{1}_{\chi}\left(Z^{*}, S^{*}\right)$, where $W^{*}:=-Q+\rho\left(\tilde{Z}-Z^{*}\right)=\theta^{*} \pi_{\Omega}\left(Z^{*}+S^{*}-D\right)$.

Proof. See Appendix A. 2 for the proof. $\square$
3. Convergence of ADMIP. When $\rho_{k}=\rho>0$ for all $k \geq 1$, the convergence of ADMIP directly follows from the standard convergence theory of ADMM -see a recent survey paper [6] for the proof of convergence. In the rest of the paper, we will focus on the case where $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a monotonically increasing sequence, and we prove that ADMIP primal-dual iterate sequence $\left\{\left(L_{k}, S_{k}, Y_{k}\right)\right\}_{k \in \mathbb{Z}_{+}}$converges under mild conditions on the penalty sequence $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$. We first establish a sequence of results that extend the similar results in [22] to the case of constrained subproblems and partial splitting of variables. Define $\left\{\hat{Y}_{k}\right\}_{k \in \mathbb{Z}_{+}}$as

$$
\begin{equation*}
\hat{Y}_{k+1}:=Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k}\right) \tag{3.1}
\end{equation*}
$$

The subproblem in Step 5 of ADMIPis equivalent to

$$
\begin{equation*}
\min _{Z, S}\left\{\xi\|S\|_{1}+\left\langle-Y_{k}, Z-L_{k+1}\right\rangle+\frac{\rho_{k}}{2}\left\|Z-L_{k+1}\right\|_{F}^{2}: \frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}\right\} \tag{3.2}
\end{equation*}
$$

In Lemma 2.1 we show that the optimal solution of this problem can be written in closed form in terms of $\theta^{*}$ such that $\phi\left(\theta^{*}\right)=\delta$. Let $\theta_{k}$ denote the value of $\theta^{*}$ when Lemma 2.1 is applied to the instance in (3.2). Then the proof of Lemma 2.1) implies that $\theta_{k}$ is the optimal dual corresponding to the constraint in (3.2).

Lemma 3.1. Let $f(\cdot):=\|\cdot\|_{*}, g(\cdot):=\xi\|\cdot\|_{1}$ and let $\left\{L_{k}, Z_{k}, S_{k}, Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$denote the ADMIP iterates corresponding to the penalty sequence $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$and let $\left\{\hat{Y}_{k}\right\}_{k \in \mathbb{Z}_{+}}$denote the sequence defined in (3.1). Then for all $k \geq 1,-Y_{k} \in \partial g\left(S_{k}\right)$ and $-\hat{Y}_{k} \in \partial f\left(L_{k}\right)$. Thus, $\left\{Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$and $\left\{\hat{Y}_{k}\right\}_{k \in \mathbb{Z}_{+}}$are bounded sequences. Moreover, $\pi_{\Omega}\left(Y_{k}\right)=Y_{k}$ for all $k \geq 1$.

Proof. See Appendix A. 3 for the proof. $\square$
Before discussing the convergence properties of ADMIP in Theorem 3.3, we need to state a technical result in Lemma 3.2 which will play a key role in proving the main result of this paper: Theorem 3.3.

Lemma 3.2. Suppose $\delta>0$. Let $\left\{L_{k}, Z_{k}, S_{k}, Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$denote the ADMIP iterates corresponding to the non-decreasing sequence of penalty multipliers, $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}} . \operatorname{Let}\left(L^{*}, L^{*}, S^{*}\right) \in \operatorname{argmin}_{L, Z, S}\left\{\|L\|_{*}+\xi\|S\|_{1}\right.$ : $\left.\frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}, L=Z\right\}$ denote any optimal solution, $Y^{*} \in \mathbb{R}^{m \times n}$ and $\theta^{*} \geq 0$ denote any optimal Lagrangian duals corresponding to the constraints $L=Z$ and $\frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}$, respectively. Then $\left\{\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}\right\}_{k \in \mathbb{Z}_{+}}$is a non-increasing sequence and

$$
\begin{array}{cl}
\sum_{k \in \mathbb{Z}_{+}}\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}<\infty, & \sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2}<\infty \\
\sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-1}\left\langle-Y_{k+1}+Y^{*}, S_{k+1}-S^{*}\right\rangle<\infty, & \sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-1}\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle<\infty \\
\sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-1}\left\langle Y^{*}-Y_{k+1}, L^{*}+S^{*}-Z_{k+1}-S_{k+1}\right\rangle<\infty
\end{array}
$$

Proof. See Appendix A. 4 for the proof. C
The partial split formulation (2.1) is equivalent to

$$
\min _{L, Z, S \in \mathbb{R}^{m \times n}}\left\{\|L\|_{*}+\xi\|S\|_{1}: L=Z, \frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}\right\}
$$

The Lagrangian function for this formulation is given by

$$
\begin{equation*}
\mathcal{L}(L, Z, S ; Y, \theta)=\|L\|_{*}+\xi\|S\|_{1}+\langle Y, L-Z\rangle+\frac{\theta}{2}\left(\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2}-\delta^{2}\right) . \tag{3.3}
\end{equation*}
$$

THEOREM 3.3. Suppose $\delta>0$. Let $\left\{L_{k}, Z_{k}, S_{k}, Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$denote the ADMIP iterates corresponding to the penalty multiplier sequence $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$. Let $\left\{\theta_{k}\right\}_{k \in \mathbb{Z}_{+}}$be the sequence such that $\theta_{k}$ is the optimal dual corresponding to the constraint in (3.2).
(i) Suppose $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a non-decreasing sequence such that $\sum_{k \in \mathbb{Z}_{+}} \frac{1}{\rho_{k}}=\infty$. Then $L^{*}:=\lim _{k \in \mathbb{Z}_{+}} Z_{k}=$ $\lim _{k \in \mathbb{Z}_{+}} L_{k}$ and $S^{*}:=\lim _{k \in \mathbb{Z}_{+}} S_{k}$ exist; and $\left(L^{*}, S^{*}\right)$ are optimal for the SPCP problem.
(ii) Suppose $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a non-decreasing sequence such that $\sum_{k \in \mathbb{Z}_{+}} \frac{1}{\rho_{k}^{2}}=\infty$. Then, in the case that $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F} \neq \delta,\left(Y^{*}, \theta^{*}\right):=\lim _{k \in \mathbb{Z}_{+}}\left(Y_{k}, \theta_{k}\right)$ exists, and $\left(L^{*}, L^{*}, S^{*}, Y^{*}, \theta^{*}\right)$ is a saddle point of the Lagrangian function $\mathcal{L}$ in (3.3). Otherwise, i.e. when $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}=\delta,\left\{Y_{k}, \theta_{k}\right\}_{k \in \mathbb{Z}_{+}}$has a limit point $\left(Y^{*}, \theta^{*}\right)$, such that $\left(Y^{*}, \theta^{*}\right) \in \operatorname{argmax}_{Y, \theta}\left\{\mathcal{L}\left(L^{*}, L^{*}, S^{*} ; Y, \theta\right): \theta \geq 0\right\}$.
The condition $\sum_{k \in \mathbb{Z}_{+}} \frac{1}{\rho_{k}}=\infty$ is similar to the condition in Theorem 2 in [22] that is needed to show that algorithm I-ALM converges to an optimal solution of the robust PCA problem. Let $\Omega=\{(i, j): 1 \leq$ $i \leq m, 1 \leq j \leq n\}$, and $D=L^{0}+S^{0}+N^{0}$ be given such that $\left(L^{0}, S^{0}, N^{0}\right)$ satisfies the assumptions of Theorem 1.2 and $\left\|S^{0}\right\|_{F}>\sqrt{C m n} \delta$. Then, with very high probability, $\left\|D-L^{*}\right\|_{F}>\delta$, where $C$ is the numerical constant defined in Theorem [1.2. Therefore, in practice, one is unlikely to encounter the case where $\left\|D-L^{*}\right\|_{F}=\delta$.

Proof. Lemma 3.2 and the fact that $L_{k+1}-Z_{k+1}=\frac{1}{\rho_{k}}\left(Y_{k+1}-Y_{k}\right)$ for all $k \geq 1$, together imply that

$$
\infty>\sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2}=\sum_{k \in \mathbb{Z}_{+}}\left\|L_{k+1}-Z_{k+1}\right\|_{F}^{2} .
$$

Thus, $\lim _{k \in \mathbb{Z}_{+}}\left(L_{k}-Z_{k}\right)=0$.
Let $\left(L^{\#}, L^{\#}, S^{\#}\right) \in \operatorname{argmin}_{L, Z, S}\left\{\|L\|_{*}+\xi\|S\|_{1}: \quad \frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}, L=Z\right\}$ denote any optimal solution, $Y^{\#} \in \mathbb{R}^{m \times n}$ and $\theta^{\#} \geq 0$ denote any Lagrangian dual optimal solutions corresponding to $L=Z$ and $\frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}$ constraints, respectively, and $f^{*}:=\left\|L^{\#}\right\|_{*}+\xi\left\|S^{\#}\right\|_{1}$.

Since $\left(Z_{k}, S_{k}\right) \in \chi$ for all $k \geq 1$, or equivalently $\mathbf{1}_{\chi}\left(Z_{k}, S_{k}\right)=0$ for all $k \geq 1$, it follows that

$$
\begin{align*}
& \left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1} \\
& =\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1}+\mathbf{1}_{\chi}\left(Z_{k}, S_{k}\right) \\
& \leq\left\|L^{\#}\right\|_{*}+\xi\left\|S^{\#}\right\|_{1}+\mathbf{1}_{\chi}\left(L^{\#}, S^{\#}\right)+\left\langle\hat{Y}_{k}, L^{\#}-L_{k}\right\rangle+\left\langle Y_{k}, S^{\#}-S_{k}\right\rangle-\left\langle Y_{k}, L^{\#}+S^{\#}-Z_{k}-S_{k}\right\rangle, \\
& =f^{*}+\left\langle-\hat{Y}_{k}+Y^{\#}, L_{k}-L^{\#}\right\rangle+\left\langle-Y_{k}+Y^{\#}, S_{k}-S^{\#}\right\rangle+\left\langle Y^{\#}-Y_{k}, L^{\#}+S^{\#}-Z_{k}-S_{k}\right\rangle \\
& \quad+\left\langle Y^{\#}, Z_{k}-L_{k}\right\rangle \tag{3.4}
\end{align*}
$$

where the inequality follows from Lemma 3.1 and the fact that $\left(Y_{k}, Y_{k}\right) \in \partial \mathbf{1}_{\chi}\left(Z_{k}, S_{k}\right)$-see Lemma 2.2 and (3.4) follows from rearranging the terms and the fact that $\left(L^{\#}, S^{\#}\right) \in \chi$.

From Lemma 3.2, we have that

$$
\sum_{k \in \mathbb{Z}_{+}} \rho_{k-1}^{-1}\left(\left\langle-\hat{Y}_{k}+Y^{\#}, L_{k}-L^{\#}\right\rangle+\left\langle-Y_{k}+Y^{\#}, S_{k}-S^{\#}\right\rangle+\left\langle Y^{\#}-Y_{k}, L^{\#}+S^{\#}-Z_{k}-S_{k}\right\rangle\right)<\infty
$$

First consider the case where $\sum_{k \in \mathbb{Z}_{+}} \frac{1}{\rho_{k}}=\infty$. There exists $\mathcal{K} \subset \mathbb{Z}_{+}$such that

$$
\begin{equation*}
\lim _{k \in \mathcal{K}}\left(\left\langle-\hat{Y}_{k}+Y^{\#}, L_{k}-L^{\#}\right\rangle+\left\langle-Y_{k}+Y^{\#}, S_{k}-S^{\#}\right\rangle+\left\langle Y^{\#}-Y_{k}, L^{\#}+S^{\#}-Z_{k}-S_{k}\right\rangle\right)=0 \tag{3.5}
\end{equation*}
$$

Therefore, (3.4), (3.5) and $\lim _{k \in \mathbb{Z}_{+}}\left(Z_{k}-L_{k}\right)=0$ together imply that

$$
\limsup _{k \in \mathcal{K}}\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1} \leq f^{*}
$$

Hence, $\left\{\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1}\right\}_{k \in \mathcal{K}}$ is a bounded sequence. Therefore, there exists $\mathcal{K}^{*} \subset \mathcal{K} \subset \mathbb{Z}_{+}$such that $\left\{\left(L_{k}, S_{k}\right)\right\}_{k \in \mathcal{K}^{*}}$ has a limit. Let $\left(L^{*}, S^{*}\right):=\lim _{k \in \mathcal{K}^{*}}\left(L_{k}, S_{k}\right)$. Since $\lim _{k \in \mathbb{Z}_{+}}\left(Z_{k}-L_{k}\right)=0$ and $\left(Z_{k}, S_{k}\right) \in \chi$ for all $k \geq 1$, we have $\left(L^{*}, S^{*}\right)=\lim _{k \in \mathcal{K}^{*}}\left(Z_{k}, S_{k}\right) \in \chi$. Taking the limit of both sides of (3.4) along $\mathcal{K}^{*}$ gives

$$
\left\|L^{*}\right\|_{*}+\xi\left\|S^{*}\right\|_{1}=\lim _{k \in \mathcal{K}^{*}}\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1} \leq f^{*}
$$

and since $\left(L^{*}, S^{*}\right) \in \chi$, we conclude that $\left(L^{*}, S^{*}\right) \in \operatorname{argmin}\left\{\|L\|_{*}+\xi\|S\|_{1}:(L, S) \in \chi\right\}$.
Note that

$$
\left(L^{*}, L^{*}, S^{*}\right) \in \underset{L, Z, S}{\operatorname{argmin}}\left\{\|L\|_{*}+\xi\|S\|_{1}: \frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}, L=Z\right\} .
$$

Let $\bar{Y} \in \mathbb{R}^{m \times n}$ and $\bar{\theta} \geq 0$ denote any Lagrangian dual optimal solutions corresponding to $L=Z$ and $\frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}$ constraints, respectively. Lemma 3.1 implies that $\left\{Y_{k}\right\}$ is a bounded sequence. Thus, from Lemma 3.2 it follows that $\left\{\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-\bar{Y}\right\|_{F}^{2}\right\}_{k \in \mathbb{Z}_{+}}$is a bounded, non-increasing sequence, and therefore, has a unique limit point; hence, every subsequence of this sequence converges to the same limit. Combining this result with the facts that $\lim _{k \in \mathcal{K}^{*}} Z_{k}=L^{*}$ and $\left\{Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a bounded sequence, it follows that

$$
\begin{aligned}
\lim _{k \in \mathbb{Z}_{+}}\left\|Z_{k}-L^{*}\right\|_{F}^{2} & =\lim _{k \in \mathbb{Z}_{+}}\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-\bar{Y}\right\|_{F}^{2} \\
& =\lim _{k \in \mathcal{K}^{*}}\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-\bar{Y}\right\|_{F}^{2} \\
& =\lim _{k \in \mathcal{K}^{*}}\left\|Z_{k}-L^{*}\right\|_{F}^{2} \\
& =0
\end{aligned}
$$

Since $\lim _{k \in \mathbb{Z}_{+}}\left\|Z_{k}-L^{*}\right\|_{F}=0$ and $\lim _{k \in \mathbb{Z}_{+}}\left(Z_{k}-L_{k}\right)=0$, it follows that $\lim _{k \in \mathbb{Z}_{+}} L_{k}=\lim _{k \in \mathbb{Z}_{+}} Z_{k}=L^{*}$.
Lemma 2.1 applied to the sub-problem in Step 5 of ADMIP corresponding to the $k$-th iteration gives

$$
\begin{align*}
S_{k+1} & =\operatorname{sgn}\left(\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right) \odot \max \left\{\left|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right|-\xi \frac{\left(\rho_{k}+\theta_{k}\right)}{\rho_{k} \theta_{k}} E, \mathbf{0}\right\}  \tag{3.6}\\
Z_{k+1} & =\pi_{\Omega}\left(\frac{\theta_{k}}{\rho_{k}+\theta_{k}}\left(D-S_{k+1}\right)+\frac{\rho_{k}}{\rho_{k}+\theta_{k}} q\left(L_{k+1}\right)\right)+\pi_{\Omega^{c}}\left(q\left(L_{k+1}\right)\right) \tag{3.7}
\end{align*}
$$

where $q\left(L_{k+1}\right):=\left(L_{k+1}+\frac{1}{\rho_{k}} Y_{k}\right)$. Here, $\theta_{k}=0$, when $\left\|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right\|_{F} \leq \delta$; otherwise, $\theta_{k}>0$ is the unique solution of the equation $\phi_{k}(\theta)=\delta$, where

$$
\begin{equation*}
\phi_{k}(\theta):=\left\|\min \left\{\frac{\xi}{\theta} E, \frac{\rho_{k}}{\rho_{k}+\theta}\left|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right|\right\}\right\|_{F} . \tag{3.8}
\end{equation*}
$$

Since $\lim _{k \in \mathbb{Z}_{+}} L_{k}=L^{*},\left\{Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a bounded sequence and $\rho_{k} \nearrow \infty$, we have that $\lim _{k \in \mathbb{Z}_{+}} q\left(L_{k+1}\right)=$ $\lim _{k \in \mathbb{Z}_{+}} L_{k+1}+\frac{1}{\rho_{k}} Y_{k}=L^{*}$. Next, we establish $\left\{S_{k}\right\}_{k \in \mathbb{Z}_{+}}$has a unique limit point $S^{*}$.
(i) First suppose $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F} \leq \delta$. Recall that we have shown that there exists a sub-sequence $\mathcal{K}^{*} \subset \mathbb{Z}_{+}$such that

$$
\lim _{k \in \mathcal{K}^{*}}\left(L_{k}, S_{k}\right)=\left(L^{*}, S^{*}\right) \in \underset{L, S}{\operatorname{argmin}}\left\{\|L\|_{*}+\xi\|S\|_{1}:\left\|\pi_{\Omega}(L+S-D)\right\|_{F} \leq \delta\right\} .
$$

Since $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F} \leq \delta,\left(L^{*}, \mathbf{0}\right)$ is a feasible solution, it follows $\left\|L^{*}\right\|_{*}+\xi\left\|S^{*}\right\| \leq\left\|L^{*}\right\|_{*}$. Consequently, $S^{*}=\mathbf{0}$.

$$
\begin{align*}
& \left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1} \\
& =\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1}+\mathbf{1}_{\chi}\left(Z_{k}, S_{k}\right) \\
& \leq\left\|L^{*}\right\|_{*}+\xi\|\mathbf{0}\|_{1}+\mathbf{1}_{\chi}\left(L^{*}, \mathbf{0}\right)-\left\langle-\hat{Y}_{k}, L^{*}-L_{k}\right\rangle-\left\langle-Y_{k}, \mathbf{0}-S_{k}\right\rangle-\left\langle Y_{k}, L^{*}+\mathbf{0}-Z_{k}-S_{k}\right\rangle, \\
& =\left\|L^{*}\right\|_{*}+\left\langle\hat{Y}_{k}, L^{*}-L_{k}\right\rangle+\left\langle Y_{k}, Z_{k}-L^{*}\right\rangle \tag{3.9}
\end{align*}
$$

where the inequality follows from Lemma 3.1 and the fact that $\left(Y_{k}, Y_{k}\right) \in \partial \mathbf{1}_{\chi}\left(Z_{k}, S_{k}\right)$ (see Lemma 2.2 for details).
Since the sequences $\left\{Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$and $\left\{\hat{Y}_{k}\right\}_{k \in \mathbb{Z}_{+}}$are both bounded and $\lim _{k \in \mathbb{Z}_{+}} L_{k}=\lim _{k \in \mathbb{Z}_{+}} Z_{k}=L^{*}$, taking the limit of both sides of (3.9), we get

$$
\begin{aligned}
\left\|L^{*}\right\|_{*}+\xi \lim _{k \in \mathbb{Z}_{+}}\left\|S_{k}\right\|_{1} & =\lim _{k \in \mathbb{Z}_{+}}\left\|L_{k}\right\|_{*}+\xi\left\|S_{k}\right\|_{1} \\
& \leq \lim _{k \in \mathbb{Z}_{+}}\left\|L_{k}\right\|_{*}+\left\langle\hat{Y}_{k}, L^{*}-L_{k}\right\rangle+\left\langle Y_{k}, Z_{k}-L^{*}\right\rangle=\left\|L^{*}\right\|_{*}
\end{aligned}
$$

Therefore, $\lim _{k \in \mathbb{Z}_{+}}\left\|S_{k}\right\|_{1}=0$, which implies that $\lim _{k \in \mathbb{Z}_{+}} S_{k}=\mathbf{0}$. Hence, $S^{*}=\lim _{k \in \mathbb{Z}_{+}} S_{k}$.
(ii) Next, suppose $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}>\delta$. Since $\lim _{k \in \mathbb{Z}_{+}}\left\|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right\|_{F}=\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}>\delta$, there exists $K \in \mathbb{Z}_{+}$such that for all $k \geq K,\left\|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right\|_{F}>\delta$. For all $k \geq K, \phi_{k}(\cdot)$, defined in (3.8), is a continuous and strictly decreasing function of $\theta$ for $\theta \geq 0$. Hence, for all $k \geq K$, the inverse function $\phi_{k}^{-1}($.$) exists in an open neighborhood containing \delta$. Thus, $\phi_{k}(0)=\left\|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right\|_{F}>\delta$ for all $k \geq K$ and $\lim _{\theta \rightarrow \infty} \phi_{k}(\theta)=0$ imply that $\theta_{k}=\phi_{k}^{-1}(\delta)>0$ for all $k \geq K$. Moreover, $\phi_{k}(\theta) \leq$ $\phi(\theta):=\left\|\frac{\xi}{\theta} E\right\|_{F}$ implies that for all $k \geq 1$,

$$
\begin{equation*}
\theta_{k}=\phi_{k}^{-1}(\delta) \leq \phi^{-1}(\delta)=\frac{\xi \sqrt{m n}}{\delta} \tag{3.10}
\end{equation*}
$$

Since $\left\{\theta_{k}\right\}_{k \geq K}$ is a bounded sequence, it has a convergent subsequence $\mathcal{K}_{\theta} \subset \mathbb{Z}_{+}$, i.e., $\theta^{*}:=\lim _{k \in \mathcal{K}_{\theta}} \theta_{k}$ exists. We also have $\phi_{k}(\theta) \rightarrow \phi_{\infty}(\theta)$ pointwise for all $0 \leq \theta \leq \frac{\xi \sqrt{m n}}{\delta}$, where

$$
\begin{equation*}
\phi_{\infty}(\theta):=\left\|\min \left\{\frac{\xi}{\theta} E,\left|\pi_{\Omega}\left(D-L^{*}\right)\right|\right\}\right\|_{F} \tag{3.11}
\end{equation*}
$$

Since $\phi_{k}\left(\theta_{k}\right)=\delta$ for all $k \geq K$, we have

$$
\begin{equation*}
\delta=\lim _{k \in \mathcal{K}_{\theta}} \phi_{k}\left(\theta_{k}\right)=\lim _{k \in \mathcal{K}_{\theta}}\left\|\min \left\{\frac{\xi}{\theta_{k}} E, \frac{\rho_{k}}{\rho_{k}+\theta_{k}}\left|\pi_{\Omega}\left(D-q\left(L_{k+1}\right)\right)\right|\right\}\right\|_{F}=\phi_{\infty}\left(\theta^{*}\right) \tag{3.12}
\end{equation*}
$$

Note that $\phi_{\infty}(\cdot)$ is also a continuous and strictly decreasing function of $\theta$ for $\theta \geq 0$. Moreover, $\phi_{\infty}(0)=\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}>\delta$ implies that $\phi_{\infty}$ is invertible around $\delta$, i.e. $\phi_{\infty}^{-1}$ exists in a neighborhood containing $\delta$, and $\phi_{\infty}^{-1}(\delta)>0$. Thus, $\theta^{*}=\phi_{\infty}^{-1}(\delta)$. Since $\mathcal{K}_{\theta}$ is an arbitrary subsequence and $\theta^{*}=\phi_{\infty}^{-1}(\delta)$ does not depend on $\mathcal{K}_{\theta}$, we can conclude that

$$
\begin{equation*}
\lim _{k \in \mathbb{Z}_{+}} \theta_{k}=\phi_{\infty}^{-1}(\delta)=\theta^{*} \tag{3.13}
\end{equation*}
$$

Since $\theta^{*}=\lim _{k \in \mathbb{Z}_{+}} \theta_{k}$, taking the limit on both sides of (3.6), we get

$$
\begin{equation*}
S^{*}:=\lim _{k \in \mathbb{Z}_{+}} S_{k+1}=\operatorname{sgn}\left(\pi_{\Omega}\left(D-L^{*}\right)\right) \odot \max \left\{\left|\pi_{\Omega}\left(D-L^{*}\right)\right|-\frac{\xi}{\theta^{*}} E, \mathbf{0}\right\} \tag{3.14}
\end{equation*}
$$

and this completes the first part of the theorem.
Now, suppose $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$is strictly increasing and $\sum_{k=1}^{\infty} \frac{1}{\rho_{k}^{2}}=\infty$. We need two results in order to establish the convergence of the duals. From Lemma 3.2, we have $\sum_{k \in \mathbb{Z}_{+}}\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}<\infty$. From the definition of $\hat{Y}_{k}$ in (3.1), it follows that

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-2}\left\|\hat{Y}_{k+1}-Y_{k+1}\right\|_{F}^{2}=\sum_{k \in \mathbb{Z}_{+}}\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}<\infty \tag{3.15}
\end{equation*}
$$

Since $\sum_{k \in \mathbb{Z}_{+}} \frac{1}{\rho_{k}^{2}}=\infty$, there exists a sub-sequence $\overline{\mathcal{K}} \subset \mathbb{Z}_{+}$such that $\lim _{k \in \overline{\mathcal{K}}}\left\|\hat{Y}_{k+1}-Y_{k+1}\right\|_{F}^{2}=0$. Hence, $\lim _{k \in \overline{\mathcal{K}}} \rho_{k}^{2}\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}=0$, i.e.

$$
\begin{equation*}
\lim _{k \in \overline{\mathcal{K}}} \rho_{k}\left(Z_{k+1}-Z_{k}\right)=0 \tag{3.16}
\end{equation*}
$$

Using (A.18), A.19) and A.20 from the proof of Lemma 3.1 in Appendix A.3 we get

$$
\begin{align*}
& 0 \in \partial\left\|L_{k+1}\right\|_{*}+\theta_{k} \pi_{\Omega}\left(Z_{k+1}+S_{k+1}-D\right)+\rho_{k}\left(Z_{k+1}-Z_{k}\right)  \tag{3.17}\\
& 0 \in \xi \partial\left\|S_{k+1}\right\|_{1}+\theta_{k} \pi_{\Omega}\left(Z_{k+1}+S_{k+1}-D\right) \tag{3.18}
\end{align*}
$$

We will establish the convergence of the duals by considering two cases.
(i) Suppose $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F} \neq \delta$. Note that from A.20), it follows that $Y_{k}=\theta_{k-1} \pi_{\Omega}\left(Z_{k}+S_{k}-D\right)$ for all $k \geq 1$. First suppose that $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}<\delta$. Since

$$
\lim _{k \in \mathbb{Z}_{+}}\left\|\pi_{\Omega}\left(D-\left(L_{k+1}+\frac{1}{\rho_{k}} Y_{k}\right)\right)\right\|_{F}=\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}<\delta
$$

there exists $K \in \mathbb{Z}_{+}$such that for all $k \geq K,\left\|\pi_{\Omega}\left(D-\left(L_{k+1}+\frac{1}{\rho_{k}} Y_{k}\right)\right)\right\|_{F}<\delta$. Thus, from Lemma 2.1 for all $k \geq K, \theta_{k}=0, S_{k+1}=0, Z_{k+1}=L_{k+1}+\frac{1}{\rho_{k}} Y_{k}$, which implies that $\theta^{*}=\lim _{k \in \mathbb{Z}_{+}} \theta_{k}=0$ and, since $S^{*}=\lim _{k \in \mathbb{Z}_{+}} S_{k}=0$ and $\lim _{k \in \mathbb{Z}_{+}} Z_{k}=L^{*}$,

$$
Y^{*}=\lim _{k \in \mathbb{Z}_{+}} Y_{k}=\lim _{k \in \mathbb{Z}_{+}} \theta_{k-1} \pi_{\Omega}\left(Z_{k}+S_{k}-D\right)=\mathbf{0}
$$

Next, suppose that $\left\|\pi_{\Omega}\left(D-L^{*}\right)\right\|_{F}>\delta$. In this case, we have established in (3.13) that $\theta^{*}=$ $\lim _{k \in \mathbb{Z}_{+}} \theta_{k}$ exists. Hence,

$$
\lim _{k \in \mathbb{Z}_{+}} Y_{k}=\lim _{k \in \mathbb{Z}_{+}} \theta_{k-1} \pi_{\Omega}\left(Z_{k}+S_{k}-D\right)=\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right)=Y^{*}
$$

exists.
Taking the limit of (3.17) and (3.18) along $\overline{\mathcal{K}} \subset \mathbb{Z}_{+}$defined in (3.16); and using the fact that $\lim _{k \in \overline{\mathcal{K}}} \rho_{k}\left(Z_{k+1}-Z_{k}\right)=0$, we get

$$
\begin{align*}
& 0 \in \partial\left\|L^{*}\right\|_{*}+\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right)  \tag{3.19}\\
& 0 \in \xi \partial\left\|S^{*}\right\|_{1}+\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right) \tag{3.20}
\end{align*}
$$

Thus, it follows that the primal variables $\left(L^{*}, S^{*}\right)$ and dual variables $Y^{*}=\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right)$ and $\theta^{*}$ satisfy KKT optimality conditions for the problem

$$
\min _{L, Z, S}\left\{\|L\|_{*}+\xi\|S\|_{1}: \frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}, L=Z\right\}
$$

Hence, $\left(L^{*}, L^{*}, S^{*}, Y^{*}, \theta^{*}\right)$ is a saddle point of the Lagrangian function

$$
\mathcal{L}(L, Z, S ; Y, \theta)=\|L\|_{*}+\xi\|S\|_{1}+\langle Y, L-Z\rangle+\frac{\theta}{2}\left(\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2}-\delta^{2}\right) .
$$

(ii) Next, consider the case where $\left\|D-L^{*}\right\|_{F}=\delta$. Fix $k>0$. $\theta_{k}=0$ if $\left\|D-\left(L_{k+1}+\frac{1}{\rho_{k}} Y_{k}\right)\right\|_{F} \leq \delta$; otherwise, $\theta_{k}>0$. Also, from (3.10) it follows that $\theta_{k} \leqq \frac{\xi \sqrt{m n}}{\delta}$. Since $\left\{\theta_{k}\right\}_{k \in \mathbb{Z}_{+}}$is a bounded sequence, there exists a further subsequence $\mathcal{K}_{\theta}$ of the sequence $\overline{\mathcal{K}}$ defined in (3.16) such that $\theta^{*}:=\lim _{k \in \mathcal{K}_{\theta}} \theta_{k-1}$ and $Y^{*}:=$ $\lim _{k \in \mathcal{K}_{\theta}} \theta_{k-1} \pi_{\Omega}\left(Z_{k}+S_{k}-D\right)=\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right)$ exist. Thus, taking the limit of (3.17), (3.18) along $\mathcal{K}_{\theta} \subset \mathbb{Z}_{+}$and using the facts that $\lim _{k \in \overline{\mathcal{K}}} \rho_{k}\left(Z_{k+1}-Z_{k}\right)=0$ and $L^{*}=\lim _{k \in \mathbb{Z}_{+}} L_{k}=\lim _{k \in \mathbb{Z}_{+}} Z_{k}$, $S^{*}=\lim _{k \in \mathbb{Z}_{+}} S_{k}$ exist, we conclude that $\left(L^{*}, L^{*}, S^{*}, Y^{*}, \theta^{*}\right)$ is a saddle point of the Lagrangian function $\mathcal{L}(L, Z, S ; Y, \theta)$.
$\square$
4. Numerical experiments. We conducted two sets of numerical experiments with ADMIP to solve SPCP problems. In the first set of experiments we solved randomly generated instances of the SPCP problem. In this setting, we conducted three different tests. First, we compared ADMIP with ADMM for different values of the fixed penalty $\rho$; second, we conducted a set of experiments to understand how ADMIP runtime
scales as a function of the problem parameters and size; and third, we compared ADMIP with ASALM [29]. ASALM is an ADMM algorithm, tailored for the SPCP problem, with a fixed penalty $\rho$. For each dual update, ASALM updates three blocks of primal variables, while ADMIP updates two blocks. In the second set of experiments, we compared ADMIP and ASALM on the foreground detection problem, where the goal is to extract the moving objects from a noisy and corrupted airport security video [21]. All the numerical experiments were conducted on a Dell M620 server computing node running on RedHat Enterprise Linux 6 (RHEL 6). Each numerical test was carried out using MATLAB R2013a ( 64 bit) with 16 GB RAM available on a single core of Intel Leon E5-2665 2.40 GHz processor. The MATLAB code for ADMIP ${ }^{2}$ is available at http://www2.ie.psu.edu/aybat/codes.html and the code for ASALM is available on request from the authors of 29].

```
Algorithm \(\operatorname{ADMIP}\left(Z_{0}, Y_{0},\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}\right)\)
    input: \(Z_{0} \in \mathbb{R}^{m \times n}, Y_{0} \in \mathbb{R}^{m \times n},\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}} \subset \mathbb{R}_{++}\)such that \(\rho_{k+1} \geq \rho_{k}, \rho_{k} \rightarrow \infty\)
    \(k \leftarrow 0\)
    while \(k \geq 0\) do
        Compute \(\operatorname{svd}\left(Z_{k}-Y_{k} / \rho_{k}\right)\) such that \(Z_{k}-Y_{k} / \rho_{k}=U \operatorname{Diag}(\sigma) V^{T}\)
        \(L_{k+1} \leftarrow U \operatorname{Diag}\left(\min \left\{\sigma-\frac{1}{\rho_{k}} \mathbf{1}, 0\right\}\right) V^{T}\)
        \(C \leftarrow L_{k+1}+\rho_{k}^{-1} Y_{k}\)
        \(\theta^{*} \leftarrow\) ThetaSearch \(\left(|D-C|, \Omega, \delta, \rho_{k}\right)\)
        \(S_{k+1} \leftarrow \operatorname{sgn}\left(\pi_{\Omega}(D-C)\right) \odot \max \left\{\left|\pi_{\Omega}(D-C)\right|-\xi \frac{\left(\rho_{k}+\theta^{*}\right)}{\rho_{k} \theta^{*}} E, \mathbf{0}\right\}\)
        \(Z_{k+1} \leftarrow \pi_{\Omega}\left(\frac{\theta^{*}}{\rho_{k}+\theta^{*}}\left(D-S^{*}\right)+\frac{\rho_{k}}{\rho_{k}+\theta^{*}} C\right)+\pi_{\Omega^{c}}(C)\)
        \(Y_{k+1} \leftarrow Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k+1}\right)\)
        \(k \leftarrow k+1\)
    end while
```

Fig. 4.1. Pseudocode for $A D M I P$
4.1. Implementation details. The optimal solution of the Step 4 subproblem corresponding to the $k$-th iteration is given by

$$
\begin{equation*}
L_{k+1}=U \operatorname{Diag}\left(\min \left\{\sigma-\frac{1}{\rho_{k}} \mathbf{1}, 0\right\}\right) V^{T} \tag{4.1}
\end{equation*}
$$

where $q\left(Z_{k}\right)=Z_{k}-Y_{k} / \rho_{k}=U \mathbf{D i a g}(\sigma) V^{T}$ and 1 denotes a vector of all ones. Computing the full SVD of $q\left(Z_{k}\right)$ is expensive for large instances. However, we do not need to compute the full SVD, because only the singular values that are larger than $1 / \rho_{k}$ and the corresponding singular vectors are needed. In order to exploit this fact, we used a modified version of LANSVD [20]3 that comes with treshold option to compute only those singular vectors with singular values greater than a given threshold value $\tau>0$. Note that we set $\tau=1 / \rho_{k}$ in the $k$-th ADMIP iteration.

The bottleneck step in the $k$-th iteration of ASALM, which is an ADMM algorithm with constant penalty $\rho>0$, also involves computing a low-rank matrix $L_{k+1}$. Indeed, first, a matrix $Q_{k}$ is computed with complexity comparable to that of computing $q\left(Z_{k}\right)$ in ADMIP. Next, $L_{k+1}$ is computed as in (4.1), where $U \operatorname{diag}(\sigma) V^{T}$ denotes the SVD of $Q_{k}$, and $\rho_{k}=\rho$ for all $k$. Thus, the overall per-iteration complexity of ASALM is comparable to that of ADMIP. The ASALM code provided by the authors of [29] calls the original LANSVD function of PROPACK which does not have the threshold option; consequently, the ASALM code computes $L_{k+1}$ by first estimating its rank, say $r$, and computing the leading $r$ singular values of $Q_{k}$, i.e. $\sigma_{1} \geq \sigma_{2} \geq \ldots \geq \sigma_{r}$. If the $r$-th singular value $\sigma_{r} \leq 1 / \rho$, then $L_{k+1}$ is computed using singular-value shrinkage as in (4.1); otherwise, the estimate $r$ is revised by setting $r=\min \{2 r, n\}$, and the leading $r$ singular values of $Q_{k}$ are computed from scratch, i.e. the first $r$ that were computed previously are simply ignored. This process is repeated until $\sigma_{r} \leq 1 / \rho$. In order to improve the efficiency of the ASALM code and make it comparable to ADMIP, we used the modified LANSVD function with the threshold option in both

[^2]```
Subroutine ThetaSearch \((A, \Omega, \delta, \rho)\)
    output: \(\theta^{*} \in \mathbb{R}_{+}\), input: \(A \in \mathbb{R}_{+}^{m \times n}, \Omega \subset\{1, \ldots, m\} \times\{1, \ldots, n\}, \delta>0, \rho>0\)
    if \(\left\|\pi_{\Omega}(A)\right\|_{F} \leq \delta\) then
        \(\theta^{*} \leftarrow 0\)
    else
        Compute \(0 \leq a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(|\Omega|)}\) by sorting \(\left\{A_{i j}:(i, j) \in \Omega\right\}\)
        \(a_{(0)} \leftarrow 0\)
        \(\bar{k} \leftarrow \max \left\{j: a_{(j)} \leq \frac{\xi}{\rho}, 0 \leq j \leq|\Omega|\right\}\)
        if \(\bar{k}==|\Omega|\) then
            \(\theta^{*} \leftarrow \rho\left(\frac{\left\|\pi_{\Omega}(A)\right\|_{F}}{\delta}-1\right)\)
        else
            \(j^{*} \leftarrow \bar{k}\)
            for \(j=\bar{k}+1, \ldots,|\Omega|\) do
                \(\phi_{j} \leftarrow \sqrt{\left(1-\frac{\xi}{\rho} a_{(j)}^{-1}\right)^{2} \sum_{i=0}^{j} a_{(i)}^{2}+(|\Omega|-j)\left(a_{(j)}-\frac{\xi}{\rho}\right)^{2}}\)
                if \(\phi_{j} \leq \delta\) then
                    \(j^{*} \leftarrow j\)
                end if
            end for
            if \(j^{*}==|\Omega|\) then
                \(\theta^{*} \leftarrow \rho\left(\frac{\left\|\pi_{\Omega}(A)\right\|_{F}}{\delta}-1\right)\)
            else
                Compute unique \(\theta^{*}>0\) by finding the roots of \(\left(\frac{\rho}{\rho+\theta^{*}}\right)^{2} \sum_{i=0}^{j^{*}} a_{(i)}^{2}+(|\Omega|-1)\left(\frac{\xi}{\theta^{*}}\right)^{2}\)
            end if
        end if
    end if
```

Fig. 4.2. ThetaSearch: Subroutine for computing the optimal dual $\theta^{*}$

ADMIP and ASALM to compute low-rank SVDs more efficiently. This modification significantly reduced the total number singular values computed by ASALM when compared to the code provided by the authors of [29].

For all three algorithms, ADMIP, ADMM, and ASALM, we set the initial iterate $\left(Z_{0}, Y_{0}\right)=(\mathbf{0}, \mathbf{0})$. For ADMIP the penalty multiplier sequence $\left\{\rho_{k}\right\}_{k \in \mathbb{Z}_{+}}$was chosen as follows:

$$
\begin{equation*}
\rho_{0}=\rho_{1}=1.25 / \sigma_{\max }\left(\pi_{\Omega}(D)\right), \quad \rho_{k+1}=\min \left\{\kappa \rho_{k}, \bar{\rho}+k\right\}, \quad k \geq 1 \tag{4.2}
\end{equation*}
$$

where $\kappa=1.25, \bar{\rho}=1000 \rho_{0}$, and $\pi_{\Omega}(\cdot)$ is defined in (1.5). Note that for ADMM and ASALM, $\rho_{k}=\rho$ for some $\rho>0$ for all $k \geq 1$.

See Figure 4.1 for an implementable pseudocode for ADMIP: line 5 follows from 4.1), and lines 8 and 9 follow from Lemma [2.1] since $\theta^{*}$ computed in line 7 satisfies the conditions given in Lemma 2.1 with $Q=-Y_{k}$, $\tilde{Z}=L_{k+1}$, and $\rho=\rho_{k}$. Subroutine ThetaSearch in Figure 4.2 uses the procedure outlined in the proof of Lemma 2.1] to compute $\theta^{*}$ in $\mathcal{O}(|\Omega| \log (|\Omega|))$ time. Also, note that the roots of the quartic equation in line 21 of Figure 4.1 can be computed in closed form using the formula first shown by Lodovico Ferrari, and later published in Cardano's Ars Magna in 1545 [7.
4.2. Random SPCP problems. For a given sparsity coefficient $c_{s} \in\{0.05,0.1\}$ and a rank coefficient $c_{r} \in\{0.05,0.1\}$, the data matrix $D=L^{0}+S^{0}+N^{0}$ was generated as follows:
i. $L^{0}=U V^{T}$, with $U \in \mathbb{R}^{n \times r}, V \in \mathbb{R}^{n \times r}$ for $r=\left\lceil c_{r} n\right\rceil$, and for all $i, j, U_{i j}, V_{i j}$, were independently drawn from a Gaussian distribution with mean 0 and variance 1.
ii. $\Lambda \subset\{(i, j): 1 \leq i, j \leq n\}:=I$ was chosen uniformly at random such that its cardinality $|\Lambda|=\left\lceil c_{s} n^{2}\right\rceil$, iii. For each $i, j, S_{i j}^{0}$ was independently drawn from a uniform distribution over the interval $\left[-\sqrt{\frac{8 r}{\pi}}, \sqrt{\frac{8 r}{\pi}}\right]$. iv. For each $i, j, N_{i j}^{0}$ was independently drawn from a Gaussian distribution with mean 0 and variance $\varrho^{2}$. This construction ensures that, on average, the the magnitude of the non-zero entries of the sparse component $S^{0}$ is of the same order as the entries of the low-rank component $L^{0}$, i.e. $\mathbb{E}\left[\left|L_{i_{1} j_{1}}^{0}\right|\right]=\mathbb{E}\left[\left|S_{i_{2} j_{2}}^{0}\right|\right]$ for all $\left(i_{1}, j_{1}\right) \in I$ and for all $\left(i_{2}, j_{2}\right) \in \Lambda$.

Let $\Omega \subset\{1, \ldots, n\} \times\{1, \ldots, n\}$ denote the set indices of the observable entries of $D$, and let $\mathrm{SR}=\frac{|\Omega|}{\mathrm{n}^{2}}$ denote the sampling ratio of $D$. Then, the signal-to-noise ratio is given by

$$
\begin{equation*}
\mathrm{SNR}=10 \log _{10}\left(\frac{\mathrm{E}\left[\left\|\pi_{\Omega}\left(\mathrm{L}^{0}+\mathrm{S}^{0}\right)\right\|_{\mathrm{F}}^{2}\right]}{\mathrm{E}\left[\left\|\pi_{\Omega}\left(\mathrm{N}^{0}\right)\right\|_{\mathrm{F}}^{2}\right]}\right)=10 \log _{10}\left(\frac{\mathrm{c}_{\mathrm{r}} \mathrm{n}+\mathrm{c}_{\mathrm{s}} \frac{8 \mathrm{r}}{3 \pi}}{\varrho^{2}}\right) . \tag{4.3}
\end{equation*}
$$

In all the numerical test problems, the value for the noise variance $\varrho^{2}$ was set to ensure a certain SNR level, i.e. $\varrho^{2}=\left(c_{r} n+c_{s} \frac{8 r}{3 \pi}\right) 10^{-\mathrm{SNR} / 10}$. We set $\delta=\sqrt{(n+\sqrt{8 n})} \varrho$ (see [29]).


Fig. 4.3. Iteration complexity of ADMM as a function $\rho$
4.2.1. ADMM vs ADMIP. We created 5 random problem instances of size $n=500$, for each of the two choices of $c_{s}$ and $c_{r}$ such that $\mathrm{SNR}=80 \mathrm{~dB}$ using the procedure described above in Section 4.2 Both ADMM and ADMIP were terminated when the following primal-dual stopping condition holds

$$
\begin{equation*}
\frac{\left\|L_{k+1}-Z_{k+1}\right\|_{F}}{\|D\|_{F}} \leq \operatorname{tol}_{p}, \quad \frac{\rho_{k}\left\|Z_{k+1}-Z_{k}\right\|_{F}}{\|D\|_{F}} \leq \operatorname{tol}_{d} \tag{4.4}
\end{equation*}
$$

See Section 3.3.1 in [5] for a detailed discussion of this stopping condition. In our experiments, we set tol $_{p}=$ $\boldsymbol{t o l}_{d}=8.9 \times 10^{-5}$ for both ADMIP and ADMM. For each $c_{s} \in\{0.05,0.1\}, c_{r} \in\{0.05,0.1\}$, and penalty parameter $\rho \in\{0.025 i: 1 \leq i \leq 50\} \subset[0.025,1.25]$, we used ADMM to solve 5 random instances. We plot the performance of ADMM as a function of $\rho$ in Figure 4.3. The solid line corresponds to the average over the five instances, and the dashed lines around the solid lines plot the maximum and minimum values over the 5 random
instances. The results of our experiments comparing ADMM with ADMIP are summarized in Table 4.1. For each random problem instance, the reported ADMM performance corresponds to the $\rho^{*}$ value that minimizes the number of iterations required for termination. The last column in Table 4.1 reports the range of $\rho^{*}$ over 5 random instances. The column labeled iter (resp. cpu) lists the minimum/average/maximum number of total number of iterations (resp. computation time in seconds) required to solve the 5 instances. The columns labeled relL and relS list the average relative error in the estimate of the low-rank component $\left\|L^{\text {sol }}-L^{0}\right\|_{F} /\left\|L^{0}\right\|_{F}$ and the estimate of the sparse component $\left\|S^{\text {sol }}-S^{0}\right\|_{F} /\left\|S^{0}\right\|_{F}$, respectively, where ( $L^{\text {sol }}, S^{\text {sol }}$ ) is the output of the particular algorithm considered. It is clear from Table 4.1that ADMIP requires significantly fewer iterations. Moreover, the range of optimal fixed penalty $\rho^{*}$ for ADMM shifts as problem parameters $c_{s}$ and $c_{r}$ change, making it even harder to estimate $\rho^{*}$. On the other hand, ADMIP does not require tuning of any problem dependent parameter.

Table 4.1
Comparison of ADMIP and ADMM

| Parameters | Algorithm | iter | cpu | relL | relS | $\rho^{*}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{c}_{\mathbf{s}}=0.05$ | ADMIP | 13/18.6/26 | 2.1/5.9/11.8 | $4.7 \mathrm{E}-5$ | 2.2E-4 | n/a |
| $\mathrm{c}_{\mathbf{r}}=0.05$ | ADMM | 68/88.6/101 | 16.8/22.5/25.1 | $3.4 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | [0.15, 0.225] |
| $\mathrm{c}_{\mathbf{s}}=0.1$ | ADMIP | 19/20.4/22 | 3.3/3.6/3.9 | 3.5E-5 | $1.3 \mathrm{E}-4$ | n/a |
| $\mathrm{c}_{\mathrm{r}}=0.05$ | ADMM | 63/69.2/77 | 17.7/20.0/21.7 | 3.6E-5 | $1.4 \mathrm{E}-4$ | [0.125, 0.15] |
| $\mathrm{c}_{\mathrm{s}}=0.05$ | ADMIP | 14/14/14 | 2.2/2.3/2.5 | $4.9 \mathrm{E}-5$ | $1.4 \mathrm{E}-4$ | n/a |
| $\mathrm{c}_{\mathrm{r}}=0.1$ | ADMM | 61/63/65 | 18.3/18.7/19.4 | $4.8 \mathrm{E}-5$ | $1.8 \mathrm{E}-4$ | [0.075, 0.1] |
| $\mathrm{c}_{\mathrm{s}}=0.1$ | ADMIP | 23/23/23 | 4.2/4.2/4.3 | $5.4 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | n/a |
| $\mathrm{c}_{\mathrm{r}}=0.1$ | ADMM | 62/65.4/69 | 19.6/21.5/19.4 | 5.3E-5 | 1.9E-4 | [0.075, 0.075] |

4.2.2. Performance of ADMIP as a function of problem parameters. Table 4.2 and Table 4.3 report the results of the numerical experiments that we conducted to determine how the run times and other performance measures for ADMIP scale with the problem size n , the rank of the low-rank component $\left\lceil c_{r} n\right\rceil$, the number of non-zero entries of the sparse component $\left\lceil c_{s} n^{2}\right\rceil$, the sampling ratio SR , and the SNR. For this set of experiments, we set the tolerances in (4.4) to $\boldsymbol{t o l}_{\mathbf{p}}=\operatorname{tol}_{\mathbf{d}}=1 \times 10^{-4}$.

The column labeled iter, lsv, cpu, relL and relS list, respectively, the number of iterations required to solve the instance, the average number of leading singular values computed per iteration by ADMIP, the total cpu time in second, the relative error in the low rank component $L^{0}$, and the relative error in the low rank component $S^{0}$, averaged over the 5 random instances. Table 4.2 corresponds to 80 dB , and Table 4.3 corresponds to 40 dB . The results in Table 4.2 and Table 4.3 show that the number of partial SVDs ranges from 11 to 29 when SNR is $80 d B$, and from 20 to 37 when SNR is $40 d B$. Moreover, the relative error of the solution depends only on SNR value, and almost independent of all the other parameters.
4.2.3. ASALM vs ADMIP. We created 5 random problem instances of size $n=500$, for each of the two choices of $c_{s}, c_{r}$, SNR and SR using the procedure described in Section 4.2, and we compared ADMIP with ASALM $[29$ on these random problems. In these numerical tests, we set tol $=0.05$, and terminated ADMIP using the stopping condition

$$
\begin{equation*}
\frac{\left\|\left(L_{k+1}, S_{k+1}\right)-\left(L_{k}, S_{k}\right)\right\|_{F}}{\left\|\left(L_{k}, S_{k}\right)\right\|_{F}+1} \leq \text { tol } \varrho . \tag{4.5}
\end{equation*}
$$

We terminated ASALM either when it computed a solution with a smaller relative error compared to the ADMIP solution for the same problem instance or when an iterate satisfied (4.5). Note that this experimental setup favors ASALM over ADMIP. The results for the two algorithms are displayed in Table 4.4, where the reported statistics iter, cpu, lsv, relL, and relS are defined in Section 4.2.2. From the results in Table 4.4 we see that for all of the problem classes, ASALM requires about twice as many iterations for convergence. But, the cpu time for ASALM is considerably larger; this difference can be explained by the fact that on average ASALM computes a larger number of leading singular values per iteration as compared to ADMIP. This is clear from the lsv statistics reported for both algorithms. The results in Table 4.4 also show that although the relative errors in the low-rank and sparse components produced by ADMIP and ASALM were of the same order, the error of ADMIP solutions were consistently lower than those of the ASALM solutions.

Table 4.2
Performance of ADMIP on random test problems with missing data, $\operatorname{SNR}(D)=80 d B$

|  |  | SR=100\% |  |  |  |  | SR=90\% |  |  |  |  | SR=80\% |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| n | $\left(\mathbf{c}_{\mathbf{s}}, \mathbf{c}_{\mathbf{r}}\right)$ | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS |
| 500 | (0.05,0.05) | 11.6 | 35.2 | 2.2 | $4.1 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | 13.2 | 35.1 | 2.4 | $4.0 \mathrm{E}-5$ | $1.3 \mathrm{E}-4$ | 29.0 | 78.5 | 9.7 | $7.2 \mathrm{E}-5$ | $4.1 \mathrm{E}-4$ |
|  | (0.1,0.05) | 17.2 | 34.8 | 2.9 | $4.3 \mathrm{E}-5$ | $1.8 \mathrm{E}-4$ | 17.8 | 34.8 | 2.9 | $4.8 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 19.0 | 34.7 | 2.7 | $5.6 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ |
|  | (0.05,0.1) | 13.0 | 58.0 | 2.2 | $5.8 \mathrm{E}-5$ | $1.8 \mathrm{E}-4$ | 15.6 | 58.0 | 2.5 | $7.0 \mathrm{E}-5$ | $1.9 \mathrm{E}-4$ | 19.8 | 58.0 | 2.9 | $8.3 \mathrm{E}-5$ | $2.0 \mathrm{E}-4$ |
|  | (0.1,0.1) | 21.2 | 58.0 | 3.6 | $6.4 \mathrm{E}-5$ | $2.2 \mathrm{E}-4$ | 23.0 | 58.0 | 4.1 | 7.2E-5 | $2.2 \mathrm{E}-4$ | 25.0 | 58.0 | 4.2 | $1.3 \mathrm{E}-4$ | $3.6 \mathrm{E}-4$ |
| 1000 | (0.05,0.05) | 11.0 | 61.4 | 6.7 | $4.5 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 12.0 | 61.1 | 6.7 | $5.4 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | 14.0 | 60.6 | 6.8 | $4.9 \mathrm{E}-5$ | $1.4 \mathrm{E}-4$ |
|  | (0.1,0.05) | 17.0 | 60.2 | 11.3 | $4.2 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 17.8 | 60.1 | 9.9 | $4.6 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | 18.8 | 60.0 | 9.3 | $5.5 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ |
|  | (0.05,0.1) | 13.4 | 105.0 | 8.5 | $5.6 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 15.0 | 105.0 | 7.6 | $7.5 \mathrm{E}-5$ | $2.0 \mathrm{E}-4$ | 19.0 | 105.0 | 9.3 | $8.3 \mathrm{E}-5$ | $1.9 \mathrm{E}-4$ |
|  | (0.1,0.1) | 21.4 | 105.0 | 13.0 | $6.3 \mathrm{E}-5$ | $2.2 \mathrm{E}-4$ | 23.0 | 105.0 | 12.0 | $7.0 \mathrm{E}-5$ | $2.1 \mathrm{E}-4$ | 25.0 | 105.0 | 13.0 | $8.8 \mathrm{E}-5$ | $2.2 \mathrm{E}-4$ |
| 1500 | (0.05,0.05) | 11.0 | 86.6 | 13.2 | $4.5 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 12.0 | 86.2 | 17.9 | $5.2 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | 14.0 | 85.4 | 17.9 | $4.9 \mathrm{E}-5$ | $1.3 \mathrm{E}-4$ |
|  | (0.1,0.05) | 17.0 | 84.6 | 21.1 | $4.2 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 17.6 | 84.5 | 26.0 | $4.7 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ | 18.4 | 84.4 | 26.5 | $5.9 \mathrm{E}-5$ | $1.7 \mathrm{E}-4$ |
|  | (0.05,0.1) | 13.4 | 153.0 | 22.2 | $5.5 \mathrm{E}-5$ | $1.6 \mathrm{E}-4$ | 15.0 | 153.0 | 24.5 | 7.2E-5 | $1.9 \mathrm{E}-4$ | 19.0 | 153.0 | 36.3 | $8.0 \mathrm{E}-5$ | $1.9 \mathrm{E}-4$ |
|  | (0.1,0.1) | 21.0 | 153.0 | 34.5 | 6.3E-5 | $2.2 \mathrm{E}-4$ | 23.0 | 153.0 | 35.6 | 7.0E-5 | $2.2 \mathrm{E}-4$ | 25.0 | 153.0 | 47.8 | 8.7E-5 | $2.2 \mathrm{E}-4$ |

Table 4.3
Performance of ADMIP on random test problems with missing data, $\operatorname{SNR}(D)=40 d B$

|  |  | SR=100\% |  |  |  |  | SR=90\% |  |  |  |  | SR=80\% |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| n | $\left(\mathbf{c}_{\mathbf{s}}, \mathbf{c}_{\mathbf{r}}\right)$ | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS |
| 500 | (0.05,0.05) | 29.8 | 178.2 | 19.2 | $6.7 \mathrm{E}-3$ | $3.6 \mathrm{E}-2$ | 27.2 | 153.2 | 14.6 | $6.8 \mathrm{E}-3$ | $3.8 \mathrm{E}-2$ | 30.4 | 136.9 | 13.8 | $7.0 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ |
|  | (0.1,0.05) | 34.0 | 161.3 | 19.1 | $7.5 \mathrm{E}-3$ | $2.8 \mathrm{E}-2$ | 31.2 | 137.7 | 14.9 | $7.6 \mathrm{E}-3$ | $3.0 \mathrm{E}-2$ | 34 | 124.1 | 14.8 | $7.9 \mathrm{E}-3$ | $3.2 \mathrm{E}-2$ |
|  | (0.05,0.1) | 26.2 | 168.1 | 14.6 | $8.1 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ | 28 | 148.4 | 13.4 | $8.9 \mathrm{E}-3$ | $4.4 \mathrm{E}-2$ | 33 | 129.8 | 13.4 | $1.0 \mathrm{E}-2$ | $5.0 \mathrm{E}-2$ |
|  | (0.1,0.1) | 29.8 | 152.4 | 14.9 | $9.4 \mathrm{E}-3$ | $3.4 \mathrm{E}-2$ | 32 | 139.7 | 15.0 | $1.0 \mathrm{E}-2$ | $3.7 \mathrm{E}-2$ | 36.8 | 130.5 | 15.3 | $1.2 \mathrm{E}-2$ | $4.2 \mathrm{E}-2$ |
| 1000 | (0.05,0.05) | 20.0 | 279.8 | 52.8 | $6.8 \mathrm{E}-3$ | $3.6 \mathrm{E}-2$ | 21.0 | 250.5 | 48.4 | $6.8 \mathrm{E}-3$ | $3.8 \mathrm{E}-2$ | 23.0 | 228.3 | 50.7 | $7.0 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ |
|  | (0.1,0.05) | 25.0 | 251.8 | 62.3 | $7.6 \mathrm{E}-3$ | $2.8 \mathrm{E}-2$ | 26.0 | 229.8 | 56.8 | $7.6 \mathrm{E}-3$ | $3.0 \mathrm{E}-2$ | 27.0 | 200.7 | 49.9 | $7.9 \mathrm{E}-3$ | $3.2 \mathrm{E}-2$ |
|  | (0.05,0.1) | 21.8 | 290.1 | 55.1 | $8.1 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ | 23.0 | 255.6 | 50.5 | $8.9 \mathrm{E}-3$ | $4.4 \mathrm{E}-2$ | 26.0 | 220.2 | 42.4 | $1.0 \mathrm{E}-2$ | $5.0 \mathrm{E}-2$ |
|  | (0.1,0.1) | 26.8 | 269.7 | 63.0 | $9.4 \mathrm{E}-3$ | $3.4 \mathrm{E}-2$ | 28.0 | 245.3 | 61.6 | $1.0 \mathrm{E}-2$ | $3.6 \mathrm{E}-2$ | 29.0 | 214.1 | 48.3 | $1.2 \mathrm{E}-2$ | $4.1 \mathrm{E}-2$ |
| 1500 | (0.05,0.05) | 20.0 | 417.2 | 174.0 | $6.8 \mathrm{E}-3$ | $3.7 \mathrm{E}-2$ | 21.0 | 374.9 | 165.0 | $6.8 \mathrm{E}-3$ | $3.8 \mathrm{E}-2$ | 21.0 | 314.8 | 130.4 | $7.1 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ |
|  | (0.1,0.05) | 25.0 | 376.8 | 198.1 | $7.6 \mathrm{E}-3$ | $2.9 \mathrm{E}-2$ | 26.0 | 343.6 | 189.1 | $7.7 \mathrm{E}-3$ | $3.0 \mathrm{E}-2$ | 26.0 | 287.0 | 148.4 | $8.0 \mathrm{E}-3$ | $3.2 \mathrm{E}-2$ |
|  | (0.05,0.1) | 22.2 | 440.1 | 190.0 | $8.1 \mathrm{E}-3$ | $4.1 \mathrm{E}-2$ | 23.0 | 381.7 | 170.2 | $8.8 \mathrm{E}-3$ | $4.5 \mathrm{E}-2$ | 26.0 | 329.1 | 150.6 | $1.0 \mathrm{E}-2$ | $5.0 \mathrm{E}-2$ |
|  | (0.1,0.1) | 27.0 | 412.9 | 211.3 | $9.4 \mathrm{E}-3$ | $3.4 \mathrm{E}-2$ | 28.0 | 365.4 | 204.5 | $1.0 \mathrm{E}-2$ | $3.7 \mathrm{E}-2$ | 29.0 | 318.7 | 164.4 | $1.2 \mathrm{E}-2$ | $4.1 \mathrm{E}-2$ |

Table 4.4
Comparison of ADMIP and ASALM

|  |  |  | SR=100\% |  |  |  |  | SR=90\% |  |  |  |  | SR=80\% |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SNR | $\left(\mathbf{c}_{\mathbf{s}}, \mathrm{c}_{\mathbf{r}}\right)$ | Algorithm | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS | iter | lsv | cpu | relL | relS |
|  | (0.05, | ADMIP | 12 | 86.2 | 12.5 | $3.5 \mathrm{E}-5$ | $1.3 \mathrm{E}-4$ | 13 | 85.8 | 12.8 | $3.9 \mathrm{E}-5$ | $1.3 \mathrm{E}-4$ | 15 | 85.1 | 13.7 | $4.1 \mathrm{E}-5$ | $1.3 \mathrm{E}-4$ |
|  | (0.05, 0.05) | ASALM | 28.4 | 123.9 | 68.7 | $4.6 \mathrm{E}-5$ | $4.8 \mathrm{E}-4$ | 29.6 | 138.3 | 76.9 | $5.0 \mathrm{E}-5$ | $5.1 \mathrm{E}-4$ | 33.4 | 146.1 | 50.4 | $5.5 \mathrm{E}-5$ | $4.7 \mathrm{E}-4$ |
|  |  | ADMIP | 18 | 84.4 | 17.7 | $3.7 \mathrm{E}-5$ | $1.4 \mathrm{E}-4$ | 18 | 84.4 | 17.1 | 4.4E-5 | $1.5 \mathrm{E}-4$ | 19.2 | 84.2 | 16.9 | $4.9 \mathrm{E}-5$ | $1.4 \mathrm{E}-4$ |
| 80 dB | (0.1, 0.05 ) | ASALM | 32.4 | 177.6 | 109.9 | $4.7 \mathrm{E}-5$ | $3.2 \mathrm{E}-4$ | 37.2 | 187.1 | 127.0 | $4.8 \mathrm{E}-5$ | $2.9 \mathrm{E}-4$ | 42 | 194.0 | 83.8 | 5.6E-5 | $2.9 \mathrm{E}-4$ |
| 80 dB |  | ADMIP | 14.2 | 153.0 | 15.9 | $4.9 \mathrm{E}-5$ | $1.4 \mathrm{E}-4$ | 16 | 153.0 | 18.6 | 5.8E-5 | $1.6 \mathrm{E}-4$ | 19 | 153.0 | 20.4 | $8.0 \mathrm{E}-5$ | $1.9 \mathrm{E}-4$ |
|  | (0.05, 0.1) | ASALM | 29.2 | 203.2 | 86.2 | $7.7 \mathrm{E}-5$ | $6.6 \mathrm{E}-4$ | 32.8 | 220.0 | 112.5 | 8.6E-5 | $6.6 \mathrm{E}-4$ | 41 | 228.4 | 79.1 | $9.3 \mathrm{E}-5$ | 5.6E-4 |
|  | (0.1, 0 | ADMIP | 21 | 153.0 | 26.0 | $6.3 \mathrm{E}-5$ | $2.2 \mathrm{E}-4$ | 23 | 153.0 | 26.5 | 7.0E-5 | $2.2 \mathrm{E}-4$ | 25 | 153.0 | 27.1 | $8.7 \mathrm{E}-5$ | $2.2 \mathrm{E}-4$ |
|  | (0.1, 0.1) | ASALM | 34.8 | 272.0 | 148.4 | $8.0 \mathrm{E}-5$ | $4.6 \mathrm{E}-4$ | 43 | 282.5 | 197.1 | $8.3 \mathrm{E}-5$ | $3.9 \mathrm{E}-4$ | 55 | 285.6 | 138.5 | $9.5 \mathrm{E}-5$ | 3.6E-4 |
|  |  | ADMIP | 7 | 89.9 | 10.5 | $3.5 \mathrm{E}-3$ | $1.4 \mathrm{E}-2$ | 8 | 88.8 | 7.7 | $3.7 \mathrm{E}-3$ | $1.5 \mathrm{E}-2$ | 8 | 88.8 | 7.7 | $4.3 \mathrm{E}-3$ | 1.6E-2 |
|  | (0.05, 0.05) | ASALM | 15 | 205.3 | 42.1 | $4.6 \mathrm{E}-3$ | $3.0 \mathrm{E}-2$ | 18 | 210.3 | 45.1 | 5.1E-03 | $3.3 \mathrm{E}-02$ | 20 | 207.1 | 45.8 | 5.8E-3 | $3.7 \mathrm{E}-2$ |
|  | (0.1, 0.05) | ADMIP | 9 | 87.9 | 12.1 | $3.8 \mathrm{E}-3$ | $1.5 \mathrm{E}-2$ | 9.8 | 87.3 | 9.1 | 4.1E-3 | $1.5 \mathrm{E}-2$ | 10 | 87.2 | 9.2 | $4.7 \mathrm{E}-3$ | 1.6E-2 |
| 40dB | (0.1, 0. | ASALM | 20 | 292.2 | 78.4 | $6.1 \mathrm{E}-3$ | $2.7 \mathrm{E}-2$ | 24 | 296.6 | 81.4 | $6.8 \mathrm{E}-03$ | $2.9 \mathrm{E}-02$ | 28 | 285.5 | 85.5 | $7.4 \mathrm{E}-3$ | $3.1 \mathrm{E}-2$ |
| 40 dB |  | ADMIP | 8 | 153.0 | 12.5 | $5.1 \mathrm{E}-3$ | $1.9 \mathrm{E}-2$ | 8.2 | 153.0 | 9.0 | $6.0 \mathrm{E}-3$ | $2.1 \mathrm{E}-2$ | 9 | 153.0 | 9.4 | 7.6E-3 | $2.5 \mathrm{E}-2$ |
|  | (0.05, 0 | ASALM | 16 | 267.3 | 47.1 | $5.7 \mathrm{E}-3$ | $3.2 \mathrm{E}-2$ | 20 | 280.5 | 53.5 | $6.9 \mathrm{E}-03$ | $3.7 \mathrm{E}-02$ | 24 | 289.7 | 65.0 | 8.2E-3 | $4.0 \mathrm{E}-2$ |
|  |  | ADMIP | 9 | 153.0 | 14.6 | $6.1 \mathrm{E}-3$ | $2.0 \mathrm{E}-2$ | 10 | 153.0 | 10.9 | 6.9E-3 | $2.2 \mathrm{E}-2$ | 11 | 153.0 | 11.9 | 8.2E-3 | $2.5 \mathrm{E}-2$ |
|  | (0.1, 0.1) | ASALM | 23 | 364.6 | 96.7 | 7.0E-3 | $2.9 \mathrm{E}-2$ | 28 | 373.5 | 102.1 | 7.8E-03 | 3.1E-02 | 35.8 | 370.7 | 124.1 | 8.9E-3 | $3.2 \mathrm{E}-2$ |



FIG. 4.4. Background extraction from a video with $\mathbf{S N R}=20 \mathrm{~dB}$ and $\mathbf{S R}=100 \%$ using ADMIP


Fig. 4.5. Background extraction from a video with $\mathbf{S N R}=20 \mathrm{~dB}$ and $\mathbf{S R}=60 \%$ using $A D M I P$
4.3. Foreground detection problem. Extracting the almost still background from a sequence of frames in a noisy video is an important task in video surveillance, and it can be formulated as SPCP problem. Let $X_{t}$ denote the $t$-th video frame, and $x_{t} \in \mathbb{R}^{R}$ is obtained by stacking the columns of $X_{t}$, where $R$ is the resolution. Suppose the background is completely stationary, and there is no measurement noise. Then $x_{t}=b+f_{t}$, where $b$ denotes the background and $f_{t}$ denotes the sparse foreground in the $t$-th frame. Let $D=\left[x_{1}, \ldots, x_{T}\right]=b \mathbf{1}^{\top}+\left[f_{1}, \ldots, f_{T}\right]$, i.e. rank 1 matrix + sparse matrix. In real videos, the background is never completely stationary, and there is always measurement noise; therefore, we expect that $D$ can be decomposed into the sum of three matrices $D=L^{0}+S^{0}+N^{0}$, where $L^{0}$ is a low rank and $S^{0}$ is a sparse matrix that represent the background and the foreground, respectively, and $N^{0}$ is a dense noise matrix.

Table 4.5
ADMIP vs ASALM: Recovery statistics for foreground detection on a noisy video, $\mathbf{S N R}=20 d B$

|  | ASALM |  |  | ADMIP $(\kappa=1.5)$ |  |  | ADMIP $(\kappa=1.25)$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SR | svd | lsv | cpu | svd | lsv | cpu | svd | lsv | cpu |
| $\mathbf{1 0 0 \%}$ | 91 | 64.7 | 198.8 | 16 | 142.5 | 105.9 | 26 | 63.3 | 192.2 |
| $\mathbf{6 0 \%}$ | 154 | 6.5 | 152.2 | 15 | 15.6 | 63.2 | 24 | 14.8 | 110.3 |

We used ADMIP and ASALM to extract the foreground in an airport surveillance video consisting of $T=201$ grayscale $144 \times 176$ frames [21], i.e $R=25,344$. In order to test the reconstruction performance of both algorithms under missing data, we created a test video by masking some of the pixels, i.e. we assumed that the sensors corresponding to these positions were malfunctioning, and therefore, not acquiring the signal. We also injected artificial white noise to the remaining pixels in order to create a video with prescribed SNR. Let SR denote the fraction of observed pixels. The locations $\Omega$ of the observed pixels were chosen uniformly at random from the set $\{1, \ldots, T\} \times\{1, \ldots, R\}$ such that the cardinality $|\Omega|=\lceil\mathrm{SR}$ T R $\rceil$. We created a noisy test video with $\mathrm{SNR}=20 \mathrm{~dB}$ by setting $\varrho=\left\|\pi_{\Omega}(D)\right\|_{F} /\left(\sqrt{|\Omega|} 10^{\mathrm{SNR} / 20}\right)$, and then for all $(i, j) \in \Omega$ by resetting $D_{i j}=D_{i j}+N_{i j}$, where each $N_{i j}$ were independently drawn from a Gaussian distribution with mean zero and variance $\varrho^{2}$. ADMIP and ASALM were terminated according to (4.5), where tol is $5 \times 10^{-6}$ for both ADMIP and ASALM.

We compared the performance of ADMIP with ASALM on the video problem with full data $\mathrm{SR}=100 \%$, and with partial data $\mathrm{SR}=60 \%$. On each problem instance, we ran ADMIP with $\kappa=1.5$ and $\kappa=1.25$, where $\kappa$ is the parameter that controls of the rate of growth of $\rho_{k}$ in (4.2). The frames recovered by ASALM were very similar to those of ADMIP due to same stopping condition used; therefore, we only show the frames recovered by ADMIP. The first rows in Figure 4.4 and Figure 4.5 display the 35 -th, 100 -th and 125 -th frames of the noisy surveillance video [21] for $S R=100 \%$ and $S R=60 \%$, respectively. The second and third rows display the recovered background and foreground images of the selected frames, respectively, using ADMIP. Both ADMIP and ASALM were able to recover the foreground and the background fairly accurately with only $60 \%$ of the pixels functioning. Even though the visual quality of recovered background and foreground are very similar for both algorithms, the statistics reported in Table 4.5 shows that both iteration count and cpu time of ADMIP are smaller than those of ASALM. Note that, although ADMIP with $\kappa=1.5$ has the least cpu time, the values for the lsv statistic for ADMIP with $\kappa=1.5$ is significantly higher than the corresponding values for ASALM and ADMIP with $\kappa=1.25$. Indeed, for large problem sizes, ADMIP has two different computational bottleneck. The first one is the computation of the low rank term $L_{k+1}$. For larger values of $\kappa$, the parameter $\rho_{k}$ grows faster; therefore, it follows from (4.1) that the number of leading singular values computed in each iteration grows. On the other hand, in order to compute $S_{k+1}$, we need to sort $|\Omega|$ numbers. This sorting operation with $\mathcal{O}(|\Omega| \log (|\Omega|))$ complexity becomes a computational bottleneck when $|\Omega|$ is large, especially when $\mathrm{SR}=100 \%$. Moreover, large values for $\kappa$ reduces the number of iterations, and consequently, the number of sortings required. From the numerical experiments, it appears that the sorting is a computationally more critical step; therefore, $\kappa=1.5$ reduces the overall cpu time in comparison to $\kappa=1.25$.

In our preliminary numerical experiments, we noticed that the recovered background frames are almost noise free even when the input video was very noisy, and all the noise shows up in the recovered foreground images. This was observed for both ADMIP and ASALM. Hence, in order to eliminate the noise seen in the
recovered foreground frames and enhance the quality of the recovered frames, we post-process $\left(L^{\text {sol }}, S^{\text {sol }}\right)$ of ADMIP as follows:

$$
\begin{equation*}
S_{\text {post }}^{\text {sol }}:=\underset{S}{\operatorname{argmin}}\left\{\|S\|_{1}:\left\|S+L^{\text {sol }}-D\right\|_{F} \leq \delta\right\} \tag{4.6}
\end{equation*}
$$

The fourth rows of Figure 4.4 and Figure 4.5 show the post-processed foreground frames.
5. Conclusions. In this paper, we propose an alternating direction method of multipliers with increasing penalty parameter sequence, ADMIP, for solving stable PCA problems. We prove that primal-dual iterate sequence converges to an optimal pair when the sequence of penalty parameters $\left\{\rho_{k}\right\}$ in non-decreasing, and unbounded. We also report numerical results comparing ADMIP with constant penalty ADMM on synthetic random test problems and on foreground-background separation problems. The results clearly show that ADMIP is able to solve huge problems involving million variables much more effectively when compared to the constant penalty ADMM. To the best of our knowledge, ADMIP is the first variable penalty ADMM that is guaranteed to converge to a primal-dual optimal pair when penalties are not bounded, the objective function is non-smooth and its subdifferential is not uniformly bounded. However, the proof of convergence of ADMIP iterates heavily leverages the problem structure. In future work, we plan to extend ADMIP to solve a more general set of convex optimization problems of the form $\min \{f(x)+g(y): A x+B y=b\}$, where $f$ and $g$ are non-smooth closed convex functions, and investigate the growth rate conditions on unbounded $\left\{\rho_{k}\right\}$ that guarantee primal and dual convergence.
6. Acknowledgements. We would like to thank to Min Tao for providing the code ASALM.

## Appendix A. Proofs.

A.1. Proof of Lemma 2.1. Suppose $\delta>0$. Let $\left(Z^{*}, S^{*}\right)$ be an optimal solution to problem $\left(P_{n s}\right), \theta^{*}$ denote the optimal Lagrangian multiplier for the constraint $(Z, S) \in \chi$ written as $\frac{1}{2}\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2} \leq \frac{\delta^{2}}{2}$ and $\pi_{\Omega}^{*}$ denotes the adjoint operator of $\pi_{\Omega}$. Note that $\pi_{\Omega}^{*}=\pi_{\Omega}$. Then the KKT conditions for this problem are given by

$$
\begin{align*}
Q+\rho\left(Z^{*}-\tilde{Z}\right)+\theta^{*} \pi_{\Omega}\left(Z^{*}+S^{*}-D\right) & =0  \tag{A.1}\\
\xi G+\theta^{*} \pi_{\Omega}\left(Z^{*}+S^{*}-D\right) & =0, \quad G \in \partial\left\|S^{*}\right\|_{1}  \tag{A.2}\\
\left\|\pi_{\Omega}\left(Z^{*}+S^{*}-D\right)\right\|_{F} & \leq \delta  \tag{A.3}\\
\theta^{*} & \geq 0  \tag{A.4}\\
\theta^{*}\left(\left\|\pi_{\Omega}\left(Z^{*}+S^{*}-D\right)\right\|_{F}-\delta\right) & =0 \tag{A.5}
\end{align*}
$$

where (A.1) and (A.2) follow from the fact that $\pi_{\Omega} \pi_{\Omega}=\pi_{\Omega}$.
From (A.1) and (A.2), we get

$$
\begin{equation*}
\pi_{\Omega^{c}}\left(Z^{*}\right)=\pi_{\Omega^{c}}(q(\tilde{Z})), \quad \pi_{\Omega^{c}}(G)=\mathbf{0} \tag{A.6}
\end{equation*}
$$

and

$$
\left[\begin{array}{cc}
\left(\rho+\theta^{*}\right) I & \theta^{*} I  \tag{A.7}\\
\theta^{*} I & \theta^{*} I
\end{array}\right]\left[\begin{array}{c}
\pi_{\Omega}\left(Z^{*}\right) \\
\pi_{\Omega}\left(S^{*}\right)
\end{array}\right]=\left[\begin{array}{c}
\pi_{\Omega}\left(\theta^{*} D+\rho q(\tilde{Z})\right) \\
\pi_{\Omega}\left(\theta^{*} D-\xi G\right)
\end{array}\right]
$$

where $q(\tilde{Z})=\tilde{Z}-\rho^{-1} Q$. From (A.7) it follows that

$$
\left[\begin{array}{cc}
\left(\rho+\theta^{*}\right) I & \theta^{*} I  \tag{A.8}\\
0 & \left(\frac{\rho \theta^{*}}{\rho+\theta^{*}}\right) I
\end{array}\right]\left[\begin{array}{c}
\pi_{\Omega}\left(Z^{*}\right) \\
\pi_{\Omega}\left(S^{*}\right)
\end{array}\right]=\left[\begin{array}{c}
\pi_{\Omega}\left(\theta^{*} D+\rho q(\tilde{Z})\right) \\
\frac{\rho \theta^{*}}{\rho+\theta^{*}} \pi_{\Omega}(D-q(\tilde{Z}))-\xi \pi_{\Omega}(G)
\end{array}\right]
$$

From the second equation in (A.8), we get

$$
\begin{equation*}
\xi \frac{\left(\rho+\theta^{*}\right)}{\rho \theta^{*}} \pi_{\Omega}(G)+\pi_{\Omega}\left(S^{*}\right)+\pi_{\Omega}(q(\tilde{Z})-D)=0 \tag{A.9}
\end{equation*}
$$

The equation (A.9) and $\pi_{\Omega^{c}}(G)=\mathbf{0}$ are precisely the first-order optimality conditions for the "shrinkage" problem

$$
\min _{S \in \mathbb{R}^{m \times n}}\left\{\xi \frac{\left(\rho+\theta^{*}\right)}{\rho \theta^{*}}\|S\|_{1}+\frac{1}{2}\left\|S+\pi_{\Omega}(q(\tilde{Z})-D)\right\|_{F}^{2}\right\} .
$$

The expression for $S^{*}$ in (2.4) is the optimal solution to this "shrinkage" problem, and $Z^{*}$ given in (2.5) follows from the first equation in (A.6) and the first row of A.8). Hence, given optimal Lagrangian dual $\theta^{*}$, $S^{*}$ and $Z^{*}$ computed from equations (2.4) and (2.5), respectively, satisfy KKT conditions (A.1) and (A.2).

Next, we show how to compute the optimal dual $\theta^{*}$. We consider two cases.
(i) Suppose $\left\|\pi_{\Omega}(D-q(\tilde{Z}))\right\|_{F} \leq \delta$. In this case, let $\theta^{*}=0$. Setting $\theta^{*}=0$ in (2.4) and (2.5), we find $S^{*}=\mathbf{0}$ and $Z^{*}=q(\tilde{Z})$. By construction, $S^{*}, Z^{*}$ and $\theta^{*}$ satisfy conditions (A.1) and (A.2). It is easy to check that this choice of $\theta^{*}=0$ trivially satisfies the rest of the conditions as well. Hence, $\theta^{*}=0$ is an optimal lagrangian dual.
(ii) Next, suppose $\left\|\pi_{\Omega}(D-q(\tilde{Z}))\right\|_{F}>\delta$. From (2.5), we have

$$
\begin{equation*}
\pi_{\Omega}\left(Z^{*}+S^{*}-D\right)=\frac{\rho}{\rho+\theta^{*}} \pi_{\Omega}\left(S^{*}+q(\tilde{Z})-D\right) \tag{A.10}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\left\|\pi_{\Omega}\left(Z^{*}+S^{*}-D\right)\right\|_{F} & =\frac{\rho}{\rho+\theta^{*}}\left\|\pi_{\Omega}\left(S^{*}+q(\tilde{Z})-D\right)\right\|_{F} \\
& =\frac{\rho}{\rho+\theta^{*}}\left\|\pi_{\Omega}\left(\max \left\{|D-q(\tilde{Z})|-\xi \frac{\left(\rho+\theta^{*}\right)}{\rho \theta^{*}} E, \mathbf{0}\right\}-|D-q(\tilde{Z})|\right)\right\|_{F} \\
& =\frac{\rho}{\rho+\theta^{*}}\left\|\pi_{\Omega}\left(\min \left\{\xi \frac{\left(\rho+\theta^{*}\right)}{\rho \theta^{*}} E,|D-q(\tilde{Z})|\right\}\right)\right\|_{F} \\
& =\left\|\min \left\{\frac{\xi}{\theta^{*}} E, \frac{\rho}{\rho+\theta^{*}}\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|\right\}\right\|_{F} \tag{A.11}
\end{align*}
$$

where the second equation is obtained after substituting (2.4) for $S^{*}$ and then componentwise dividing the resulting expression inside the norm by $\operatorname{sgn}(D-q(\tilde{Z}))$. Define $\phi: \mathbb{R}_{+} \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\phi(\theta):=\left\|\min \left\{\frac{\xi}{\theta} E, \frac{\rho}{\rho+\theta}\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|\right\}\right\|_{F} \tag{A.12}
\end{equation*}
$$

It is easy to show that $\phi$ is a strictly decreasing function of $\theta$. Since $\phi(0)=\left\|\pi_{\Omega}(D-q(\tilde{Z}))\right\|_{F}>\delta$ and $\lim _{\theta \rightarrow \infty} \phi(\theta)=0$, there exists a unique $\theta^{*}>0$ such that $\phi\left(\theta^{*}\right)=\delta$. Moreover, since $\theta^{*}>0$ and $\phi\left(\theta^{*}\right)=\delta$, (A.11) implies that $Z^{*}, S^{*}$ and $\theta^{*}$ satisfy the rest of KKT conditions (A.3), A.4) and A.5) as well. Thus, the unique $\theta^{*}>0$ that satisfies $\phi\left(\theta^{*}\right)=\delta$ is the optimal Lagrangian dual.
We now show that $\theta^{*}$ can be computed in $\mathcal{O}(|\Omega| \log (|\Omega|))$ time. Let $A:=\left|\pi_{\Omega}(D-q(\tilde{Z}))\right|$ and $0 \leq a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(|\Omega|)}$ be the $|\Omega|$ elements of the matrix $A$ corresponding to the indices $(i, j) \in \Omega$ sorted in increasing order, which can be done in $\mathcal{O}(|\Omega| \log (|\Omega|))$ time. Defining $a_{(0)}:=0$ and $a_{(|\Omega|+1)}:=\infty$, we then have for all $j \in\{0,1, \ldots,|\Omega|\}$ that

$$
\begin{equation*}
\frac{\rho}{\rho+\theta} a_{(j)} \leq \frac{\xi}{\theta} \leq \frac{\rho}{\rho+\theta} a_{(j+1)} \Leftrightarrow \frac{1}{\xi} a_{(j)}-\frac{1}{\rho} \leq \frac{1}{\theta} \leq \frac{1}{\xi} a_{(j+1)}-\frac{1}{\rho} . \tag{A.13}
\end{equation*}
$$

Let $\bar{k}:=\max \left\{j: a_{(j)} \leq \frac{\xi}{\rho}, 0 \leq j \leq|\Omega|\right\}$, and for all $\bar{k}<j \leq|\Omega|$ define $\theta_{j}:=\frac{1}{\frac{1}{\xi} a_{(j)}-\frac{1}{\rho}}$. Then for all $\bar{k}<j \leq|\Omega|$, we have

$$
\begin{equation*}
\phi\left(\theta_{j}\right)=\sqrt{\left(\frac{\rho}{\rho+\theta_{j}}\right)^{2} \sum_{i=0}^{j} a_{(i)}^{2}+(|\Omega|-j)\left(\frac{\xi}{\theta_{j}}\right)^{2}} . \tag{A.14}
\end{equation*}
$$

Also define $\theta_{\bar{k}}:=\infty$ and $\theta_{|\Omega|+1}:=0$ so that $\phi\left(\theta_{\bar{k}}\right):=0$ and $\phi\left(\theta_{|\Omega|+1}\right)=\phi(0)=\|A\|_{F}>\delta$. Note that $\left\{\theta_{j}\right\}_{\{\bar{k}<j \leq|\Omega|\}}$ contains all the points at which $\phi(\theta)$ may not be differentiable for $\theta \geq 0$. Define $j^{*}:=\max \left\{j: \phi\left(\theta_{j}\right) \leq \delta, \bar{k} \leq j \leq|\Omega|\right\}$. Then $\theta^{*}$ is the unique solution of the system

$$
\begin{equation*}
\sqrt{\left(\frac{\rho}{\rho+\theta}\right)^{2} \sum_{i=0}^{j^{*}} a_{(i)}^{2}+\left(|\Omega|-j^{*}\right)\left(\frac{\xi}{\theta}\right)^{2}}=\delta \text { and } \theta>0 \tag{A.15}
\end{equation*}
$$

since $\phi(\theta)$ is continuous and strictly decreasing in $\theta$ for $\theta \geq 0$. Solving the equation in (A.15) requires finding the roots of a fourth-order polynomial (also known as a quartic function). Lodovico Ferrari showed in 1540 that the roots of quartic functions can be solved in closed form. Thus, it follows that $\theta^{*}>0$ can be computed in $\mathcal{O}(1)$ operations.
Note that if $\bar{k}=|\Omega|$, then $\theta^{*}$ is the solution of the equation

$$
\begin{equation*}
\sqrt{\left(\frac{\rho}{\rho+\theta^{*}}\right)^{2} \sum_{i=1}^{|\Omega|} a_{(i)}^{2}}=\delta \tag{A.16}
\end{equation*}
$$

i.e. $\theta^{*}=\rho\left(\frac{\|A\|_{F}}{\delta}-1\right)=\rho\left(\frac{\left\|\pi_{\Omega}(D-q(\tilde{Z}))\right\|_{F}}{\delta}-1\right)$.

Hence, we have proved that problem ( $P_{n s}$ ) can be solved efficiently when $\delta>0$.
Now, suppose $\delta=0$. Since $\pi_{\Omega}\left(Z^{*}+S^{*}-D\right)=0$, problem $\left(P_{n s}\right)$ can be written as

$$
\begin{equation*}
\min _{Z, S \in \mathbb{R}^{m \times n}} \quad \xi \rho^{-1}\left\|\pi_{\Omega}(S)\right\|_{1}+\frac{1}{2}\left\|\pi_{\Omega}(D-S-q(\tilde{Z}))+\pi_{\Omega^{c}}(Z-q(\tilde{Z}))\right\|_{F}^{2} . \tag{A.17}
\end{equation*}
$$

Then (2.7) and $Z^{*}=\pi_{\Omega}\left(D-S^{*}\right)+\pi_{\Omega^{c}}(q(\tilde{Z}))$ trivially follow from first-order optimality conditions for the above problem.
A.2. Proof of Lemma 2.2. Let $W^{*}:=-Q+\rho\left(\tilde{Z}-Z^{*}\right)$. Then A.1), A.4) and (A.5) in the proof of Lemma 2.1 imply that $W^{*}=\theta^{*} \pi_{\Omega}\left(Z^{*}+S^{*}-D\right)$. From the first-order optimality conditions of $\left(P_{n s}\right)$ in (2.3), we have that $\left(W^{*}, W\right) \in \partial \mathbf{1}_{\chi}\left(Z^{*}, S^{*}\right)$ for some $W \in \partial \xi\left\|S^{*}\right\|_{1}$. From (A.1) and (A.2), it follows that $W^{*} \in \partial \xi\left\|S^{*}\right\|_{1}$. The definition of $\chi$, chain rule on subdifferential (see Theorem 23.9 in [26]), and $W^{*} \in \partial \xi\left\|S^{*}\right\|_{1}$ together imply that $\left(W^{*}, W^{*}\right) \in \partial \mathbf{1}_{\chi}\left(Z^{*}, S^{*}\right)$.
A.3. Proof of Lemma 3.1, Since $L_{k+1}$ is the optimal solution to the subproblem in Step 4 of ADMIP corresponding to the $k$-th iteration, it follows that

$$
\begin{equation*}
0 \in \partial\left\|L_{k+1}\right\|_{*}+Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k}\right) \tag{A.18}
\end{equation*}
$$

Let $\theta_{k} \geq 0$ denote the optimal Lagrange multiplier for the quadratic constraint in Step 5 sub-problem in the $k$-th iteration. Since $\left(Z_{k+1}, S_{k+1}\right)$ is the optimal solution, the first-order optimality conditions imply that

$$
\begin{array}{r}
0 \in \xi \partial\left\|S_{k+1}\right\|_{1}+\theta_{k} \pi_{\Omega}\left(Z_{k+1}+S_{k+1}-D\right), \\
-Y_{k}+\rho_{k}\left(Z_{k+1}-L_{k+1}\right)+\theta_{k} \pi_{\Omega}\left(Z_{k+1}+S_{k+1}-D\right)=0 . \tag{A.20}
\end{array}
$$

From (A.18), it follows that $-\hat{Y}_{k+1} \in \partial\left\|L_{k+1}\right\|_{*}$. From A.19) and A.20), it follows that $-Y_{k+1} \in$ $\xi \partial\left\|S_{k+1}\right\|_{1}$. Since $\partial\|L\|_{*}$ and $\partial\|S\|_{1}$ are uniformly bounded sets for all $L, S \in \mathbb{R}^{m \times n}$, it follows that $\left\{\hat{Y}_{k}\right\}_{k \in \mathbb{Z}_{+}}$and $\left\{Y_{k}\right\}_{k \in \mathbb{Z}_{+}}$are bounded sequences. Moreover, A.20) implies that $\pi_{\Omega}\left(Y_{k}\right)=Y_{k}$ for all $k \geq 1$.
A.4. Proof of Lemma 3.2, For all $k \geq 0$, since $Y_{k+1}=Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k+1}\right)$ and and $\hat{Y}_{k+1}:=$ $Y_{k}+\rho_{k}\left(L_{k+1}-Z_{k}\right)$, we have that $Y_{k+1}-\hat{Y}_{k+1}=\rho_{k}\left(Z_{k}-Z_{k+1}\right)$. Using these relations, we obtain the following equality

$$
\begin{align*}
& \rho_{k}^{-1}\left\langle Y_{k+1}-Y_{k}, Y_{k+1}-Y^{*}\right\rangle \\
& =\rho_{k}\left\langle L_{k+1}-L^{*}, Z_{k}-Z_{k+1}\right\rangle+\left\langle L_{k+1}-L^{*}, \hat{Y}_{k+1}-Y^{*}\right\rangle+\left\langle L^{*}-Z_{k+1}, Y_{k+1}-Y^{*}\right\rangle \tag{A.21}
\end{align*}
$$

Moreover, we also have

$$
\begin{align*}
& \left\|Z_{k+1}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k+1}-Y^{*}\right\|_{F}^{2} \\
& =\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}-\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}-\rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2} \\
& \quad+2\left\langle Z_{k+1}-L^{*}, Z_{k+1}-Z_{k}\right\rangle+2 \rho_{k}^{-2}\left\langle Y_{k+1}-Y_{k}, Y_{k+1}-Y^{*}\right\rangle  \tag{A.22}\\
& =\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}-\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}-\rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2} \\
& \quad+2\left\langle Z_{k+1}-L_{k+1}, Z_{k+1}-Z_{k}\right\rangle-2 \rho_{k}^{-1}\left(\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, L^{*}-Z_{k+1}\right\rangle\right), \\
& =\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}-\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}-\rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2} \\
& \quad-2 \rho_{k}^{-1}\left(\left\langle Y_{k+1}-Y_{k}, Z_{k+1}-Z_{k}\right\rangle+\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, L^{*}-Z_{k+1}\right\rangle\right) \tag{A.23}
\end{align*}
$$

where the second equality follows from rewriting the last term in A.22) using (A.21), and the last equality follows from the relation $L_{k+1}-Z_{k+1}=\rho_{k}^{-1}\left(Y_{k+1}-Y_{k}\right)$.

Since $Y^{*}$ and $\theta^{*}$ are optimal Lagrangian dual variables, we have

$$
\left(L^{*}, L^{*}, S^{*}\right)=\underset{L, Z, S}{\operatorname{argmin}}\|L\|_{*}+\xi\|S\|_{1}+\left\langle Y^{*}, L-Z\right\rangle+\frac{\theta^{*}}{2}\left(\left\|\pi_{\Omega}(Z+S-D)\right\|_{F}^{2}-\delta^{2}\right)
$$

From first-order optimality conditions, we get

$$
\begin{aligned}
& 0 \in \partial\left\|L^{*}\right\|_{*}+Y^{*} \\
& 0 \in \xi \partial\left\|S^{*}\right\|_{1}+\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right) \\
& 0=-Y^{*}+\theta^{*} \pi_{\Omega}\left(L^{*}+S^{*}-D\right)
\end{aligned}
$$

Hence, $-Y^{*} \in \partial\left\|L^{*}\right\|_{*}$ and $-Y^{*} \in \xi \partial\left\|S^{*}\right\|_{1}$. Moreover, from Lemma 3.1 we also have that $-Y_{k} \in \partial \xi\left\|S_{k}\right\|_{1}$ for all $k \geq 1$. Since $\xi\|\cdot\|_{1}$ is convex, it follows that

$$
\begin{align*}
& \left\langle-Y_{k+1}+Y_{k}, S_{k+1}-S_{k}\right\rangle \geq 0  \tag{A.24}\\
& \left\langle-Y_{k+1}+Y^{*}, S_{k+1}-S^{*}\right\rangle \geq 0 \tag{A.25}
\end{align*}
$$

Since $\rho_{k+1} \geq \rho_{k}$ for all $k \geq 1$, first adding (A.24) to (A.23), then adding and subtracting (A.25), we get

$$
\begin{align*}
& \left\|Z_{k+1}-L^{*}\right\|_{F}^{2}+\rho_{k+1}^{-2}\left\|Y_{k+1}-Y^{*}\right\|_{F}^{2} \\
& \leq\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}-\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}-\rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2} \\
& \quad-2 \rho_{k}^{-1}\left(\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, S_{k+1}-S^{*}\right\rangle\right) \\
& \quad-2 \rho_{k}^{-1}\left(\left\langle Y_{k+1}-Y_{k}, Z_{k+1}+S_{k+1}-Z_{k}-S_{k}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, L^{*}+S^{*}-Z_{k+1}-S_{k+1}\right\rangle\right) . \tag{A.26}
\end{align*}
$$

Lemma 2.2 applied to the Step 5 sub-problem corresponding to the $k$-th iteration gives $\left(Y_{k+1}, Y_{k+1}\right) \in$ $\partial \mathbf{1}_{\chi}\left(Z_{k+1}, S_{k+1}\right)$. Using an argument similar to that used in the proof of Lemma 2.2, one can also show that $\left(Y^{*}, Y^{*}\right) \in \partial \mathbf{1}_{\chi}\left(L^{*}, S^{*}\right)$. Moreover, since $-Y^{*} \in \partial \xi\left\|S^{*}\right\|_{1},-Y^{*} \in \partial\left\|L^{*}\right\|_{*}$, and $-Y_{k} \in \partial \xi\left\|S_{k}\right\|_{1}$, $-\hat{Y}_{k} \in \partial\left\|L_{k}\right\|_{*}$ for all $k \geq 1$, we have that for all $k \geq 0$,

$$
\begin{aligned}
\left\langle Y_{k+1}-Y_{k}, Z_{k+1}+S_{k+1}-Z_{k}-S_{k}\right\rangle & \geq 0 \\
\left\langle-Y_{k+1}+Y^{*}, L^{*}+S^{*}-Z_{k+1}-S_{k+1}\right\rangle & \geq 0 \\
\left\langle-Y_{k+1}+Y^{*}, S_{k+1}-S^{*}\right\rangle & \geq 0 \\
\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle & \geq 0
\end{aligned}
$$

This set of inequalities and (A.26) together imply that $\left\{\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}\right\}_{k \in \mathbb{Z}_{+}}$is a non-increasing
sequence. Using this fact, rewriting A.26) and summing over $k \in \mathbb{Z}_{+}$, we get

$$
\begin{aligned}
& \sum_{k \in \mathbb{Z}_{+}}\left\|Z_{k+1}-Z_{k}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k+1}-Y_{k}\right\|_{F}^{2} \\
& +2 \sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-1}\left(\left\langle-\hat{Y}_{k+1}+Y^{*}, L_{k+1}-L^{*}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, S_{k+1}-S^{*}\right\rangle\right) \\
& +2 \sum_{k \in \mathbb{Z}_{+}} \rho_{k}^{-1}\left(\left\langle Y_{k+1}-Y_{k}, Z_{k+1}+S_{k+1}-Z_{k}-S_{k}\right\rangle+\left\langle-Y_{k+1}+Y^{*}, L^{*}+S^{*}-Z_{k+1}-S_{k+1}\right\rangle\right) \\
\leq & \sum_{k \in \mathbb{Z}_{+}}\left(\left\|Z_{k}-L^{*}\right\|_{F}^{2}+\rho_{k}^{-2}\left\|Y_{k}-Y^{*}\right\|_{F}^{2}-\left\|Z_{k+1}-L^{*}\right\|_{F}^{2}-\rho_{k+1}^{-2}\left\|Y_{k+1}-Y^{*}\right\|_{F}^{2}\right)<\infty .
\end{aligned}
$$

This inequality is sufficient to prove the rest of the lemma.

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[^1]:    ${ }^{1}$ In an earlier preprint, we named it as NSA algorithm.

[^2]:    ${ }^{2}$ In an earlier preprint, we named it as Non-Smooth Augmented Lagrangian (NSA) algorithm.
    ${ }^{3}$ The modified version is available from http://svt.stanford.edu/code.html

