A PIECEWISE CONSERVATIVE METHOD FOR UNCONSTRAINED CONVEX OPTIMIZATION

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ABSTRACT. We consider a continuous-time optimization method based on a dynamical system, where a massive particle starting at rest moves in the conservative force field generated by the objective function, without any kind of friction. We formulate a restart criterion based on the mean dissipation of the kinetic energy, and we prove a global convergence result for strongly-convex functions. Using the Symplectic Euler discretization scheme, we obtain an iterative optimization algorithm. We have considered a discrete mean dissipation restart scheme, but we have also introduced a new restart procedure based on ensuring at each iteration a decrease of the objective function greater than the one achieved by a step of the classical gradient method. For the discrete conservative algorithm, this last restart criterion is capable of guaranteeing a qualitative convergence result. We apply the same restart scheme to the Nesterov Accelerated Gradient (NAG-C), and we use this restarted NAG-C as benchmark in the numerical experiments. In the smooth convex problems considered, our method shows a faster convergence rate than the restarted NAG-C. We propose an extension of our discrete conservative algorithm to composite optimization: in the numerical tests involving non-strongly convex functions with ℓ^1 -regularization, it has better performances than the well known efficient Fast Iterative Shrinkage-Thresholding Algorithm, accelerated with an adaptive restart scheme.

1. INTRODUCTION

Convex optimization is of primary importance in many fields of Applied Mathematics. In this paper we are interested in unconstrained minimization problems of the form

 $\min_{x \in \mathbb{R}^n} f(x),$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function. We will further assume that ∇f is Lipschitzcontinuous and that f is strongly convex. The simplest algorithm for the numerical resolution of this minimization problem is the classical gradient descent. In the second half of the last century other important first-order algorithms were introduced in order to speed up the convergence of the gradient descent: Polyak proposed his *heavy ball method* (see [17], [18]), and Nesterov introduced a new class of *accelerated gradient descent methods* (see [12], [14]). For a complete introduction to the subject, we refer the reader to [4] and [5].

The approach of blending the study of optimization methods with Dynamical Systems considerations has been fruitfully followed in several recent works, where Dynamical Systems tools were employed to study existing optimization methods and to introduce new ones: in [24] the authors derived an ODE for modeling the Nesterov Accelerated Gradient algorithm; in [20] and [21] the authors studied accelerated methods (Nesterov and Polyak) through *high-resolution* ODEs. Other contributions in this direction come from [1] and [2]. Almost all the ODEs obtained in the aforementioned papers can be reduced to the form

$$\ddot{x} + \nabla f(x) = -B(x, t)\dot{x},\tag{1}$$

where B(x,t) is a symmetric positive definite matrix, possibly depending on t. The term $-B(x,t)\dot{x}$ in (1) represents the contribution of a generalized viscous friction. If, for example, the matrix B does not depend on the time t, then the convergence of any solution of (1) to the minimizer of f is guaranteed by the dissipation of the total mechanical energy $H = \frac{1}{2}|\dot{x}|^2 + f(x)$, which plays the role of Lyapunov function. Indeed, by differentiation of the energy H along any solution of (1), we obtain $\frac{d}{dt}H(t) = -\dot{x}^T B(x)\dot{x} < 0$, as long as $\dot{x} \neq 0$. The choice of the matrix B(x,t) is of primary importance as shown in [2] and [24].

We recall that the Dynamical System approach was first undertaken in [24], where the authors proved that the Nesterov method for non-strongly convex functions (NAG-C) can be modeled by considering the solutions of (1), with the dissipative term of the form $-B(x,t)\dot{x} = -\frac{3}{t}\dot{x}$ and with initial velocity equal to zero. Moreover, they proved that the objective function achieves a decay $O(t^{-2})$ along these curves. In order to avoid oscillations of the solutions, which otherwise would slow down the convergence, the authors introduced an adaptive restart strategy that consists in resetting the velocity equal to zero in correspondence of local maxima of the kinetic energy $E_K := \frac{1}{2}|\dot{x}|^2$. In [24] it was shown that this adaptively restarted method has a linear convergence rate when the objective function is strongly convex. In this paper we will focus on the conservative mechanical ODE

$$\ddot{x} + \nabla f(x) = 0, \tag{2}$$

and we will design a piecewise conservative method based on an adaptive restart strategy: the resulting continuous-time algorithm achieves a linear convergence rate when the objective function is strongly convex. As well as the aforementioned restarted method proposed in [24], our method does not make use of the constant of strong convexity of the objective function. This is a relevant point, since in practice estimating this quantity may be a very challenging task.

The presence of viscosity friction in the dynamics studied in [24] yields to dissipate the kinetic energy from the very beginning of the motion. In alternative, if we consider dynamics (2) with initial velocity equal to zero and starting point $x(0) = x_0$ far from the minimizer x^* , it could be a good idea to let the system evolve without damping (i.e., conservatively) for an amount of time ΔT , so that the solution may be free to get closer to the minimizer, without being decelerated by the viscosity friction. This is the idea that underlies the restarted method that we consider in this paper, based on the maximization of the mean dissipation of the kinetic energy as a stopping criterion.

The introduction of this new restart strategy is motivated by the fact that the maximization of the kinetic energy (employed in [24]) is not suitable for the conservative dynamics (2). Indeed, in [24] the proof of the uniform upper bound for the restart time (which is the cornerstone of the linear convergence result) heavily relies on the presence of the viscosity friction. On the other hand, in the conservative dynamics, the efficacy of the maximization of the kinetic energy as restart criterion depends on the dimension of the ambient space. In the one-dimensional case the solution converges to a local minimizer in a single restart iteration, as shown in Section 2. Unfortunately, in the multi-dimensional case and for a general f, it is not possible to prove that a local maximum of the kinetic energy is reached in a finite amount of time. For this reason in [25], where the conservative dynamics with maximization of the kinetic energy was investigated, the authors proved a linear convergence result for strongly convex objective functions by assuming *a priori* the existence of a uniform upper bound for the restart time.

The original restart strategy (based on the *mean dissipation* of the kinetic energy) that we develop in Section 3 allows us to manage the absence of viscosity friction in the system. Indeed, we can prove that the restart time is always finite in the case that the objective function is coercive. Moreover, when dealing with strongly convex functions, we prove that the restart time is uniformly bounded, and hence we can strengthen the linear convergence result of [25]. We also show that the trajectory obtained has finite length.

In Section 4 we derive a discrete-time optimization algorithm by applying the Symplectic Euler scheme to (2), yielding an update rule of the form $x_{k+1} = x_k - \alpha \nabla f(x_k) + (x_k - x_{k-1})$ where $\alpha > 0$ is the step-size.

We introduce two restart criteria for the discrete algorithm based on the maximization of the mean dissipation of the discrete kinetic energy, referred as RCM-mmd (Algorithm 1 and Algorithm 2). We also consider the case of restarting the discrete conservative algorithm when the discrete kinetic energy is maximized: this yields to the algorithm investigated in [22], where the authors proved some partial results for quadratic objective functions. Finally, we design a restart criterion by imposing that, at each iteration, the decrease of the objective function is greater or equal than the per-iteration-decrease achieved by the classical gradient descent method with the same step-size. We end up obtaining a discrete method referred as RCM-grad (Algorithm 3), similar to those described in [25]. Moreover, we observe that this reasoning holds also for the Nesterov Accelerated Gradient with the gradient restart scheme (NAG-C-restart) proposed in [15] and recently employed in [11]. In other words, both RCM-grad and NAG-C-restart achieve at each iteration an effective acceleration of the gradient method. This fact allows us, as a by-product, to prove a qualitative global convergence result for RCM-grad: to the best of our knowledge, this is the first convergence result for a method based on the discretization of conservative dynamics. This method is suitable both for strongly and non-strongly convex optimization, since it does not require an a priori estimate of the strong convexity constant of the objective function. We recall that other important contributions in this direction for Nesterov-like restarted methods come from [13], [6], [7].

In Section 5 we have planned a quite extensive comparison to experimentally evaluate the performance of the convergence rate between the different discrete-time restart methods and the different versions of the Nesterov Accelerated Gradient. In particular, we use as benchmark NAG-C-restart, since it was shown to achieve high performances in both strongly and non-strongly convex optimization (see [15]), and it does not require the knowledge of the constant of strong convexity. We also give some insights on possible extensions of our method for composite optimization problems. We carry out numerical experiments in presence of ℓ^1 -regularization and we compare our method with the restarted FISTA proposed in [15].

2. One-dimensional case and quadratic functions

In this Section we introduce the piecewise conservative method with restart strategy based on the maximization of the kinetic energy $E_K := \frac{1}{2}|\dot{x}|^2$. More precisely, given a $C_L^{1,1}$ -function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e., a function of class C^1 such that its gradient ∇f is Lipschitz-continuous with constant L > 0) to be minimized, and given a starting point $x_0 \in \mathbb{R}^n$, we consider the solution of the ODE

$$\ddot{x} + \nabla f = 0, \tag{3}$$

with $x(0) = x_0$ and $\dot{x}(0) = 0$. We recall that (3) preserves the total mechanical energy $H(x,\dot{x}) := \frac{1}{2}|\dot{x}|^2 + f(x)$, and we observe that at the initial instant the total mechanical energy coincides with the potential energy. Hence, during the motion, part of the initial potential energy is transformed into kinetic energy, and, if we aim at minimizing f, a natural strategy could be to wait until the kinetic energy attains a local maximum. At this point, we reset the velocity equal to zero, and we repeat the whole procedure. This continuoustime method was investigated in [25], where the authors proved that, when f is coercive and any critical point is a minimizer, the set of the minimizers of f is globally asymptotically stable for the trajectories of the restarted system. However, the main difficulty in establishing convergence rates lies in the estimate of the restart time. As we show in this section, in the one-dimensional case the system converges to a local minimizer in a single iteration, i.e., in a finite amount of time. On the other hand, in the multi-dimensional case the weakness of this criterion is inherent in our inability to prove that the restart time is finite for a generic strongly convex function. In Section 3 we modify the restart strategy in order to overcome this issue. We use $\mathscr{S}_{\mu,L}^{1,1}$ to denote the class of functions in $C_L^{1,1}$ that are μ -strongly convex, i.e., there exists $\mu > 0$ such that $x \mapsto f(x) - \frac{1}{2}\mu |x|^2$ is convex.

We start by investigating the one-dimensional case, when $f : \mathbb{R} \to \mathbb{R}$ is a $C_L^{1,1}$ function. We consider the following Cauchy problem:

$$\begin{cases} \ddot{x} + f'(x) = 0, \\ x(0) = x_0, \\ \dot{x}(0) = 0, \end{cases}$$
(4)

and we reset the velocity equal to zero whenever the kinetic energy $E_K = \frac{1}{2}|\dot{x}|^2$ achieves a local maximum. We prove that this continuous-time method arrives to a local minimizer of f at the first restart.

Proposition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C_L^{1,1}$ function and let us assume that f is coercive. For every $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$, let $x : [0, +\infty) \to \mathbb{R}$ be the solution of Cauchy problem (4). Then, there exists $\overline{t} \in (0, +\infty)$ such that the kinetic energy function $E_K : t \mapsto \frac{1}{2} |\dot{x}(t)|^2$ has a local maximum at \overline{t} . Moreover, for every $\overline{t} \in (0, +\infty)$ such that E_K has a local maximum at \overline{t} , the point $x(\overline{t})$ is a local minimizer of f.

The proof of Proposition 2.1 is postponed in Appendix A. Under the same assumptions and notations of Proposition 2.1, we can compute an explicit expression for the instant \bar{t} when the solution of (4) visits for the first time the local minimizer $x^* = x(\bar{t})$. We may assume that $x_0 < x^*$. For every $y \in [x_0, x^*]$ and for $t \in [0, \bar{t}]$, from the conservation of the total mechanical energy it follows that the solution of (4) visits the point y with velocity $v_y = \sqrt{2(f(x_0) - f(y))}$. Thus, we obtain that

$$\bar{t} = \int_{x_0}^{x^*} \frac{1}{\sqrt{2(f(x_0) - f(y))}} dy.$$
(5)

We observe that the hypothesis $f'(x_0) \neq 0$ guarantees that the singularity at x_0 in (5) is integrable, and thus that \bar{t} is finite.

When the objective function $f : \mathbb{R} \to \mathbb{R}$ is in $\mathscr{S}_{\mu,L}^{1,1}$ we can give an upper bound to \bar{t} that does not depend on the initial position x_0 . We prove this in the following Proposition.

Proposition 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $\mathscr{S}_{\mu,L}^{1,1}$. Let x^* be the unique minimizer of f and let us choose $x_0 \in \mathbb{R}$ such that $x_0 \neq x^*$. Let $t \mapsto x(t)$ be the solution of the Cauchy problem (4) and let \bar{t} be the instant when the solution visits for the first time the point x^* . Then the following inequality holds:

$$\bar{t} \le \frac{\pi}{2\sqrt{\mu}}.\tag{6}$$

The proof of Proposition 2.2 is postponed in Appendix B. The statement of Proposition 2.2 is sharp: inequality (6) is achieved for quadratic functions. On the other hand, if the function f is not strongly convex, the visiting time \bar{t} depends, in general, on the initial position. As we are going to show in the following example, it may happen that the closer the starting point is to the minimizer, the longer it takes to arrive at.

Example 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{4}x^4$. Clearly, f is strictly convex (but not strongly) and $x^* = 0$ is the unique minimizer. Let us choose $x_0 > 0$. Then we have that

$$\bar{t} = \int_0^{x_0} \frac{\sqrt{2}}{\sqrt{x_0^4 - y^4}} dy = \int_0^{x_0} \frac{\sqrt{2}}{\sqrt{x_0^2 + y^2} \sqrt{x_0^2 - y^2}} dy$$
$$\geq \int_0^{x_0} \frac{1}{x_0 \sqrt{x_0^2 - y^2}} dy = \frac{\pi}{2x_0}.$$

This shows that, in general, we can not give a priori an upper bound for \bar{t} . However, this does not mean that methods designed with this approach are not suitable for the optimization of non-strongly convex functions. Indeed, the visiting time \bar{t} is finite, and this guarantees that the continuous-time method converges in a finite amount of time. This is not true, for example, in the case of the classical gradient flow.

The multidimensional case is much more complicated. We now focus on quadratic objective functions and, as we will see, also in this basic case our global knowledge is quite unsatisfactory. On the other hand, the study of quadratic functions leads to useful considerations that we try to apply to more general cases. Let us consider the Cauchy problem

$$\begin{cases} \ddot{x} + \nabla f(x) = 0, \\ x(0) = x_0, \\ \dot{x}(0) = 0. \end{cases}$$
(7)

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The main difference with respect to the one-dimensional case lies in the fact that, in general, the solution $t \mapsto x(t)$ of (7) never visits a local minimizer of f. The example below shows this phenomenon.

Example 2.4. Let us consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = \frac{a^2}{2}x_1^2 + \frac{b^2}{2}x_2^2$, where a, b > 0. Let us set $x(0) = (x_{0,1}, x_{0,2})^T \in \mathbb{R}^2$. Then the solution of Cauchy problem (7) is

$$t \mapsto x(t) = (x_{0,1}\cos(at), x_{0,2}\cos(bt))^T$$
.

If $x_{0,1}, x_{0,2} \neq 0$ and if the ratio a/b is not a rational number, then $x(t) \neq (0,0)^T$ for every $t \in [0, +\infty)$. This also shows that, when the dimension is larger than one, Proposition 2.1 fails. Indeed, it is easy to check that the kinetic energy function $t \mapsto \frac{1}{2} |\dot{x}(t)|^2$ has many local maxima, but the solution never visits any local minimizer of f.

When $f : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex quadratic function, we can estimate the decrease of the objective function after each arrest.

Lemma 2.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function of the form

$$f(x) = \frac{1}{2}x^T A x,$$

where A is a symmetric and positive definite matrix. Let $x_0 \in \mathbb{R}^n$ be the starting point of Cauchy problem (7). Let $0 < \lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of A. Then, the following inequality is satisfied:

$$f\left(x\left(\frac{\pi}{2\sqrt{\lambda_n}}\right)\right) \le \cos^2\left(\frac{\pi}{2}\sqrt{\frac{\lambda_1}{\lambda_n}}\right)f(x_0).$$
(8)

Remark 2.6. Let us assume that the kinetic energy function has at least one local maximizer and let $t_1 \in (0, +\infty)$ be the smallest. Then Lemma 2.5 implies that

$$f(x(t_1)) \le \cos^2\left(\frac{\pi}{2}\sqrt{\frac{\lambda_1}{\lambda_n}}\right)f(x_0).$$

Indeed, we have that $t_1 \geq \frac{\pi}{2\sqrt{\lambda_n}}$, since the time derivative of the kinetic energy function is non-negative at $t = \frac{\pi}{2\sqrt{\lambda_n}}$. Hence, if we iterate the evolution-restart procedure k times (assuming that the kinetic energy function always attains a local maximum) and if we call $x^{(k)}$ the restart point after the k-th iteration, we have that

$$f(x^{(k)}) \le \left[\cos^2\left(\frac{\pi}{2}\sqrt{\frac{\lambda_1}{\lambda_n}}\right)\right]^k f(x_0).$$

So, in terms of evolution-restart iterations, we have that the value of the objective function decreases at exponential rate. However, since we do not have an upper bound on the restart time, we do not know the rate of decrease in terms of the evolution time. As we explain in the next Section, we can overcome this problem by designing alternative restart criteria. For example, in the particular case of quadratic functions, we can keep the free-evolution amount of time constant and equal to $\Delta T = \frac{\pi}{2\sqrt{\lambda_n}}$. Let $t \mapsto \tilde{x}(t)$ be the curve obtained with this procedure, then, owing to the proof of Lemma 2.5, we have that

$$f(\tilde{x}(t)) \leq \left[\cos^2\left(\frac{\pi}{2}\sqrt{\frac{\lambda_1}{\lambda_n}}\right)\right]^{\lfloor \frac{t}{\Delta T} \rfloor} f(x_0),$$

where $|\cdot|$ denotes the integer part.

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3. An Alternative restart criterion

The restart criterion that we have considered so far consists in waiting until the kinetic energy reaches a local maximum. This idea has already been introduced in [24] in order to improve the convergence rate of the solutions of the ODE modeling the Nesterov method. Indeed, the solutions of the ODE considered in [24] exhibit undesirable oscillations, that can be avoided by means of this adaptive restart strategy. As a matter of fact, in [24] it is proved that the restarted Nesteov ODE achieves a linear convergence rate when the objective function is strongly convex. The proof of this result consists basically of two main steps (i.e., the estimate of the decrease of the objective function at each evolution-restart iteration, and the uniform upper bound for the restart time), and in both of them the presence of the viscosity friction plays a crucial role. On the other hand, in the conservative ODE that we study in the present paper there is no viscosity term, hence we cannot adapt to our case the arguments employed in [24]. In order to manage the absence of friction, in this section we introduce an original restart strategy, alternative to the one considered so far. We prove that the resulting continuous-time method achieves linear convergence rate when the objective function is strongly convex, and we show that the curve obtained has finite length. It is important to recall that our method, as well as the restarted method studied in [24], does not make use of the constant of strong convexity of the objective function.

In this section we propose a restart criterion based on the maximization of the mean dissipation. If we arrest the conservative evolution at the instant t > 0, then the value of the kinetic energy $E_K(t)$ at the instant t equals the decrease of the objective function. The idea behind this alternative restart criterion is that we arrest the conservative evolution of the system when the mean dissipation $t \mapsto E_K(t)/t$ reaches a local maximum. We use the notation $C_L^{2,1}$ to denote the functions in C^2 whose gradient is Lipschitz with constant L > 0. The symbol $\mathscr{S}_{\mu,L}^{2,1}$ is used to indicate the functions in $C_L^{2,1}$ that are strongly convex with constant $\mu > 0$.

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a $C_L^{1,1}$ function and let us consider the Cauchy problem

$$\begin{cases} \ddot{x} + \nabla f(x) = 0, \\ x(0) = x_0, \\ \dot{x}(0) = 0. \end{cases}$$
(9)

Let us define the function $r:[0,+\infty)\to [0,+\infty)$ as

$$r(t) = \begin{cases} 0 & t = 0, \\ \frac{E_K(t)}{t} & t > 0, \end{cases}$$
(10)

where E_K is the kinetic energy function relative to the solution of Cauchy problem (9). We observe that r is differentiable at t = 0, since we have that

$$E_K(t) = \frac{1}{2} |\nabla f(x_0)|^2 t^2 + o(t^2), \qquad (11)$$

as $t \to 0$. If we take the derivative of r with respect to the time, we obtain that

$$\frac{d}{dt}r = \frac{tE_K(t) - E_K(t)}{t^2}$$
(12)

for every t > 0. With a simple computation, we can check that the derivative of r can be continuously extended at t = 0. We have that

$$\frac{d}{dt}r = \begin{cases} \frac{1}{2}|\nabla f(x_0)|^2 & t = 0, \\ \frac{t\dot{E}_K(t) - E_K(t)}{t^2} & t > 0. \end{cases}$$

We observe that the derivative of r at t = 0 is positive, hence it remains non-negative in an interval $[0, \varepsilon)$. The Maximum Mean Dissipation criterion consists of restarting the evolution when the function $t \mapsto r(t)$ reaches a local maximum. The restart time is

$$t_a = \inf\{t : tE_K(t) - E_K(t) < 0\}.$$
(13)



FIGURE 1. Mean dissipation. The black graph represents a typical profile of the kinetic energy function, in the case it attains a local maximum. The slope of the segments represents the mean dissipation that we obtain when we stop the evolution in a given instant. The picture shows that stopping the evolution in correspondence of a local maximum of the kinetic energy function does not guarantee the highest mean dissipation.

We observe that, if $\bar{t} \in (0, +\infty)$ is a local maximizer of the kinetic energy, then we have that

$$\bar{t}E_K(\bar{t}) - E_K(\bar{t}) = -E_K(\bar{t}) < 0.$$

This means that a local maximizer of the kinetic energy can not be a maximizer of the mean dissipation r. This fact is described in Figure 1.

We can prove that the restart time t_a is finite. We remark that the following result holds even if the function f is not convex.

Lemma 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C_L^{1,1}$ coercive function and let us take $x_0 \in \mathbb{R}^n$. Let $t \mapsto E_K(t)$ be the kinetic energy function of the solution of Cauchy problem (9). Then there exists $\hat{t} \in (0, +\infty)$ such that

$$\hat{t}E_K(\hat{t}) - E_K(\hat{t}) < 0.$$

Proof. We argue by contradiction. Let us assume that

$$t\dot{E}_K(t) - E_K(t) \ge 0, \tag{14}$$

for every $t \ge 0$. Using (12), we deduce that the mean dissipation $t \mapsto r(t)$ is non-decreasing for every $t \ge 0$. Let $t_1 > 0$ be any instant such that the kinetic energy is positive, i.e., $E_K(t_1) > 0$. Then we have that

$$E_K(t) \ge \frac{E_K(t_1)}{t_1}t,\tag{15}$$

for every $t > t_1$. This is impossible since the kinetic energy is always bounded from above if the function f is coercive.

Using the idea of the proof of Lemma 3.1, we can estimate from above the restart time of the Maximum Mean Dissipation. Indeed, the derivative of the mean dissipation is positive at t = 0, and then it remains non-negative in the interval $[0, t_a]$. Then, using the same notations as in the proof above, for every $t \in [t_1, t_a]$ the kinetic energy function E_K satisfies inequality (15). On the other hand, from the conservation of the energy it follows that

$$f(x_0) - f^* \ge E_K(t)$$

where f^* is the minimum value of the objective function f. This implies that

$$f(x_0) - f^* \ge \frac{E_K(t_1)}{t_1} t_a,$$

that can be rewritten as

$$t_a \le \frac{t_1}{E_K(t_1)} (f(x_0) - f^*).$$
(16)

Remark 3.2. Inequality (16) implies that the restart condition of the Maximum Mean Dissipation is met after a finite amount of time, as soon as f is a coercive function in $C_L^{1,1}$. On the other hand, to the best of our knowledge, it is not possible to exclude that the kinetic energy E_K could grow monotonically, without assuming maximum. This means that in general, when dealing with conservative system (9) together with the restart strategy based on the maximization of the kinetic energy, it is not possible to prove an upper bound for the restart time. For this reason, in [25] the linear convergence result for strongly convex objective functions is proved under the assumption of the *a priori* existence of a uniform upper bound for the restart time. On the other hand, using the Maximum Mean Dissipation, when the objective function is in $\mathscr{S}_{\mu_2 L}^{1,1}$ we can bound the restart time with a quantity that depends only on the constants μ and L.

In the case of a strongly convex function, we can give uniform estimates for the restart time t_a .

Proposition 3.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $\mathscr{S}^{1,1}_{\mu,L}$. For every $x_0 \in \mathbb{R}^n$, let us consider Cauchy Problem (9) with starting point x_0 and let $t \mapsto E_K(t)$ be the kinetic energy function of the solution. Let t_a be the stopping time defined in (13). Then the following estimate s hold: $\sqrt{\mu}$

$$\iota_a > \frac{1}{8L},\tag{11}$$

(17)

$$t_a \le \mathcal{T}_R := 32 \frac{L}{\mu \sqrt{\mu}}.$$
(18)

We postpone the proof of Proposition 3.3 since we need some technical lemmas. In the following lemma we recall the Polyak-Lojasiewicz inequality for strongly-convex functions (see [16], and [14], Theorem 2.1.10] for its proof).

Lemma 3.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and let us assume that f is μ -strongly convex, with $\mu > 0$. Let x^* be the unique minimizer of f. Then, for every $x \in \mathbb{R}^n$, the following Polyak-Lojasiewicz inequality holds:

$$f(x) - f(x^*) \le \frac{1}{2\mu} |\nabla f(x)|^2.$$
 (19)

In the following lemma, we give an estimate of the growth of the kinetic energy function $t \mapsto E_K(t)$ when t is small.

Lemma 3.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $\mathscr{S}^{1,1}_{\mu,L}$. For every $x_0 \in \mathbb{R}^n$, let us consider Cauchy Problem (9) with starting point x_0 and let $t \mapsto E_K(t)$ be the kinetic energy function of the solution. Then, for every $0 \le t \le \sqrt{\mu}/(2L)$, the following inequality holds:

$$\frac{1}{8}|\nabla f(x_0)|^2 t^2 \le E_K(t) \le \frac{25}{32}|\nabla f(x_0)|^2 t^2.$$
(20)

Proof. We recall that L is a Lipschitz constant for $x \mapsto \nabla f(x)$. Using the conservation of the mechanical energy, we deduce that

$$\begin{aligned} |\nabla f(x(t)) - \nabla f(x_0)| &\leq L |x(t) - x_0| \leq L \int_0^t |\dot{x}(u)| \, du \\ &\leq L \int_0^t \sqrt{2} \sqrt{f(x_0) - f^*} \, du = L \sqrt{2} \sqrt{f(x_0) - f^*} \, t. \end{aligned}$$

Owing to (19), we obtain that

$$|\nabla f(x(t)) - \nabla f(x_0)| \le \frac{L}{\sqrt{\mu}} |\nabla f(x_0)|t.$$
(21)

and

Using this fact, we deduce that

$$\left|\dot{x}(t) + t\nabla f(x_0)\right| = \left|\int_0^t \left(-\nabla f(x(s)) + \nabla f(x_0)\right) ds\right| \le \int_0^t \frac{L}{\sqrt{\mu}} |\nabla f(x_0)| s \, ds.$$

Hence we have that

$$|\dot{x}(t) + t\nabla f(x_0)| \le \frac{L}{2\sqrt{\mu}} |\nabla f(x_0)| t^2.$$

$$\tag{22}$$

Using (22) and the triangular inequality, we obtain that

$$|\nabla f(x_0)|t - \frac{L}{2\sqrt{\mu}}|\nabla f(x_0)|t^2 \le |\dot{x}(t)| \le |\nabla f(x_0)|t + \frac{L}{2\sqrt{\mu}}|\nabla f(x_0)|t^2.$$

Therefore, if $t \leq \sqrt{\mu}/L$, we have that

$$\frac{1}{2}|\nabla f(x_0)|t \le |\dot{x}(t)|$$

On the other hand, if $t \leq \sqrt{\mu}/(2L)$, we have that

$$|\dot{x}(t)| \le \frac{5}{4} |\nabla f(x_0)|t$$

This concludes the proof.

We now prove Proposition 3.3.

Proof of Proposition 3.3. The proof of inequality (17) is based on the study of the sign of the quantity $t \mapsto t \dot{E}_K(t) - E_K(t)$. First of all, we observe that

$$\dot{x}(t) = -t\nabla f(x_0) - \int_0^t \left(\nabla f(x(s)) - \nabla f(x_0)\right) \, ds.$$
(23)

Therefore, we deduce that

$$\begin{split} \dot{E}_K(t) &= \ddot{x}(t) \cdot \dot{x}(t) = -\nabla f(x(t)) \cdot \dot{x}(t) \\ &= \nabla f(x(t)) \cdot \left(t \nabla f(x_0) + \int_0^t \left(\nabla f(x(s)) - \nabla f(x_0) \right) \, ds \right) \\ &= \left(\nabla f(x(t)) - \nabla f(x_0) \right) \cdot \left(t \nabla f(x_0) + \int_0^t \left(\nabla f(x(s)) - \nabla f(x_0) \right) \, ds \right) \\ &+ |\nabla f(x_0)|^2 t + \nabla f(x_0) \cdot \int_0^t \left(\nabla f(x(s)) - \nabla f(x_0) \right) \, ds. \end{split}$$

Owing to (21), we obtain that:

$$\dot{E}_K(t) \ge |\nabla f(x_0)|^2 t - \frac{3}{2} \frac{L}{\sqrt{\mu}} |\nabla f(x_0)|^2 t^2 - \frac{1}{2} \frac{L^2}{\mu} |\nabla f(x_0)|^2 t^3.$$

Using inequality (20), we have that

$$t\dot{E}_K(t) - E_K(t) \ge |\nabla f(x_0)|^2 t^2 \left(\frac{7}{32} - \frac{3}{2}\frac{L}{\sqrt{\mu}}t - \frac{1}{2}\frac{L^2}{\mu}t^2\right),$$

for $t \leq \sqrt{\mu}/(2L)$. With a simple computation, we obtain that $t\dot{E}_K(t) - E_K(t) > 0$ when $t \leq \tilde{t}$, where

$$\tilde{t} := \frac{\sqrt{\mu}}{8L}.\tag{24}$$

By the definition of the stopping time t_a , we deduce that $t_a > \tilde{t}$. This proves (17).

We now prove (18). Using (17) and the definition of the stopping time t_a , we have that

$$\frac{E_K(t_a)}{t_a} \ge \frac{E_K(\tilde{t})}{\tilde{t}},$$

where \tilde{t} is defined in (24). The last inequality can be rewritten as

$$t_a \le \frac{E_K(t_a)}{E_K(\tilde{t})}\tilde{t}.$$
(25)

Using the conservation of the energy and inequality (20), we obtain that

$$\frac{E_K(t_a)}{E_K(\tilde{t})} \le \frac{8(f(x_0) - f^*)}{|\nabla f(x_0)|^2 \tilde{t}^2} \le \frac{4}{\mu} \frac{1}{\tilde{t}^2},\tag{26}$$

where in the second inequality we used Lemma 3.4. Combining (25) and (26), and using the definition of \tilde{t} given in (24), we obtain that $t_a \leq \mathcal{T}_R$, where we set

$$\mathcal{T}_R := 32 \frac{L}{\mu \sqrt{\mu}}.$$

This concludes the proof.

The framework for analyzing our restarted scheme is similar to that employed in [24], however we point out that the proofs are different since the ODE considered in [24] and the conservative system (9) are structurally different.

For this stopping criterion we have proved that the restart time is uniformly bounded by \mathcal{T}_R . In the following result, we provide an estimate about the value of the kinetic energy at the restart instant.

Lemma 3.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C_L^{2,1}$ function. For every $x_0 \in \mathbb{R}^n$, let us consider Cauchy Problem (9) with starting point x_0 and let $t \mapsto E_K(t)$ be the kinetic energy function of the solution. Then the following inequality holds:

$$E_K(t_a) \ge \frac{1}{2L} |\nabla f(x(t_a))|^2, \qquad (27)$$

where t_a is the stopping time defined in (13).

Proof. Owing to the definition, we have that t_a is a local maximizer for the function $t \mapsto r(t)$, where $r : [0, +\infty) \to [0, +\infty)$ is the Mean Dissipation function defined in (10). Recalling that $t_a \dot{E}_K(t_a) = E_K(t_a)$, we have that

$$\frac{d^2}{dt^2}r(t_a) = \frac{\ddot{E}_K(t_a)}{t_a} \le 0.$$

On the other hand, we have

$$\ddot{E}_K(t_a) = |\nabla f(x(t_a))|^2 - \dot{x}(t_a)^T \nabla^2 f(x(t_a)) \dot{x}(t_a) \le 0.$$
(28)

By the hypothesis, the matrix $LId - \nabla^2 f(x)$ is positive definite for every $x \in \mathbb{R}^n$. Using this fact in (28), we obtain that

$$2LE_K(t_a) - |\nabla f(x(t_a))|^2 \ge 0,$$

and this concludes the proof.

We conclude this section providing an estimate about the decrease of the objective function with the convergence result. Moreover, we prove that the piecewise conservative method produces a curve that has finite length.

Theorem 3.7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function in $\mathscr{S}^{2,1}_{\mu,L}$. Let $x^* \in \mathbb{R}^n$ be the unique minimizer of f, and let $x_0 \in \mathbb{R}^n$ be the starting point. Let $t \mapsto \tilde{x}(t)$ be the curve obtained applying the following iterative procedure:

• set $t_0 = 0$ and consider the forward solution of $\ddot{\tilde{x}} + \nabla f(\tilde{x}) = 0$, with $\tilde{x}(t_0) = x_0$ and $\dot{\tilde{x}}(t_0) = 0$;

• for every $k \ge 1$, let t_k be the instant when the mean dissipation

$$t \mapsto \frac{|\tilde{x}(t)|^2}{2(t-t_{k-1})}, \quad t > t_{k-1}$$

attains a local maximum for the first time, and set $x_k = \tilde{x}(t_k)$. Then consider the forward solution of $\ddot{\tilde{x}} + \nabla f(\tilde{x}) = 0$, with $\tilde{x}(t_k) = x_k$ and $\dot{\tilde{x}}(t_k) = 0$.

Then for every $t \ge 0$ the following inequality is satisfied:

$$f(\tilde{x}(t)) - f(x^*) \le \left(1 + \frac{\mu}{L}\right)^{-\left\lfloor \frac{t}{\tau_R} \right\rfloor} (f(x_0) - f(x^*)),$$
(29)

where \mathcal{T}_R is defined in (18). Moreover, we can prove the following upper bound for the length of the curve $t \mapsto \tilde{x}(t)$:

$$\int_{0}^{\infty} |\dot{\tilde{x}}(t)| \, dt \le 4\sqrt{2} \frac{L}{\mu} \mathcal{T}_{R} \sqrt{f(x_{0}) - f(x^{*})}.$$
(30)

Proof. We begin by proving (29). Let $t_1 > 0$ be the first stopping instant. Owing to the conservation of the total mechanical energy, we have that

$$f(x_0) - f(x(t_1)) = E_K(t_1).$$

On the other hand, combining Lemma 3.6 and Lemma 3.4, we obtain that

$$E_K(t_1) \ge \frac{1}{2L} |\nabla f(x(t_1))|^2 \ge \frac{\mu}{L} \left(f(x(t_1)) - f(x^*) \right)$$

Therefore, we deduce that

$$f(x(t_1)) - f(x^*) \le \left(1 + \frac{\mu}{L}\right)^{-1} (f(x_0) - f(x^*)).$$
(31)

The last inequality gives an estimate of the decrease-per-iteration of the objective function. Owing to Proposition 3.3, we have that the stopping time is always bounded by \mathcal{T}_R . This means that in the interval [0, t] the number k of restart iterations is greater or equal than $\left|\frac{t}{\mathcal{T}_R}\right|$, where $\lfloor \cdot \rfloor$ denotes the integer part. Using inequality (31), we obtain that

$$f(\tilde{x}(t)) - f(x^*) \le \left(1 + \frac{\mu}{L}\right)^{-k} \left(f(x_0) - f(x^*)\right)$$
$$\le \left(1 + \frac{\mu}{L}\right)^{-\left\lfloor\frac{t}{\tau_R}\right\rfloor} \left(f(x_0) - f(x^*)\right)$$

This proves (29).

We now study the length of the curve $t \mapsto \tilde{x}(t)$ at each evolution interval $[t_k, t_{k+1}]$ for $k \ge 0$. Using the conservation of the total mechanical energy and Proposition 3.3, we have that

$$\int_{t_k}^{t_{k+1}} |\dot{\tilde{x}}(t)| \, dt \le \mathcal{T}_R \sqrt{2(f(\tilde{x}(t_k) - f(x^*))}.$$
(32)

Moreover, owing to (31), we obtain that

$$f(\tilde{x}(t_k) - f(x^*) \le \left(1 + \frac{\mu}{L}\right)^{-k} (f(x_0) - f(x^*))$$
(33)

for every $k \ge 0$. Combining (32) and (33), we have that

$$\int_{0}^{\infty} |\dot{\tilde{x}}(t)| \, dt = \sum_{k=0}^{\infty} \int_{t_{k}}^{t_{k+1}} |\dot{\tilde{x}}(t)| \, dt \le 4\sqrt{2} \frac{L}{\mu} \mathcal{T}_{R} \sqrt{f(x_{0}) - f(x^{*})}.$$
so the proof.

This concludes the proof.

Remark 3.8. In the case of quadratic functions, we can compare our convergence result with the one proved in [25]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be of the form

$$f(x) = \frac{1}{2}x^T A x,$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, and let $0 < \lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of A. Let $x_0 \in \mathbb{R}^n$ be the starting point and let $t \mapsto \bar{x}(t)$ the curve obtained following the construction proposed in [25]. Owing to Theorem 2 and Lemma 4 in [25], the following estimate holds:

$$f(\bar{x}(t)) \le \left(1 + \frac{\lambda_1}{\lambda_n}\right)^{-\left\lfloor \frac{t}{\tau_R'} \right\rfloor} (f(x_0) - f(x^*))$$

where

$$\mathcal{T}_R' = \frac{2n\pi}{\sqrt{\lambda_1}}.$$

On the other hand, if we consider the curve $t \mapsto \tilde{x}(t)$ obtained with our restart procedure, inequality (29) holds with

$$\mathcal{T}_R = 32 \frac{\lambda_n}{\lambda_1 \sqrt{\lambda_1}}.$$

Hence we observe that \mathcal{T}'_R is affected by the dimension of the problem, while \mathcal{T}_R is sensitive to the condition number of the matrix A.

Remark 3.9. It is important to observe that the estimate expressed in (30) is invariant if we multiply the objective function f by a factor $\nu > 0$. This is not the case for the estimate in (29), because the multiplication of f by a factor $\nu > 0$ does affect the parametrization of the curve produced by the method, while the trajectory remains unchanged. For these reasons, we believe that, when dealing with continuous-time optimization methods, the study of the length of trajectories may be a useful tool.

4. DISCRETE VERSION OF THE METHOD

In this section we develop a discrete version of the continuous-time algorithm that we have described so far. The basic idea is to rewrite the second order ODE

$$\ddot{x} + \nabla f(x) = 0, \ x(0) = x_0, \ \dot{x}(0) = 0,$$

as a first order ODE, by doubling the variables:

$$\begin{cases} \dot{x} = v, & x(0) = x_0, \\ \dot{v} = -\nabla f(x), & v(0) = 0. \end{cases}$$
(34)

The differential equation (34) is a time-independent Hamiltonian system and for its discretization we use the Symplectic Euler scheme, due to its well-known suitability (see e.g. [8, 21]), yielding to the following recurrence sequence for $k \ge 0$ given (x_0, v_0) :

$$\begin{cases} v_{k+1} = v_k - h \,\nabla f(x_k), \\ x_{k+1} = x_k + h \,v_{k+1}, \end{cases}$$
(35)

where h > 0 is the discretization step, and where x_0 is the starting point and $v_0 = 0$. We recall that, in general, the Symplectic Euler scheme for time-independent Hamiltonian systems leads to implicit discrete systems. However, for the particular Hamiltonian function $\mathcal{H}(x,v) = \frac{1}{2}|v|^2 + f(x)$, the discrete system (35) is explicit. Combining the equations of (35), we have that for $k \ge 0$

$$x_{k+1} = x_k - h^2 \nabla f(x_k) + h v_k.$$
(36)

This shows that the sequence defined in (35) consists of an iteration of the classical gradient descent method with step h^2 , plus the momentum term hv_k .

Remark 4.1. We observe that we can rewrite (36) as follows for $k \ge 0$:

$$x_{k+1} = x_k - h^2 \nabla f(x_k) + (x_k - x_{k-1}), \qquad (37)$$

with $x_{-1} = x_0 - hv_0$.

The last expression is very similar to the update rule of the heavy-ball method:

$$x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta (x_k - x_{k-1}).$$

It is important to recall that the local-convergence result for the heavy-ball method proved in [18] requires that $0 \le \beta < 1$. This means that we can not apply the aforementioned theorem to the sequence obtained using (36). However, this is not an issue, since, as well as in the continuous-time case, the convergence of our method relies on a proper restart scheme.

Remark 4.2. We point out that the discretization of conservative system (34) should not be understood as a method for providing an accurate approximation of the continuous-time solution. Nevertheless, it is natural to ask what is the structure of the update rule obtained with an higher-order symplectic scheme. For instance, if we apply the symplectic secondorder Störmer-Verlet scheme (see [8, Chapter VI]) to the conservative system (34), a simple computation shows that the sequence $(x_k)_k$ produced satisfies the so called *leapfrog scheme*, i.e., the same recurrence scheme as in (37), with initialization $x_{-1} = x_0 - hv_0 - \frac{h^2}{2}\nabla f(x_0)$. In other words, we obtain that the sequence $(x_k)_k$ produced by the Störmer-Verlet method satisfies the same recursive relation as the one produced by the Symplectic Euler scheme, but with a different trigger.

In order to design a discrete restart procedure, a first natural attempt is to formulate a discrete-time version of the Maximum Mean Dissipation. We recall that in the continuous-time setting this procedure consists in restarting the evolution as soon as the quantity

$$r(t) := \frac{|v(t)|^2}{2t}$$
(38)

attains a local maximum. Hence we can restart the discrete system (35) as soon as the following condition is met:

$$\frac{|v_k|^2}{k-l} > \frac{|v_{k+1}|^2}{k+1-l},\tag{39}$$

where l is 0 or it is the index when the latest restart has occurred. On the other hand, a local maximum can be characterized by a change of sign of the first derivative

$$\dot{r}(t) = -\frac{|v(t)|^2 + 2t\nabla f(x(t)) \cdot v(t)}{2t}$$

thus we can restart the discrete evolution when

$$|v_{k+1}|^2 + 2(k+1-l)\nabla f(x_{k+1}) \cdot v_{k+1} > 0, \qquad (40)$$

where l is 0 or it is the index when the latest restart has occurred. We call Restart-Conservative Method with maximum mean dissipation (RCM-mmd-r) the procedure given by (35) with restart condition (39). We call Restart-Conservative Method with differential maximum mean dissipation (RCM-mmd-dr) the procedure given by (35) with restart condition (40). These methods are described respectively in Algorithm 1 and Algorithm 2.

Now we consider an alternative restart strategy, for which we can prove a *qualitative* global convergence result. Indeed a natural request for our discrete algorithm is that, at each iteration, the decrease of the objective function is greater or equal than the decrease achieved by the gradient descent method with the same step. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 -convex function and let us define $z_{k+1} = x_k - h^2 \nabla f(x_k)$, then, owing to the convexity of f, we have that

$$f(z_{k+1}) \ge f(z_{k+1} + hv_k) - \nabla f(z_{k+1} + hv_k) \cdot hv_k.$$

Recalling that $x_{k+1} = z_{k+1} + hv_k$, we deduce that as long as the following inequality holds

$$\nabla f(x_{k+1}) \cdot v_k \le 0,\tag{41}$$

then we have that

$$f(x_k - h^2 \nabla f(x_k)) \ge f(x_{k+1}).$$

We can use inequality (41) to design a restart criterion for the sequence defined in (35): when (41) is violated, i.e.,

$$\nabla f(x_{k+1}) \cdot v_k > 0,$$

then we set

$$x_{k+1} = x_k - h^2 \nabla f(x_k), \quad v_{k+1} = -h \nabla f(x_k).$$

Algorithm 1 Restart-Conservative Method with maximum mean dissipation (RCM-mmd-r)

1: $x \leftarrow x_0 - h^2 \nabla f(x_0)$ 2: $v \leftarrow -h\nabla f(x_0)$ 3: $i \leftarrow 1$ 4: $l \leftarrow i - 1$ 5: while $i \leq max_iter$ do $i \gets i+1$ 6: $v' \leftarrow v - h \nabla f(x)$ 7: $x' \leftarrow x + hv'$ 8: if $|v'|^2/(i-l) < |v|^2/(i-1-l)$ then 9: $x \leftarrow x - h^2 \nabla f(x)$ 10: $v \leftarrow -h\nabla f(x)$ 11: $l \leftarrow i - 1$ 12:else 13: $x \leftarrow x'$ 14: 15: $v \leftarrow v'$ end if 16: 17: end while

Algorithm 2 Restart-Conservative Method with differential maximum mean dissipation (RCM-mmd-dr)

1: $x \leftarrow x_0 - h^2 \nabla f(x_0)$ 2: $v \leftarrow -h\nabla f(x_0)$ 3: $i \leftarrow 1$ 4: $l \leftarrow i - 1$ 5: while $i \leq max_{iter} do$ $i \leftarrow i+1$ 6: $v' \leftarrow v - h\nabla f(x)$ 7: $x' \leftarrow x + hv'$ 8: if $|v'|^2 + 2(i-l)\nabla f(x') \cdot v' > 0$ then 9: $x \leftarrow x - h^2 \nabla f(x)$ 10: $v \leftarrow -h\nabla f(x)$ 11: $l \leftarrow i - 1$ 12:else 13: $x \leftarrow x'$ 14:15: $v \leftarrow v - h\nabla f(x)$ end if 16: 17: end while

We call this procedure *Restart-Conservative Method with gradient restart* (RCM-grad) and we present its implementation in Algorithm 3. This method coincides with the one described in [25].

Remark 4.3. It is interesting to observe that the discrete restart condition

$$\nabla f(y_k + hv_k) \cdot v_k > 0 \tag{42}$$

is the discrete-time analogue of the inequality

$$\nabla f(x(t)) \cdot \dot{x}(t) \ge 0,$$

which is satisfied as soon as the kinetic energy function $E(t) = \frac{1}{2}|\dot{x}(t)|^2$ stops growing. This fact suggests that we may also consider a discrete restart strategy based on the maximization of the kinetic energy. Namely, we can restart the evolution of the discrete system (35) as soon

Algorithm 3 Restart-Conservative Method with gradient restart (RCM-grad)

1: $x \leftarrow x_0$ 2: $v \leftarrow 0$ 3: while $i \leq max_iter$ do $x' \leftarrow x - h^2 \nabla f(x) + hv$ 4: if $\nabla f(x') \cdot v > 0$ then 5: $x \leftarrow x - h^2 \nabla f(x)$ 6: $v \leftarrow -h\nabla f(x)$ 7: 8: else 9: $x \leftarrow x'$ $v \leftarrow v - h\nabla f(x)$ 10: end if 11: $i \leftarrow i + 1$ 12:13: end while

as

$$\frac{1}{2}|v_k|^2 > \frac{1}{2}|v_{k+1}|^2.$$
(43)

This restarted algorithm was proposed in [22], where the authors proved some partial results when dealing with quadratic objectives.

The convergence of RCM-grad for strictly convex functions in $C_L^{1,1}$ descends directly from the convergence of the gradient method, as shown in the following result.

Theorem 4.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C_L^{1,1}$ strictly convex function that admits a unique minimizer $x^* \in \mathbb{R}^n$. Let $(x_k)_{k \ge 0} \subset \mathbb{R}^n$ be the sequence produced by RCM-grad with time-step $0 < h < \frac{\sqrt{2}}{\sqrt{L}}$. Then the sequence converges to the minimizer x^* .

Proof. Owing to the construction of RCM-grad, we have that, for every $k \ge 0$ the following inequality holds:

$$f(x_{k+1}) - f(x_k) \le f(x_k - h^2 \nabla f(x_k)) - f(x_k).$$

By the fact that f is a convex function in $C_L^{1,1}$, we deduce that $f(x+\nu) \leq f(x) + \langle \nabla f(x), \nu \rangle + \frac{L}{2}|\nu|^2$ for every $x, \nu \in \mathbb{R}^n$ (see, e.g., [14, Theorem 2.1.5]). This implies that

$$f(x_{k+1}) - f(x_k) \le -\omega(h) |\nabla f(x_k)|^2,$$
(44)

where

$$\omega(h) = h^2 \left(1 - \frac{L}{2} h^2 \right).$$

Since we have that $\omega(h) > 0$ when $0 < h < \frac{\sqrt{2}}{\sqrt{L}}$, then $f(x_{k+1}) \leq f(x_k)$ for every $k \geq 0$. Moreover, we observe that the function f is coercive, since it is assumed to be strictly convex and to admit a minimizer. Therefore, the sequence $(x_k)_{k\geq 0} \subset \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ is bounded. Hence we can repeat the argument of [14, Theorem 2.1.14] to deduce that $f(x_k) \to f(x^*)$ as $k \to \infty$. Using again the boundedness of the sequence $(x_k)_{k\geq 0}$ and the fact that f admits a unique minimizer, we obtain that $x_k \to x^*$ as $k \to \infty$.

Remark 4.5. If we define $e_k := f(x_k) - f(x^*)$ for every $k \ge 0$, and if we assume $f \in \mathscr{S}_{\mu,L}^{1,1}$, then, combining (19) with (44), we obtain $e_{k+1} \le e_k(1-2\mu\omega(h))$ for $0 < h < \sqrt{2/L}$. Choosing $\bar{h} = 1/\sqrt{L}$ we have as a byproduct the linear convergence estimate $e_{k+1} \le e_k(1-\frac{\mu}{L})$. We point out that Theorem 4.4 and the previous estimate should be understood as a *qualitative* global convergence result. Indeed, on one hand, we may deduce that the convergence rate of RCM-grad is *at least* as fast as the convergence rate of the classical gradient descent. On the other hand, this estimate of the performances of RCM-grad is very pessimistic, as shown in the numerical experiments of Section 5. To the best of our knowledge, Theorem 4.4 is the first convergence result for an optimization algorithm based on the discretization of the conservative dynamics, although it does not provide the convergence rate observed in the numerical experiments developed in the paper.

4.1. Choice of the time-step. The proof of the convergence of RCM-grad holds true for any choice of the time-step h such that the gradient method with step-size h^2 is convergent. In this subsection, we provide considerations about the choice of time-step h by studying the one-dimensional quadratic case. Let us fix a > 0 and let us consider $f(x) = \frac{1}{2}ax^2$. In this case the sequences $(x_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ are recursively defined as

$$\begin{cases} v_{k+1} = v_k - ha \, x_k, \\ x_{k+1} = x_k + h \, v_{k+1}. \end{cases}$$
(45)

It is easy to check that the following discrete conservation holds:

$$\frac{1}{2}v_k^2 + \frac{a}{2}x_k^2 - \frac{1}{2}ah\,x_kv_k = \frac{a}{2}x_0^2.$$

This implies that the sequence of points $(x_k, v_k)_{k\geq 0} \in \mathbb{R}^2$ lies on the following conic curve in the (x, v)-plane:

$$\frac{1}{2}v^2 + \frac{a}{2}x^2 + \frac{1}{2}ah\,xv = c.$$
(46)

It is natural to set h such that the curve defined by (46) is compact. Using the characterization of conic curves in the plane, we obtain that

$$h < \frac{2}{\sqrt{a}}.\tag{47}$$

Another natural request is to impose that $|x_1| < |x_0|$ and that $x_0x_1 \ge 0$. Using (45), we have that $x_1 = (1 - ah^2)x_0$, so we impose that $1 > 1 - ah^2 > 0$, and we deduce that

$$h < \frac{1}{\sqrt{a}}.\tag{48}$$

For a generic $C_L^{1,1}$ convex function $f : \mathbb{R}^n \to \mathbb{R}$, an heuristic rule for designing h could be to use (48), where a is a constant that bounds from above the Lipschitz constant of ∇f .

4.2. Nesterov Accelerated Gradient methods with restart. We recall that the most efficient algorithms for convex optimization problems belong to the the family of the Nesterov Accelerated Gradient methods (see [12], [14]). We use the acronym NAG to refer to this family. Namely, when the problem consists in minimizing a function in $\mathscr{S}_{\mu,L}^{1,1}$ whose constant μ is known, the most performing algorithm is called NAG-SC and it is defined as

$$\begin{cases} y_{k+1} = x_k - s \nabla f(x_k), \\ x_{k+1} = y_{k+1} + \frac{1 - \sqrt{\mu s}}{1 + \sqrt{\mu s}} (y_{k+1} - y_k), \end{cases}$$
(49)

with $0 < s \leq \frac{1}{L}$.

On the other hand, when dealing with a non-strongly convex function in C_L^1 , the most suitable algorithm is NAG-C, whose update rule is

$$\begin{cases} y_{k+1} = x_k - s\nabla f(x_k), \\ x_{k+1} = y_{k+1} + \frac{k}{k+3}(y_{k+1} - y_k), \end{cases}$$
(50)

where, as above, $0 < s \leq \frac{1}{L}$.

In order to boost the convergence of NAG-C via adaptive restart, in [15] O'Donoghue and Candès suggest some schemes reproducing in a discrete form the requirement that f(x(t))is monotone decreasing along the curve $t \mapsto x(t)$, solution of an ODE with suitable friction term. More precisely, they proposed to restart (50) as soon as $f(y_{k+1}) > f(y_k)$ (function scheme), or as soon as $\nabla f(x_{k+1}) \cdot (y_{k+1} - y_k) > 0$ (gradient scheme). The intuitive idea that lies behind the latter scheme is to restart the evolution when the momentum and the negative direction of the gradient form an obtuse angle. We recall that the update of y_{k+1} coincides with a step of the gradient method, namely $y_{k+1} = x_k - s \nabla f(x_k)$. Hence we have that the step of NAG-C is given by

$$x_{k+1} = y_{k+1} + w_k,$$

where

$$w_k = \beta_k (y_{k+1} - y_k)$$
 and $\beta_k = \frac{k}{k+3}$.

Using these facts, the gradient restart scheme for NAG-C can be better motivated. Indeed, as done before for the conservative algorithm, we can impose that each iteration of NAG-C achieves a greater decrease of the objective function than a step of the gradient method:

$$f(y_{k+1}) \ge f(y_{k+1} + w_k). \tag{51}$$

If we apply *verbatim* the reasoning done before about the restart of the conservative method, then for every C^1 -convex function we have that

$$f(y_{k+1}) \ge f(y_{k+1} + w_k) - \nabla f(y_{k+1} + w_k) \cdot w_k.$$

Recalling that $x_{k+1} = y_{k+1} + w_k$ and that $w_k = \beta_k (y_{k+1} - y_k)$, we deduce that (51) is satisfied as long as the following condition holds:

$$\nabla f(x_{k+1}) \cdot (y_{k+1} - y_k) \le 0.$$
(52)

If we restart the method as soon as (52) is violated, we recover the gradient restart scheme proposed by O'Donoghue and Candès in [15]. In conclusion, this proves that the NAG-C with the gradient restart scheme achieves, at *each iteration*, an effective acceleration with respect to the classical gradient descent. In the experiments reported in [15], the authors show that the gradient restart scheme has better performances than the function restart scheme. For these reasons, in the numerical tests reported in Section 5 we used NAG-C with gradient restart as benchmark for the convergence rate. From now on, we refer to this method as NAG-C-restart. An important feature of NAG-C-restart is that it is suitable for strongly convex minimization when the strong convexity parameter of the objective is not available, as observed in [13] in the framework of composite optimization. This aspect was further studied in [6], where the authors proved linear convergence results for restarted accelerated methods when the objective function satisfies a local quadratic growth condition. In [7] this was extended to accelerated coordinate descent methods. Finally, we observe that the gradient restart scheme has been recently used in [11] in order to accelerate the convergence of the Optimized Gradient Method introduced in [10].

5. Numerical tests

In this section we describe the numerical experiments that we used to test the efficiency of our method. We used different variants of NAG as comparison: in particular, NAG-C-restart (see Subsection 4.2) is the benchmark of our numerical tests.

5.1. Quadratic function. We considered a quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \frac{1}{2}x^T A x + b^T x$$

where n = 1000, A is a symmetric positive definite matrix and $b \in \mathbb{R}^n$ is sampled using $\mathcal{N}(0, 1)$. The eigenvalues of A are randomly chosen using an uniform distribution over [0.03, 15]. The Lipschitz constant of ∇f is λ_{\max} , the largest eigenvalue of A. The function f is μ -strongly convex for every $0 < \mu \leq \lambda_{\min}$, where λ_{\min} is the minimum eigenvalue of A. For each experiment, we run the following algorithms:

- NAG-SC with $s = \frac{1}{\lambda_{\text{max}}}$ and $\mu = \lambda_{\text{min}}$. This is the sharpest possible setting of the
- parameters for the given problem; NAG-SC with $s = \frac{1}{\lambda_{\text{max}}}$ and $\mu = \frac{\lambda_{\text{min}}}{3}$. This simulates an underestimation of the strongly-convexity constant;
- NAG-C-restart with s = 1/λ_{max};
 RCM-grad with h = 1/√λ_{max};



FIGURE 2. Quadratic case. At the top we report the result of a single experiment, at the bottom the average over 50 repetitions of the experiment. The plots at left-hand side shows the decay of the objective function achieved by RCM-grad (blue), RCM-mmd-dr (magenta), NAG-SC with exact strongly-convexity constant (NAG-SC-1, black), NAG-SC with underestimated strongly-convexity constant (NAG-SC-2, dashed), and NAG-C-restart (red). At right-hand side we compare the convergence rate of the Restart-Conservative method with different restart schemes. We observe that on average RCM-grad and RCM-mmd-dr have a slightly better performances than the benchmark NAG-C-restart.

- RCM-mmd-dr with $h = \frac{1}{\sqrt{\lambda_{\max}}}$; RCM-mmd-r with $h = \frac{1}{\sqrt{\lambda_{\max}}}$; RCM-kin with $h = \frac{1}{\sqrt{\lambda_{\max}}}$.

The results are described in Figure 2. This test shows that, among the Restart-Conservative Methods, RCM-grad and RCM-mmd-dr are the most performing. Moreover, RCM-grad and RCM-mmd-dr achieve a faster convergence rate than NAG-SC-2 when a sharp estimate of the strongly-convexity constant is not available. We also observe that both RCM-grad and RCM-mmd-dr have on average slightly better performances than NAG-C-restart. However, when the strongly-convexity constant is known, NAG-SC has better performances than the other algorithms. Finally, RCM-mmd-r and RCM-kin have a slower convergence rate than other Restart-Conservative methods.

5.2. Logistic regression. We considered a typical logistic regression problem. First of all, we randomly generated the vector $x_0 \in \mathbb{R}^n$ using $\mathcal{N}(0, 0.01)$. Then we independently sampled the entries of the vector $y = (y_1, \ldots, y_m)^T \in \{0, 1\}^m$ using the law

$$\mathbb{P}(Y_i = 1) = \frac{1}{1 + e^{-a_i^T x_0}},$$



FIGURE 3. Logistic regression. At the top we report the result of a single experiment, at the bottom the average over 50 repetitions of the experiment. The plots at left-hand side shows the decay of the norm of the gradient of the objective function achieved by RCM-grad (blue), RCM-mmd (magenta), NAG-C-restart (red), and the classical gradient descent (black). At righthand side we compare the convergence rate of RCM with different restart schemes. RCM-grad and RCM-mmd-dr seem to have faster convergence rate than the benchmark NAG-C-restart.

where $A = (a_1, \ldots, a_n)$ was a $n \times m$ matrix with i.i.d. entries generated with the $\mathcal{N}(0, 1)$ distribution. Supposing that y and A were known, we tried to recover x_0 using the loglikelyhood maximization. This is equivalent to the minimization of the function

$$f(x) = \sum_{i=1}^{m} \left((1 - y_i) a_i^T x + \log \left(1 + e^{-a_i^T x} \right) \right)$$
(53)

We set n = 100 and m = 500. Let L be the Lipschitz constant of the function ∇f . We recall that function (53) is convex but not strongly convex. We minimized the right-hand-side of (53) using the following algorithms:

- Classical gradient descent method with step-size $s = \frac{1}{L}$;
- NAG-C-restart and $s = \frac{1}{L}$;
- RCM-grad with $h = \frac{1}{\sqrt{L}}$;
- RCM-mmd-dr with $h = \frac{1}{\sqrt{L}}$; RCM-mmd-r with $h = \frac{1}{\sqrt{L}}$; RCM-kin with $h = \frac{1}{\sqrt{L}}$;

The results of the experiment are presented in Figure 3. We observe that the most performing methods are RCM-grad and RCM-mmd-dr and they show similar behavior in both single and average runs. Moreover, RCM-grad and RCM-mmd-dr have faster convergence rates than NAG-C-restart. Among the RCM methods the RCM-kin and RCM-mmd-r are the slowest.



FIGURE 4. LogSumExp function. At the top we report the result of a single experiment, at the bottom the average over 50 repetitions of the experiment. The plots at left-hand side shows the decay of the norm of the gradient of the objective function achieved by RCM-grad (blue), RCM-mmd (magenta), NAG-C-restart (red), and the classical gradient descent (black). At righthand side we compare the convergence rate of RCM with different restart schemes. RCM-grad and RCM-mmd-dr seem to be faster than the benchmark NAG-C-restart.

5.3. LogSumExp. We considered the non-strongly convex function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \rho \log\left(\sum_{i=1}^{m} \exp\left(\frac{a_i^T x - b_i}{\rho}\right)\right),\tag{54}$$

where $A = (a_1, \ldots, a_m)$ was a $n \times m$ matrix whose entries were independently generated using the normal distribution $\mathcal{N}(0,1)$. The vector $b \in \mathbb{R}^m$ was sampled using $\mathcal{N}(0,1)$. We set n = 50, m = 200 and $\rho = 1$. Let L be the Lipschitz constant of the function ∇f . We minimized f using the following algorithms:

- Classical gradient descent method with step-size $s = \frac{1}{L}$;
- NAG-C-restart with $s = \frac{1}{L}$;
- RCM-grad with $h = \frac{1}{\sqrt{L}}$;
- RCM-mmd-dr with $h = \frac{1}{\sqrt{L}}$; RCM-mmd-r with $h = \frac{1}{\sqrt{L}}$;
- RCM-kin with $h = \frac{1}{\sqrt{L}};$

The results are shown in Figure 4. We observe that in the presented single run the RCM-grad shows the best performance. In the average RCM-grad and RCM-mmd-dr exhibit very similar behaviors and have a slightly better performances than the benchmark NAG-C-restart.

We want to conclude this section with some considerations about the non-smooth case. We try to give heuristic ideas to generalize our method to the minimization of *composite* functions (see, for example, [13] for an introduction to the subject). Namely, we consider functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = g(x) + \Psi(x),$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz-continuous convex function. The main obstruction to the direct application of RCM for the minimization of f is due to the fact that, in general, the gradient ∇f may not be well-defined. In order to avoid this inconvenient, we introduce the map $\partial^- f : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$\partial^{-} f(x) = \operatorname{argmin} \left\{ ||v||_{2} : v \in \partial f(x) \right\},$$
(55)

where $\partial f(x) \subset \mathbb{R}^n$ is the sub-differential of f at the point x, and $|| \cdot ||_2$ denotes the Euclidean norm. The good definition of the map $\partial^- f$ descends from general properties of convex functions (see, for example, the textbooks [9], [19]). Hence, the first modification consists in replacing ∇f with $\partial^- f$.

The second modification to the original RCM is suggested by physical intuition. Let us imagine that a small massive ball subject to the gravity force is constrained to move on the graph of the function f. The graph is *sharp-shaped* in correspondence of the nondifferentiability points of the function f. If a physical ball crosses these regions, we expect a loss of kinetic energy due to the inelastic collision between the ball and the sharp surface of the graph. Then, for example, we can reset the velocity equal to zero whenever the sequence crosses a non-differentiability region. This intuition can be motivated by the fact that the quantity $\partial^- f$ usually has sudden variation in correspondence of non-differentiability points of f. Hence, when we cross these regions, the information carried by the momentum can be of little use, if not misleading.

From now on, we suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}$ has the form:

$$\Psi(x) = \sum_{i=1}^{n} |x_i|.$$

For this choice of Ψ , we propose in Algorithm 4 a variant of RCM.

Algorithm 4 Restart-Conservative Method for ℓ^1 -composite optimization (RCM-COMPgrad)

1: $x \leftarrow x_0$ 2: $v \leftarrow 0$ 3: while $i < max_iter$ do $x' \leftarrow x - h^2 \partial^- f(x) + hv$ $4 \cdot$ if $\partial^- f(x') \cdot v > 0$ then 5: $x' \leftarrow x - h^2 \partial^- f(x)$ 6: $v \leftarrow -h\partial^- f(x)$ 7: else 8: $v \leftarrow v - h\partial^- f(x')$ 9: end if 10: 11: for j = 1, ..., n do if $x'_i x_j < 0$ then 12: $x'_i \leftarrow 0$ 13: $v \leftarrow 0$ 14: end if 15:end for 16: $x \leftarrow x'$ 17: $i \leftarrow i + 1$ 18:19: end while

In the lines 11–16 of Algorithm 4 we check if the sequence has crossed the set where the function f is not differentiable, i.e., the set $\{x \in \mathbb{R}^n : x_1 \cdots x_n = 0\}$. If it has, we reset the velocity equal to 0. As done for RCM-grad, we can replace the gradient restart criterion at

line 5 with the alternative restart procedures described in Section 4. Similarly as before, we call RCM-COMP-kin, RCM-COMP-mmd-r and RCM-COMP-mmd-dr the methods obtained using the alternative restart criteria. We just recall that, in the case of RCM-mmd-dr, in (40) we need to replace $\nabla f(x_{k+1})$ with $\partial^- f(x_{k+1})$.

For the experiments concerning the ℓ^1 -composite optimization, we use as benchmark the restarted version of FISTA proposed in [15]: as done for the NAG-C, in their paper O'Donoghue and Candès proposed an adaptive restart procedure to accelerate the convergence of FISTA. We refer to this algorithm as FISTA-restart. We recall that FISTA was originally introduced in [3].

5.4. Quadratic with ℓ^1 -regularization. We considered the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{2}x^{T}Ax + b^{T}x + \gamma \sum_{i=1}^{n} |x_{i}|,$$
(56)

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ were constructed as in Subsection 5.1. We set $\gamma = \frac{1}{4} ||b||_{\infty}$, in order to guarantee that the minimizer is not the origin. Let $\lambda_{\rm max}$ be the greatest eigenvalue of A. We minimized (56) using the following algorithms:

- FISTA with step-size $s = \frac{1}{\lambda_{\text{max}}}$;

shows an asymptotic convergence rate very similar to FISTA-restart.

- FISTA with step-size $s = \frac{1}{\lambda_{\text{max}}}$; FISTA-restart with step-size $s = \frac{1}{\lambda_{\text{max}}}$; RCM-COMP-grad with step-size $h = \frac{1}{\sqrt{\lambda_{\text{max}}}}$; RCM-COMP-mmd-dr with step-size $h = \frac{1}{\sqrt{\lambda_{\text{max}}}}$; RCM-COMP-mmd-r with step-size $h = \frac{1}{\sqrt{\lambda_{\text{max}}}}$; RCM-COMP-kin with step-size $h = \frac{1}{\sqrt{\lambda_{\text{max}}}}$;
- The results are shown in Figure 5. We measured the convergence rate by considering the decay of $||\partial^{-}f||$ along the sequences generated by the methods. We observe that in this problem the most performing algorithm is the benchmark FISTA-restart and the worst is the original FISTA without restart. Among the RCM algorithms RCM-mmd-r and RCM-mmd-dr exhibit similar convergence rate while RCM -kin is the slowest. Finally, RCM-COMP-grad

5.5. Logistic with ℓ^1 -regularization. We considered the function $g: \mathbb{R}^n \to \mathbb{R}$ defined as

$$g(x) = \sum_{i=1}^{m} \left((1 - y_i) a_i^T x + \log \left(1 + e^{-a_i^T x} \right) \right),$$

where $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m}$ and $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$ were constructed as in Subsection 5.2. We studied the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = g(x) + \gamma \sum_{i=1}^{n} |x_i|.$$
(57)

We set $\gamma = \frac{1}{2} ||\nabla g(0)||_{\infty}$, in order to guarantee that the minimizer of f is not the origin. Let L be the Lipschitz constant of the function ∇g . We minimized (57) using the following algorithms:

- FISTA with step-size s = ¹/_L;
 FISTA-restart with step-size s = ¹/_L;
- RCM-COMP-grad with step-size $\bar{h} = \frac{1}{\sqrt{L}}$;
- RCM-COMP-mmd-dr with step-size $h = \frac{1}{\sqrt{L}}$;
- RCM-COMP-mmd-r with step-size $h = \frac{1}{\sqrt{L}}$;
- RCM-COMP-kin with step-size $h = \frac{1}{\sqrt{L}}$.



FIGURE 5. Quadratic function with ℓ^1 -regularization. At the top we report the result of a single experiment, at the bottom the average over 100 repetitions of the experiment. The plots at left-hand side shows the decay of $||\partial^- f||$ achieved by RCM-COMP-grad (blue), RCM-COMP-mmd-dr (magenta), FISTA-restart (red), and FISTA (black). At right-hand side we compare the convergence rate of the Restart-Conservative method with different restart schemes. The benchmark method FISTA-restart has the best performances. Among the Restart-Conservative family, RCM-COMP-grad shows the fastest convergence rate.

The results are shown in Figure 6. We measured the convergence rate by considering the decay of $||\partial^- f||$ along the sequences generated by the methods. We recall that in this test the smooth part of the objective function is a non-strongly convex function. We observe that RCM-COMP-grad and RCM-COMP-mmd-r exhibit the best performances while the original FISTA is the worst performing method. In this case RCM-COMP-mmd-dr shows on average performances very close to the benchmark FISTA-restart and both exhibit the same convergence rate.

5.6. LogSumExp with ℓ^1 -regularization. We considered the function $g : \mathbb{R}^n \to \mathbb{R}$ defined as

$$g(x) = \rho \log\left(\sum_{i=1}^{m} \exp\left(\frac{a_i^T x - b_i}{\rho}\right)\right),$$

where $A = (a_1, \ldots, a_m) \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$ were constructed as in Subsection 5.3. We set $\rho = 1$. We studied the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = g(x) + \gamma \sum_{i=1}^{n} |x_i|.$$
 (58)

We set $\gamma = \frac{1}{2} ||\nabla g(0)||_{\infty}$, in order to guarantee that the minimizer of f is not the origin. Let L be the Lipschitz constant of the function ∇g . We minimized (57) using the following algorithms:



FIGURE 6. Logistic with ℓ^1 -regularization. At the top we report the result of a single experiment, at the bottom the average over 100 repetitions of the experiment. The plots at left-hand side shows the decay of $||\partial^{-}f||$ achieved by RCM-COMP-grad (blue), RCM-COMP-mmd-dr (magenta), FISTA-restart (red), and FISTA (black). At right-hand side we compare the convergence rate of the Restart-Conservative method with different restart schemes. RCM-COMP-grad and RCM-COMP-mmd-r (green) show better convergence rate than the benchmark FISTA-restart.

- FISTA with step-size $s = \frac{1}{L}$;
- FISTA-restart with step-size s = 1/L;
 RCM-COMP-grad with step-size h = 1/√L;
- RCM-COMP-mmd-dr with step-size $h = \frac{1}{\sqrt{L}}$;
- RCM-COMP-mmd-r with step-size h = ¹/_{√L};
 RCM-COMP-kin with step-size h = ¹/_{√L}.

The results are shown in Figure 7. We measured the convergence rate by considering the decay of $||\partial^{-} f||$ along the sequences generated by the methods. We observe that RCM-COMP-grad is the most performing method. Moreover on average, all the other RCM methods show performances very close to the benchmark FISTA-restarted.

6. Conclusions

In a series of recent works (see, e.g., [23], [24], [1], [20], [2], [21]) the connection between ODEs with suitable friction term and discrete optimization algorithms with suitable momentum term has been investigated from both theoretical and computational point of view. Conversely, in the present work we investigate optimization method derived from a conservative ODE (i.e., without friction) with suitable restarting criteria.

In the first part of the paper we propose a continuous-time optimization method for convex functions starting from a conservative ODE coupled with an original restart procedure based on the mean dissipation of the kinetic energy. Here, our main contribution consists



FIGURE 7. LogSumExp with ℓ^1 -regularization. At the top we report the result of a single experiment, at the bottom the average over 100 repetitions of the experiment. The plots at left-hand side shows the decay of $||\partial^- f||$ achieved by RCM-COMP-grad (blue), RCM-COMP-mmd-dr (magenta), FISTA-restart (red), and FISTA (black). At right-hand side we compare the convergence rate of the Restart-Conservative method with different restart schemes. RCM-COMP-grad is the most performing and it shows better convergence rate than the benchmark FISTA-restart.

in proving a linear convergence result. Indeed, with our stopping procedure we can prove the boundedness of the restart time, and therefore we can strengthen the linear convergence theorem of [25], where the authors assumed the boundedness in the hypotheses. As well as in [24], where a restarted dissipative dynamics was investigated, our continuous-time method does not require an estimate of the strong convexity parameter.

In the second part, a discrete algorithm is derived (Restart-Conservative Method, RCM) and various discrete restart criteria are considered, some of them proposed in [22] and [25]. A new contribution is the qualitative global convergence result for RCM with gradient restart (RCM-grad). To the best of our knowledge, no global convergence result was available for optimization methods obtained by the discretization of the conservative dynamics.

The numerical tests show that the Restart Conservative methods can effectively compete with the most performing existing algorithms. We used as benchmark the restarted versions of NAG-C and FISTA proposed in [15]. We recall that these methods do not make use of the constant of strong convexity of the objective, and they are suitable both for strongly and non-strongly convex optimization. In the smooth case, in the experiments with non-strongly convex functions, RCM-grad and RCM-mmd-dr have similar performances and they both show a faster convergence rate than NAG-C-restart (see Figures 3, 4). In the non-smooth case, when minimizing a non-strongly convex function with ℓ^1 -regularization, the experiments show that RCM-COMP-grad outperforms FISTA-restart. Moreover, RCM-COMP-mmd-dr shows performances similar to FISTA-restart (see Figures 6, 7).

Acknowledgments. The Authors want to thank Prof. G. Savaré for the helpful suggestions and discussions.

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Appendix A. Proof of Proposition 2.1

Proof. Let us prove that the function $t \mapsto E_K(t)$ has a local maximum in $[0, +\infty)$. By contradiction, if $t \mapsto E_K(t)$ has no local maxima, then $t \mapsto E_K(t)$ is injective (otherwise we can apply twice Weierstrass Theorem and we can find a local maximum). Since $t \mapsto E_K(t)$ is continuous, it has to be strictly increasing. This implies that $t \mapsto \dot{x}(t)$ can not change sign and hence that it is monotone as well. Moreover, it follows that $t \mapsto x(t)$ is monotone as well. Since both x(t) and $\dot{x}(t)$ remain bounded for every $t \in [0, +\infty)$, there exist $x_{\infty}, v_{\infty} \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} x(t) = x_{\infty} \text{ and } \lim_{t \to +\infty} \dot{x}(t) = v_{\infty}$$

On the other hand, v_{∞} should be zero, and this is a contradiction.

Let \overline{t} be a point of local maximum for the kinetic energy function $t \mapsto E_K(t)$. This implies that $|\dot{x}(\overline{t})| > 0$. The conservation of the total mechanical energy ensures that the function $t \mapsto f(x(t))$ attains a local minimum at \overline{t} . Using the Implicit Function Theorem, we obtain that $t \mapsto x(t)$ is a local homeomorphism around \overline{t} . This implies that $x(\overline{t})$ is a point of local minimum for f.

Appendix B. Proof of Proposition 2.2

Proof. Without loss of generality, we can assume that $x^* = 0$ and that $x_0 > 0$. We define a strongly convex function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) := \frac{1}{2}\mu|x - x^*|^2 = \frac{1}{2}\mu|x|^2.$$

We claim that, for every $y \in [0, x_0]$, the following inequality is satisfied:

$$f(x_0) - f(y) \ge g(x_0) - g(y).$$
(59)

Indeed, we have that

$$f(x_0) - g(x_0) = f(y) - g(y) + \int_y^{x_0} (f'(u) - g'(u)) \, du \ge f(y) - g(y),$$

since $f'(u) - g'(u) \ge 0$ for every $u \ge 0$. Combining (5) and (59) we obtain:

$$t_1 = \int_0^{x_0} \frac{1}{\sqrt{2(f(x_0) - f(y))}} dy \le \int_0^{x_0} \frac{1}{\sqrt{2(g(x_0) - g(y))}} dy$$
$$= \int_0^{x_0} \frac{1}{\sqrt{\mu(x_0^2 - y^2)}} dy = \frac{\pi}{2\sqrt{\mu}}$$

This completes the proof.

Appendix C. Proof of Lemma 2.5

Proof. Up to a linear orthonormal change of coordinates, we can assume that the function f is of the form

$$f(x) = \sum_{i=1}^{n} \lambda_i \frac{x_i^2}{2}$$

Hence, the differential system (7) becomes

$$\begin{cases} \ddot{x}_1 + \lambda_1 x_1 = 0, \\ \vdots \\ \ddot{x}_n + \lambda_n x_n = 0, \end{cases}$$

i.e., the components evolve independently one of each other. If the Cauchy datum is

$$x(0) = (x_{1,0}, \dots, x_{n,0})$$
 and $\dot{x}(0) = 0$,

then we can compute the expression of the kinetic energy function $E_K: t \mapsto \frac{1}{2} |\dot{x}(t)|^2$:

$$E_K(t) = \sum_{i=1}^n \lambda_i \frac{x_{i,0}^2}{2} \sin^2(\sqrt{\lambda_i}t).$$

For every $0 \le t \le \frac{\pi}{2\sqrt{\lambda_n}}$, we have that

$$0 \le \sin(\sqrt{\lambda_1}t) \le \ldots \le \sin(\sqrt{\lambda_n}t),$$

and then we deduce that

$$E_K(t) \ge \left(\sum_{i=1}^n \lambda_i \frac{x_{i,0}^2}{2}\right) \sin^2(\sqrt{\lambda_1}t),$$

for every $t \in [0, \pi/(2\sqrt{\lambda_n})]$. Evaluating the last inequality for $t = \frac{\pi}{2\sqrt{\lambda_n}}$ and using the conservation of the energy, we obtain the thesis.

References

- H. Attouch, J. Peypouquet, P. Redont: Fast convex optimization via inertial dynamics with Hessian driven damping. *Journal of Differential Equations*, 261:5734–5783, 2016.
- [2] H. Attouch, Z. Chbani, J. Peypouquet, P. Redont: Fast convergence of inertial dynamics and algorithms with asymptotic vanishing viscosity. *Mathematical Programming*, 168:123–175, 2018.
- [3] A. Beck, M. Teboulle: A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM Journal on Imaging Sciences, 2:183–202, 2009.
- [4] A. Beck Introduction to nonlinear optimization : theory, algorithms, and applications with MATLAB. SIAM, Philadelphia, 2014.
- [5] S. Boyd, L. Vandenberghe: Convex optimization. Cambridge University Press, 2004.
- [6] O. Fercoq, Z. Qu: Adaptive restart of accelerated gradient methods under local quadratic growth condition. IMA Journal of Numerical Analysis, 39(4):2069–2095, 2019.
- [7] O. Fercoq, Z. Qu: Restarting the accelerated coordinate descent method with a rough strong convexity estimate. *Computational Optimization and Applications*, 75:63–91, 2020.
- [8] E. Hairer, C. Lubic, G. Wanner: Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations. Springer-Verlag Berlin Heidelberg, 2006.
- [9] J.-B. Hiriart-Urruty, C. Lemaréchal: Fundamentals of convex analysis. Springer Science and Business Media, 2012.
- [10] D. Kim, J.A. Fessler: On the convergence analysis of the Optimized Gradient Method. Journal of Optimization Theory and Applications, 172:187–205, 2017.
- [11] D. Kim, J.A. Fessler: Adaptive restart of the Optimized Gradient Method for convex optimization. Journal of Optimization Theory and Applications, 178:240–263, 2018.
- [12] Y. Nesterov: A method of solving a convex programming problem with convergence rate O(1/k²). Soviet Mathematics Doklady, 27:372–376, 1983.
- [13] Y. Nesterov: Gradient methods for minimizing composite functions. *Mathematical Programming*, 140:125–161, 2013.
- [14] Y. Nesterov: Lectures on Convex Optimization. Springer International Publishing, 2018.
- [15] B. O'Donoghue, E. Candès: Adaptive restart for accelerated gradient schemes. Foundations of Computational Mathematics, 15(3):715-732, 2015.
- [16] B.T. Polyak: Gradient method for the minimization of functionals. USSR Computational Mathematics and Mathematical Physics, 3(4):864–878, 1963.
- [17] B.T. Polyak: Some methods of speeding up the convergence of iteration methods. USSR Computational Mathematics and Mathematical Physics, 4(5):1–17, 1964.
- [18] B.T. Polyak: Introduction to optimization. Optimization Software, 1987.
- [19] R.T. Rockafellar: Convex Analysis. Princeton University Press, 1997.
- [20] B. Shi, S.S. Du, M.I. Jordan, W.J. Su: Understanding the acceleration phenomenon via high-resolution differential equations. arXiv preprint, arXiv:1810.08907, 2018.
- [21] B. Shi, S.S. Du, M.I. Jordan, W.J. Su: Acceleration via symplectic discretization of high-resolution differential equations Advances in Neural Information Processing Systems, 32:5744–5752, 2019.
- [22] B. Shi, S.S Iyengar: Mathematical Theories of Machine Learning Theory and Applications; Ch. 8, 63-85. Springer Nature Switzerland AG, 2020, and arXiv:1708.08035v3.
- [23] W.J. Su, S. Boyd, E. Candès: A differential equation for modeling Nesterov's accelerated gradient method: theory and insights. Advances in Neural Information Processing Systems: 2510–2518, 2014.
- [24] W.J. Su, S. Boyd, E. Candès: A differential equation for modeling Nesterov's accelerated gradient method: theory and insights. *Journal of Machine Learning Research*, 17(153):1–43, 2016.
- [25] A.R. Teel, J.I. Poveda, J. Le: First-order optimization algorithms with resets and Hamiltonian flows. 2019 IEEE 58th Conference on Decision and Control (CDC): 5838–5843, 2019.

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