

REAL EIGENVALUES OF NONSYMMETRIC TENSORS

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ABSTRACT. This paper discusses the computation of real \mathbb{Z} -eigenvalues and \mathbb{H} -eigenvalues of nonsymmetric tensors. A general nonsymmetric tensor has finitely many \mathbb{Z} -eigenvalues, while there may be infinitely many ones for special tensors. In the contrast, every nonsymmetric tensor has finitely many \mathbb{H} -eigenvalues. We propose Lasserre type semidefinite relaxation methods for computing such eigenvalues. For every nonsymmetric tensor that has finitely many real \mathbb{Z} -eigenvalues, we can compute all of them; each of them can be computed by solving a finite sequence of semidefinite relaxations. For every nonsymmetric tensor, we can compute all its real \mathbb{H} -eigenvalues; each of them can be computed by solving a finite sequence of semidefinite relaxations. Various examples are demonstrated.

1. INTRODUCTION

For positive integers m and n_1, n_2, \dots, n_m , an m -order and (n_1, n_2, \dots, n_m) -dimensional real tensor is an array in the space $\mathbb{R}^{n_1 \times n_2 \times \dots \times n_m}$. Every tensor \mathcal{A} from this space can be indexed as

$$(1.1) \quad \mathcal{A} = (\mathcal{A}_{i_1 i_2 \dots i_m}), \quad 1 \leq i_j \leq n_j, \quad j = 1, 2, \dots, m.$$

When $n_1 = \dots = n_m = n$, \mathcal{A} is called an m -order n -dimensional tensor. In such case, the tensor space $\mathbb{R}^{n_1 \times n_2 \times \dots \times n_m}$ is denoted as $\mathbb{T}^m(\mathbb{R}^n)$. A tensor in $\mathbb{T}^m(\mathbb{R}^n)$ is said to be *symmetric* if its entries are invariant under permutations of indices (i_1, i_2, \dots, i_m) . The subspace of symmetric tensors in $\mathbb{T}^m(\mathbb{R}^n)$ is denoted as $\mathbb{S}^m(\mathbb{R}^n)$. By replacing the real field \mathbb{R} by the complex field \mathbb{C} , the tensor spaces $\mathbb{T}^m(\mathbb{C}^n)$ and $\mathbb{S}^m(\mathbb{C}^n)$ are similarly defined. Using the notation as in Qi [29], for $\mathcal{A} \in \mathbb{T}^m(\mathbb{C}^n)$ and $x := (x_1, \dots, x_n)$, we denote

$$(1.2) \quad \begin{cases} \mathcal{A}x^m & := \sum_{1 \leq i_1, \dots, i_m \leq n} \mathcal{A}_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}, \\ \mathcal{A}x^{m-1} & := \left(\sum_{1 \leq i_2, \dots, i_m \leq n} \mathcal{A}_{j i_2 \dots i_m} x_{i_2} \dots x_{i_m} \right)_{j=1, \dots, n}. \end{cases}$$

Note that $\mathcal{A}x^{m-1}$ is an n -dimensional vector. Now we give some definitions of tensor eigenvalues that are introduced in [3, 4, 16, 29].

Definition 1.1. For $\mathcal{A} \in \mathbb{T}^m(\mathbb{C}^n)$, a number $\lambda \in \mathbb{C}$ is called a \mathbb{Z} -eigenvalue of \mathcal{A} if there exists a vector $u \in \mathbb{C}^n$ such that

$$(1.3) \quad \mathcal{A}u^{m-1} = \lambda u, \quad u^T u = 1.$$

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(The superscript T denotes the transpose.) Such u is called a Z -eigenvector associated with λ , and such (λ, u) is called a Z -eigenpair.

Definition 1.2. For $\mathcal{A} \in \mathbb{T}^m(\mathbb{C}^n)$, a number $\lambda \in \mathbb{C}$ is called an H -eigenvalue of \mathcal{A} if there exists $0 \neq u \in \mathbb{C}^n$ such that

$$(1.4) \quad \mathcal{A}u^{m-1} = \lambda u^{[m-1]}.$$

(The symbol $u^{[m-1]}$ denotes the vector such that $(u^{[m-1]})_i = (u_i)^{m-1}$ for $i = 1, \dots, n$). Such u is called an H -eigenvector associated with λ , and such (λ, u) is called an H -eigenpair.

When λ is a real Z -eigenvalue (resp., H -eigenvalue), the associated Z -eigenvector (resp., H -eigenvector) is not necessarily real. The (λ, u) is called a real Z -eigenpair (resp., H -eigenpair) if both λ and u are real. In the paper, we only discuss real eigenvalues. Throughout the paper, for convenience, we call λ a *real* Z -eigenvalue (resp., H -eigenvalue) if *both* λ and u are real.

Tensor eigenvalues have broad applications in sciences and engineering. They were introduced in Lim [16] and Qi [28]. The Z -eigenvalue and H -eigenvalues are useful in signal processing, control, and diffusion imaging (cf. [2, 19, 27, 30, 31]). For an introduction to the theory and applications of tensor computations, we refer to [11, 18, 26].

When \mathcal{A} is a real symmetric tensor, Cui et al. [3] discussed how to compute all real Z -eigenvalues and H -eigenvalues. There also exists work on computing partial eigenvalues, e.g., the biggest and smallest ones. For the case $(m, n) = (3, 2)$, Qi et al. [33] discussed how to compute largest Z -eigenvalues. Shifted power methods are proposed for computing largest Z -eigenvalues (cf. [12, 35]). In [20], a semidefinite relaxation method was proposed to find best rank-1 approximations, which can also be used for computing largest Z -eigenvalues. As shown in [9, 35], it is NP-hard to compute extreme eigenvalues of tensors. For nonnegative tensors, the largest H -eigenvalues can be computed by methods based on the Perron-Frobenius theorem (cf. [4, 25]).

When \mathcal{A} is a general nonsymmetric tensor, there is little work on computing all real Z -eigenvalues and H -eigenvalues. An elementary approach for this task is to solve the polynomial systems (1.3) and (1.4) directly, for getting all complex solutions by classical symbolic methods. This approach is typically quite expensive and not practical, because of typically high complexity of symbolic computations.

There are fundamental differences between symmetric and nonsymmetric tensor eigenvalues. It is known that every symmetric tensor has finitely many Z -eigenvalues (cf. [3]). However, this may not be true for nonsymmetric tensors. A nonsymmetric tensor may have no real Z -eigenvalues, or may have infinitely many real ones. We show such facts by the following examples.

Example 1.3. Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^2)$ such that $\mathcal{A}_{ijkl} = 0$ except

$$\mathcal{A}_{1112} = \mathcal{A}_{1222} = 1, \mathcal{A}_{2111} = \mathcal{A}_{2122} = -1.$$

By the definition, (λ, x) is a Z-eigenpair if and only if

$$\begin{cases} (x_1^2 + x_2^2)x_2 = \lambda x_1, \\ -(x_1^2 + x_2^2)x_1 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

One can check that the above does not have a real solution, so \mathcal{A} has no real Z-eigenpairs. By the definition, (λ, x) is an H-eigenpair if and only if

$$\begin{cases} (x_1^2 + x_2^2)x_2 = \lambda x_1^3, \\ -(x_1^2 + x_2^2)x_1 = \lambda x_2^3, \\ (x_1, x_2) \neq (0, 0). \end{cases}$$

One can similarly check that \mathcal{A} has no real H-eigenpairs.

Example 1.4. Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^2)$ such that $\mathcal{A}_{ijkl} = 0$ except

$$\mathcal{A}_{1111} = \mathcal{A}_{2112} = 1.$$

Then, (λ, x) is a Z-eigenpair of \mathcal{A} if and only if

$$(1.5) \quad \begin{cases} x_1^3 & = \lambda x_1, \\ x_1^2 x_2 & = \lambda x_2, \\ x_1^2 + x_2^2 & = 1. \end{cases}$$

One can check that every $\lambda \in [0, 1]$ is a real Z-eigenvalue, with 4 real Z-eigenvectors $(\pm\sqrt{\lambda}, \pm\sqrt{1-\lambda})$.

However, we would like to remark that the above examples are not general cases. In fact, every generic nonsymmetric tensor has finitely many Z-eigenvalues, and its number can be given by explicit formula. This is shown by Cartwright and Sturmfels [5]. Moreover, every nonsymmetric tensor has finitely many H-eigenvalues. By these facts, it is generally a well-posed question to compute all real Z-eigenvalues and H-eigenvalues.

In this paper, we propose numerical methods for computing all real Z-eigenvalues (if there are finitely many ones) and all H-eigenvalues. For symmetric tensors, the Z-eigenvalues and H-eigenvalues are critical values of some polynomial optimization problems. This property was significantly used in [3] for computing all real eigenvalues. The method in [3] is based on Jacobian SDP relaxations [23], which are specially designed for solving polynomial optimization. Indeed, the same kind of method can be used to compute all local minima of polynomial optimization [24]. However, the method in [3] are not suitable for computing eigenvalues of nonsymmetric tensors, because their eigenvalues are no longer critical values of polynomial optimization problems.

This paper is organized as follows. Section 2 gives some preliminaries on polynomial optimization and tensor eigenvalues. Section 3 proposes Lasserre type semidefinite relaxations for computing real Z-eigenvalues. If there are finitely many ones, all the real Z-eigenvalues can be found, and each of them can be computed by solving a finite sequence of semidefinite relaxations. Section 4 proposes Lasserre type semidefinite relaxations for computing all real H-eigenvalues. Each of them can be computed by solving a finite sequence of semidefinite relaxations. Numerical examples are shown in Section 5.

2. PRELIMINARIES

In this section, we review some basics in polynomial optimization. We refer to [14, 15] for surveys in the area. In the space \mathbb{R}^n , the symbol $\|\cdot\|$ denotes the standard Euclidean norm. Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients and in variables $x := (x_1, \dots, x_n)$, and let $\mathbb{R}[x]_d$ be the set of real polynomials in x whose degrees are at most d . For a polynomial tuple $h = (h_1, h_2, \dots, h_s)$, the ideal generated by h is the set

$$I(h) := h_1 \cdot \mathbb{R}[x] + h_2 \cdot \mathbb{R}[x] + \dots + h_s \cdot \mathbb{R}[x].$$

The k -th truncation of $I(h)$ is the set

$$I_k(h) := h_1 \cdot \mathbb{R}[x]_{k-\deg(h_1)} + \dots + h_s \cdot \mathbb{R}[x]_{k-\deg(h_s)}.$$

The complex and real algebraic varieties of h are respectively defined as

$$\mathcal{V}_{\mathbb{C}}(h) := \{x \in \mathbb{C}^n \mid h(x) = 0\}, \quad \mathcal{V}_{\mathbb{R}}(h) := \mathcal{V}_{\mathbb{C}}(h) \cap \mathbb{R}^n.$$

A polynomial p is said to be sum of squares (SOS) if there exist $p_1, p_2, \dots, p_r \in \mathbb{R}[x]$ such that $p = p_1^2 + p_2^2 + \dots + p_r^2$. The set of all SOS polynomials is denoted as $\Sigma[x]$. For a given degree m , denote

$$\Sigma[x]_m := \Sigma[x] \cap \mathbb{R}[x]_m.$$

The quadratic module generated by a polynomial tupe $g = (g_1, \dots, g_t)$ is the set

$$Q(g) := \Sigma[x] + g_1 \cdot \Sigma[x] + \dots + g_t \cdot \Sigma[x].$$

The k -th truncation of the quadratic module $Q(g)$ is the set

$$Q_k(g) := \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{2k-\deg(g_1)} + \dots + g_t \cdot \Sigma[x]_{2k-\deg(g_t)}.$$

Note that if $g = \emptyset$ is an empty tuple, then $Q(g) = \Sigma[x]$ and $Q_k(g) = \Sigma[x]_{2k}$.

Let \mathbb{N} be the set of nonnegative integers. For $x := (x_1, \dots, x_n)$, $\alpha := (\alpha_1, \dots, \alpha_n)$ and a degree d , denote

$$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \dots + \alpha_n, \quad \mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}.$$

Denote by $\mathbb{R}^{\mathbb{N}_d^n}$ the space of all real vectors y that are indexed by $\alpha \in \mathbb{N}_d^n$. For $y \in \mathbb{R}^{\mathbb{N}_d^n}$, we can write it as

$$y = (y_\alpha), \quad \alpha \in \mathbb{N}_d^n.$$

For $f = \sum_{\alpha \in \mathbb{N}_d^n} f_\alpha x^\alpha \in \mathbb{R}[x]_d$ and $y \in \mathbb{R}^{\mathbb{N}_d^n}$, we define the operation

$$(2.1) \quad \langle f, y \rangle := \sum_{\alpha \in \mathbb{N}_d^n} f_\alpha y_\alpha.$$

For an integer $t \leq d$ and $y \in \mathbb{R}^{\mathbb{N}_d^n}$, denote the t -th truncation of y as

$$(2.2) \quad y|_t := (y_\alpha)_{\alpha \in \mathbb{N}_t^n}.$$

Let $q \in \mathbb{R}[x]$ with $\deg(q) \leq 2k$. For each $y \in \mathbb{R}^{\mathbb{N}_{2k}^n}$, $\langle qp^2, y \rangle$ is a quadratic form in $\text{vec}(p)$, the coefficient vector of the polynomial p with $\deg(pp^2) \leq 2k$. Let $L_q^{(k)}(y)$ be the symmetric matrix such that

$$(2.3) \quad \langle qp^2, y \rangle = \text{vec}(p)^T \left(L_q^{(k)}(y) \right) \text{vec}(p).$$

The matrix $L_q^{(k)}(y)$ is called the k -th localizing matrix of q generated by y . It is linear in y . For instance, when $n = 2$, $k = 2$ and $q = x_1x_2 - x_1^2 - x_2^2$,

$$L_{x_1x_2-x_1^2-x_2^2}^{(2)}(y) = \begin{pmatrix} y_{11} - y_{20} - y_{02} & y_{21} - y_{30} - y_{12} & y_{12} - y_{21} - y_{03} \\ y_{21} - y_{30} - y_{12} & y_{31} - y_{40} - y_{22} & y_{22} - y_{31} - y_{13} \\ y_{12} - y_{21} - y_{03} & y_{22} - y_{31} - y_{13} & y_{13} - y_{22} - y_{04} \end{pmatrix}.$$

If $q = (q_1, \dots, q_r)$ is a tuple of polynomials, we then define

$$L_q^{(k)}(y) := \left(L_{q_1}^{(k)}(y), \dots, L_{q_r}^{(k)}(y) \right).$$

When $q = 1$ (the constant 1 polynomial), $L_1^{(k)}(y)$ is called the k -th moment matrix generated by y , and we denote

$$(2.4) \quad M_k(y) := L_1^{(k)}(y).$$

For instance, when $n = 2$ and $k = 2$,

$$M_2(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix}.$$

For a degree d , denote the monomial vector

$$(2.5) \quad [x]_d := [1 \ x_1 \ \dots \ x_n \ x_1^2 \ x_1x_2 \ \dots \ x_n^2 \ \dots \ x_1^m \ \dots \ x_n^m]^T.$$

As shown in Example 1.4, there may be infinitely many Z-eigenvalues. But this is not the case for a general nonsymmetric tensor. In (1.3), a real Z-eigenpair (λ, u) of a tensor \mathcal{A} is called *isolated* if there exists $\varepsilon > 0$ such that no other real Z-eigenpair (μ, v) satisfies $|\lambda - \mu| + \|u - v\| < \varepsilon$. Similarly, a real Z-eigenvalue λ is called *isolated* if there exists $\varepsilon > 0$ such that no other real Z-eigenvalue μ satisfies $|\lambda - \mu| < \varepsilon$. In practice, we need to check whether a Z-eigenvalue or Z-eigenpair is isolated or not. Denote

$$F(\lambda, x) := \begin{bmatrix} x^T x - 1 \\ \mathcal{A}x^{m-1} - \lambda x \end{bmatrix}.$$

Then, (λ, u) is a Z-eigenpair of \mathcal{A} if and only if $F(\lambda, u) = 0$. Let $J(\lambda, x)$ be the Jacobian matrix of the vector function $F(\lambda, x)$ with respect to (λ, x) .

Lemma 2.1. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and λ be a real Z-eigenvalue of \mathcal{A} .*

- (i) *If u is a real Z-eigenvector associated with λ and $J(\lambda, u)$ is nonsingular, then (λ, u) is isolated.*
- (ii) *If u_1, \dots, u_N are the all real Z-eigenvectors of \mathcal{A} associated to λ and each $J(\lambda, u_i)$ is nonsingular, then λ is an isolated Z-eigenvalue of \mathcal{A} .*

Proof. (i) We prove it by a contradiction argument. Suppose otherwise the Z-eigenpair (λ, u) is not isolated. Then there exists a sequence $\{(\lambda_l, u^{(l)})\}_{l=1}^\infty$ of Z-eigenpairs such that each $(\lambda_l, u^{(l)}) \neq (\lambda, u)$ and $(\lambda_l, u^{(l)}) \rightarrow (\lambda, u)$. Let

$$d^{(l)} = \begin{bmatrix} \lambda_l - \lambda \\ u^{(l)} - u \end{bmatrix}.$$

By the second order Taylor expansion, we have

$$0 = F(\lambda_l, u^{(l)}) = F(\lambda, u) + J(\lambda, u)d^{(l)} + O(\|d^{(l)}\|^2).$$

Note that $F(\lambda, u) = 0$ and each $\|d^{(l)}\| \neq 0$. The above implies that

$$J(\lambda, u)\left(d^{(l)}/\|d^{(l)}\|\right) = O(\|d^{(l)}\|).$$

Since each $d^{(l)}/\|d^{(l)}\|$ has unit length, we can generally assume that $d^{(l)}/\|d^{(l)}\| \rightarrow \hat{d}$. Then $\|\hat{d}\| = 1$ and $\hat{d} \neq 0$. The above implies that

$$J(\lambda, u)\hat{d} = 0,$$

contradicting the nonsingularity of $J(\lambda, u)$. So, (λ, u) is isolated.

(ii) Suppose otherwise that λ is not isolated. Then there exists a sequence $\{\lambda_l\}$ of distinct real Z-eigenvalues such that $\lambda_l \rightarrow \lambda$. Each λ_l has a real Z-eigenvector $u^{(l)}$. Since $u^{(l)}$ has unit length, we can generally assume $u^{(l)} \rightarrow \hat{u}$. Thus, $(\lambda_l, u^{(l)}) \rightarrow (\lambda, \hat{u})$. Clearly, (λ, \hat{u}) is also a real Z-eigenpair, but it is not isolated. By the assumption, \hat{u} is one of u_1, \dots, u_N . So, one of (λ, u_i) is not isolated, which is a contradiction. \square

3. COMPUTING Z-EIGENVALUES

Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ be a tensor. Recall that (λ, u) is a Z-eigenpair of \mathcal{A} if $\mathcal{A}u^{m-1} = \lambda u$ and $u^T u = 1$. So,

$$\lambda = \lambda u^T u = u^T \mathcal{A}u^{m-1} = \mathcal{A}u^m.$$

Hence, u is a Z-eigenvector if and only if

$$\mathcal{A}u^{m-1} = (\mathcal{A}u^m)u, \quad u^T u = 1,$$

and the associated Z-eigenvalue is $\mathcal{A}u^m$. A general nonsymmetric tensor in $\mathbb{T}^m(\mathbb{R}^n)$ has finitely many Z-eigenvalues, as shown in [5]. For special tensors, there might be infinitely many ones (cf. Example 1.4).

In this section, we aim at computing all real Z-eigenvalues when there are finitely many ones. Let h the polynomial tuple:

$$(3.1) \quad h = (\mathcal{A}x^{m-1} - (\mathcal{A}x^m)x, x^T x - 1).$$

Then, u is a Z-eigenvector of \mathcal{A} if and only if $h(u) = 0$. Let $Z(\mathbb{R}, \mathcal{A})$ denote the set of real Z-eigenvalues of \mathcal{A} . If it is a finite set, we list $Z(\mathbb{R}, \mathcal{A})$ monotonically as

$$\lambda_1 < \lambda_2 < \dots < \lambda_N.$$

We aim at computing them sequentially, from the smallest to the biggest.

3.1. The smallest Z-eigenvalue. To compute the smallest Z-eigenvalue λ_1 , we consider the polynomial optimization problem

$$(3.2) \quad \min f(x) := \mathcal{A}x^m \quad \text{s.t.} \quad h(x) = 0,$$

where h is as in (3.1). Note that u is a Z-eigenvector if and only if $h(u) = 0$, with the Z-eigenvalue $f(u)$. The optimal value of (3.2) is λ_1 , if it exists. Let

$$(3.3) \quad k_0 = \lceil (m+1)/2 \rceil.$$

Lasserre's hierarchy [13] of semidefinite relaxations for solving (3.2) is

$$(3.4) \quad \begin{cases} f_1^{1,k} := \min & \langle f, y \rangle \\ & \text{s.t. } \langle 1, y \rangle = 1, L_h^{(k)}(y) = 0, \\ & M_k(y) \succeq 0, y \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases}$$

for the orders $k = k_0, k_0 + 1, \dots$. See (2.3)-(2.4) for the notation $L_h^{(k)}(y)$ and $M_k(y)$. In the above, $X \succeq 0$ means that the matrix X is positive semidefinite. The dual optimization problem of (3.4) is

$$(3.5) \quad \begin{cases} f_1^{2,k} := \max & \gamma \\ & \text{s.t. } f - \gamma \in I_{2k}(h) + \Sigma[x]_{2k}. \end{cases}$$

As in [13], it can be shown that for all k

$$f_1^{2,k} \leq f_1^{1,k} \leq \lambda_1$$

and the sequences $\{f_1^{1,k}\}$ and $\{f_1^{2,k}\}$ are monotonically increasing.

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $Z(\mathbb{R}, \mathcal{A})$ be the set of its real Z-eigenvalues. Then we have:*

- (i) *The set $Z(\mathbb{R}, \mathcal{A}) = \emptyset$ if and only if the semidefinite relaxation (3.4) is infeasible for some order k .*
- (ii) *If $Z(\mathbb{R}, \mathcal{A}) \neq \emptyset$ and λ_1 is the smallest Z-eigenvalue, then*

$$(3.6) \quad \lim_{k \rightarrow \infty} f_1^{2,k} = \lim_{k \rightarrow \infty} f_1^{1,k} = \lambda_1.$$

If, in addition, $Z(\mathbb{R}, \mathcal{A})$ is a finite set, then for all k sufficiently big

$$(3.7) \quad f_1^{2,k} = f_1^{1,k} = \lambda_1.$$

- (iii) *Suppose y^* is a minimizer of (3.4). If there exists $t \leq k$ such that*

$$(3.8) \quad \text{rank } M_{t-k_0}(y^*) = \text{rank } M_t(y^*),$$

then $f_1^{1,k} = \lambda_1$ and there are $r := \text{rank } M_t(y^)$ distinct real Z-eigenvectors u_1, \dots, u_r associated with λ_1 .*

- (iv) *Suppose $Z(\mathbb{R}, \mathcal{A})$ is a finite set. If there are finitely many real Z-eigenvectors associated with λ_1 , then, for all k big enough and for every minimizer y^* of (3.4), (3.8) is satisfied for some $t \leq k$.*

Proof. (i) “if” direction: this is obvious. If \mathcal{A} has a real Z-eigenpair (λ, u) , then $[u]_{2k}$ (see (2.5) for the notation) is feasible for (3.4), a contradiction.

“only if” direction: If \mathcal{A} has no real Z-eigenvalues, then the equation $h(x) = 0$ has no real solutions. By Positivstellensatz (cf. [1]), $-1 \in I(h) + \Sigma[x]$. So, when k is big enough, $-1 \in I_{2k}(h) + \Sigma[x]_{2k}$, and then (3.5) is unbounded from above. By weak duality, (3.4) must be infeasible, for all k big enough.

(ii) Note that $x^T x - 1$ is a polynomial in the tuple h . So, $-(x^T x - 1)^2 \in I(h)$ and the set $-(x^T x - 1)^2 \geq 0$ is compact. The ideal $I(h)$ is archimedean (cf. [13]). The asymptotic convergence (3.6) can be implied by Theorem 4.2 of [13].

Next, we prove the finite convergence (3.7) when $Z(\mathbb{R}, \mathcal{A}) \neq \emptyset$ is a finite set. Write $Z(\mathbb{R}, \mathcal{A}) = \{\lambda_1, \dots, \lambda_N\}$, with $\lambda_1 < \dots < \lambda_N$. Let $b_1, \dots, b_N \in \mathbb{R}[t]$ be the

univariate real polynomials in t such that $b_i(\lambda_j) = 0$ when $i \neq j$ and $b_i(\lambda_j) = 1$ when $i = j$. For $i = 1, \dots, N$, let

$$s_i := (\lambda_i - \lambda_1) \left(b_i(f(x)) \right)^2.$$

Let $s := s_1 + \dots + s_N$. Then, $s \in \Sigma[x]_{2k_1}$ for some $k_1 > 0$. The polynomial

$$\hat{f} := f - \lambda_1 - s$$

vanishes identically on $\mathcal{V}_{\mathbb{R}}(h)$. By Real Nullstellensatz (cf. [1, Corollary 4.1.8]), there exist an integer $\ell > 0$ and $q \in \Sigma[x]$ such that

$$\hat{f}^{2\ell} + q \in I(h).$$

For all $\varepsilon > 0$ and $c > 0$, we can write

$$\hat{f} + \varepsilon = \phi_\varepsilon + \theta_\varepsilon \quad \text{where}$$

$$\phi_\varepsilon = -c\varepsilon^{1-2\ell}(\hat{f}^{2\ell} + q), \quad \theta_\varepsilon = \varepsilon \left(1 + \hat{f}/\varepsilon + c(\hat{f}/\varepsilon)^{2\ell} \right) + c\varepsilon^{1-2\varepsilon}q.$$

By Lemma 2.1 of [21], when $c \geq \frac{1}{2\ell}$, there exists k_2 such that, for all $\varepsilon > 0$,

$$\phi_\varepsilon \in I_{2k_2}(h), \quad \theta_\varepsilon \in \Sigma[x]_{2k_2}.$$

Hence, we can get

$$f - (\lambda_1 - \varepsilon) = \phi_\varepsilon + \sigma_\varepsilon,$$

where $\sigma_\varepsilon = \theta_\varepsilon + s \in \Sigma[x]_{2k_2}$ for all $\varepsilon > 0$. This implies that, for all $\varepsilon > 0$, $\gamma = \lambda_1 - \varepsilon$ is feasible in (3.5) for the order k_2 . Thus, we get $f_1^{2,k_2} \geq \lambda_1$. Note that $f_1^{2,k} \leq f_1^{1,k} \leq \lambda_1$ for all k and the sequence $\{f_1^{2,k}\}$ is monotonically increasing. So, (3.7) must be true when $k \geq k_2$.

(iii) Note that $M_t(y^*) \succeq 0$ and $L_h^{(t)}(y^*) = 0$, because $t \leq k$. When (3.8) is satisfied, by Theorem 1.1 of [6], there exist $r := \text{rank } M_t(y^*)$ vectors $u_1, \dots, u_r \in \mathcal{V}_{\mathbb{R}}(h)$ such that

$$y^*|_{2t} = c_1[u_1]_{2t} + \dots + c_r[u_r]_{2t},$$

with numbers $c_1, \dots, c_r > 0$. The condition $\langle 1, y^* \rangle = 1$ implies that

$$c_1 + \dots + c_r = 1.$$

By the notation $\langle \cdot, \cdot \rangle$ as in (2.1), we can see that $\langle f, [u_i]_{2k} \rangle = f(u_i)$, so

$$f_1^{1,k} = \langle f, y^* \rangle = c_1 f(u_1) + \dots + c_r f(u_r),$$

$$f_1^{1,k} \leq f(u_i) \quad i = 1, \dots, r,$$

because each $[u_i]_{2k}$ is feasible for (3.4). Thus,

$$f_1^{1,k} = f(u_1) = \dots = f(u_r).$$

Also note that $f_1^{1,k} \leq \lambda_1$ and each u_i is a Z-eigenvector. So,

$$f_1^{1,k} = f(u_1) = \dots = f(u_r) = \lambda_1.$$

(iv) When $Z(\mathbb{R}, \mathcal{A})$ is a finite set, the sequences $\{f_1^{1,k}\}$ and $\{f_1^{2,k}\}$ have finite convergence to λ_1 , by item (iii). If (3.2) has finitely many minimizers, i.e., λ_1 has finitely many real Z-eigenvectors, the rank condition (3.8) must be satisfied when k is sufficiently big. This can be implied by Theorem 2.6 of [22]. \square

Remark 3.2. (1) The rank condition (3.8) can be used as a criterion to check whether $f_1^{1,k} = \lambda_1$ or not. If it is satisfied, then we can get r distinct minimizers u_1, \dots, u_r of (3.2), i.e., each u_i is \mathbf{Z} -eigenvector of \mathcal{A} . The vectors u_i can be computed by the method in Henrion and Lasserre [7].

(2) Suppose (3.8) holds. If $\text{rank } M_k(y^*)$ is maximum among the set of all optimizers of (3.4), then we can get all mimizers of (3.2) (cf. [15, §6.6]), i.e., we can get all real \mathbf{Z} -eigenvectors associated to λ_1 . When (3.4)-(3.5) are solved by primal-dual interior point methods, typically we can get all \mathbf{Z} -eigenvectors associated to λ_1 , if there are finitely many ones.

3.2. Bigger \mathbf{Z} -eigenvalues. Suppose the i th smallest real \mathbf{Z} -eigenvalue λ_i is isolated and is known. We want to determine whether the next bigger one λ_{i+1} exists or not. If it exists, we show how to compute it. Let $\delta > 0$ be a small number. Consider the polynomial optimization problem

$$(3.9) \quad \begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0, f(x) \geq \lambda_i + \delta. \end{cases}$$

Clearly, the optimal value of (3.9) is the smallest \mathbf{Z} -eigenvalue that is greater than or equal to $\lambda_i + \delta$. Let

$$(3.10) \quad \omega(\lambda_i + \delta) := \min\{\lambda \in Z(\mathbb{R}, \mathcal{A}) : \lambda \geq \lambda_i + \delta\}.$$

Lasserre's hierarchy of semidefinite relaxations for solving (3.9) is

$$(3.11) \quad \begin{cases} f_{i+1}^{1,k} := \min & \langle f, y \rangle \\ \text{s.t.} & \langle 1, y \rangle = 1, L_h^{(k)}(y) = 0, \\ & M_k(y) \succeq 0, L_{f-\lambda_i-\delta}^{(k)}(y) \succeq 0, y \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases}$$

for the orders $k = k_0, k_0 + 1, \dots$. The dual problem of (3.11) is

$$(3.12) \quad \begin{cases} f_{i+1}^{2,k} := \max & \gamma \\ \text{s.t.} & f - \gamma \in I_{2k}(h) + Q_k(f - \lambda_i - \delta). \end{cases}$$

Similar to Theorem 3.1, we have the following convergence result.

Theorem 3.3. *Let $\mathcal{A} \in \mathbf{T}^m(\mathbb{R}^n)$ and $Z(\mathbb{R}, \mathcal{A})$ be the set of real \mathbf{Z} -eigenvalues of \mathcal{A} . Suppose $\lambda_i \in Z(\mathbb{R}, \mathcal{A})$. Then we have:*

- (i) *The intersection $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \emptyset$ if and only if the semidefinite relaxation (3.11) is infeasible for some order k .*
- (ii) *If $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) \neq \emptyset$, then*

$$(3.13) \quad \lim_{k \rightarrow \infty} f_{i+1}^{2,k} = \lim_{k \rightarrow \infty} f_{i+1}^{1,k} = \omega(\lambda_i + \delta).$$

If, in addition, $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty)$ is a finite set, then

$$(3.14) \quad f_{i+1}^{2,k} = f_{i+1}^{1,k} = \omega(\lambda_i + \delta)$$

for all k sufficiently big.

- (iii) *Suppose y^* is a minimizer of (3.11). If (3.8) is satisfied for some $t \leq k$, then $f_{i+1}^{1,k} = \omega(\lambda_i + \delta)$ and there are $r := \text{rank} M_t(y^*)$ distinct real \mathbf{Z} -eigenvectors u_1, \dots, u_r , associated with the \mathbf{Z} -eigenvalue $\omega(\lambda_i + \delta)$.*

- (iv) Suppose $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty)$ is a finite set and $\omega(\lambda_i + \delta)$ has finitely many Z-eigenvectors. Then, for all k big enough, and for every minimizer y^* of (3.11), there exists $t \leq k$ satisfying (3.8).

Proof. The proof is mostly same as for Theorem 3.1. In the following, we only list the differences.

(i) For the “only if” direction: If $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \emptyset$, then (3.9) is infeasible. By Positivstellensatz, $-1 \in I(h) + Q(f - \lambda_i - \delta)$. The resting proof is same as for Theorem 3.1(i).

(ii) The ideal $I(h)$ is archmedean, and so is $I(h) + Q(f - \lambda_i - \delta)$. The asymptotic convergence (3.13) can be implied by Theorem 4.2 of [13]. To prove the finite convergence (3.14), we follow the same proof as for Theorem 3.1(ii). Suppose $Z(\mathbb{R}, \mathcal{A}) \cap [\lambda_i + \delta, +\infty) = \{\nu_1, \dots, \nu_L\}$. Construct the polynomial s same as there and let

$$\hat{f} = f - \omega(\lambda_i + \delta) - s.$$

Then \hat{f} vanishes identically on the set $\{x \in \mathbb{R}^n : h(x) = 0, f(x) - \lambda_i - \delta \geq 0\}$. By Real Nullstellensatz (cf. [1, Corollary 4.1.8]), there exist an integer $\ell > 0$ and $q \in Q(f - \lambda_i - \delta)$ such that $\hat{f}^{2\ell} + q \in I(h)$. The resting proof is same, except replacing $\Sigma[x]$ by $Q(f - \lambda_i - \delta)$, and $\Sigma[x]_{2k}$ by $Q_k(f - \lambda_i - \delta)$.

(iii)-(iv) The proof is same as for Theorem 3.1(iii)-(iv). \square

The convergence of semidefinite relaxations of (3.11)-(3.12) can be checked by the condition (3.8). When it is satisfied, the Z-eigenvectors u_1, \dots, u_r can be computed by the method in [7]. Typically, we can get all Z-eigenvectors if primal-dual interior-point methods are used to solve the semidefinite programs. We refer to Remark 3.2.

Next, we show how to use $\omega(\lambda_i + \delta)$ to determine λ_{i+1} . Assume that λ_i is isolated, otherwise there are infinitely many Z-eigenvalues and it is impossible to get all of them. If λ_i is the biggest Z-eigenvalue, then we stop; otherwise, the next bigger one λ_{i+1} exists. For such case, if $\delta > 0$ in (3.9) is small enough, then $\omega(\lambda_i + \delta)$ as in (3.10) equals λ_{i+1} . Consider the optimization problem

$$(3.15) \quad \begin{cases} \nu_i := \max & f(x) \\ \text{s.t.} & h(x) = 0, f(x) \leq \lambda_i + \delta. \end{cases}$$

The optimal value of (3.15) is the biggest Z-eigenvalue of \mathcal{A} that is smaller than or equal to $\lambda_i + \delta$, i.e.,

$$\nu_i = \max\{\lambda \in Z(\mathbb{R}, \mathcal{A}) : \lambda \leq \lambda_i + \delta\}.$$

The next bigger Z-eigenvalue λ_{i+1} can be determined by the following theorem.

Theorem 3.4. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $\delta > 0$. Suppose λ_i is a Z-eigenvalue of \mathcal{A} and λ_{max} is the biggest one.*

- (i) *Suppose λ_i is an isolated Z-eigenvalue of \mathcal{A} . If $\nu_i = \lambda_i$ and $\lambda_{max} > \lambda_i$, then $\omega(\lambda_i + \delta) = \lambda_{i+1}$, which is the smallest Z-eigenvalue bigger than λ_i .*
(ii) *If $\nu_i = \lambda_i$ and (3.11) is infeasible for some k , then $\lambda_i = \lambda_{max}$.*

Proof. (i) We have seen that $\nu_i = \lambda_i$ is the biggest Z-eigenvalue of \mathcal{A} that is smaller than or equal to $\lambda_i + \delta$. Since $\lambda_i = \nu_i$ is isolated and $\lambda_{max} > \lambda_i$, the smallest Z-eigenvalue bigger than λ_i is λ_{i+1} , and $\lambda_{i+1} > \lambda_i + \delta$. By (3.10), $\omega(\lambda_i + \delta)$ is the smallest Z-eigenvalue that is greater than or equal to $\lambda_i + \delta$. There are no Z-eigenvalues in the open interval $(\lambda_i, \lambda_{i+1})$. So, $\omega(\lambda_i + \delta) = \lambda_i + \delta$.

(ii) When (3.11) is infeasible for some k , by Theorem 3.3(i), all the real Z-eigenvalues are smaller than $\lambda_i + \delta$. Note that ν_i is the biggest Z-eigenvalue that is smaller than or equal to $\lambda_i + \delta$. If $\lambda_i = \nu_i$, then λ_i must be the biggest Z-eigenvalue, i.e., $\lambda_i = \lambda_{max}$. \square

The problem (3.15) is a polynomial optimization. Its optimal value ν_i can also be computed by solving Lasserre type semidefinite relaxations that are similar to (3.11)-(3.12). When λ_i is isolated, for $\delta > 0$ sufficiently small, we must have $\nu_i = \lambda_i$, no matter if λ_{i+1} exists or not. This is because ν_i is the smallest Z-eigenvalue greater than or equal to $\lambda_i + \delta$. Lemma 2.1 can be used to verify that λ_i is isolated.

3.3. An algorithm for computing all real Z-eigenvalues. Assume all the real Z-eigenvalues of the tensor \mathcal{A} are isolated. We compute all of them, from the smallest to the biggest. First, we compute λ_1 if it exists, by solving (3.4)-(3.5). After getting λ_1 , we solve the hierarchy of (3.11)-(3.12) and then determine λ_2 . If λ_2 does not exist, we stop; otherwise, we then determine λ_3 . Repeating this procedure, we can get all the real Z-eigenvalues.

The following algorithm can be applied to get Z-eigenvalues.

Algorithm 3.5. *Compute real Z-eigenvalues of a tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$.*

- Step 0: *Choose a small positive value for δ (e.g., 0.05).*
- Step 1: *If (3.4) is infeasible for some order k , then \mathcal{A} has no real Z-eigenvalues and stop. Otherwise, solve it to get the smallest Z-eigenvalue λ_1 . Let $i = 1$.*
- Step 2: *Solve (3.15) for ν_i . If $\nu_i = \lambda_i$, go to Step 3. Otherwise, reduce the value of δ (e.g., let $\delta := \delta/5$) and compute ν_i . Repeat until we get $\nu_i = \lambda_i$.*
- Step 3: *If (3.11) is infeasible for some order k , the largest Z-eigenvalue is λ_i and stop. Otherwise, solve it for $\omega(\delta_i + \delta)$. Let $\lambda_{i+1} := \omega(\delta_i + \delta)$, $i := i + 1$ and go to Step 2.*

For a generic tensor \mathcal{A} , it has finitely many Z-eigenvalues, and all of them are isolated. So, Algorithm 3.5 terminates after a finite number of steps, for almost all nonsymmetric tensors in $\mathbb{T}^m(\mathbb{R}^n)$.

4. COMPUTING ALL H-EIGENVALUES

In this section, we compute all H-eigenvalues for nonsymmetric tensors. Unlike Z-eigenvalues, the number of H-eigenvalues is always finite.

Proposition 4.1. *Every tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{C}^n)$ has $n(m-1)^{n-1}$ complex H-eigenvalues, including their multiplicities.*

Proof. Let $\mathcal{I} \in \mathbb{T}^m(\mathbb{R}^n)$ be the identity tensor whose only non-zero entries are $\mathcal{I}_{i \dots i} = 1$, with $i = 1, 2, \dots, n$. Recall that λ is an H-eigenvalue if and only if there exists $0 \neq u \in \mathbb{C}^n$ such that $\mathcal{A}u^{m-1} = \lambda u^{[m-1]}$, that is, $(\mathcal{A} - \lambda \mathcal{I})u^{m-1} = 0$. By the definition of resultant (cf. [34]), which we denote by Res , λ is an H-eigenvalue if and only if

$$(4.1) \quad Res((\mathcal{A} - \lambda \mathcal{I})x^{m-1}) = 0.$$

The resultant $Res((\mathcal{A} - \lambda \mathcal{I})x^{m-1})$ is homogeneous in the entries of \mathcal{A} and λ . It has degree $D := n(m-1)^{n-1}$. We can expand it as

$$Res((\mathcal{A} - \lambda \mathcal{I})x^{m-1}) = p_0(\mathcal{A}) + p_1(\mathcal{A})\lambda + \dots + p_D(\mathcal{A})\lambda^D.$$

By the homogeneity of Res , $p_D(\mathcal{A}) = Res(-\mathcal{I}x^{m-1}) \neq 0$, because the homogeneous polynomial system $-\mathcal{I}x^{m-1} = 0$ has no nonzero complex solutions. This means that $Res((\mathcal{A} - \lambda \mathcal{I})x^{m-1})$ is not constantly zero, and its degree is D . Therefore, (4.1) has D complex roots including multiplicities. Hence, \mathcal{A} has D complex H-eigenvalues. \square

Recall that (λ, u) is a real H-eigenpair of a tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ if and only if $0 \neq u \in \mathbb{R}^n$ and $\mathcal{A}u^{m-1} = \lambda u^{[m-1]}$. Let m_0 be the biggest even number not bigger than m , i.e.,

$$m_0 = 2\lceil(m-1)/2\rceil.$$

Note that $m-1 \leq m_0 \leq m$. We can normalize such u as

$$(4.2) \quad (u_1)^{m_0} + \dots + (u_n)^{m_0} = 1.$$

Under this normalization, the H-eigenvalue λ can be given as

$$\lambda = \lambda(u^{[m_0-m+1]})^T u^{[m-1]} = (u^{[m_0-m+1]})^T \mathcal{A}u^{m-1}.$$

Let h be the polynomial tuple

$$(4.3) \quad h := \left(\mathcal{A}x^{m-1} - ((x^{[m_0-m+1]})^T \mathcal{A}x^{m-1})x^{[m-1]}, \sum_{i=1}^n (x_i)^{m_0} - 1 \right).$$

Then, u is an H-eigenvector normalized as in (4.2) if and only if $h(u) = 0$. Since \mathcal{A} has finitely many H-eigenvalues, we can order its real ones monotonically as

$$\mu_1 < \mu_2 < \dots < \mu_N,$$

if at least one real H-eigenvalue exists. We call μ_i the i th smallest H-eigenvalue.

4.1. The smallest H-eigenvalue. In this subsection, we show how to determine μ_1 . Let h be the polynomial tuple as in (4.3), then μ_1 equals the optimal value of the optimization problem

$$(4.4) \quad \begin{cases} \min & f(x) := (x^{[m_0-m+1]})^T \mathcal{A}x^{m-1} \\ \text{s.t.} & h(x) = 0. \end{cases}$$

Lasserre's hierarchy [13] of semidefinite relaxations for solving (4.4) is

$$(4.5) \quad \begin{cases} \rho_1^{1,k} := \min & \langle f, z \rangle \\ \text{s.t.} & \langle 1, z \rangle = 1, L_h^{(k)}(z) = 0, \\ & M_k(z) \succeq 0, z \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases}$$

for the orders $k = k_0, k_0 + 1, \dots$, where

$$(4.6) \quad k_0 := \lceil (m_0 + m - 1)/2 \rceil.$$

The dual optimization problem of (4.5) is

$$(4.7) \quad \begin{cases} \rho_1^{2,k} := \max & \gamma \\ \text{s.t.} & f - \gamma \in I_{2k}(h) + \Sigma[x]_{2k}. \end{cases}$$

As can be shown in [13], $\rho_1^{2,k} \leq \rho_1^{1,k} \leq \mu_1$ for all k , and the sequences $\{\rho_1^{1,k}\}$ and $\{\rho_1^{2,k}\}$ are monotonically increasing.

Theorem 4.2. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $H(\mathbb{R}, \mathcal{A})$ be the set of its real \mathbb{H} -eigenvalues. Then we have:*

- (i) *The set $H(\mathbb{R}, \mathcal{A}) = \emptyset$ if and only if the semidefinite relaxation (4.5) is infeasible for some order k .*
- (ii) *If $H(\mathbb{R}, \mathcal{A}) \neq \emptyset$, then for all k sufficiently large*

$$(4.8) \quad \rho_1^{1,k} = \rho_1^{2,k} = \mu_1.$$

- (iii) *Let k_0 be as in (4.6). Suppose z^* is a minimizer of (4.5). If there exists an integer $t \leq k$ such that*

$$(4.9) \quad \text{rank } M_{t-k_0}(z^*) = \text{rank } M_t(z^*),$$

then $\rho_1^{1,k} = \mu_1$ and there are $r := \text{rank } M_t(z^)$ distinct real \mathbb{H} -eigenvectors u_1, \dots, u_r associated with μ_1 and normalized as in (4.2).*

- (iv) *Suppose \mathcal{A} has finitely many \mathbb{H} -eigenvectors associated to μ_1 . Then, for all k big enough and for every minimizer z^* of (4.5), there exists an integer $t \leq k$ satisfying (4.9).*

Proof. It can be proved in the same way as for Theorem 3.1. The only difference is that $H(\mathbb{R}, \mathcal{A})$ is always a finite set, by Proposition 4.1. For cleanness of the paper, we omit the proof here. \square

The rank condition (4.9) is a criterion for checking the convergence of Lasserre's hierarchy of (4.5) and (4.7). When it is satisfied, the \mathbb{H} -eigenvectors u_1, \dots, u_r can be computed by the method in [7]. Typically, we can get all \mathbb{H} -eigenvectors if primal-dual interior-point methods are used to solve the semidefinite relaxations. We refer to Remark 3.2.

4.2. Bigger \mathbb{H} -eigenvalues. Suppose the i th smallest real \mathbb{H} -eigenvalue μ_i exists and is known. We want to determine whether the next bigger one μ_{i+1} exists or not. If it exists, we show how to compute it.

Let $\delta > 0$ be a small number. Consider the optimization problem

$$(4.10) \quad \begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0, f(x) \geq \mu_i + \delta, \end{cases}$$

where f, h are same as in (4.4). The optimal value of (4.10) is the smallest \mathbb{H} -eigenvalue of \mathcal{A} that is greater than or equal to $\mu_i + \delta$. Denote

$$(4.11) \quad \varpi(\mu_i + \delta) := \min\{\mu \in H(\mathbb{R}, \mathcal{A}) : \mu \geq \mu_i + \delta\}.$$

Lasserre's hierarchy of semidefinite relaxations for solving (4.10) is

$$(4.12) \quad \begin{cases} \rho_{i+1}^{1,k} := \min & \langle f, z \rangle \\ \text{s.t.} & \langle 1, z \rangle = 1, L_h^{(k)}(z) = 0, \\ & M_k(z) \succeq 0, L_{f-\mu_i-\delta}^{(k)}(z) \succeq 0, z \in \mathbb{R}^{\mathbb{N}_{2k}^n}, \end{cases}$$

for the orders $k = k_0, k_0 + 1, \dots$. The dual problem of (4.12) is

$$(4.13) \quad \begin{cases} \rho_{i+1}^{2,k} := \max & \gamma \\ \text{s.t.} & f - \gamma \in I_{2k}(h) + Q_k(f - \mu_i - \delta). \end{cases}$$

The properties of relaxations (4.12)-(4.13) are as follows.

Theorem 4.3. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $H(\mathbb{R}, \mathcal{A})$ be the set of its real \mathbb{H} -eigenvalues. Assume that $\mu_i \in H(\mathbb{R}, \mathcal{A})$. Then we have:*

- (i) *The intersection $H(\mathbb{R}, \mathcal{A}) \cap [\mu_i + \delta, +\infty) = \emptyset$ if and only if the semidefinite relaxation (4.12) is infeasible for some order k .*
- (ii) *If $H(\mathbb{R}, \mathcal{A}) \cap [\mu_i + \delta, +\infty) \neq \emptyset$, then for all k sufficiently large*

$$(4.14) \quad \rho_{i+1}^{2,k} = \rho_{i+1}^{1,k} = \varpi(\mu_i + \delta).$$
- (iii) *Let z^* be a minimizer of (4.12). If (4.9) is satisfied for some $t \leq k$, then there exists $r := \text{rank} M_t(z^*)$ \mathbb{H} -eigenvectors u_1, \dots, u_r that are normalized as in (4.2) and are associated with $\varpi(\mu_i + \delta)$.*
- (iv) *Suppose \mathcal{A} has finitely many \mathbb{H} -eigenvectors that are associated with $\varpi(\mu_i + \delta)$ and are normalized as in (4.2). Then, for all k big enough and for all minimizer z^* of (4.12), there exists $t \leq k$ satisfying (4.9).*

Proof. It can be proved in the same way as for Theorem 3.3. Note that $H(\mathbb{R}, \mathcal{A})$ is always a finite set, by Proposition 4.1. \square

In the following, we show how to use $\varpi(\mu_i + \delta)$ to determine μ_{i+1} . Consider the maximization problem

$$(4.15) \quad \begin{cases} v_i := \max & f(x) \\ \text{s.t.} & h(x) = 0, f(x) \leq \mu_i + \delta. \end{cases}$$

The optimal value of (4.15) is the biggest \mathbb{H} -eigenvalue of \mathcal{A} that is smaller than or equal to $\mu_i + \delta$, i.e.,

$$v_i = \max\{\mu \in H(\mathbb{R}, \mathcal{A}) : \mu \leq \mu_i + \delta\}.$$

The next bigger \mathbb{H} -eigenvalue μ_{i+1} can be determined by the following theorem.

Theorem 4.4. *Let $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$ and $\delta > 0$. Suppose $\mu_i \in H(\mathbb{R}, \mathcal{A})$ and μ_{max} is the maximum \mathbb{H} -eigenvalue. Let $\varpi(\lambda_i + \delta)$ be as in (4.11).*

- (i) *If $v_i = \mu_i$ and $\mu_{max} > \mu_i$, then $\varpi(\lambda_i + \delta) = \mu_{i+1}$.*
- (ii) *If $v_i = \mu_i$ and (4.12) is infeasible for some k , then $\mu_i = \mu_{max}$.*

Proof. The proof is the same as for Theorem 3.4. Note that \mathcal{A} has finitely many \mathbb{H} -eigenvalues, and μ_i is always an isolated one. \square

Since (4.15) is a polynomial optimization, the optimal value v_i can also be computed by solving Lasserre type semidefinite relaxations that are similar to (4.12)-(4.13). For $\delta > 0$ sufficiently small, we must have $v_i = \mu_i$, no matter if μ_{i+1} exists or not. This is because v_i is the biggest H-eigenvalue that is less than or equal to $\mu_i + \delta$.

4.3. An algorithm for all H-eigenvalues. We can compute all real H-eigenvalues of a tensor \mathcal{A} sequentially, from the smallest one to the biggest one, if they exist. A similar version of Algorithm 3.5 can be applied.

Algorithm 4.5. *Compute all real H-eigenvalues of a tensor $\mathcal{A} \in \mathbb{T}^m(\mathbb{R}^n)$.*

- Step 0: *Choose a small positive value for δ (e.g., 0.05).*
- Step 1: *If (4.5) is infeasible for some order k , then \mathcal{A} has no real H-eigenvalues and stop. Otherwise, compute the smallest one μ_1 by solving the hierarchy of (4.5). Let $i := 1$.*
- Step 2: *Solve (4.15) for its optimal value v_i . If $v_i = \mu_i$, go to Step 3; otherwise, reduce δ (e.g., $\delta = \delta/5$) and compute v_i . Repeat until we get $v_i = \mu_i$.*
- Step 3: *If (4.12) is infeasible for some order k , the largest H-eigenvalue is μ_i and stop. Otherwise, compute $\varpi(\mu_i + \delta)$ by solving the hierarchy of (4.12). Let $\mu_{i+1} := \varpi(\mu_i + \delta)$, $i := i + 1$ and go to Step 2.*

5. NUMERICAL EXAMPLES

In this section, we give numerical examples for how to compute real Z-eigenvalues and H-eigenvalues for nonsymmetric tensors. The computation is implemented in MATLAB 7.10 in a Dell Linux Desktop with 8GB memory and Intel(R) CPU 2.8GHz. The software Gloptipoly 3 [8] is used to solve the semidefinite relaxations. For computational results, only four decimal digits are displayed, for cleanness of the presentation.

For odd ordered tensors, the Z-eigenvalues always appear in \pm pairs, so only nonnegative Z-eigenvalues are shown for them. For Algorithm 3.5 to compute all of them, the real Z-eigenvalues of the tensor need to be all isolated. General nonsymmetric tensors have finitely many Z-eigenvalues, so they are all isolated. In applications, for particular tensors, Lemma 2.1 can be used to verify that all real Z-eigenvalues are isolated, once they are computed. In all our examples, we used it to check the isolatedness. The computations of all real eigenvalues in our examples took from a few to a couple of seconds.

Example 5.1. ([17, Example 3]) *Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^2)$ with entries $\mathcal{A}_{i_1 i_2 i_3 i_4} = 0$ except*

$$\mathcal{A}_{1111} = 25.1, \mathcal{A}_{1212} = 25.6, \mathcal{A}_{2121} = 24.8, \mathcal{A}_{2222} = 23.$$

Applying Algorithms 3.5 and 4.5, we get all the real Z-eigenvalues and H-eigenvalues correctly. The computed eigenvalues, as well as their eigenvectors, are shown in Table 1.

TABLE 1. Z/H-eigenpairs of the tensor in Example 5.1

i	1	2	3
Z-eigenvalue λ_i	23.000	25.1000	
Z-eigenvectors	$\pm(0, 1)$	$\pm(1, 0)$	
H-eigenvalue μ_i	23.0000	25.1000	49.2687
H-eigenvector	$\pm(0, 1)$	$\pm(1, 0)$	$\pm(0.8527, \pm 0.8285)$

TABLE 2. Z/H-eigenvalues of the tensor in Example 5.4

n	Z-eigenvalues (≥ 0)	H-eigenvalues
$n = 2$	10.5518	none
$n = 3$	0.2336, 1.6614, 10.5063	-2.5615, 0.3456
$n = 4$	3.3651, 8.8507, 10.4981	-6.2888, -0.7048, 2.8947, 5.9245

Example 5.2. ([32, Example 1]) Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^3)$ with the entries $\mathcal{A}_{i_1 i_2 i_3} = 0$ except

$$\begin{aligned} \mathcal{A}_{111} &= 0.4333, & \mathcal{A}_{121} &= 0.4278, & \mathcal{A}_{131} &= 0.4140, & \mathcal{A}_{211} &= 0.8154, & \mathcal{A}_{221} &= 0.0199, \\ \mathcal{A}_{231} &= 0.5598, & \mathcal{A}_{311} &= 0.0643, & \mathcal{A}_{321} &= 0.3815, & \mathcal{A}_{331} &= 0.8834, & \mathcal{A}_{112} &= 0.4866, \\ \mathcal{A}_{122} &= 0.8087, & \mathcal{A}_{132} &= 0.2073, & \mathcal{A}_{212} &= 0.7641, & \mathcal{A}_{222} &= 0.9924, & \mathcal{A}_{232} &= 0.8752, \\ \mathcal{A}_{312} &= 0.6708, & \mathcal{A}_{322} &= 0.8296, & \mathcal{A}_{332} &= 0.125, & \mathcal{A}_{113} &= 0.3871, & \mathcal{A}_{123} &= 0.0769, \\ \mathcal{A}_{133} &= 0.3151, & \mathcal{A}_{213} &= 0.1355, & \mathcal{A}_{223} &= 0.7727, & \mathcal{A}_{233} &= 0.4089, & \mathcal{A}_{313} &= 0.9715, \\ \mathcal{A}_{323} &= 0.7726, & \mathcal{A}_{333} &= 0.5526. \end{aligned}$$

By Algorithms 3.5 and 4.5, we get all its nonnegative Z-eigenvalues

$$0.2331, \quad 0.4869, \quad 2.7418,$$

and all its real H-eigenvalues

$$1.3586, \quad 1.4985, \quad 1.5226, \quad 4.7303.$$

Example 5.3. ([10, §4.1]) Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^3)$ with $\mathcal{A}_{i_1 i_2 i_3} = 0$ except

$$\begin{aligned} \mathcal{A}_{111} &= 0.0072, & \mathcal{A}_{121} &= -0.4413, & \mathcal{A}_{131} &= 0.1941, & \mathcal{A}_{211} &= -0.4413, & \mathcal{A}_{221} &= 0.0940, \\ \mathcal{A}_{231} &= 0.5901, & \mathcal{A}_{311} &= 0.1941, & \mathcal{A}_{321} &= -0.4099, & \mathcal{A}_{331} &= -0.1012, & \mathcal{A}_{112} &= -0.4413, \\ \mathcal{A}_{122} &= 0.0940, & \mathcal{A}_{132} &= -0.4099, & \mathcal{A}_{212} &= 0.0940, & \mathcal{A}_{222} &= 0.2183, & \mathcal{A}_{232} &= 0.2950, \\ \mathcal{A}_{312} &= 0.5901, & \mathcal{A}_{322} &= 0.2950, & \mathcal{A}_{332} &= 0.2229, & \mathcal{A}_{113} &= 0.1941, & \mathcal{A}_{123} &= 0.5901, \\ \mathcal{A}_{133} &= -0.1012, & \mathcal{A}_{213} &= -0.4099, & \mathcal{A}_{223} &= 0.2950, & \mathcal{A}_{233} &= 0.2229, & \mathcal{A}_{313} &= -0.1012, \\ \mathcal{A}_{323} &= 0.2229, & \mathcal{A}_{333} &= -0.4891. \end{aligned}$$

By Algorithms 3.5 and 4.5, we get all its nonnegative Z-eigenvalues

$$0.0000, \quad 0.5774$$

and all its real H-eigenvalues

$$0.0000, \quad 0.7875.$$

Example 5.4. ([20, Example 3.19]) Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3} = \tan\left(i_1 - \frac{i_2}{2} + \frac{i_3}{3}\right).$$

Applying Algorithms 3.5 and 4.5, we get all the real Z-eigenvalues and H-eigenvalues. They are reported in Table 2, for $n = 2, 3, 4$. There are no real H-eigenvalues for the case $n = 2$.

TABLE 3. Z/H-eigenvalues of the tensor in Example 5.7

n	Z-eigenvalues(≥ 0)	H-eigenvalues
2	0.4721	0.5138, 1.2654
3	0.6158	0.5196, 2.0800, 2.2995, 2.4335
4	0.7682	0.5199, 2.0964, 2.2980, 2.3991, 2.9454, 4.4609, 4.9588, 5.4419
5	0.8384	0.5199, 2.0978, 2.2997, 2.3860, 2.4010, 2.9658, 4.4713, 4.4902, 4.6880, 4.7008, 5.0136, 5.7891, 6.0668, 7.3250, 7.3469, 8.8555

Example 5.5. Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}_{i_1 \dots i_4} = \arctan(i_1 i_2^2 i_3 i_4).$$

By Algorithms 3.5 and 4.5, we get all its real Z-eigenvalues

$$-0.2700, \quad 0.0003, \quad 13.8286,$$

and all its H-eigenvalues

$$-0.3662, \quad 0.0005, \quad 41.4705.$$

Example 5.6. Consider the tensor $\mathcal{A} \in \mathbb{T}^4(\mathbb{R}^3)$ such that

$$\mathcal{A}_{i_1 \dots i_4} = (1 + i_1 + 2i_2 + 3i_3 + 4i_4)^{-1}.$$

By Algorithms 3.5 and 4.5, we get all its real Z-eigenvalues

$$0.0000, \quad 0.0002, \quad 0.4572,$$

and all its real H-eigenvalues

$$0.0000, \quad 0.0005, \quad 1.3581.$$

Example 5.7. Consider the tensor $\mathcal{A} \in \mathbb{T}^5(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1, \dots, i_5} = \left(\sum_{j=1}^5 (-1)^{j-1} \exp(i_j) \right)^{-1}.$$

For $n = 2, 3, 4, 5$, all the real Z-eigenvalues and H-eigenvalues are found by Algorithms 3.5 and 4.5. They are shown in Table 3.

Example 5.8. Consider the tensor $\mathcal{A} \in \mathbb{T}^3(\mathbb{R}^n)$ such that

$$\mathcal{A}_{i_1 i_2 i_3} = \frac{1}{10} \left(i_1 + 2i_2 + 3i_3 - \sqrt{i_1^2 + 2i_2^2 + 3i_3^2} \right).$$

For the values $n = 2, 3, 4$, we get all the real Z-eigenvalues and H-eigenvalues, by Algorithms 3.5 and 4.5. They are reported in Table 4.

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TABLE 4. Z/H-eigenvalues of the tensor in Example 5.8

n	Z-eigenvalues(≥ 0)	H-eigenvalues
2	0.0024, 0.0038, 1.4928	0.0060, 2.0960
3	0.0067, 0.0161, 3.6417	-0.0401, -0.0243, 0.0086, 0.0235, 0.1568, 0.6635, 1.4958, 6.2378
4	0.0000, 0.0107, 0.0396, 6.9922	-0.0240, -0.0087, 0.0000, 0.0102, 0.0258, 0.0437, 1.5761, 2.6824, 4.1089, 5.8270, 13.7960

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