Optimal stochastic extragradient schemes for pseudomonotone stochastic variational inequality problems and their variants

Aswin Kannan and Uday V. Shanbhag^{*}

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Abstract

We consider the stochastic variational inequality problem in which the map is expectationvalued in a component-wise sense. Much of the available convergence theory and rate statements for stochastic approximation schemes are limited to monotone maps. However, non-monotone stochastic variational inequality problems are not uncommon and are seen to arise from product pricing, fractional optimization problems, and subclasses of economic equilibrium problems. Motivated by the need to address a broader class of maps, we make the following contributions: (i) We present an extragradient-based stochastic approximation scheme and prove that the iterates converge to a solution of the original problem under either pseudomonotonicity requirements or a suitably defined acute angle condition. Such statements are shown to be generalizable to the stochastic mirror-prox framework; (ii) Under strong pseudomonotonicity, we show that the mean-squared error in the solution iterates produced by the extragradient SA scheme converges at the optimal rate of $\mathcal{O}\left(\frac{1}{K}\right)$, statements that were hitherto unavailable in this regime. Notably, we optimize the initial steplength by obtaining an ϵ -infimum of a discontinuous nonconvex function. Similar statements are derived for mirror-prox generalizations and can accommodate monotone SVIs under a weak-sharpness requirement. Finally, both the asymptotics and the empirical rates of the schemes are studied on a set of variational problems where it is seen that the theoretically specified initial steplength leads to significant performance benefits.

1 Introduction

Several applications arising in engineering, science, finance, and economics lead to a broad range of optimization and equilibrium problems. Under suitable convexity assumptions, the equilibrium conditions of such problems may be compactly stated as as a variational inequality problem [1,2, Ch. 1]. Recall that given a set $X \subseteq \mathbb{R}^n$ and a map $F : \mathbb{R}^n \to \mathbb{R}^n$, the variational inequality problem, denoted by $\operatorname{VI}(X, F)$, requires an $x^* \in X$ such that $(x - x^*)^T F(x^*) \ge 0$ for all $x \in X$. In a multitude of settings, either the evaluation of the map F(x) is corrupted by error or its components are expectation valued, a consequence of the original model being a stochastic optimization or equilibrium problem. Consequently, $F_i(x) \triangleq \mathbb{E}[F_i(x,\xi)]$ for $i = 1, \dots, n$. Note that $\xi : \Omega \to \mathbb{R}^d$, $F : X \times \mathbb{R}^d \to \mathbb{R}^n$ and (Ω, F, P) is the associated probability space. As a result, the stochastic variational problem requires finding a vector $x^* \in X$ such that

$$(x - x^*)^T \mathbb{E}[F(x^*, \xi(\omega))] \ge 0, \quad \forall x \in X.$$

$$(1)$$

^{*}The first author is at IBM research while the second is affiliated with Industrial and Manufacturing Engineering, The Pennsylvania State University, University Park, PA, 16802 Email: {aswinkannan1987@gmail.com,udaybag@psu.edu}. This research has been partially supported by NSF Awards 1246887 (CAREER), 1538193, and 1408366.

Throughout, we use $F(x; \omega)$ to refer to $F(x, \xi(\omega))$ and refer to our problem as SVI(X, F). We begin by providing some motivation for weakening the monotonicity requirement.

1.1 Motivation

We draw motivation from three classes of problems.

(a) Competitive exchange economy. We consider a competitive exchange economy [3] in which there is a collection of n goods with an associated price vector $p \in \mathbb{R}^{n}_{++}$ and a positive budget w. The consumption vector $F(p, w; \omega)$ specifies the uncertain consumption level and the consumption has to satisfy budget constraints in an expected-value sense, as specified by

$$\mathbb{E}[p^T F(p, w; \omega)] \le w.$$

This demand function is assumed to be homogeneous with degree zero or $F(\lambda p, \lambda w; \omega) = F(p, w; \omega)$ for any positive λ . Furthermore, $F(p, w) \triangleq \mathbb{E}[F(p, w; \omega)]$. An additional condition satisfied by $F(p, w; \omega)$ is the (expected) Weak Weak Axiom of revealed preference (EWWA), which requires that for all pairs (p_1, w_1) and (p_2, w_2)

$$p_2^T \mathbb{E}[F(p_1, w_1; \omega)] \le w_2 \implies p_1^T \mathbb{E}[F(p_2, w_2; \omega)] \ge w_1.$$

This axiom is an expected-value variant of WWA which itself represents a weakening of the Weak Axiom of revealed preference [4]. Before proceeding, we provide some intuition for this axiom. At prices p_2 and budget w_2 , an agent chooses a bundle $F(p_2, w_2)$. If at the same prices, she can also afford $F(p_1, w_1)$, then we have that $p_2^T F(p_1, w_1) \leq w_2$. Consequently, the consumer believes that the bundle $F(p_2, w_2)$ is at least as good as $F(p_1, w_1)$. If at p_1 and w_1 , the bundle $F(p_2, w_2)$ is cheaper than the chosen bundle $F(p_1, w_1)$, it follows that she can afford a bundle b such that b contains more of each commodity than $F(p_2, w_2)$. It may then be concluded that the agent prefers b to $F(p_2, w_2)$. But $F(p_2, w_2)$ is at least as good as $F(p_1, w_1)$, implying that the bundle b is preferable to $F(p_1, w_1)$. But bundle b is cheaper than $F(p_1, w_1)$, contradicting the choice of $F(p_1, w_1)$ and one can conclude that $F(p_2, w_2)$ cannot be cheaper than $F(p_1, w_1)$ or $p_1^T F(p_2, w_2) \geq w_1$. If $w_1 = w_2$, we have the following:

$$p_2^T F(p_1, w) \le w \implies p_1^T F(p_2, w) \ge w.$$

By the budget identity, $p_1^T F(p_1, w) = p_2^T F(p_2, w) = w$, implying that

$$(p_2 - p_1)^T F(p_1, w) \le 0 \implies (p_2 - p_1)^T F(p_2, w) \le 0.$$

It follows that $F(\bullet, w)$ is a pseudomonotone map in (\bullet) for any positive w. Formally, the property of pseudomonotonicity can be defined as follows.

Definition 1 (Pseudomonotonicity). A continuous mapping $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is pseudomonotone on X if for all $x, y \in X$, $(x - y)^T F(y) \ge 0 \implies (x - y)^T F(x) \ge 0$.

We now present how one may model the notion of equilibrium in a general consumption sector with a finite set of agents, denoted by \mathcal{A} . An agent $a \in \mathcal{A}$ is characterized by an endowment $e_a \in \mathbb{R}^n_{++}$ and a demand function $F_a(p, p^T e_a)$, implying that the consumption of agent a is given by $\varphi_a(p) = p^T F_a(p, p^T e_a)$. The aggregate demand, given by the function F(p), is defined as F(p) = $\sum_{a \in \mathcal{A}} F_a(p, p^T e_a)$. Note that F(p) is homogeneous in p with degree zero and by the individual budget identities, we have Walras' law; for all p, $p^T F(p) = p^T e = p^T (\sum_a e_a)$, where e denotes the sector-wide initial endowment. The demand function F(p) satisfies the WWA if

$$p_2^T F(p_1) \le p_2^T e \implies p_1^T F(p_2) \ge p_1^T e.$$

The WWA can be presented in a more convenient fashion if Z(p) = F(p) - e. While the consumption sector is captured by Z(p), the set Y represents the set of technology available and $y \in Y$ represents either input (negative) or output (positive) based on sign. The set Y satisfies two requirements: (i) *Free disposal* : goods may be arbitrarily wasted without using further inputs or $-\mathbb{R}_n^+ \subseteq Y$; and (ii) "No free lunch": production requires some inputs or $Y \cap \mathbb{R}_+^n = \{0\}$. Consequently, an equilibrium of the economy (Y, Z) is given by a $p \in \mathcal{P}$ such that

(a)
$$Z(p) \in Y$$
 and (b) $p^T y \leq 0$, for all $y \in Y$.

Condition (a) implies that demand is met at equilibrium while (b) implies that firms maximize profits by choosing plans y = Z(p). In fact, by [3, Th. 1], p is an equilibrium of the economy (Y, Z)if and only if p is a solution of VI(Q, Z), where $Q \triangleq \mathcal{P} \cap Y^{\circ}$ and $Y^{\circ} \triangleq \{d : d^T y \ge 0, y \in Y\}$. But by leveraging the WWA, $Z(p) = \mathbb{E}[Z(p; \xi(\omega))]$ is a pseudomonotone map from the EWWA, leading to a pseudomonotone stochastic variational inequality problem.

(b) Stochastic fractional programming. Fractional programs involve the optimization of metrics such as lift-to-drag ratios in aircraft design [5], fuel economy to engine performance ratios in automotive design [6], and signal-to-noise ratios in wireless networks [7]; these problems can often be cast as pseudomonotone. More recently, efforts in financial engineering optimize the Omega ratio [8,9] which quantifies the ratio of gain probability to loss probability. All of the above problems fall under the umbrella of "fractional programming" and we consider the stochastic generalization of this problem:

$$\min_{x \in X} \quad h(x) \triangleq \mathbb{E}\left[\frac{f(x;\xi(\omega))}{g(x;\xi(\omega))}\right],\tag{2}$$

where $f, g: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $\xi: \Omega \to \mathbb{R}^d$. While h(x) cannot be guaranteed to be pseudoconvex in general, in automotive problems [6], $f(x; \xi(\omega))$ corresponds to the uncertain time taken to accelerate from 0 to v^{\max} miles per hour while $g(x; \xi(\omega))$ denotes the uncertain fuel economy. The design space x corresponds to engine design specifications such as gear ratios and transmission switching levels. Consequently, the equilibrium conditions are given by a pseudomonotone stochastic variational inequality problem. We present an extended version of Lemma 2.1 from [10] as a definition (proof omitted). It provides conditions for the pseudoconvexity of h(x) under some basic assumptions. Note that h(x) is pseudoconvex if and only if -h(x) is pseudoconcave.

Lemma 1 (Stochastic pseudoconvex function). Suppose the following hold. (i) $f: X \times \mathbb{R}^d \to \mathbb{R}$ is a nonnegative convex function in a.s. fashion; (ii) $g: X \times \mathbb{R}^d \to \mathbb{R}$ is a positive concave (strictly concave) function in an a.s. sense; and (iii) $f(\bullet; \omega), g(\bullet; \omega) \in C^1$ in a.s. sense, then the function $h: X \to \mathbb{R}$, given by $h(x) = \mathbb{E}[f(x, \omega)/g(x, \omega)]$, is a pseudoconvex (strongly pseudoconvex) function.

(c) Stochastic product pricing. Consider an oligopolistic market with a set of substitutable goods in which firms compete in prices. In such Bertrand markets [11, 12], the quantity sold by a particular firm is contingent on the prices set by the firms (and possibly other product attributes)

and this firm-specific demand is captured by the Generalized Extreme Value (GEV) model [12–14]. and the Multiplicative Competitive Interaction (MCI) model [15, 16] have been very useful in characterizing consumer demand based on price and product attributes. The *multinomial logit* is a commonly used GEV model that possesses some tractability and finds application in revenue management problems in product pricing. We begin by defining the product pricing problem for firm j:

$$\max_{p_j \in \mathcal{P}_j} f_j(p) \text{ where } f_j(p) \triangleq \mathbb{E}\left[p_j \zeta_j(p;\omega)\right],$$

 p_j denotes the price set by firm j and $\zeta_j(p;\omega)$ denotes the demand for product j, defined as

$$\zeta_j(p;\omega) \triangleq \frac{e^{-\alpha_j(\omega)p_j}}{c + \sum_{i=1}^N e^{-\alpha_i(\omega)p_i}},$$

where $\alpha_j(\omega)$ denote positive parameters for j = 1, ..., N. The resulting revenue function has been shown to be pseudoconcave (see [17]). The relevance of this observation can be traced to the knowledge that under a pseudoconvexity assumption, given p_{-j}^* , p_j^* is a stationary point of this problem if and only if p_j^* is a global minimizer of this problem. Consequently, any solution to the collection of pseudomonotone variational inequality problems is a Nash-Bertrand equilibrium. Note that in Cournot or quantity games, suitably specified inverse demand functions also lead to pseudoconcave revenue functions [18, Th. 3.4].

1.2 Stochastic approximation schemes

The stochastic counterpart of the variational inequality has received relatively less attention compared to its deterministic counterpart. Early efforts focused on the use of sample average approximation (SAA) techniques and developed consistency statements of the resulting estimators [19]. In fact, such techniques have been applied towards the computation of solutions stochastic variational inequality problems [20]. More recently, such avenues have been utilized to develop confidence regions with suitable central limit results [21, 22]. An alternative approach inspired by the seminal work by Robbins and Monro [23] is that of stochastic approximation [24–26]. Via averaging techniques [27-30], optimal rates in function values can be derived (also see related work [31, 32] and prior work [33] on averaging). In the last decade, there has been a surge of interest in the development of techniques for stochastic convex optimization with a focus on optimal constant steplengths [34], composite problems [35, 36] and nonconvexity [37, 38]. However, in the context of stochastic variational inequality problems, much of the prior work has been in the context of monotone operators. Almost-sure convergence of the solution iterates was first proven by Jiang and Xu [39] under either strong/strict monotonicity or a variant of the acute angle condition, while regularized schemes for addressing merely monotone but Lipschitzian maps were subsequently developed by Koshal et al. [40]. In [41], Yousefian et al. weakened the Lipschitzian requirement by developing an SA scheme that combined local smoothing and iterative regularization. From a rate standpoint, there has been relatively less in the context of SVIs. A particularly influential paper by Tauvel et al. [42] proved that the mirror-prox stochastic approximation scheme admits the optimal rate of $\mathcal{O}(1/\sqrt{K})$ in terms of the gap function when employing averaging over monotone SVIs. In related work [43], Yousefian et al. develop optimal extragradient-based robust smoothing schemes for monotone SVIs in non-Lipshitzian regimes. Noteworthy amongst past efforts is the development of a class of accelerated techniques for deterministic and stochastic variational inequality problems [44]. We summarize much of the prior results in Table 1. We believe that this is amongst the first efforts to contend with this problem class but it is worth noting that subsequent to the conference version of this paper [45], there have been at least two papers that have considered the solution of stochastic pseudomonotone variational inequality problems. Of these, the first¹ utilizes a similar extragradient scheme with a.s. and rate statements that incorporates variable batch size [46], leading to an improved rate of $\mathcal{O}(\frac{1}{K})$ in terms of dist²(\bar{x}_K, X^*). In addition, the second author of this paper has also recently jointly coauthored work on a block-coordinate variant of such schemes that incorporates a novel averaging scheme in the context of stochastic mirror-prox schemes [47].

Ref.	Applicability	SA Algorithm	Avg.	Metric	Rate	a.s.
	Strongly monotone					
[39]	Strongly monotone, Lipschitz map	Proj. based	Ν	Iterates	-	Y
[48]	Strongly monotone, non-Lipschitz map	Proj. based + self-tuned step.	Ν	MSE (Soln. Iter.)	-	Y
	Monotone + Single Proj.					
[39]	Monotone, acute-angle condn.	Proj. based	Ν	Iterates	-	Y
[40]	Monotone, Lipschitz map	Proj. based + Regularization	Ν	Iterates	-	Y
[41]	Monotone, non-Lipschitz	Proj. based + Regularization + smoothing	Ν	Iterates -		Y
[49]	$ \begin{array}{l} \text{Monotone,} \\ \text{Lipschitz} \\ X \triangleq \cap_{i=1}^m X_i \end{array} $	Proj-based +Regularization	Υ	Gap fn. $\mathcal{O}\left(\frac{\ln(k)}{\sqrt{k}}\right)$		Y
	Monotone + Extragradient-based					
[42]	$Monotone, \ F(x) - F(y)\ _{*} \le L\ x - y\ + B$	Mirror-prox	Y	Gap fn.	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	N
[43]	Monotone, Non-Lipschitz, Bounded map	Extragradient + Rand. smoothing	Y	Gap fn.	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	Y
[50]	Monotone Non-Lipschitz	Extragradient	Υ	Gap fn.	$\mathcal{O}\left(rac{1}{k^{1/6}} ight)$	Y
[44]	Monotone, $F(x) = \nabla G(x) + H(x),$ G is L_G -smooth, H is L_H -Lipschitz	Mirror-prox + accelerated	Υ	-	$\mathcal{O}\left(\frac{L_G}{k^2} + \frac{L_H}{k} + \frac{c}{\sqrt{k}}\right)$	N
	Pseudomonotone + Extragradient-based					
[46]	Pseudomonotone Lipschitz	Extragradient +var. redn	Υ	$ \begin{bmatrix} r(x_k)^2 \\ (\text{exp. squared-resid}) \end{bmatrix} $	$\mathcal{O}\left(\frac{1}{k}\right)$	Y
Sec. 3.3	Strongly pseudo. or Strictly pseudo. or Pseudomonotplus	Extragradient, Mirror-prox	Ν			Y
Sec. 3.4	Strongly pseudo. or monotone+weak-sharp	Extragradient, Mirror-prox	Ν	MSE (Soln. Iter)	$\mathcal{O}\left(\frac{1}{k}\right)$	Y

Table 1: SA based approaches for Stochastic Variational Inequality Problems (bold represents the current work)

1.3 Contributions and outline:

This paper makes the following contributions:

¹Note that this paper references our conference paper and a preprint of the current paper.

(i) Almost-sure convergence . In Section 2, we consider a *stochastic* extragradient method and show that the generated sequence of iterates converges to a solution in an almost-sure sense. We refine these statements to a subclass of non-monotone problems and extend the convergence statement to the mirror-prox regime. To the best of our knowledge, there appears to be no prior a.s. convergence theory for this class of SVIs.

(ii) Rate analysis . Under slightly stronger settings of pseudomonotonicty, in Section 3, we prove that the extragradient scheme displays the optimal rate for strongly pseudomonotone maps. Additionally, a similar statement is proved for the mirror-prox generalization as well as for problems characterized by the weak-sharpness property. In particular, we emphasize that our work derives rate estimates for the *iterates* without resorting to averaging, in contrast with available statements for monotone SVIs that provide rate statements in terms of the gap function. Notably, in all three cases, we further refine the bound by selecting a suitable initial steplength by deriving an (ϵ -) infimum of a nonconvex discontinuous function in terms of problem-specific constants (the strong pseudomonotonicity constant, Lipschitz constant, compactness measures, etc.) Again, this appears to be amongst the first rate statements in the regime of pseudomonotone problems.

(iii) Numerical Results: In Section 4, based on a test suite of problems, we examine the empirical behavior of our schemes and note the benefits seen from optimizing the initial steplength.

2 Extragradient-based stochastic approximation schemes

2.1 Background and Assumptions

Given an $x_0 \in X$ in a traditional SA scheme and a steplength sequence $\{\gamma_k\}$, a sequence $\{x_k\}$ is constructed via the following update rule:

$$x_{k+1} := \Pi_X(x_k - \gamma_k(F(x_k) + w_k)), \quad k \ge 0,$$
 (SA)

where w_k is defined as $w_k \triangleq F(x; \omega_k) - F(x_k)$. We consider a *stochastic* extragradient scheme akin to that presented in [42]. Given an $x_0 \in X$ and a steplength sequence $\{\gamma_k\}$, this scheme comprises of two steps for $k \ge 0$:

$$x_{k+1/2} := \Pi_X (x_k - \gamma_k (F(x_k) + w_k)),$$

$$x_{k+1} := \Pi_X (x_k - \gamma_k (F(x_{k+1/2}) + w_{k+1/2})),$$
(ESA)

where $w_{k+1/2} \triangleq F(x_{k+1/2}; \omega_{k+1/2}) - F(x_{k+1/2})$. At any iteration k, the history \mathcal{F}_k and $\mathcal{F}_{k+1/2}$ are defined as $\mathcal{F}_k \triangleq \sigma \{x^0, \omega_0, \omega_{1/2}, \omega_1, \dots, \omega_{k-1/2}\}$ and $\mathcal{F}_{k+1/2} \triangleq \mathcal{F}_k \cup \{\omega_k\}$, respectively. Next, we define the property of pseudomonotonicity and its variants and denote the solution set of SVI(X, F) by X^* :

Definition 2 (Pseudomonotonicity and variants). Consider a continuous mapping $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Then, the following hold:

- (i) F is pseudomonotone on X if for all $x, y \in X$, $(x y)^T F(y) \ge 0 \implies (x y)^T F(x) \ge 0$.
- (ii) F is pseudomonotone-plus on X if it is pseudo-monotone on X and for all vectors x and y in $X, (x y)^T F(y) \ge 0, (x y)^T F(x) = 0 \implies F(x) = F(y).$
- (iii) F is strictly pseudomonotone on X, if for all $x, y \in X$, $(x-y)^T F(y) \ge 0 \implies (x-y)^T F(x) > 0$ where $x \ne y$.

- (iv) F is strongly pseudomonotone on X if for all $x, y \in X$, there exists $\sigma > 0$ such that $(x y)^T F(y) \ge 0 \implies (x y)^T F(x) \ge \sigma ||x y||^2$, where $x \ne y$.
- (v) The acute angle relation holds if for any $x \in X \setminus X^*$ and $x^* \in X^*$,

$$(x - x^*)^T F(x) > 0. (3)$$

(vi) The weak sharpness property holds if there exists an $\alpha > 0$ such that

$$(x - x^*)^T F(x^*) \ge \alpha \min_{x^* \in X^*} ||x - x^*||, \quad \forall x \in X, \quad \forall x^* \in X^*.$$

It is worth recalling that strict pseudomonotonicity implies the acute angle condition. Next, we make the following assumptions on the conditional first and second moments:

Assumption 1 (Unbiasedness and boundedness of conditional second moments). At an iteration k, the following hold in an a.s. sense:

- (A1) The conditional first moments $\mathbb{E}[w_k \mid \mathcal{F}_k]$ and $\mathbb{E}[w_{k+1/2} \mid \mathcal{F}_{k+1/2}]$ are zero;
- (A2) The conditional second moments are bounded a.s. in that there exists a $\nu > 0$ such that $\mathbb{E}[||w_k||^2 | \mathcal{F}_k] \leq \nu^2$ and $\mathbb{E}[||w_{k+1/2}||^2 | \mathcal{F}_{k+1/2}] \leq \nu^2$ for all k.

We now provide assumptions on steplength sequences consistent with most SA schemes.

Assumption 2 (Square summability and non-summability of steplength sequences). The diminishing sequence $\{\gamma_k\}$ satisfies the following:

(A3) The steplength sequence is square-summable: $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$.

(A4) The steplength sequence is non-summable: $\sum_{k=0}^{\infty} \gamma_k = \infty$.

We impose a further requirement on the map given by the following:

Assumption 3 (Lipschitz continuity and boundedness of F). (A5) F(x) is Lipschitz continuous and bounded over X i.e. there exist positive scalars L and B such that for all $x, y \in X$ $||F(x) - F(y)|| \le L||x - y||$ and $||F(x)|| \le \frac{B}{2}$.

We use the following super-martingale convergence results [51, Lemma 10,11, page 49,50].

Lemma 2. Let V_k be a sequence of nonnegative random variables adapted to σ -algebra \mathcal{F}_k and such that $\mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \leq (1 - \delta_k)V_k + \psi_k$, $\forall k \geq 0$, a.s. where $0 \leq \delta_k \leq 1, \psi_k \geq 0$, and $\sum_{k=0}^{\infty} \delta_k = \infty, \sum_{k=0}^{\infty} \psi_k < \infty$, and $\lim_{k \to \infty} \frac{\psi_k}{\delta_k} = 0$. Then, $V_k \to 0$ in a.s. sense.

Lemma 3. Let V_k, u_k, ψ_k and γ_k be nonnegative random variables adapted to σ algebra \mathcal{F}_k . If a.s $\sum_{k=0}^{\infty} \delta_k < \infty, \sum_{k=0}^{\infty} \psi_k < \infty$, and $\mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \leq (1 + \delta_k)V_k - u_k + \psi_k$, $\forall k \geq 0$, then V_k is convergent in an a.s. sense and $\sum_{k=0}^{\infty} u_k < \infty$ in an a.s. sense.

2.2 An extragradient SA scheme

In this subsection, we prove that the iterates generated by the (ESA) scheme converge to the solution set of the original problem in an almost sure sense by leveraging ideas drawn from the proof of the deterministic version presented in [1, Lemma 12.1.10]. A challenge in this scheme arises due to the two independent stochastic errors respectively from the two sub-steps at every iteration and the lack of a direct expression of x_{k+1} in terms of $F(x_k)$ unlike the standard projection scheme. We begin by relating any two successive iterates in Lemma 4.

Lemma 4. Consider the stochastic variational inequality problem defined by SVI(X, F) and let x^* denote any solution of SVI(X, F). Suppose Assumption (A5) holds and consider the sequence of the iterates be generated by the extragradient scheme (ESA) and let $u_k \triangleq 2\gamma_k F(x_k)^T (x_k - x^*)$. Then, the following holds for any iterate k:

$$\|x_{k+1} - x^*\|^2 \le \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - u_k - 2\gamma_k w_{k+1/2}^T (x_k - x^*) + \gamma_k^2 t_k,\tag{4}$$

where the scalar $t_k \geq 0$ is an appropriately defined scalar.

Proof. Let $y_k = x_k - \gamma_k (F(x_{k+1/2}) + w_{k+1/2})$. Then,

$$\|x_{k+1} - x^*\|^2 = \|\Pi_X(y_k) - x^*\|^2 = \|y_k - x^*\|^2 + \|\Pi_X(y_k) - y_k\|^2 + 2(\Pi_X(y_k) - y_k)^T(y_k - x^*).$$

Note that
$$2||y_k - \Pi_X(y_k)||^2 + 2(\Pi_X(y_k) - y_k)^T(y_k - x^*)$$
$$= 2||y_k - \Pi_X(y_k)||^2 + 2(\Pi_X(y_k) - y_k)^T(y_k - \Pi_X(y_k) + \Pi_X(y_k) - x^*)$$
$$= 2||y_k - \Pi_X(y_k)||^2 - 2||y_k - \Pi_X(y_k)||^2 + 2(\Pi_X(y_k) - y_k)^T(\Pi_X(y_k) - x^*)$$
$$= 2(\Pi_X(y_k) - y_k)^T(\Pi_X(y_k) - x^*) \le 0,$$

where the last inequality follows from the projection property. As a consequence, we have that

$$\|y_k - \Pi_X(y_k)\|^2 + 2(\Pi_X(y_k) - y_k)^T(y_k - x^*) \le -\|y_k - \Pi_X(y_k)\|^2.$$
(5)

By invoking (5) in the expansion of $||x_{k+1} - x^*||^2$, we obtain

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|y_k - x^*\|^2 + \|\Pi_X(y_k) - y_k\|^2 + 2(\Pi_X(y_k) - x^*)^T(y_k - x^*) \\ &\leq \|y_k - x^*\|^2 - \|y_k - \Pi_X(y_k)\|^2 \\ &= \|x_k - \gamma_k(F(x_{k+1/2}) + w_{k+1/2}) - x^*\|^2 - \|x_k - \gamma_k(F(x_{k+1/2}) + w_{k+1/2}) - x_{k+1}\|^2 \\ &= \|x_k - x^*\|^2 + \gamma_k^2 \|F(x_{k+1/2}) + w_{k+1/2})\|^2 - 2\gamma_k(x_k - x^*)^T(F(x_{k+1/2}) + w_{k+1/2}) \\ &- \|x_{k+1} - x_k\|^2 - \gamma_k^2 \|F(x_{k+1/2}) + w_{k+1/2})\|^2 + 2\gamma_k(x_k - x_{k+1})^T(F(x_{k+1/2}) + w_{k+1/2}) \\ &= \|x_k - x^*\|^2 - \|x_k - x_{k+1}\|^2 + 2\gamma_k(x^* - x_{k+1})^T(F(x_{k+1/2}) + w_{k+1/2}). \end{aligned}$$

By adding and subtracting $x_k^T(F(x_{k+1/2}) + w_{k+1/2})$, we obtain

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq \|x_k - x^*\|^2 - \|x_k - x_{k+1}\|^2 + 2\gamma_k (x^* - x_{k+1})^T (F(x_{k+1/2}) + w_{k+1/2}) \\ &= \|x_k - x^*\|^2 - \|x_k - x_{k+1}\|^2 + 2\gamma_k (x^* - x_k)^T (F(x_{k+1/2}) + w_{k+1/2}) \\ &+ 2\gamma_k (x_k - x_{k+1})^T (F(x_{k+1/2}) + w_{k+1/2}) \\ &\leq \|x_k - x^*\|^2 - \|x_k - x_{k+1}\|^2 + 2\gamma_k (x^* - x_k)^T (F(x_{k+1/2}) + w_{k+1/2}) \\ &+ \|x_k - x_{k+1}\|^2 + \gamma_k^2 \|F(x_{k+1/2}) + w_{k+1/2}\|^2 \\ &= \|x_k - x^*\|^2 + 2\gamma_k (x^* - x_k)^T F(x_k) + 2\gamma_k (x^* - x_k)^T (F(x_{k+1/2}) - F(x_k)) \right)_{\text{Term a}} \\ &+ 2\gamma_k (x^* - x_k)^T w_{k+1/2} + \gamma_k^2 \|F(x_{k+1/2}) + w_{k+1/2}\|^2. \end{aligned}$$

Next, we observe that Term a can be bounded as follows:

Term
$$\mathbf{a} \leq \frac{\gamma_k^2}{\beta} \|x^* - x_k\|^2 + \beta \|F(x_{k+1/2} - F(x_k))\|^2 \leq \frac{\gamma_k^2}{\beta} \|x^* - x_k\|^2 + \beta L^2 \|x_{k+1/2} - x_k\|^2$$

 $\leq \frac{\gamma_k^2}{\beta} \|x^* - x_k\|^2 + \beta L^2 \|\Pi_X(x_k - \gamma_k(F(x_k) + w_k)) - \Pi_X(x_k)\|^2$
 $= \frac{\gamma_k^2}{\beta} \|x^* - x_k\|^2 + \gamma_k^2 \beta L^2 \|F(x_k) + w_k\|^2.$

As a consequence, we have that

$$\|x_{k+1} - x^*\|^2 \leq \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - 2\gamma_k (x_k - x^*)^T F(x_k) + 2\gamma_k (x^* - x_k)^T w_{k+1/2} + \gamma_k^2 \left(\|F(x_{k+1/2})\|^2 + \|w_{k+1/2}\|^2 + \beta L^2 \|F(x_k)\|^2 + \beta L^2 \|w_k\|^2\right) + 2\gamma_k^2 \left(\beta L^2 w_k^T F(x_k) + w_{k+1/2}^T F(x_{k+1/2})\right) \leq \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - 2\gamma_k (x_k - x^*)^T F(x_k) + 2\gamma_k (x^* - x_k)^T w_{k+1/2} + \gamma_k^2 \underbrace{\left(\frac{B^2(1 + \beta L^2)}{4} + \|w_{k+1/2}\|^2 + \beta L^2 \|w_k\|^2 + 2w_{k+1/2}^T F(x_{k+1/2}) + 2\beta L^2 w_k^T F(x_k)\right)}_{k+1/2}$$
(6)

where the last expression follows by invoking the boundedness of F over X.

While the above Lemma relates iterates for general mappings F, we begin with an analysis on pseudomonotone-plus problems.

Proposition 1 (a.s. convergence of ESA). Consider SVI(X, F) where F is assumed to be a pseudomonotone-plus mapping. Suppose assumptions (A1)–(A5) hold. Then, the extragradient scheme (ESA) generates a sequence $\{x_k\}$ such that $\{x_k\}$ is bounded a.s. and any limit point of $\{x_k\}$ is a solution of SVI(X, F) in an a.s. sense.

Proof. Consider the recursion (6). Taking expectations conditioned on \mathcal{F}_k , we have that

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - u_k - 2\gamma_k \mathbb{E}[(x_k - x^*)^T w_{k+1/2} \mid \mathcal{F}_k] \\
+ \gamma_k^2 \left(\frac{B^2(1 + \beta L^2)}{4} + \mathbb{E}[\|w_{k+1/2}\|^2 \mid \mathcal{F}_k] + \beta L^2 \mathbb{E}[\|w_k\|^2 \mid \mathcal{F}_k]\right) \\
+ \gamma_k^2 \left(\mathbb{E}[2w_{k+1/2}^T F(x_{k+1/2}) \mid \mathcal{F}_k] + 2\beta L^2 \mathbb{E}[w_k^T F(x_k) \mid \mathcal{F}_k]\right) \\
= \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - u_k - 2\gamma_k \mathbb{E}[\mathbb{E}[(x_k - x^*)^T w_{k+1/2} \mid \mathcal{F}_{k+1/2}] \mid \mathcal{F}_k] \\
+ \gamma_k^2 \left(\frac{B^2(1 + \beta L^2)}{4} + \mathbb{E}[\mathbb{E}[\|w_{k+1/2}\|^2 \mid \mathcal{F}_{k+1/2}] \mid \mathcal{F}_k] + \beta L^2 \mathbb{E}[\|w_k\|^2 \mid \mathcal{F}_k]\right) \\
+ \gamma_k^2 \left(\mathbb{E}[\mathbb{E}[2w_{k+1/2}^T F(x_{k+1/2}) \mid \mathcal{F}_{k+1/2}] \mid \mathcal{F}_k] + 2\beta L^2 \mathbb{E}[w_k^T F(x_k) \mid \mathcal{F}_k]\right) \\
\leq (1 + \delta_k) \|x_k - x^*\|^2 - u_k + \psi_k, \tag{7}$$
where $\delta_k \triangleq \frac{\gamma_k^2}{\beta}$ and $\psi_k \triangleq \gamma_k^2 \left(\frac{(B^2 + 8\nu^2)(1 + \beta L^2)}{4}\right)$,

and the inequalities follow from using the tower law and assumption (A1). The remainder of the proof requires application of the super-martingale convergence theorem (Lemma 3). This requires that $u_k \geq 0$ for all k, a fact that follows from noting that F is a pseudomonotone map over X, implying that

$$(x_k - x^*)^T F(x^*) \ge 0 \implies \overbrace{(x_k - x^*)^T F(x_k)}^{\triangleq \frac{u_k}{\gamma_k}} \ge 0$$

From the solution property of $F(x^*)^T(x_k - x^*) \ge 0$ and from the pseudomonotonicity of F(x), $u_k = 2\gamma_k F(x_k)^T(x_k - x^*) \ge 0$. Using assumption (A3), it is observed that $\sum_k \psi_k < \infty$. Invoking Lemma 3, we have that $\{\|x_k - x^*\|^2\}$ is a convergent sequence in an a.s. sense. Then in an a.s. sense, it follows that $\{x_k\}_{k\ge 0}$ is a bounded sequence and has a convergent subsequence $\{x_k\}_{k\in\mathcal{K}}$. We proceed by contradiction; suppose x_k converges to \hat{x} along subsequence \mathcal{K} where \hat{x} is not necessarily a solution to SVI(X, F). Since $\{\delta_k\}$ is summable in an a.s. sense, $\{u_k\}$ is summable a.s. and from the non-summability of γ_k , we have that a.s., the following implication holds.

$$\sum_{k \in \mathcal{K}} u_k = \sum_{k \in \mathcal{K}} 2\gamma_k F(x_k)^T (x_k - x^*) < \infty \implies \lim_{k \in \mathcal{K}} F(x_k)^T (x_k - x^*) = 0.$$

Since $x_k \xrightarrow[k \to \infty]{a.s.} \hat{x}$ along the subsequence \mathcal{K} from (7) and by the continuity of F over X, we obtain

$$F(\hat{x})^T(\hat{x} - x^*) = 0.$$
(8)

By recalling that x^* is a solution of VI(X, F) and by invoking the pseudomonotone-plus property of F together with (8), we have

$$[F(x^*)^T(\hat{x} - x^*) \ge 0 \text{ and } F(\hat{x})^T(\hat{x} - x^*) = 0] \Longrightarrow F(\hat{x}) = F(x^*).$$
 (9)

Therefore from (9), the following holds:

$$\forall x \in X, \quad F(\hat{x})^T (x - \hat{x}) = F(x^*)^T (x - \hat{x}) = F(x^*)^T (x - x^*) + F(x^*)^T (x^* - \hat{x}) \\ \geq F(x^*)^T (x^* - \hat{x}) = F(\hat{x})^T (x^* - \hat{x}) = 0,$$

where the last equality follows from (8). It follows that \hat{x} is a solution to SVI(X, F) and any limit point of $\{x_k\}$ is a solution of SVI(X, F) in an a.s. sense.

Next, we extend the convergence theory to accommodate variants of pseudomonotonicity as well as problems satisfing the acute angle property.

Proposition 2 (a.s. convergence of ESA under weaker conditions). Consider SVI(X, F) and let assumptions (A1-A5) hold. Consider one of the following statements:

- (i) F satisfies the acute angle relation at X^* given by (3).
- (ii) F is strictly pseudomonotone on X.
- (iii) F is strongly pseudomonotone on X.
- (iv) F is pseudomonotone on X and is given by the gradient of $\mathbb{E}[f(x,\omega)]$.

Then, the extragradient scheme (ESA) generates a sequence $\{x_k\}$ such that $\{x_k\}$ is bounded a.s. and any limit point of $\{x_k\}$ is a solution of SVI(X, F) in an a.s. sense.

Proof. We begin from (7) in the proof of Prop. 1 and instead of the pseudomonotone-plus property, we impose properties imposed by (i) - (iv).

(i) From (3), $u_k = 2\gamma_k F(x_k)^T (x_k - x^*) > 0$ and δ_k and ψ_k are summable since γ_k is a square summable sequence. Invoking Lemma 3, we have that $\sum_k u_k < \infty$ a.s. and $\{||x_k - x^*||^2\}$ is a convergent sequence in an a.s. sense, implying that $\{x_k\}_{k\geq 0}$ is a bounded sequence in an a.s. sense. It follows that $\{x_k\}$ has a convergent subsequence indexed by \mathcal{K} with limit point \hat{x} . We proceed by contradiction and assume that $\hat{x} \notin X^*$. Recall that $\sum_k \gamma_k = \infty$ and $\sum_{k\in\mathcal{K}} u_k < \infty$, implying that $\lim_{k\to\infty} F(x_k)^T (x_k - x^*) = 0$ in an a.s. sense. Consequently, by continuity of F and by recalling that $x_k \xrightarrow[a.s.]{k\to\infty} \hat{x}$ along \mathcal{K} , we have that $F(\hat{x})^T (\hat{x} - x^*) = 0$. But this contradicts the acute angle property and it follows that $\hat{x} \in X^*$.

- (ii) Since F is strictly pseudomonotone, F satisfies the acute angle relation and the result follows.
- (iii) Since F is strongly pseudomonotone, F is strictly pseudomonotone and the result follows.

(iv) From the first part of the proof of Prop. 1, since the map is pseudomonotone we have that $||\{x_k - x^*||^2\}$ is a convergent sequence in an a.s. sense, implying that in an a.s. sense, $\{x_k\}_{k\geq 0}$ is a bounded sequence and has a convergent subsequence with index set \mathcal{K} and limit point \hat{x} . We proceed by contradiction and assume that $\hat{x} \notin X^*$. By the pseudoconvexity of f(x), for any $x_1, x_2 \in X, \nabla f(x_1)^T(x_2 - x_1) \geq 0 \implies f(x_2) \geq f(x_1)$, implying that

$$\nabla f(x^*)^T(\hat{x} - x^*) \ge 0 \implies f(\hat{x}) \ge f(x^*).$$
(10)

But from (8), we have that $\nabla f(\hat{x})^T(x^* - \hat{x}) = 0$ implying that $f(\hat{x}) \leq f(x^*)$. It follows that $f(\hat{x}) = f(x^*)$ and \hat{x} is a global minimizer of $\mathbb{E}[f(x,\xi)]$ over X and a solution of SVI(X,F).

Next, we consider a monotone regime where VI(X, F) satisfies a weak sharpness property. Prior to providing our main convergence statement, we provide an intermediate lemma.

Proposition 3 (a.s. convergence under weak-sharpness and monotonicity). Let (A1-A5) hold. Consider SVI(X, F) where F is a continuous monotone map over X. Suppose the weak sharpness property holds for the mapping F and the solution set X^* with parameter α . Then, the extragradient scheme (ESA) generates a sequence $\{x_k\}$ such that $\{x_k\}$ converges to a solution of SVI(X, F)in an a.s. sense. *Proof.* We begin by restating (7) as follows:

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \le \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - u_k + \gamma_k^2 \left(\frac{(B^2 + 8\nu^2)(1 + \beta L^2)}{4}\right).$$
(11)

where $-u_k = -2\gamma_k F(x_k)^T(x_k - x^*) \leq -2\gamma_k \alpha \operatorname{dist}(x_k, X^*)$ by the weak-sharpness property. This further implies that

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 \mid \mathcal{F}_k] \le \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - \gamma_k \alpha \operatorname{dist}(x_k, X^*) + \gamma_k^2 \left(\frac{(B^2 + 8\nu^2)(1 + \beta L^2)}{4}\right).$$
(12)

From the super-martingale convergence Lemma, it can be seen that the following hold in an a.s. sense: (i) $\{\|x_k - x^*\|^2\}$ is a convergent sequence; and (ii) $\sum_k \gamma_k \alpha \operatorname{dist}(x_k, X^*) < \infty$. We proceed to show that $\operatorname{dist}(x_k, X^*) \xrightarrow{k \to \infty} 0$; equivalently, this implies that almost every subsequence $\{x_k\}_{k \in \mathcal{K}} \to X^*$ as $k \to \infty$. Suppose this is false. Then for $\omega \in \Omega_1 \subset \Omega$ and $\mu(\Omega_1) > 0$ (i.e. finite probability), $\{x_k\}_{k \in \mathcal{K}(\omega)} \to \widehat{x}(\omega)$, $\operatorname{dist}(x_k, X^*) \xrightarrow{k \to \infty} v(\omega) > 0$ where $\mathcal{K}(\omega)$ denotes a random subsequence with limit point $\widehat{x}(\omega)$, $v(\omega)$ is a random positive scalar and $\inf_{\omega \in \Omega_1} v(\omega) \ge \overline{v}$. Then for every $\omega \in \Omega_1$ and for $\overline{v} > 0$, there exists a $K(\omega)$ such that $\|x_k - \widehat{x}(\omega)\| \le \overline{v}/2$ for $k \in \mathcal{K}(\omega)$ and $k \ge K(\omega)$. It follows that that for $\omega \in \Omega_1$ (with finite probability), we have that

$$\sum_{k \in \mathcal{K}(\omega)} \gamma_k \operatorname{dist}(x_k, X^*) \geq \sum_{k \geq K(\omega), k \in \mathcal{K}(\omega)} \gamma_k \operatorname{dist}(x_k, X^*) = \infty.$$

But this contradicts that claim that $\sum_{k=1}^{\infty} \gamma_k \operatorname{dist}(x_k, X^*) < \infty$ almost surely. Consequently, $\operatorname{dist}(x_k, X^*) \xrightarrow{k \to \infty} 0$.

Remark: A natural question emerges as to the relevance of this result, given that monotone maps have been studied in the past (cf. [40, 42]). First, we present statements that show that the original sequence converges almost surely to a point in the solution set, rather than showing the averaged counterpart coverges to the solution set. Second, in contrast with [40], we do not resort to regularization in deriving almost-sure convergence statements. Third, we proceed to show that under the prescribed assumptions, we obtain the optimal rate of convergence in the solution estimators, rather than the gap function. Finally, we do not require that X be bounded for claiming the a.s. convergence of solution iterates in this regime.

2.3 Mirror-prox generalizations

Given a point and a set, the Euclidean projection computes the closest point in this set by using the Euclidean norm as the distance metric. A generalization to this operation [34, 52] utilizes a class of distance functions that include the Euclidean norm as a special case. Given a distance metric s(x), the prox function V(x, z) takes the form:

$$V(x,z) \triangleq s(z) - \left[s(x) + \nabla s(x)^T (z-x)\right], \tag{13}$$

where s(x) is assumed to be a strongly convex and differentiable function in x. The resulting prox subproblem is given by the following:

$$\mathbf{P}_X(x,r) \triangleq \arg\min_{z \in X} \left(r^T z + V(x,z) \right).$$
(14)

It may be observed that when $s(x) = \frac{1}{2} ||x||^2$, then $V(x, z) = \frac{1}{2} ||x-z||^2$. Furthermore, if $r = \gamma F(x)$, it can be shown that (14) represents the standard gradient projection. Recent work [53] proposes prox generalizations to the extragradient scheme for monotone as well as pseudomonotone deterministic variational inequality problems and forms the inspiration for our analysis. In monotone regimes, stochastic variants to these prox schemes have been suggested in [42] while inexact oracles regimes have also been recently examined in [54]. However, those settings derive error bounds under a monotone setting. We consider a prox-based generalization of the extragradient scheme for stochastic variational problems (referred to as mirror-prox in [42]). Our contribution lies in showing that the sequence of iterates converges a.s. to the solution set under pseudomonotone settings as shown earlier under appropriate choices of steplengths. Formally, the mirror prox stochastic approximation (MPSA) scheme is defined as follows:

$$x_{k+1/2} := \mathbf{P}_X(x_k, \gamma_k F(x_k; \omega_k)), \qquad k \ge 0
 x_{k+1} := \mathbf{P}_X(x_k, \gamma_k F(x_{k+1/2}; \omega_{k+1/2})), \qquad k \ge 0.$$
(MPSA)

From the strong convexity of s(x), it can be seen that there exists a positive scalar $\theta \ge 1$ such that

$$V(x,z) \ge \frac{\theta}{2} ||x-z||^2.$$
 (15)

Next, we recall the definition of the dual norm.

$$\|x\|_* = \sup\left\{z^T x \mid \|z\| \le 1\right\}.$$
(16)

Based on the dual norm, we provide a modified statement for Assumptions (A2) and (A5), given by the following.

Assumption 4 (Dual Norms).

- (A6) For any $x, y \in X$, there exist L_* and B_* such that $||F(x) F(y)||_* \le L_* ||x y||$ and $||F(x)||_* \le \frac{B_*}{2}$.
- (A7) The conditional second moment of the error is bounded with respect to the dual norm as specified by $\mathbb{E}\left[\|w_k\|_*^2 \mid \mathcal{F}_k\right] \leq \nu_*^2$ and $\mathbb{E}\left[\|w_{k+1/2}\|_*^2 \mid \mathcal{F}_{k+1/2}\right] \leq \nu_*^2$.

Under the assumption that $\nabla s(x)$ is Lipschitz continuous with a positive constant L_V , the following holds [55, Lemma 1.2.3]:

$$V(x,z) \le L_V^2 \|x - z\|^2.$$
(17)

We use the following result from [53].

Lemma 5. Let $X \subseteq \mathbb{R}^n$ be a convex set and $p: X \to \mathbb{R}$ be a differentiable convex function. If u^* is an optimal solution of $\min\{p(u) + V(\tilde{x}; u) : u \in X\}$, the following holds:

$$p(u^*) + V(u^*; u) + V(\tilde{x}; u^*) \le p(u) + V(\tilde{x}; u)$$
 for all $u \in X$.

The next lemma relates prox functions over successive iterates and is the generalization of Lemma 4 from standard extragradient schemes to the mirror-prox regime.

Lemma 6. Consider SVI(X, F) and suppose Assumption (A6) holds. Suppose x^* is any solution of SVI(X, F). Consider the iterates generated by (MPSA). Then the following holds for any k:

$$V(x_{k+1}, x^*) \le \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - 2\gamma_k w_{k+1/2}^T(x_k - x^*) - u_k + \gamma_k^2 t_k,$$

where $u_k = \gamma_k F(x_k)^T (x_k - x^*)$, c is a positive scalar, and t_k is defined accordingly.

Proof. From the definition of the iterates, with x_{k+1} being the solution to the second proxsubproblem (MPSA) and using Lemma 5, we obtain that for all $x \in X$,

$$V(x_k, x) \ge \gamma_k F(x_{k+1/2}, \omega_{k+1/2})^T (x_{k+1} - x) + V(x_k, x_{k+1}) + V(x_{k+1}, x),$$
(18)

where $\gamma_k F(x_{k+1/2}, \omega_{k+1/2})^T(x_{k+1} - x)$ takes the form of p(u) defined earlier. Adding and subtracting x_k from the first term on the right hand side, we have

$$V(x_k, x) \ge \gamma_k F(x_{k+1/2}; \omega_{k+1/2})^T (x_k - x) + \gamma_k F(x_{k+1/2}; \omega_{k+1/2})^T (x_{k+1} - x_k) + V(x_k, x_{k+1}) + V(x_{k+1}, x^*).$$
(19)

Setting $x = x^*$, adding and subtracting $\gamma_k F(x_k)^T(x_k - x^*)$ from the first term on the right, (19) can be rewritten as follows.

$$V(x_k, x^*) \ge \gamma_k F(x_k)^T (x_k - x^*) + \gamma_k w_{k+1/2}^T (x_k - x^*) + \gamma_k \left(F(x_{k+1/2}) - F(x_k) \right)^T (x_k - x^*) + \gamma_k F(x_{k+1/2}; \omega_{k+1/2})^T (x_{k+1} - x_k) + V(x_k, x_{k+1}) + V(x_{k+1}, x^*).$$
(20)

Rearranging and completing squares, for any positive β and c, the following holds:

$$V(x_{k}, x^{*}) + \frac{\gamma_{k}^{2}}{\beta} \|x_{k} - x^{*}\|^{2} + \beta \|F(x_{k+1/2}) - F(x_{k})\|_{*}^{2} + c\gamma_{k}^{2} \|F(x_{k+1/2}; \omega_{k+1/2})\|_{*}^{2} + \frac{1}{c} \|x_{k+1} - x_{k}\|^{2}$$

$$\geq u_{k} + \gamma_{k} w_{k+1/2}^{T}(x_{k} - x^{*}) + V(x_{k}, x_{k+1}) + V(x_{k+1}, x^{*}), \qquad (21)$$

where $u_k = \gamma_k F(x_k)^T (x_k - x^*)$. By invoking the property $V(x_k, x_{k+1}) \ge \frac{\theta}{2} ||x_k - x_{k+1}||^2$, we have

$$V(x_{k+1}, x^*) \le \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - \gamma_k w_{k+1/2}^T(x_k - x^*) - u_k + \beta L_*^2 \|x_{k+1/2} - x_k\|^2 + c\gamma_k^2 \left(\|F(x_{k+1/2})\|_*^2 + \|w_{k+1/2}\|_*^2 + 2w_{k+1/2}^T F(x_{k+1/2})\right).$$
(22)

From the definition of the iterates, with $x_{k+1/2}$ being the solution to the first prox-subproblem (MPSA), from Lemma 5 for any $\tilde{x} \in X$, we have that

$$\gamma_k x_{k+1/2}^T \left(F(x_k) + w_k \right) + V(x_k, x_{k+1/2}) + V(\tilde{x}, x_{k+1/2}) \le \gamma_k x_k^T F(x_k + w_k) + V(\tilde{x}, x_k).$$

By choosing $\tilde{x} = x_k$, we obtain that

$$\gamma_k x_{k+1/2}^T \left(F(x_k) + w_k \right) + V(x_k, x_{k+1/2}) \le \gamma_k x_k^T F(x_k + w_k) + V(x_k, x_k) = \gamma_k x_k^T F(x_k + w_k),$$

since $V(x_k, x_k) = 0$ and $V(x_k, x_{k+1/2}) \ge 0$. A consequence of this inequality is that

$$\frac{\theta}{2} \|x_{k+1/2} - x_k\|^2 \leq V(x_k, x_{k+1/2}) \leq \gamma_k (x_k - x_{k+1/2})^T (F(x_k) + w_k).$$

$$\implies \frac{\theta}{2} \|x_{k+1/2} - x_k\|^2 \leq \frac{1}{2} \|x_k - x_{k+1/2}\|^2 + \frac{\gamma_k^2}{2} \|F(x_k) + w_k\|_*^2$$

$$\implies \|x_{k+1/2} - x_k\|^2 \leq \gamma_k^2 \frac{\|F(x_k) + w_k\|_*^2}{\theta - 1} = \frac{\gamma_k^2}{\theta - 1} \left(\|F(x_k)\|_*^2 + \|w_k\|_*^2 + 2F(x_k)^T w_k\right)$$

$$\le \frac{\gamma_k^2}{\theta - 1} \left(\frac{B_*^2}{4} + \|w_k\|_*^2 + 2F(x_k)^T w_k\right).$$
(23)

Substituting the above expression in (22), our claim follows.

$$V(x_{k+1}, x^*) \leq \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - \gamma_k w_{k+1/2}^T(x_k - x^*) - u_k + \gamma_k^2 \left(c\|F(x_{k+1/2})\|_*^2 + c\|w_{k+1/2}\|_*^2 + 2cw_{k+1/2}^TF(x_{k+1/2})\right) + \frac{\beta L_*^2 \gamma_k^2}{\theta - 1} \left(\frac{B_*^2}{4} + \|w_k\|^2 + 2F(x_k)^T w_k\right).$$
(24)

It can be observed that we may recover the statement of Lemma 4 by choosing V(x, z) as the squared Euclidean norm. We now proceed to use (22) to prove the almost sure convergence of the sequence produced by the (MPSA) scheme.

Proposition 4 (a.s. convergence of MPSA scheme for pseudomonotone-plus mappings). Consider the SVI(X, F) and let F be a pseudomonotone-plus map on X. Let assumptions (A1), (A2), (A3), (A4), (A6) and (A7) hold. Additionally let $\beta > 0$ and c > 0 be chosen such that $\theta/2 - (1/c) \ge 0$. Let $\{x_k\}$ denote a sequence of iterates generated by (MPSA) and suppose X^* denotes the set of solutions to the SVI(X, F). Then $\{x_k\}$ is bounded a.s. and any limit point of $\{x_k\}$ is a solution of SVI(X, F) in an a.s. sense.

Proof. Considering (24) and by taking conditional expectations with respect to \mathcal{F}_k and using the tower law,

$$\mathbb{E}[V(x_{k+1}, x^*) \mid \mathcal{F}_k] \leq \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \gamma_k \underbrace{\mathbb{E}[\mathbb{E}[w_{k+1/2}^T(x_k - x^*)|\mathcal{F}_{k+1/2}]|\mathcal{F}_k]}_{= 0} - u_k + \gamma_k^2 \left(c \|F(x_{k+1/2})\|^2 + c\mathbb{E}[\mathbb{E}[\|w_{k+1/2}\|^2|\mathcal{F}_{k+1/2}] \mid \mathcal{F}_k]\right) + 2\gamma_k^2 c \left(\mathbb{E}[\underbrace{\mathbb{E}[w_{k+1/2}^TF(x_{k+1/2}) \mid \mathcal{F}_{k+1/2}]}_{= 0} \mid \mathcal{F}_{k+1/2}] \mid \mathcal{F}_k]\right) + \frac{\gamma_k^2 \beta L_*^2}{\theta - 1} \left(\frac{B_*^2}{4} + \mathbb{E}[\|w_k\|_*^2 \mid \mathcal{F}_k] + 2\underbrace{\mathbb{E}[F(x_k)^Tw_k \mid \mathcal{F}_k]}_{= 0}\right).$$

Note that the term $-((\theta/2) - (1/c)) ||x_k - x_{k+1/2}||^2 \le 0$ by the definition of θ and c and hence is dropped from the right hand side of the above expression. Invoking assumptions (A1-A2), we have

$$\mathbb{E}[V(x_{k+1}, x^*) \mid \mathcal{F}_k] \le \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - u_k + \gamma_k^2 \left(\frac{B_*^2 + 4\nu_*^2}{4}\right) \left(\frac{c(\theta - 1) + \beta L_*^2}{\theta - 1}\right).$$

This inequality is analogous to (7) in Prop. 1 and the remainder of the proof follows by replacing $\frac{1}{2}||x-y||^2$ by V(x,y).

Corollary 1 (a.s. convergence of MPSA scheme under sub-classes of non-monotonicity). Consider the SVI(X, F). Let assumptions (A1), (A2), (A3), (A4), (A6), and (A7) hold. Consider a sequence of iterates $\{x_k\}$ generated by the MPSA scheme where γ_0 is chosen to be sufficiently small. Suppose one of the following statements hold:

- (i) F satisfies the acute angle relation at X^* given by (3).
- (ii) F is strictly pseudomonotone on X.
- (iii) F is strongly pseudomonotone on X.
- (iv) F is pseudomonotone on X and is given by the gradient of $\mathbb{E}[f(x,\omega)]$.

Then $\{x_k\}$ is bounded a.s. and any limit point of $\{x_k\}$ is a solution of SVI(X, F) in an a.s. sense. *Proof.* We begin by restating (22)

$$V(x_k, x^*) \ge \gamma_k F(x_k)^T (x_k - x^*) + \gamma_k w_{k+1/2}^T (x_k - x^*) + \gamma_k \left(F(x_{k+1/2}) - F(x_k) \right)^T (x_k - x^*) + \gamma_k F(x_{k+1/2}; \omega_{k+1/2})^T (x_{k+1} - x_k) + V(x_k, x_{k+1}) + V(x_{k+1}, x).$$

Since $u_k = \gamma_k F(x_k)^T (x_k - x^*)$, we obtain the following bound by using Young's inequality to derive an upper bound on $\gamma_k F(x_{k+1/2}; \omega_{k+1/2})^T (x_{k+1} - x_k)$ and completing squares.

$$V(x_{k}, x^{*}) + \frac{\gamma_{k}^{2}}{\beta} \|x_{k} - x^{*}\|^{2} + \beta \|F(x_{k+1/2}) - F(x_{k})\|_{*}^{2} + c\gamma_{k}^{2} \|F(x_{k+1/2}; \omega_{k+1/2})\|_{*}^{2} + \frac{1}{c} \|x_{k+1} - x_{k}\|^{2}$$

$$\geq u_{k} + \gamma_{k} w_{k+1/2}^{T} (x_{k} - x^{*}) + V(x_{k+1}, x^{*}) + V(x_{k}, x_{k+1}), \qquad (25)$$

Since $||F(x_{k+1/2};\omega_{k+1/2})||_*^2 \leq 2||F(x_{k+1/2})||_*^2 + 2||w_{k+1/2}||_*^2$ and proceeding a similar fashion to Lemma 6, we have that

$$V(x_{k+1}, x^*) \leq \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - \gamma_k w_{k+1/2}^T(x_k - x^*) - u_k + 2\gamma_k^2 c \left(\|F(x_{k+1/2})\|_*^2 + \|w_{k+1/2}\|_*^2\right) + \gamma_k^2 \beta L_*^2 \left(\|x_{k+1/2} - x_k\|^2\right).$$
(26)

By proceeding in a similar fashion to Lemma 6, we obtain the following.

$$\begin{aligned} \frac{\theta}{2} \|x_{k+1/2} - x_k\|^2 &\leq V(x_k, x_{k+1/2}) \leq \gamma_k (x_k - x_{k+1/2})^T \left(F(x_k) + w_k\right) \\ &\leq \gamma_k \|x_k - x_{k+1/2}\| \|F(x_k) + w_k\|_* \leq \frac{1}{2} \|x_k - x_{k+1/2}\|^2 + \frac{\gamma_k^2}{2} \|F(x_k) + w_k\|_*^2 \\ \implies \|x_{k+1/2} - x_k\|^2 \leq \frac{\|F(x_k) + w_k\|_*^2}{\theta - 1} \leq \frac{2\|F(x_k)\|_*^2 + 2\|w_k\|_*^2}{\theta - 1}.\end{aligned}$$

Proceeding analogously to Lemma 6, we have that

$$V(x_{k+1}, x^*) \leq \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - \gamma_k w_{k+1/2}^T (x_k - x^*) - u_k + 2c\gamma_k^2 \left(\frac{B_*^2}{4} + \|w_{k+1/2}\|_*^2\right) + \frac{2\beta L_*^2 \gamma_k^2}{\theta - 1} \left(\frac{B_*^2}{4} + \|w_k\|_*^2\right).$$
(27)

For the case of pseudomonotone-plus regimes, the proof follows along the lines of Proposition 4. The other non-monotone extensions mentioned above are straightforward and we do not provide the corresponding proofs.

Without loss of generality, the assumptions (A6) and (A7) on the dual norm $\|.\|_*$ above can be replaced with those ((A2) and (A5)) on the standard norm $\|.\|$ and the distance function s(x) can be defined with respect to the dual norm. The above proofs for MPSA follow since $\|.\|_{**} = \|.\|$.

3 Optimal rate statements

In the prior section, we proved the a.s. convergence of iterates generated by the ESA and the MPSA schemes. In this section, we consider the development of error bounds. Rate statements for the gap function have been provided in the context of monotone stochastic variational inequality problems by Tauvel et al. [42]. Here, we generalize these findings in deriving rate statements under two requirements, strong pseudomonotonicity and mere monotonicity with a weak sharpness requirement. In the remainder of this section, we assume that the steplength sequence is given by

$$\gamma_k = \frac{\gamma_0}{k},\tag{28}$$

where γ_0 is a finite scalar. It is easy to observe that this satisfies assumptions (A3)–(A4).

Proposition 5 (Rate statements under strong pseudomonotonicity). Suppose assumptions (A1)–(A5) hold. Let F be σ -strongly pseudomonotone over X and let the sequence of iterates $\{x_k\}$ be generated by (ESA). Additionally let X be compact such that for all $x \in X$, $||x|| \leq U$, where U is a positive constant. If x^* denotes a solution to the SVI(X, F) and M_B and M_{ν} are appropriately defined constants, then the following hold:

(a) For any k > 0, we have that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{M(\gamma_0)}{k},$$

where $M(\gamma_0)$ is a suitably defined positive scalar.

(b) In addition, if $\gamma_0 = (2 - \epsilon)/(2\sigma)$ where $\epsilon \in (0, 1/2)$, we have that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{1}{k} \max\left\{\frac{7(M_B + M_\nu)}{4\sigma^2}, \|x_0 - x^*\|^2\right\}.$$

Proof. (a). We begin by considering (6) as follows:

$$\|x_{k+1} - x^*\|^2 \le \left(1 + \frac{\gamma_k^2}{\beta}\right) \|x_k - x^*\|^2 - 2\gamma_k(x_k - x^*)^T F(x_k) + 2\gamma_k(x^* - x_k)^T w_{k+1/2} + \gamma_k^2 \underbrace{\left(\frac{B^2(1 + \beta L^2)}{4} + \|w_{k+1/2}\|^2 + \beta L^2 \|w_k\|^2 + 2\beta L^2 F(x_k)^T w_k + 2w_{k+1/2}^T F(x_{k+1/2})\right)}_{= \frac{1}{2}}$$

Since x^* is a solution of VI(X, F), we have that for any feasible x_k , $F(x^*)^T(x_k - x^*) \ge 0$. By recalling the definition of strong pseudomonotonicity,

$$-u_k = -2\gamma_k F(x_k)^T (x_k - x^*) \le -2\gamma_k \sigma ||x_k - x^*||^2.$$
(29)

Employing the bound (29) in (6),

$$\|x_{k+1} - x^*\|^2 \le (1 - 2\sigma\gamma_k) \|x_k - x^*\|^2 + 2\gamma_k (x^* - x_k)^T w_{k+1/2} + \frac{\gamma_k^2}{\beta} \|x_k - x^*\|^2 + \gamma_k^2 \left(\frac{B^2(1 + \beta L^2)}{4} + \|w_{k+1/2}\|^2 + \beta L^2 \|w_k\|^2 + 2\beta L^2 F(x_k)^T w_k + 2w_{k+1/2}^T F(x_{k+1/2})\right).$$
(30)

Taking expectations on both sides of (30),

$$\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq \underbrace{\mathbb{E}[(1 - 2\sigma\gamma_k)\|x_k - x^*\|^2]}_{\text{Term A}} + \underbrace{\gamma_k^2 \mathbb{E}\left[\frac{B^2(1 + \beta L^2)}{4} + \frac{\|x_k - x^*\|^2}{\beta} + 2\gamma_k(x^* - x_k)^T w_{k+1/2}\right]}_{\text{Term B}} + \gamma_k^2 \underbrace{\mathbb{E}\left[\|w_{k+1/2}\|^2 + \beta L^2\|w_k\|^2 + 2\beta L^2 F(x_k)^T w_k + 2w_{k+1/2}^T F(x_{k+1/2})\right]}_{\text{Term C}}.$$

We begin by deriving a bound on Term C by noting that the conditional first moments $\mathbb{E}[w_k \mid \mathcal{F}_k] = 0$ and $\mathbb{E}[w_{k+1/2} \mid \mathcal{F}_{k+1/2}] = 0$.

Form C =
$$\mathbb{E}\left[\mathbb{E}\left[\|w_{k+1/2}\|^2 |\mathcal{F}_{k+1/2}|\right] + \beta L^2 \mathbb{E}\left[\mathbb{E}\left[\|w_k\|^2 |\mathcal{F}_k|\right] + 2\beta L^2 \mathbb{E}\left[\mathbb{E}\left[F(x_k)^T w_k |\mathcal{F}_k\right]\right] + \mathbb{E}\left[\mathbb{E}\left[2w_{k+1/2}^T F(x_{k+1/2}) |\mathcal{F}_{k+1/2}|\right]\right] \le \gamma_k^2 (1+\beta L^2) \nu^2 \triangleq c(\beta).$$

Next, Term B can be bounded as follows:

Term
$$\mathbf{B} = \gamma_k^2 \frac{B^2(1+\beta L^2)}{4} + \gamma_k^2 \mathbb{E}\left[\frac{\|x_k - x^*\|^2}{\beta}\right] + 2\gamma_k \mathbb{E}\left[\mathbb{E}\left[(x^* - x_k)^T w_{k+1/2} | \mathcal{F}_{k+1/2}\right]\right]$$

$$\leq \gamma_k^2 \left(\frac{B^2(1+\beta L^2)}{4} + \frac{4U^2}{\beta}\right) \triangleq b(\beta).$$

To minimize $c(\beta) + b(\beta)$, it suffices to minimize the following expression.

$$s_0(\beta) = \beta \left(\frac{B^2 L^2}{4} + L^2 \nu^2\right) + \frac{4U^2}{\beta}$$

Setting $\beta^* = \frac{4U}{L\sqrt{B^2+4\nu^2}}$ by minimizing the above expression, we obtain that

$$\mathbb{E}[\|x_{k+1} - x^*\|^2] \le (1 - q_k)\mathbb{E}[\|x_k - x^*\|^2] + t_k,$$

where
$$q_k = \frac{2\sigma\gamma_0}{k}$$
, $t_k = \frac{\gamma_0^2(M_\nu + M_B)}{k^2}$, $M_B = \frac{B^2}{4}$, and $M_\nu = \nu^2 + 2UL\sqrt{B^2 + 4\nu^2}$.

By assuming that $2\sigma\gamma_0 > 1$ and by invoking Lemma 8 presented in the appendix, we obtain that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{M(\gamma_0)}{k}, \text{ where } M(\gamma_0) \triangleq \max\left\{\frac{\gamma_0^2(M_\nu + M_B)}{2\sigma\gamma_0 - \lfloor 2\sigma\gamma_0 \rfloor}, \|x_0 - x^*\|^2\right\}$$

(b). Recall that $M(\gamma_0)$ may be defined as follows:

$$M(\gamma_0) = \max\left\{t_0(\gamma_0)(M_{\nu} + M_B), \|x_0 - x^*\|^2\right\}, \text{ where } t_0(\gamma_0) = \left(\frac{\gamma_0^2}{2\sigma\gamma_0 - \lfloor 2\sigma\gamma_0 \rfloor}\right).$$

Based on Lemma 7, an ϵ -infimum of $t_0(\gamma_0)$ is achieved by choosing $\gamma_0 = \gamma$, where $\gamma = (2 - \epsilon)/(2\sigma)$ and $\epsilon \in (0, 1/2)$. Since $\lfloor 2\sigma\gamma \rfloor = 1$, it follows that $t_0(\gamma) = \frac{(2-\epsilon)^2}{4\sigma^2(1-\epsilon)}$, which may be bounded as follows when $\epsilon \in (0, 1/2)$:

$$t_0\left(\frac{(2-\epsilon)}{(2\sigma)}\right) = \frac{(2-\epsilon)^2}{4\sigma^2(1-\epsilon)} = \frac{(1-\epsilon)^2}{4\sigma^2(1-\epsilon)} + \frac{3-2\epsilon}{4\sigma^2(1-\epsilon)} \le \frac{(1-\epsilon)}{4\sigma^2} + \frac{3}{2\sigma^2} \le \frac{7}{4\sigma^2}.$$

It follows that

$$\mathbb{E}[\|x_k - x^*\|^2] \le \frac{1}{k} \max\left\{\frac{7(M_B + M_\nu)}{4\sigma^2}, \|x_0 - x^*\|^2\right\}.$$

Remark: This result is notable from several standpoints. First, in contrast to rate statements for settings that lack strong monotonicity, we provide a rate statement in terms of solution iterates, rather than in terms of the gap function. Second, our rate statement is optimal from a rate standpoint with a slightly poorer constant, in part due to the use of B^2 instead of $B^2/4$. Notably, in strongly convex optimization problems (cf. [19]), it can be seen that based on the optimal initial steplength, we have that $\mathbb{E}[||x_K - x^*||]^2 \leq \frac{1}{K} \max\left(\frac{2(B^2 + \nu^2)}{\sigma^2}, ||x_0 - x^*||^2\right)$. Next, we generalize the above rate result to prox functions with general distance functions.

Corollary 2 (Rate statements for MPSA). Suppose assumptions (A1),(A2), (A3, (A4), (A6), and -(A7) hold. Let F be σ -strongly pseudomonotone over X and let the sequence of iterates $\{x_k\}$ be generated by (MPSA). Additionally let X be compact such that for all $x \in X$, $||x|| \leq U$, where U is a positive constant. If x^* denotes a solution to the SVI(X, F) and M_B^* and M_{ν}^* are appropriately defined constants, then the following hold:

(a) For any k > 0, we have that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{M(\gamma_0)}{k},$$

where $M(\gamma_0)$ is a suitably defined positive scalar.

(b) In addition, if $\tilde{\sigma} = \frac{\sigma}{L_V^2}$ and $\gamma_0 = (2 - \epsilon)/(2\tilde{\sigma})$, where $\epsilon \in (0, 1/2)$, we have that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{1}{k} \max\left\{\frac{7(M_B^* + M_\nu^*)}{4\tilde{\sigma}^2}, \|x_0 - x^*\|^2\right\}$$

Proof. We begin by recalling the inequality given by (24):

$$V(x_{k+1}, x^*) \leq \left(1 + \frac{2\gamma_k^2}{\theta\beta}\right) V(x_k, x^*) - \left(\frac{\theta}{2} - \frac{1}{c}\right) \|x_{k+1} - x_k\|^2 - \gamma_k w_{k+1/2}^T(x_k - x^*) - u_k$$

+ $\gamma_k^2 \left(c \|F(x_{k+1/2})\|_*^2 + c \|w_{k+1/2}\|_*^2 + 2c w_{k+1/2}^T F(x_{k+1/2})\right)$
+ $\gamma_k^2 \left(\frac{\beta L_*^2}{\theta - 1} \left(\frac{B_*^2}{4} + \|w_k\|_*^2 + 2F(x_k)^T w_k\right)\right).$

Using $u_k \ge -2\gamma_k \sigma ||x_k - x^*||^2$ from (29) and by noting that $V(x_k, x^*) \le L_V^2 ||x_k - x^*||^2 \le 4L_V^2 U^2$ based on compactness and Lipschitzian properties, we have the following by taking expectations on both sides.

$$\mathbb{E}[V(x_{k+1}, x^*)] \leq \left(1 - \frac{2\sigma\gamma_k}{L_V^2}\right) \mathbb{E}[V(x_k, x^*)] - \underbrace{\left(\frac{\theta}{2} - \frac{1}{c}\right) \mathbb{E}[\|x_{k+1} - x_k\|^2]}_{\text{Term H}} + \underbrace{\mathbb{E}\left[-\gamma_k w_{k+1/2}^T(x_k - x^*) + \gamma_k^2 \left(\frac{\beta L_*^2 B_*^2}{4(\theta - 1)} + \frac{8L_V^2 U^2}{\theta\beta}\right)\right]}_{\text{Term B}} + \underbrace{\gamma_k^2 \mathbb{E}\left[\left(c\|F(x_{k+1/2})\|_*^2 + c\|w_{k+1/2}\|_*^2 + 2cw_{k+1/2}^T F(x_{k+1/2}) + \frac{\beta L_*^2}{\theta - 1} \left(\|w_k\|_*^2 + 2F(x_k)^T w_k\right)\right)\right]}_{\text{Term C}}.$$

Choosing $c = 2/\theta$ allows from dropping Term H while Terms B and C can be bounded in a fashion similar to Proposition 5, where we have

Term B
$$\leq \gamma_k^2 \left(\frac{\beta L_*^2 B_*^2}{4(\theta - 1)} + \frac{8L_V^2 U^2}{\beta \theta} \right)$$
 and Term C $\leq \gamma_k^2 \left(\frac{B_*^2 + 4\nu^2}{2\theta} + \frac{\beta L_*^2 \nu^2}{\theta - 1} \right).$

Akin to Proposition 5, we minimize the bounds on Term $B(\beta)$ + Term $C(\beta)$ and it suffices to minimize $s_0(\beta)$, defined as

$$s_0(\beta) = \beta \left(\frac{L_*^2 B_*^2}{4(\theta - 1)} + \frac{L_*^2 \nu^2}{\theta - 1} \right) + \frac{8L_V^2 U^2}{\beta \theta}.$$

Noting that $\beta^* = \frac{4L_V U}{L_*} \sqrt{\frac{2(\theta-1)}{\theta(B^2+4\nu^2)}}$ minimizes $s_0(\beta)$, we have that

$$\mathbb{E}[V(x_{k+1}, x^*)] \le (1 - q_k) \mathbb{E}[V(x_k, x^*)] + t_k, \quad \text{where}$$

$$q_k \triangleq \frac{2\sigma\gamma_0}{L_V^2 k} = \frac{2\tilde{\sigma}\gamma_0}{k}, t_k \triangleq \gamma_k^2 \left(M_B^* + M_\nu^*\right), M_B^* \triangleq \frac{B_*^2}{2\theta}, \text{ and } M_\nu^* \triangleq \left\{\frac{2\nu^2}{\theta} + 2L_*UL_V\sqrt{\frac{2(B^2 + 4\nu^2)}{\theta(\theta - 1)}}\right\}.$$

By assuming that $2\sigma\gamma_0 > 1$ and by invoking Lemma 8 presented in the appendix, we obtain that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{M(\gamma_0)}{k}, \text{ where } M(\gamma_0) \triangleq \max\left\{\frac{\gamma_0^2(M_\nu^* + M_B^*)}{2\tilde{\sigma}\gamma_0 - \lfloor 2\tilde{\sigma}\gamma_0 \rfloor}, \|x_0 - x^*\|^2\right\}.$$

(b). From Proposition 5(b), by setting $\gamma_0 = (2 - \epsilon)/(2\tilde{\sigma})$, it follows that

$$\mathbb{E}[\|x_k - x^*\|^2] \le \frac{1}{k} \max\left\{\frac{7(M_B^* + M_\nu^*)}{4\tilde{\sigma}^2}, \|x_0 - x^*\|^2\right\}$$

Remark: It can be observed that when the Euclidean norm is used as the distance function ($\theta = 2$ and $L_V = 1$), the MPSA scheme reduces to the standard extragradient scheme and we obtain the same upper bound as with the case of ESA.

We conclude this section with a rate analysis on the solution iterates under mere monotonicity of the map but under an additional requirement of weak-sharpness. We observe that the specification of the initial steplength requires globally minimizing a product of positive functions over a Cartesian product of convex sets. While there are settings where this product is indeed convex, it may also turn out to be nonconvex. Yet, we observe that the global minimizer can be tractably obtained by solving two optimization problems. Lemma 9 provides the necessary support for this result. Note that in our setting, one of these functions is a discontinuous nonconvex function and its infima are analyzed in Lemma 7.

Next, under a monotonicity and weak-sharpness requirement, the ESA scheme is shown to display the optimal rate of convergence in solution iterates. Additionally, we prescribe the optimal initial steplength that minimizes the mean-squared error by deriving the global minimizer of a nonconvex function in closed-form.

Proposition 6 (Rate statement under monotonicity and weak sharpness). Consider the SVI(X, F). Suppose assumptions (A1)–(A5) hold and let γ_k be defined as per (28). Let F(x) be a monotone map over the set X. Let the mapping F(x) and solution set X^* possess the weak-sharpness property with constant α and let X be compact such that $||x|| \leq U$ for all $x \in X$. Suppose x_k is generated by (ESA). Then the following hold.

(a) For any k > 0, we have that

$$\mathbb{E}[\operatorname{dist}^2(x_k, X^*)] \le \frac{M(\gamma_0)}{k},$$

where $M(\gamma_0)$ is a suitably defined scalar.

(b) In addition, if $\gamma_0 = (2 - \epsilon)/(2\bar{\sigma})$ and $\bar{\sigma} \triangleq \frac{\alpha}{2U}$ where $\epsilon \in (0, 1/2)$ and s_0^* is a suitably defined positive scalar, we have that

$$\mathbb{E}[\operatorname{dist}^2(x_k, X^*)] \le \frac{1}{k} \max\left\{\frac{7s_0^*}{4\bar{\sigma}^2}, \operatorname{dist}^2(x_0, X^*)\right\}.$$

Proof. (a). By taking expectations on both sides of (6) and by leveraging the property of weaksharpness, $u_k = -2\gamma_k F(x_k)^T (x_k - x^*) \leq -2\gamma_k \alpha \operatorname{dist}(x_k, X^*)$, we have

$$\mathbb{E}[\|x_{k+1} - x^*\|^2] \le \mathbb{E}[\|x_k - x^*\|^2] - 2\gamma_k \alpha \operatorname{dist}(x_k, X^*) + \mathbb{E}[t_k] + \frac{\gamma_k^2}{\beta} \mathbb{E}[\|x_k - x^*\|^2].$$
(31)

Since dist² $(x_{k+1}, X^*) \leq ||x_{k+1} - x^*||^2$ and $||x_k - x^*||^2 \leq 4U^2$, by minimizing the expression on the right of (31) in x^* over X^* , we have

$$\mathbb{E}[\operatorname{dist}^{2}(x_{k+1}, X^{*})] \leq \mathbb{E}[\operatorname{dist}^{2}(x_{k}, X^{*})] - 2\gamma_{k}\alpha \operatorname{dist}(x_{k}, X^{*}) + \mathbb{E}[t_{k}] + \frac{4\gamma_{k}^{2}U^{2}}{\beta}.$$
(32)

Since dist $(x_k, X^*) \leq 2U$, it follows that $-2\gamma_k \alpha \text{dist}(x_k, X^*) \leq -\frac{\gamma_k \alpha}{U} \text{dist}^2(x_k, X^*)$. Furthermore, bounding t_k along the lines of Proposition 5, we have

$$\mathbb{E}[\operatorname{dist}^{2}(x_{k+1}, X^{*})] \leq \left(1 - \frac{\alpha \gamma_{k}}{U}\right) \mathbb{E}[\operatorname{dist}^{2}(x_{k}, X^{*})] + \gamma_{k}^{2} \left(M_{\nu} + M_{B}\right) = (1 - q_{k}) \mathbb{E}[\operatorname{dist}^{2}(x_{k}, X^{*})] + s_{k},$$

where

$$M_{\nu}(\beta) \triangleq (1 + \beta L^{2})\nu^{2}, M_{B}(\beta) \triangleq (1 + \beta L^{2})\frac{B^{2}}{4} + \frac{4U^{2}}{\beta},$$
$$q_{k} = \frac{2\bar{\sigma}\gamma_{0}}{k}, s_{k} = \frac{s_{0}(\beta)\gamma_{0}^{2}}{k^{2}}, \bar{\sigma} = \frac{\alpha}{2U}, \text{ and } s_{0}(\beta) = M_{C}(\beta) + M_{\nu}(\beta).$$
(33)

Through the application of Lemma 8, we obtain the following bound on mean-squared error for every positive integer K:

$$\mathbb{E}[\operatorname{dist}^{2}(x_{K}, X^{*})] \leq \frac{1}{K} \max\left\{h(\gamma_{0})s_{0}(\beta), \operatorname{dist}^{2}(x_{0}, X^{*})\right\},$$

where $h(\gamma_{0}) \triangleq \frac{\gamma_{0}^{2}}{2\bar{\sigma}\gamma_{0} - \lfloor 2\bar{\sigma}\gamma_{0}\rfloor}$ and $s_{0}(\beta) \triangleq (M_{\nu}(\beta) + M_{B}(\beta))$

(b.) Suppose Γ_0 and \mathcal{Z} are sets defined as

$$\Gamma_0 \triangleq \{\gamma_0 : 2 > 2\bar{\sigma}\gamma_0 > 1\} \text{ and } \mathcal{Z} \triangleq \{\beta : \beta \ge 0\}.$$

Moreover, $h(\gamma_0)s_0(\beta)$ is a product of two positive functions and a global minimizer of this product can be obtained by getting a global minimizer of each by invoking Lemma 9. Of these, an ϵ -infimum of $h(\gamma_0)$ can be obtained as $\gamma_0^* = (2 - \epsilon)/(2\bar{\sigma})$ where $\epsilon \in (0, 1/2)$. A minimizer β^* of the convex function $s_0(\beta)$ is given by the following.

$$\min_{\beta} \quad \left(\left(1 + \beta L^2 \right) \left(\nu^2 + \frac{B^2}{4} \right) + \frac{4U^2}{\beta} \right) \implies \beta^* = \frac{4U}{L\sqrt{B^2 + 4\nu^2}},$$

implying that

$$s_0^* = \frac{B^2}{4} + \nu^2 + 2UL\sqrt{B^2 + 4\nu^2}.$$

We may then conclude that

$$\mathbb{E}[\operatorname{dist}^2(x_K, X^*)] \le \frac{1}{K} \max\left\{\frac{7s_0^*}{4\bar{\sigma}^2}, \operatorname{dist}^2(x_0, X^*)\right\}.$$

Remark: In past research, the optimality of the rate of convergence has been proved for monotone SVIs but in terms of the gap function. Our result shows that under a suitable weak-sharpness property, rate optimality also holds in terms of the solution iterates in a non-ergodic sense. Notably, we further refine the statement by selecting the initial steplength by (globally) minimizing a nonconvex function.²

4 Numerical Results

In this section, we examine the performance of the presented schemes on a suite of four test problems described in Section 4.1 while the algorithm parameters are defined in Section 4.2. In Section 4.3, we compare the performance of the ESA scheme with the MPSA schemes over the suite of test problems. Finally in Section 4.4, we compare the empirical rates with the theoretically predicted rates and quantify the benefits of optimal initial steplength.

4.1 Test Suite

The first two test problems are stochastic fractional convex quadratic and nonlinear, both of which lead to pseudomonotone stochastic variational inequality problems. The third set of test problems are stochastic variational inequality problems that represent the (sufficient) equilibrium conditions of a stochastic Nash-Cournot game. While the players maximize pseudoconcave expectation-valued functions, the resulting stochastic variational inequality problem is *not necessarily* pseudomonotone. However, some choices of parameters lead to pseudomonotone SVIs. Our fourth test problem is Watson's complementarity problem [56], which is not necessarily monotone.

(i) Fractional Convex Quadratic Problems: Maximizing or minimizing ratios in engineering settings often leads to stochastic fractional convex problems of the form: $\min_{x \in X} \mathbb{E}[f(x; \omega)/g(x)]$ where $\mathbb{E}[f(x; \omega)]$ and g(x) are strictly positive convex quadratic and linear functions, respectively, defined as $f(x; \omega) \triangleq 0.5x^T(\theta UU^T + \lambda V(\omega))x + 0.5((c + \bar{c}(\omega))^T x + 4n)^2$ and $g(x) \triangleq r^T x + t + 4n$. We note that $V(\omega)$ and $\bar{c}(\omega)$ are randomly generated from standard normal and uniform distributions, Uand c are deterministic constants generated once from the standard normal distribution, while rand t are generated once from uniform distributions. We note that $\theta = 0.025$ and $\lambda = \epsilon ||\theta UU^T||_F /$ $||V(\omega)||_F$, where $||.||_F$ denotes the Frobenius norm and $\epsilon = 0.025$. The set X is defined as $X \triangleq$ $\{x \mid Ax \leq v, 0 \leq x \leq 4\}$, where $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^{m \times 1}$ are generated once from standard normal and uniform distributions respectively. Note that $m = \lceil n/10 \rceil$ is a variable dependent integer. It is easily seen that the resulting SVI is pseudomonotone.

²We prefer not to qualify the initial steplength as "optimal" since the error bound in general is a function of γ_0 and β .

(ii) Fractional Convex Nonlinear Problems: We consider a nonlinear variant of (i) with the same parameters and numerator but an exponential denominator $g(x) = 10^4 (\lambda - e^{(r^T x + t + 4n)/2000})$, where $\lambda = e^{(8n+2)/2000}$.

(iii) Nash-Cournot Games: Next we consider a Nash-Cournot game with n selfish players, all of which sell the same commodity [18, 57] at a price given by the function of the aggregate sales as per the Cournot specification [11, 58]. Specifically, the *i*th agent solves the following problem: $\max_{x_i \in X_i} f_i(x) = \mathbb{E}[p(\bar{x}; \omega)x_i]$, where $p(\bar{x}; \omega) = (a - b^{\omega}\bar{x})^{\kappa}$, $\bar{x} = \sum_{i=1}^n x_i$, $\kappa \in (0, 1)$ and $X_i = \{x_i \mid Ax \leq v, 0 \leq x_i \leq 3n\}$. We note that $a = 100 \lceil n/3 \rceil$ while b^{ω} is generated from a uniform distribution with mean 1 and standard deviation ϵ , where $\epsilon = 0.025$. We note that A and v are also generated randomly as stated earlier. The equilibrium of this shared constraint Nash game [59] is given by a variational inequality problem. Note that agent payoffs are pseudoconcave [18, Theorem 3.4] and the (sufficient) equilibrium conditions are given by a variational inequality VI(X, F), which is not necessarily pseudomonotone and where $F(x) = (\nabla_{x_i} f_i(x))_{i=1}^n$.

(iv) Watson's Problem: Finally, we consider a stochastic variant of the ten variable non-monotone linear complementarity problem, first proposed by Watson [56]: $0 \le x \perp \mathbb{E}[(M + \epsilon M^{\omega})x + q + \epsilon q^{\omega}] \ge 0$, where M^{ω} and q^{ω} are randomly generated matrices and vectors (from the standard normal distribution) respectively and $\epsilon = 0.025$ refers to the level of noise. We omit the definition of the ten-dimensional matrix M, which can be found in [56, Example 3]. Note that $q = e_i$ and we consider ten different instances, each corresponding to a coordinate direction e_i .

4.2 Algorithm parameters and termination criteria

We conduct two sets of tests, the first of these pertains to the a.s. convergence behavior while the second set compares the empirical rate estimates with the theoretically prescribed levels. All the numerics were generated with Matlab R2012a on a Linux OS with a 2.39 GHZ processor and 16 GB of memory. For the first two test problem sets, $x_0 = 2e$ and γ_0 is 1 and 2.5 respectively while for the second two test problem sets, $x_0 = 0$ and γ_0 is 2.5 and 0.6, respectively.

(i) a.s. convergence: Here, n was varied from 10 to 30 in steps of 2 for the first three test problems while ten different instances of q were generated as stated earlier for the Watson's problem, leading to a total of 40 test instances. Recalling that x is a solution of VI(X, F) if and only if $F_X^{nat}(x) = x - \prod_X (x - F(x)) = 0$, a.s. convergence can be empirically verified based on the value of $\psi(x_k) = ||F_X^{nat}(x_k)||$. Note that our problem choices allow for evaluating the expectation, which is generally not possible in stochastic regimes.

(ii) Rate statements: When evaluating the rate estimates, we consider a modified Nash-Cournot game. The price was made affine, $\kappa = 1$ and the linear constraints were dropped. We generated ten different problem instances for n ranging from 10 to 19 and set $a = 0.1 \lceil n/10 \rceil$ and b = a/n. Note that b^{ω} was generated from a normal distribution with mean b and standard deviation ϵ , where $\epsilon = 0.025b$. The associated set and mapping are defined to be $F(x) = b(I + ee^T)x - ae$, $X = \{x \mid 0 \le x_i \le 1, i = 1, ..., n\}$, where e and I denote the vector of ones and the identity matrix. We note that $\nabla F(x) = b(I + ee^T)$, is strongly monotone (implies strongly pseudomonotone) with constant $\sigma = b$. The stochastic error can be bounded as follows:

$$\mathbb{E}[\|F(x;\omega) - F(x)\|^2] = \mathbb{E}[|b - b^{\omega}|^2] \|(I + ee^T)x\|^2 \le n(n+1)^2 \epsilon^2 = \nu^2.$$

Further, we have that $||F(x)|| = ||b(I + ee^T)x - ae|| = a ||n(I + ee^T)s - e|| \le a\sqrt{n}$, where the last inequality follows from b = a/n and $0 \le s \le e$. This implies that $B = 2a\sqrt{n}$. Since $0 \le x_i \le 1$, it follows that $||x|| \le \sqrt{n} = U$. It is easy to observe that the Lipschitz constant $L = b||I + ee^T||_F = b||I + ee^T||_F$

 $b\sqrt{n^2 - n + 4n} = a\sqrt{(n+3)/(n)}$. If x^* denotes the unique solution of VI(X, F), then the empirical error $\psi_e(x_K)$ and theoretical error $\psi_b(x_K)$ are defined as follows (see Proposition 5):

$$\psi_e(x_K) = \frac{1}{N} \sum_{j=1}^N \|x_K^j - x^*\|^2, \ \psi_b(x_K) = \frac{M(\gamma_0)}{K} \ge \mathbb{E}[\|x_K - x^*\|^2], \ M(\gamma_0) = \frac{\gamma_0^2(M_\nu + M_B)}{2\sigma\gamma_0 - 1}, \ (34)$$

where $\psi_e(x_K)$ is a result of averaging over N sample paths. Setting $\beta = 1$, we have

$$M_B = (1+L^2) (B^2/4) + 4U^2 = n(a^2+a^4+4) + 3a^4, \text{ and } M_\nu = (1+L^2) \nu^2 = (n+na^2+3a^2) (n+1)\epsilon^2.$$

4.3 Almost sure convergence behavior

In this subsection, we compare the a.s. convergence behavior of the extragradient and mirror-prox schemes under two different distance metrics. Table 2 displays $\psi(x_K)$ generated from the ESA scheme for increasing number of major iterations for the four problems of interest. We observe that in the fractional quadratic and nonlinear problems, the ESA scheme performs relatively well, barring two instances. Notably, much of the progress is made in the first 1000 iterations.

	n	Error $\psi(x_K)$				
		K = 1	K = 1000	K = 15000		
Erna Quad	10	6.017e + 00	4.690e-02	7.951e-04		
Flac. Quad.	15	7.473e + 00	1.441e-01	2.959e-02		
Ence Meelin	10	5.345e+00	2.754e-02	2.955e-03		
Frac. Nomin.	15	7.145e+00	9.433e-03	1.288e-02		
Nach game	10	1.581e + 01	4.624e-01	1.634e-01		
Ivasii game	15	2.165e+01	5.613e-01	2.377e-01		
Wataan CD	10	9.695e-01	2.329e-01	2.477e-01		
watson-CF	15	9.381e-01	1.357e-01	1.255e-01		

Table 2: Asymptotics of ESA

Next, we compare the stochastic extragradient scheme with two prox-based generalizations that employ two distance functions proposed by Nemirowski [52] given by $s_a(x) = \sum_{i=1}^n (x_i + \delta) \log(x_i + \delta)$ and $s_b(x) = \log(n) \sum_{i=1}^n x_i^{\left(1 + \frac{1}{\log(n)}\right)}$. The variants of MPSA, referred to as MPSA-a and MPSAb respectively, are studied and the results are compared with the ESA scheme in Table 3 for ten nonlinear fractional problems in the test set for progressively increasing number of major iterations. It is observed that the ESA scheme sometimes (but not always) performs better than MPSA-a from an error standpoint but each step of MPSA-a (and MPSA-b) tends to require more effort as captured by the CPU time.

n	K	Projection		Pro	ox-A	Prox-B		
		$\psi(x_K)$ Time (s)		$\psi(x_K)$ Time(s)		$\psi(x_K)$	Time (s)	
10	1000	2.754e-02	1.411e+01	1.352e-01	2.662e + 01	1.624e-01	2.229e + 01	
10	15000	2.955e-03	2.057e + 02	1.019e-01	3.969e + 02	8.953e-02	4.551e + 02	
15	1000	9.433e-03	1.388e+01	3.508e-02	4.528e + 01	2.277e-02	2.464e + 01	
	15000	1.288e-02	2.111e+02	1.578e-02	6.314e + 02	1.107e-02	8.836e + 02	
19	1000	1.030e-01	1.734e+01	1.677e-01	4.998e+01	3.652e-01	5.098e + 01	
	15000	8.360e-02	2.603e + 02	1.179e-01	8.753e + 02	2.398e-01	9.625e + 02	

Table 3: Comparison of SA schemes for frac. nonlin. problems

4.4 Error analysis and optimal choices of γ_0

While the previous results focused on asymptotics, we now compare the empirical rates with the theoretically predicted rates, as discussed in Sec. 3. In obtaining the empirical results, the initial

steplength γ_0 was set to be $(1+\sqrt{33})/(4\sigma)$ and fifteen different sample paths of ESA were generated to compute Ψ_e (34). Note that the choice of such a steplength is to ensure that $1 \leq \sigma \gamma_0 \leq 2$ to further demonstrate the alignment with the theoretical rate statement. Given that the expectation may be evaluated, we may solve the original problem to obtain an estimate of x^* . Table 4 compares the analytical bounds with empirical results for the given set of problems in increasing iterations. For the (monotone) problems considered, the theoretical bound is shown to be valid but relatively weak.

Dim (n)	K = 1		K = 100		K = 1000		K = 10000		K = 150000	
	$\psi_e(x_K)$	$\psi_b(x_K)$								
5	3.455e+00	6.007e+04	1.024e-04	6.007e+02	4.540e-05	6.007e+01	2.544e-05	6.007e+00	2.246e-05	4.005-01
6	4.382e+00	1.038e + 05	3.227e-04	1.038e+03	5.512e-05	1.038e + 02	3.779e-05	1.038e+01	3.372e-03	6.920e-01
7	5.324e + 00	1.648e + 05	3.323e-04	1.648e + 03	9.332e-05	1.648e + 02	5.180e-05	1.648e + 01	5.823e-05	1.098e+00
8	6.275e + 00	2.460e+05	2.218e-03	2.460e + 03	2.218e-03	2.460e + 02	2.218e-03	2.460e+01	2.218e-03	1.640e + 00
9	7.234e + 00	3.503e+05	5.531e-04	3.503e+03	1.397e-04	3.503e+02	1.241e-04	3.503e+01	1.055e-04	2.335e+00
10	8.197e+00	4.806e + 05	5.201e-03	4.806e + 03	5.201e-03	4.806e + 02	5.201e-03	4.806e + 01	5.201e-03	3.204e + 00

Table 4: Analytical vs Empirical Bounds for stochastic Nash-Cournot Game.

We now investigate the benefit of utilizing steplength close to the optimal γ_0 , denoted by γ_0^* . We choose $\epsilon = 0.02$ and our steplength is further given by $\gamma_0^* = ((2 - \epsilon)/2\sigma) = 0.99/\sigma$. Here, we consider the same set of problems as in the previous section and report the behavior of the proposed extragradient scheme in Table 5 for six different choices of γ_0 , ranging from $0.0017\gamma_0^*$ to $170\gamma_0^*$ in factors of 10. It can be seen that steplengths close to γ_0^* perform either the best (or close to the best) for all schemes. In fact, a poorly chosen steplength leads to significant drop off in performance.

		Empirical error ψ_e							
Dim (n)	Iteration (K)	$0.0017\gamma_{0}^{*}$	$0.017\gamma_{0}^{*}$	$0.17\gamma_{0}^{*}$	$1.7\gamma_{0}^{*}$	$17\gamma_{0}^{*}$	$170\gamma_{0}^{*}$		
5	15000	2.754e + 00	3.660e-01	1.748e-05	1.716e-05	3.084e-05	9.780e-05		
10	15000	5.406e+00	1.366e-01	1.346e-04	1.434e-04	2.113e-04	4.360e-04		
14	15000	6.940e + 00	6.410e-02	9.316e-04	9.316e-04	9.316e-04	1.486e-03		

Table 5: Optimality error for varying choices of γ_0 (Bold represents optimal γ_0)

5 Concluding remarks

Variational inequality problems represent a useful tool for modeling a range of phenomena arising in engineering, economics, and the applied sciences. As the role of uncertainty grows, there has been a growing interest in the *stochastic* variational inequality problem. However, much of the past research, particularly the algorithmic aspects, have focused on monotone stochastic variational inequality problems. In this context, we provide amongst the first results for claiming a.s. convergence of the solution iterates to the solution set produced by a (stochastic) extragradient scheme as well as mirror-prox generalizations. We also show that similar statements can be provided for monotone SVIs under a weak-sharpness requirement; notably much of the prior research for monotone SVIs uses averaging techniques in showing that the gap function convergence in an expected-value sense. Under stronger assumptions on the map, we show that both the extragradient and the mirror-prox schemes attain the optimal rate of convergence in terms of solution iterates, rather than in terms of the gap function. Importanly, we further refine the rate statement by deriving the optimal initial steplength. Notably, we see a modest degradation of the rate from strongly monotone SVIs to strongly pseudomonotone SVIs. Preliminary numerics suggest that the schemes perform well on a breadth of pseudomonotone and non-monotone problems. Furthermore, empirical observations suggest that significant benefit may accrue in terms of mean-squared error from employing the optimal initial steplength. Our work has made an initial step towards understanding how stochastic approximation schemes can be extended to regimes where pseudomonotonicity, rather than monotonicity, of the map holds. Yet, we believe much remains to be understood regarding how stochastic approximation schemes can be extended/modified to contend with far weaker requirements on the map.

6 Appendix

Lemma 7. Consider the function $t_0(\gamma_0)$ defined as

$$t_0(\gamma_0) \triangleq \frac{\gamma_0^2}{2\sigma\gamma_0 - \lfloor 2\sigma\gamma_0 \rfloor}$$

where σ_0 denotes the strong pseudomonotonicity constant. Then the following hold:

- (a) A minimizer of $t_0(\gamma_0)$ cannot exist in an interval $|2\sigma\gamma_0| \in [n, n+1]$ where n > 1.
- (b) The infimum of $t_0(\gamma_0)$ is given by the following:

$$f^* \triangleq \inf_{\gamma_0} \{ t_0(\gamma_0) \mid 1 < 2\sigma\gamma_0 < 2 \} = \frac{1}{\sigma^2}.$$

(c) Suppose an ϵ -infimum of $t_0(\gamma_0)$, denoted by f_{ϵ}^* , satisfies $f_{\epsilon}^* \leq f^* + \beta \epsilon$ for some $\beta > 0$. Then f_{ϵ}^* is achieved by $\gamma_0 = \frac{2-\epsilon}{2\sigma}$ and satisfies $f_{\epsilon}^* \leq f^* + \frac{2\epsilon}{\sigma^2}$, where $\epsilon \in (0, 1/2)$.

Proof. (a). We begin by observing that if $2\sigma\gamma_0 \in \mathbb{Z}_+$, then $t_0(\gamma_0) = +\infty$. Consequently, any minimizer of $t_0(\gamma_0)$ has to satisfy $2\gamma_0\sigma \notin \mathbb{Z}_+$. We proceed to show that $t_0(\gamma_0)$ does not admit a minimizer in (n, n + 1) where n > 1. Assume this is false and suppose there exists a minimizer γ_0^* satisfying $2\gamma_0^*\sigma \in (n, n+1)$. But there exists a $\tilde{\gamma}_0$ such that $2\sigma\tilde{\gamma}_0 \in (n-1, n)$. In fact, $t_0(\tilde{\gamma}_0) < t_0(\gamma_0^*)$ as we show next and our claim follows.

$$t_0(\tilde{\gamma}_0) = \left(\frac{\tilde{\gamma}_0^2}{2\sigma\tilde{\gamma}_0 - \lfloor 2\sigma\tilde{\gamma}_0 \rfloor}\right) < \left(\frac{(\gamma_0^*)^2}{2\sigma\gamma_0^* - \lfloor 2\sigma\gamma_0^* \rfloor}\right) = t_0(\gamma_0^*).$$

(b). From (a), it follows that if a minimizer exists, it has to satisfy $2\gamma_0^*\sigma \in (1,2)$. It follows that $t_0(\gamma_0)$ reduces to $\gamma_0^2/(2\sigma\gamma_0-1)$. Consider the following optimization problem:

$$\inf_{\gamma_0} \left\{ \frac{\gamma_0^2}{(2\sigma\gamma_0 - 1)} \mid 1 < 2\sigma\gamma_0 < 2 \right\}.$$

We observe that $t_0(\gamma_0)$ is a strictly decreasing function by noting that

$$t_0'(\gamma_0) = \frac{2\gamma_0}{2\sigma\gamma_0 - 1} - \frac{2\sigma\gamma_0^2}{(2\sigma\gamma_0 - 1)^2} = \frac{2\gamma_0}{2\sigma\gamma_0 - 1} \left(1 - \frac{\sigma\gamma_0}{2\sigma\gamma_0 - 1}\right) = \frac{2\gamma_0}{2\sigma\gamma_0 - 1} \left(\frac{\sigma\gamma_0 - 1}{2\sigma\gamma_0 - 1}\right) < 0,$$

since $\sigma \gamma_0 < 1$. It follows that the infimum is at the end-point given by $\sigma \gamma_0 = 1$ implying that

$$\inf_{\gamma_0} \left\{ \frac{\gamma_0^2}{(2\sigma\gamma_0 - 1)} \mid 1 < 2\sigma\gamma_0 < 2 \right\} = \frac{1}{\sigma^2}.$$

(c.) Suppose $\widehat{\gamma}_0 = \frac{2-\epsilon}{2\sigma}$ where $\epsilon \in (0, 1/2)$. Then we have that

$$f_{\epsilon}^* - f^* = \frac{(2-\epsilon)^2}{4\sigma^2(1-\epsilon)} - \frac{1}{\sigma^2} \le \frac{1}{\sigma^2} \left(\frac{1}{1-\epsilon} - 1\right) = \frac{1}{\sigma^2} \frac{\epsilon}{1-\epsilon} = \frac{2}{\sigma^2}\epsilon.$$

It follows that $f_{\epsilon}^* \leq f^* + \frac{2\epsilon}{\sigma^2}$.



Figure 1: A schematic of $t_0(\gamma_0)$

Example 1. Unfortunately, while one can derive an infimum of the above discontinuous optimization problem, this infimum cannot be achieved and the problem lacks a minimizer as proved in the above result. Yet, this infimum is informative in developing an approximate ϵ -solution as part (c) shows. We proceed to use this ϵ -infimum in deriving rate statements and demonstrate this result through an example. Suppose $\sigma = 0.1\sqrt{1.3}$. Then $t_0(\gamma_0)$ is shown as a solid line with discontinuities in Fig. 1 while the dashed flat line displays the infimum $1/\sigma^2$.

Lemma 8. Consider the following recursion: $a_{k+1} \leq (1 - 2c\theta/k)a_k + \frac{1}{2}\theta^2 M^2/k^2$, where θ and M are positive constants, $a_k \geq 0$, and $(1 - 2c\theta) < 0$. Then for $k \geq 1$, we have that

$$2a_k \le \frac{\max\left(\frac{\theta^2}{2c\theta - \lfloor 2c\theta \rfloor}M^2, 2a_1\right)}{k}.$$

Proof. We begin by noting that $\bar{\epsilon} > 0$ and $\kappa > 1$ as seen next.

$$\kappa = \left(1 + \frac{\bar{k} - 1}{2c\theta - \bar{k}}\right) = \left(\frac{2c\theta - 1}{2c\theta - \lfloor 2c\theta \rfloor}\right) > 1.$$

We consider the following cases for k.

Case 1: Consider k = 1. Then the following holds: $a_2 \leq (1 - 2c\theta)a_1 + \frac{1}{2}\theta^2 M^2$. If $(2c\theta - 1) > 0$, we may rearrange the inequalities as follows:

$$(2c\theta - 1)a_1 \le -a_2 + \frac{1}{2}\theta^2 M^2 \le \frac{1}{2}\theta^2 M^2 \text{ or } 2a_1 \le \frac{1}{2c\theta - 1}\theta^2 M^2 \Longrightarrow 2a_1 \le \max\left((2c\theta - 1)^{-1}\theta^2 M^2, 2a_1\right) \le \max\left((2c\theta - 1)^{-1}\theta^2 M^2\kappa, 2a_1\right),$$

where $\kappa > 1$.

Case 2: 1 < $k \leq \bar{k}$. Recall that when $k \leq \bar{k}$, we have that

$$(1 - 2c\theta/k) \le (1 - 2c\theta/\bar{k}) = (1 - (2c\theta)/(\lceil 2c\theta \rceil)) < 0.$$

Then the following holds:

$$a_{k+1} \le \left(\left(1 - \frac{2c\theta}{k} \right) a_k + \frac{1}{2} \frac{\theta^2 M^2}{k^2} \right)$$
$$a_k \left(\frac{2c\theta}{k} - 1 \right) \le -a_{k+1} + \frac{\theta^2 M^2}{2k^2} \le \frac{\theta^2 M^2}{2k^2}$$
$$\implies a_k \le \frac{\theta^2 M^2}{2k^2} \left(\frac{k}{2c\theta - k} \right) = \frac{\theta^2 M^2}{2k(2c\theta - k)}$$

By the definition of κ and $u \triangleq \max\left(\frac{\kappa\theta^2 M^2}{(2c\theta-1)}, 2a_1\right)$, we may conclude the following:

$$2a_k \leq \frac{\theta^2 M^2}{k(2c\theta - k)} = \frac{\theta^2 M^2}{k(2c\theta - 1)} \left(\frac{2c\theta - 1}{2c\theta - k}\right)$$
$$\leq \frac{\theta^2 M^2}{k(2c\theta - 1)} \left(\frac{2c\theta - k + \bar{k} - 1}{2c\theta - k}\right)$$
$$\leq \frac{\theta^2 M^2}{k(2c\theta - 1)} \left(1 + \frac{\bar{k} - 1}{2c\theta - k}\right)$$
$$\leq \frac{\theta^2 M^2}{k(2c\theta - 1)} \left(1 + \frac{\bar{k} - 1}{2c\theta - \bar{k}}\right) = \frac{\theta^2 M^2}{k(2c\theta - 1)}\kappa$$
$$\leq \frac{1}{k} \max\left(\frac{\kappa\theta^2 M^2}{(2c\theta - 1)}, 2a_1\right) = \frac{u}{k},$$

Case 3: $k > \bar{k}$. Suppose, this holds for $k > \bar{k}$, implying that $2a_k \leq \frac{\max(\theta^2 M^2(2c\theta-1)^{-1}\kappa, 2a_1)}{k}$. We proceed to show that this holds for k := k + 1 where $u \triangleq \max\left(\theta^2 M^2(2c\theta-1)^{-1}\kappa, 2a_1\right)$ and $\left(1 - \frac{2c\theta}{k}\right) > 0$ since $k > \bar{k}$:

$$\begin{aligned} a_{k+1} &\leq \left(1 - \frac{2c\theta}{k}\right) \frac{u}{2k} + \frac{1}{2} \left(\frac{\theta^2 M^2}{k^2}\right) \\ &= \left(1 - \frac{2c\theta}{k}\right) \frac{u}{2k} + \frac{(2c\theta - 1)}{2k} \left(\frac{\theta^2 M^2}{(2c\theta - 1)k}\right) \\ &\leq \left(1 - \frac{2c\theta}{k}\right) \frac{u}{2k} + \frac{(2c\theta - 1)}{2k} \left(\frac{\theta^2 M^2 \kappa}{(2c\theta - 1)k}\right) \\ &\leq \left(\frac{u}{2k} - \left(\frac{2c\theta}{k}\right) \frac{u}{2k}\right) + \left(\frac{2c\theta - 1}{2k}\right) \frac{u}{k} \\ &= \frac{u}{2k} - \frac{1}{k} \left(\frac{u}{2k}\right) \leq \frac{u}{2k} - \frac{1}{k+1} \left(\frac{u}{2k}\right) = \frac{u}{2(k+1)}. \end{aligned}$$

Lemma 9. Consider the following problem: $\min \{h(\gamma_0)g(z) \mid \gamma_0 \in \Gamma_0, z \in \mathbb{Z}\}$, where h and g are positive functions over Γ_0 and \mathbb{Z} , respectively. If $\overline{\gamma}_0$ and \overline{z} denote global minimizers of $h(\gamma_0)$ and g(z) over Γ_0 and \mathbb{Z} , respectively, then the following holds:

$$\min_{\gamma_0\in\Gamma_0, z\in\mathcal{Z}} h(\gamma_0)g(z) = h(\bar{\gamma}_0)g(\bar{z}).$$

Proof. The proof has two steps. First, we note that $\min_{\gamma_0 \in \Gamma_0, z \in \mathbb{Z}} h(\gamma_0) g(z) \ge h(\bar{\gamma}_0) g(\bar{z})$, implying that at any global minimizer (γ_0^*, z^*) ,

$$h(\gamma_0^*)g(z^*) \ge h(\bar{\gamma}_0)g(\bar{z}). \tag{35}$$

Second, since $(\bar{\gamma}_0, \bar{z}) \in \Gamma_0 \times \mathcal{Z}$, we have that $h(\gamma_0^*)g(z^*)$ has an optimal value that is no smaller than that the value associated with any feasible solution or

$$h(\gamma_0^*)g(z^*) = \min_{\gamma_0 \in \Gamma_0, z \in \mathcal{Z}} h(\gamma_0)g(z) \le h(\bar{\gamma}_0)g(\bar{z}).$$
(36)

By combining (35) and (36), the result follows.

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References

- F. Facchinei and J-S. Pang. Finite Dimensional Variational Inequalities and Complementarity Problems: Vols I and II. Springer-Verlag, NY, Inc., 2003.
- [2] I. V. Konnov. Equilibrium Models and Variational Inequalities. Elsevier, 2007.
- [3] L. Brighi and R. John. Characterizations of pseudomonotone maps and economic equilibrium. Journal of Statistics and Management Systems, 5(1-3):253-273, 2002.
- [4] R. Kihlstrom, A. Mas-Colell, and H. Sonnenschein. The demand theory of the weak axiom of revealed preference. *Econometrica*, 44(5):971–978, 1976.
- [5] A.M. Elizarov. Maximizing the lift-drag ratio of wing airfoils with a turbulent boundary layer: Exact solutions and approximations. *Doklady Physics*, 53(4):221–227, 2008.
- [6] A. Rousseau, P. Sharer, S. Pagerit, and S. Das. Trade-off between fuel economy and cost for advanced vehicle configurations. In *Proceedings of the 20th International Electric Vehicle* Symposium, Monaco, 2005.
- [7] G.R. Duensing, H.R. Brooker, and J.R. Fitzsimmons. Maximizing signal-to-noise ratio in the presence of coil coupling. *Journal of Magnetic Resonance, Series B*, 111(3):230–235, 1996.
- [8] W. Shadwick and C. Keating. A universal performance measure. Journal of Performance Measurement, 6(3):59–84, 2002.
- [9] K. Hossein, S. Thomas, and G. Raj. Omega as a performance measure. *Preliminary Report*, Duke University, June 2003.
- [10] S. Chandra. Strong pseudo-convex programming. Indian Journal of Pure and Applied Mathematics, 3(2):278–282, 1972.
- [11] B. F. Hobbs. Mill pricing versus spatial price discrimination under bertrand and cournot spatial competition. The Journal of Industrial Economics, 35(2):173–191, 1986.

- [12] S. C. Choi, W. S. Desarbo, and P. T. Harker. Product positioning under price competition. Management Science, 36(2):175–199, 1990.
- [13] L. A. Garrow and F. S. Koppelman. Multinomial and nested logit models of airline passengers' no-show and standby behaviour. *Journal of Revenue Pricing Management*, 3(3):237 253, 2004.
- [14] J. P. Newman. Normalization of network generalized extreme value models. Transportation Research Part B: Methodological, 42(10):958–969, December 2008.
- [15] G. Cliquet. Implementing a subjective mci model: An application to the furniture market. European Journal of Operational Research, 84(2):279–291, July 1995.
- [16] M. Nakanishi and L. G. Cooper. Parameter estimation for a multiplicative competitive interaction model: Least squares approach. *Journal of Marketing Research*, 11(3):303–311, 1974.
- [17] G. Gallego and M. Hu. Dynamic pricing of perishable assets under competition. Management science: journal of the Institute for operations research and the management sciences, 60(5):1241–1259, 2014.
- [18] C. Ewerhart. Cournot games with biconcave demand. Games and Economic Behavior, 85(0):37 - 47, 2014.
- [19] A. Shapiro, D. Dentcheva, and A. Ruszczynski. Lectures on stochastic programming: modeling and theory. The society for industrial and applied mathematics and the mathematical programming society, Philadelphia, USA, 2009.
- [20] H. Xu. Sample average approximation methods for a class of stochastic variational inequality problems. Asia-Pac. J. Oper. Res., 27(1):103–119, 2010.
- [21] S. Lu and A. Budhiraja. Confidence regions for stochastic variational inequalities. Math. Oper. Res., 38(3):545–568, 2013.
- [22] S. Lu. Symmetric confidence regions and confidence intervals for normal map formulations of stochastic variational inequalities. *SIAM Journal on Optimization*, 24(3):1458–1484, 2014.
- [23] H. Robbins and S. Monro. A stochastic approximation method. Annals of Mathematical Statistics, 22:400–407, 1951.
- [24] H. J. Kushner and G. G. Yin. Stochastic Approximation and Recursive Algorithms and Applications. Springer New York, 2003.
- [25] V. S. Borkar. Stochastic Approximation: A Dynamical Systems Viewpoint. Cambridge University Press, 2008.
- [26] J. C. Spall. Introduction to Stochastic Search and Optimization: Estimation, Simulation, and Control. Wiley Series in Discrete Mathematics and Optimization. Wiley, 2005.
- [27] A.S. Nemirovski and D.B. Judin. On cezari's convergence of the steepest descent method for approximating saddle point of convex-concave functions. *Soviet Mathematics-Doklady*, 19, 1978.
- [28] D. Ruppert. Efficient estimations from a slowly convergent robbins-monro process. Cornell University Technical Report, Operations Research and Industrial Engineering, 1988.

- [29] B. T. Polyak. New stochastic approximation type procedures. Automat. i Telemekh., 7:98–107, 1990.
- [30] B. T. Polyak and A. Juditsky. Acceleration of stochastic approximation by averaging. SIAM Journal on Control and Optimization, 30(4):838–855, 1992.
- [31] H. J. Kushner and J. Yang. Stochastic approximation with averaging of the iterates: Optimal asymptotic rate of convergence for general processes. SIAM J. Control Optim., 31(4):1045– 1062, July 1993.
- [32] H. J. Kushner and J. Yang. Analysis of adaptive step-size SA algorithms for parameter tracking. Automatic Control, IEEE Transactions on, 40(8):1403–1410, Aug 1995.
- [33] A. S. Nemirovski and D. B. Yudin. Problem complexity and method efficiency in optimization. Wiley-Interscience, Translated by: E. R. Dawson, 1983.
- [34] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. SIAM Journal on Optimization, 19(4):1574–1609, 2009.
- [35] S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework. SIAM Journal on Optimization, 22(4):1469–1492, 2012.
- [36] S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization, ii: Shrinking procedures and optimal algorithms. SIAM Journal on Optimization, 23(4):2061–2089, 2013.
- [37] D. Bertsekas and J. Tsitsiklis. Gradient convergence in gradient methods with errors. SIAM Journal on Optimization, 10(3):627–642, 2000.
- [38] S. Ghadimi and G. Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013.
- [39] H. Jiang and H. Xu. Stochastic approximation approaches to the stochastic variational inequality problem. *IEEE Transactions on Automatic Control*, 53(6):1462–1475, 2008.
- [40] J. Koshal, A. Nedić, and U.V. Shanbhag. Regularized iterative stochastic approximation methods for stochastic variational inequality problems. *IEEE Transactions on Automatic Control*, 58(3):594–609, 2013.
- [41] F. Yousefian, A Nedić, and U.V. Shanbhag. A regularized smoothing stochastic approximation (RSSA) algorithm for stochastic variational inequality problems. In *Proceedings of the Winter Simulation Conference (WSC)*, pages 933–944, Dec 2013.
- [42] A. Juditsky, A. Nemirovski, and C. Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm. *Stochastic Systems*, 1(1):17–58, 2011.
- [43] F. Yousefian, A. Nedić, and U. V. Shanbhag. Optimal robust smoothing extragradient algorithms for stochastic variational inequality problems. In 53rd IEEE Conference on Decision and Control, pages 5831–5836, Dec 2014.
- [44] Y. Chen, G. Lan, and Y. Ouyang. Accelerated schemes for a class of variational inequalities. Math. Program., 165(1):113–149, September 2017.

- [45] A. Kannan and U. V. Shanbhag. The pseudomonotone stochastic variational inequality problem: Analytical statements and stochastic extragradient schemes. In American Control Conference, ACC 2014, Portland, OR, USA, June 4-6, 2014, pages 2930–2935. IEEE, 2014.
- [46] A. N. Iusem, A. Jofré, R. I. Oliveira, and P. Thompson. Extragradient method with variance reduction for stochastic variational inequalities. *SIAM Journal on Optimization*, 27(2):686– 724, 2017.
- [47] F. Yousefian, A. Nedić, and U. V. Shanbhag. On stochastic mirror-prox algorithms for stochastic Cartesian variational inequalities: Randomized block coordinate and optimal averaging schemes. *Set-Valued and Variational Analysis*, 26(4):789–819, December 2018.
- [48] F. Yousefian, A. Nedić, and U. V. Shanbhag. Self-tuned stochastic approximation schemes for non-Lipschitzian stochastic multi-user optimization and Nash games. *IEEE Transactions on Automatic Control*, 61(7):1753–1766, July 2016.
- [49] A. N. Iusem, A. Jofré, and P. Thompson. Incremental constraint projection methods for monotone stochastic variational inequalities. *Mathematics of Operations Research*, https://doi.org/10.1287/moor.2017.0922, 2018.
- [50] F. Yousefian, A. Nedić, and U. V. Shanbhag. On smoothing, regularization, and averaging in stochastic approximation methods for stochastic variational inequality problems. *Math. Program.*, 165(1):391–431, 2017.
- [51] B. T. Polyak. Introduction to Optimization. Optimization Software Inc., New York, 1987.
- [52] A. Nemirovski. Prox-method with rate of convergence O(1/T) for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems. SIAM Journal on Optimization, 15(1):229–251, 2005.
- [53] C. D. Dang and G. Lan. On the convergence properties of non-euclidean extragradient methods for variational inequalities with generalized monotone operators. *Computational Optimization* and Applications, 60(2):277–310, March 2015.
- [54] P. Dvurechensky, A. Gasnikov, F. Stonyakin, and A. Titov. Generalized Mirror Prox: Solving Variational Inequalities with Monotone Operator, Inexact Oracle, and Unknown Hölder Parameters. https://arxiv.org/pdf/1806.05140.pdf, 2018.
- [55] Y. Nesterov. Introductory lectures on convex optimization : a basic course. Applied optimization. Kluwer Academic Publ., Boston, Dordrecht, London, 2004.
- [56] L. T. Watson. Solving the nonlinear complementarity problem by a homotopy method. SIAM Journal on Control and Optimization, 17(1):36–46, 1979.
- [57] A. Kannan and U. V. Shanbhag. Distributed computation of equilibria in monotone Nash games via iterative regularization techniques. SIAM Journal on Optimization, 22(4):1177– 1205, 2012.
- [58] B. Allaz and J. L. Vila. Cournot competition, forward markets and efficiency. Journal of Economic Theory, 59(1):1 – 16, 1993.
- [59] F. Facchinei and C. Kanzow. Generalized Nash equilibrium problems. 4OR, 5(3):173–210, 2007.