Distributed Stochastic Compositional Optimization Problems over Directed Networks

Shengchao Zhao and Yongchao Liu*

Abstract. We study the distributed stochastic compositional optimization problems over directed communication networks in which agents privately own a stochastic compositional objective function and collaborate to minimize the sum of all objective functions. We propose a distributed stochastic compositional gradient descent method, where the gradient tracking and the stochastic correction techniques are employed to adapt to the networks' directed structure and increase the accuracy of inner function estimation. When the objective function is smooth, the proposed method achieves the convergence rate $\mathcal{O}(k^{-1/2})$ and sample complexity $\mathcal{O}(\frac{1}{\epsilon^2})$ for finding the (ϵ) -stationary point. When the objective function is strongly convex, the convergence rate is improved to $\mathcal{O}(k^{-1})$. Moreover, the asymptotic normality of Polyak-Ruppert averaged iterates of the proposed method is also presented. We demonstrate the empirical performance of the proposed method on model-agnostic meta-learning problem and logistic regression problem.

Key words. distributed stochastic compositional optimization, directed communication networks, gradient tracking, asymptotic normality

1 Introduction

Stochastic compositional optimization problem (SCO) has received extensive attention recently for its application in machine learning, stochastic programming and financial engineering, etc, [2, 12, 21, 26, 27], which is in the form of

$$\min_{x \in \mathbb{P}^d} f(g(x)),\tag{1}$$

where $f(g(x)) = (f \circ g)(x)$ denotes the function composition, $f(z) := \mathbb{E}[F(z;\zeta)], g(x) := \mathbb{E}[G(x;\phi)], G(\cdot;\phi) : \mathbb{R}^d \to \mathbb{R}^p$ and $F(\cdot;\zeta) : \mathbb{R}^p \to \mathbb{R}$ are measurable functions parameterized by random variables ζ and ϕ respectively.

To solve the stochastic compositional optimization problem (1), one may employ the two sample based popular schemes in stochastic optimization, sample average approximation (SAA) and stochastic approximation (SA). For the SAA scheme, Dentcheva et al. [5] discuss the asymptotic behavior of the SAA problem and establish the central limit theorem for the optimal

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value. Ermoliev and Norkin [7] study the conditions for convergence in mean, almost surely of SAA problem and provide the large deviation bounds for the optimal values. The SA based method for SCO can be traced back to 1970s [6], in which penalty functions for stochastic constraints and composite regression models are considered. More recently, Wang et al. [26] present the stochastic compositional gradient descent method (SCGD) for problem (1), which is defined as follows

$$z_{k+1} = (1 - \beta_k) z_k + \beta_k G(x_k; \phi_k),$$

$$x_{k+1} = x_k - \alpha_k \nabla G(x_k; \phi_k) \nabla F(z_{k+1}; \zeta_k),$$
(2)

where stepsize α_k diminishes to zero at a faster rate than β_k , iterates z_{k+1} and x_k are the estimations of inner function value $g(x_k)$ and decision variable respectively. For nonsmooth convex problems, the SCGD [26] achieves a convergence rate of $\mathcal{O}(k^{-1/4})$ in the general case and $\mathcal{O}(k^{-2/3})$ in the strongly convex case. An accelerated variant of SCGD with improved convergence rate has been presented in [27], where an extrapolation-smoothing scheme is introduced. Moreover, the variance reduction techniques, such as SVRG and SARAH, have been merged into the compositional optimization framework [14, 16, 30]. Note the fact that the twotimescale structure of SCGD may decrease the convergence rate, Ghadimi et al. [11] propose a nested averaged stochastic approximation method to solve SCO (1), which is a single-timescale method and achieves the optimal convergence rate $\mathcal{O}(k^{-1/2})$ as methods for one-level unconstrained stochastic optimization. Chen et al. [2] propose the stochastically corrected stochastic compositional gradient method (SCSC), which is also a single-timescale method and achieves the optimal convergence rate $\mathcal{O}(k^{-1/2})$. Specially, the SCSC read as follows:

$$z_{k+1} = (1 - \beta_k)(z_k + G(x_k; \phi_k) - G(x_{k-1}; \phi_k)) + \beta_k G(x_k; \phi_k),$$

$$x_{k+1} = x_k - \alpha_k \nabla G(x_k; \phi_k) \nabla F(z_{k+1}; \zeta_k),$$

where stepsize α_k does not have to decay to zero at a faster rate than β_k . Compared with SCGD, SCSC adds an extra term $G(x_k; \phi_k) - G(x_{k-1}; \phi_k)$ in recursion (2), which may reduce the tracking variance of $g(x_k)$. We refer [1, 15, 24, 29, 30] for the new developments on multilevel compositional optimization and [4, 13] on conditional stochastic optimization.

Note that the machine learning and financial engineering problems tend to be characterized by large scale or distributed storage of data, consequently it is necessary to study the distributed stochastic compositional optimization problems. Gao and Huang [10] first consider the distributed stochastic compositional optimization problem (DSCO)

$$\min_{x \in \mathbb{R}^d} h(x) := \frac{1}{n} \sum_{j=1}^n f_j(g_j(x)),$$
(3)

where $f_j(g_j(x)) := \mathbb{E}[F_j(\mathbb{E}[G_j(x; \phi_j)]; \zeta_j)]$ is the local objective of agent j. Under the assumption that each agent only knows its own local objective function, Gao and Huang [10] propose a distributed stochastic compositional gradient descent method for problem (3), which is named GP-DSCGD:

$$z_{i,k+1} = (1 - \gamma \beta_k) z_{i,k} + \gamma \beta_k G_i(x_{i,k}; \phi_{i,k}),$$
(4)

$$\tilde{x}_{i,k+1} = \sum_{j=1}^{n} w_{i,j} x_{j,k} - \eta \nabla G_i(x_{i,k}; \phi_{i,k}) \nabla F(z_{i,k+1}; \zeta_{i,k}),$$
(5)

$$x_{i,k+1} = x_{i,k} + \beta_k (\tilde{x}_{i,k+1} - x_{i,k}), \tag{6}$$

where parameters $\gamma > 0, \beta_k > 0, \eta > 0$ are stepszie parameters and $W = \{w_{ij}\}$ is a symmetric and doubly stochastic matrix. Different from the SCGD in [26], there is an additional hyperparameter γ in computing $z_{i,k+1}$ (4), which is helpful to control the estimation variance of $z_{i,k+1}$. On the other hand, (6) can also be beneficial to control the estimation variance of $z_{i,k+1}$. When the objective function is smooth and the communication network is undirected, the proposed method achieves the optimal convergence rate $\mathcal{O}(k^{-1/2})$. Moreover, a gradient-tracking version of GP-DSCGD, named GT-DSCGD, is also proposed in [10], where the local gradient $\nabla G_i(x_{i,k}; \phi_{i,k}) \nabla F(z_{i,k+1}; \zeta_{i,k})$ in (5) is replaced with the global gradient tracker. As the two methods need the increasing batch size $\mathcal{O}(\sqrt{k})$, the corresponding sample complexity for finding the (ϵ) -stationary point is $\mathcal{O}(\frac{1}{\epsilon^3})$.

In this paper, we consider the distributed stochastic problem (3) over directed communication networks. We propose a gradient-tracking based distributed stochastic method, which incorporates the SCSC method [2] into the AB/push-pull scheme [19, 28]. The collaboration of AB scheme and SCSC induces more complex estimate errors, for example, the estimate error of inner function values is involved in the gradient tracking process of AB scheme, the errors of tracked gradient affect the iterations and then the inner function values. Therefore, the convergence analysis techniques of AB/push-pull scheme [19, 28] and SCSC are not applicable. Moreover, the techniques used in [10] are not applicable as the induced estimate error forms non-martingale-difference. As far as we are concerned, the contributions of the paper can be summarized as follows.

- We propose a distributed stochastic optimization method for DSCO over directed communication networks. To the best of our knowledge, it is the first one for distributed stochastic compositional optimization problem (3) over directed communication networks.
 (i) For the nonconvex smooth objective, it achieves the same order of convergence rate O (k^{-1/2}) as GP-DSCGD and GT-DSCGD in [10] under constant stepsize strategy. However, the sample complexity for finding the (ε)-stationary point is O (¹/_{ε²}) as the proposed method does not require increasing batch size in each iteration. (ii) For the strongly convex and smooth objective, we show that the square of the distance between the iterate and the optimal solution converges to zero with rate O (k⁻¹) under diminishing stepsize strategy, which is the optimal convergence rate as methods for one-level unconstrained stochastic optimization [23].
- We present that Polyak-Ruppert averaged iterates of the proposed method converge in distribution to a normal random vector for any agent. Research on asymptotic normality results for the SA based algorithm can be traced to the works in the 1950s [3, 8]. To the best of our knowledge, our result is the first asymptotic normality result for the stochastic approximation based method of DSCO. On the other hand, it is a complement to the asymptotic normality on the SAA scheme for stochastic compositional optimization [5]. We verify our theoretical results using two numerical examples, model-agnostic meta-learning problem and logistic regression problem.

The rest of this paper is organized as follows. Section 2 introduces the proposed method and some standard assumptions for DSCO, communication graphs and weighted matrices. Section 3 focuses on the convergence analysis of the proposed method. At last, Section 4 presents numerical results to validate the theoretic results.

Throughout this paper, we use the following notation. \mathbb{R}^d denotes the d-dimension Euclidean space endowed with norm $||x|| = \sqrt{\langle x, x \rangle}$. Denote $\mathbf{1} := (1 \ 1 \dots 1)^{\mathsf{T}} \in \mathbb{R}^n$, $\mathbf{0} := (0 \ 0 \dots 0)^{\mathsf{T}} \in \mathbb{R}^d$. $\mathbf{I}_d \in \mathbb{R}^{d \times d}$ stands for the identity matrix. $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product of matrix \mathbf{A} and \mathbf{B} . For any positive sequences $\{a_k\}$ and $\{b_k\}$, $a_k = \mathcal{O}(b_k)$ if there exists c > 0 such that $a_k \leq cb_k$. The communication relationship between agents is characterized by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. For any $i \in \mathcal{V}$, P_{ϕ_i} and P_{ζ_i} are distributions of random variables ϕ_i and ζ_i respectively.

2 AB-DSCSC Method

In this section, we propose a gradient tracking based distributed stochastically corrected stochastic compositional gradient method for DSCO over directed communication networks.

Algorithm 1 AB/push-pull based Distributed Stochastically Corrected Stochastic Compositional Gradient (AB-DSCSC):

Require: initial values $x_{i,1} \in \mathbb{R}^d$, $z_{i,1} \in \mathbb{R}^p$, $\phi_{i,1} \stackrel{iid}{\sim} P_{\phi_i}$, $\zeta_{i,1} \stackrel{iid}{\sim} P_{\zeta_i}$, $y_{i,1} = \nabla G_i(x_{i,1}; \phi_{i,1}) \nabla F_i(z_{i,1}; \zeta_{i,1})$ for any $i \in \mathcal{V}$; stepsizes $\alpha_k > 0$, $\beta_k > 0$; nonnegative weight matrices $\mathbf{A} = \{a_{ij}\}_{1 \leq i,j \leq n}$ and $\mathbf{B} = \{b_{ij}\}_{1 \leq i,j \leq n}$.

- 1: For $k = 1, 2, \cdots$ do
- 2: State update: for any $i \in \mathcal{V}$,

$$x_{i,k+1} = \sum_{j=1}^{n} a_{ij} \left(x_{j,k} - \alpha_k y_{j,k} \right).$$
(7)

3: Inner function value tracking update: for any $i \in \mathcal{V}$, draw $\phi'_{i,k+1} \stackrel{iid}{\sim} P_{\phi_i}$ to compute

$$z_{i,k+1} = (1 - \beta_k) \left(z_{i,k} + G_i(x_{i,k+1}; \phi'_{i,k+1}) - G_i(x_{i,k}; \phi'_{i,k+1}) \right) + \beta_k G_i(x_{i,k+1}; \phi'_{i,k+1}).$$
(8)

4: Gradient tracking update: for any $i \in \mathcal{V}$, draw $\phi_{i,k+1} \stackrel{iid}{\sim} P_{\phi_i}$, $\zeta_{i,k+1} \stackrel{iid}{\sim} P_{\zeta_i}$ to compute

$$y_{i,k+1} = \sum_{j=1}^{n} b_{ij} y_{j,k} + \nabla G_i(x_{i,k+1}; \phi_{i,k+1}) \nabla F_i(z_{i,k+1}; \zeta_{i,k+1}) - \nabla G_i(x_{i,k}; \phi_{i,k}) \nabla F_i(z_{i,k}; \zeta_{i,k}).$$
(9)

4: **end for**

Throughout our analysis in the paper, we make the following two assumptions on the objective function, communication graphs and weight matrices A and B.

Assumption 1. [Objective function] Let C_g, C_f, V_g, L_g and L_f be positive scalars. For $\forall i \in \mathcal{V}, \forall x, x' \in \mathbb{R}^d, \forall y, y' \in \mathbb{R}^p$,

(a) functions $G_i(\cdot; \phi_i)$ and $F_i(\cdot; \zeta_i)$ are L_g and L_f smooth, that is,

$$\|\nabla G_i(x;\phi_i) - \nabla G_i(x';\phi_i)\| \le L_g \|x - x'\|,$$

and

$$\left\|\nabla F_{i}(y;\zeta_{i}) - \nabla F_{i}(y';\zeta_{i})\right\| \leq L_{f} \left\|y - y'\right\|_{2}$$

(b)

$$\mathbb{E}\left[\nabla G_{i}(x;\phi_{i,1})\nabla F_{i}(y;\zeta_{i,1})\right] = \nabla g_{i}(x)\nabla f_{i}(y), \quad \mathbb{E}\left[G_{i}(x;\phi_{i,1}')\right] = g_{i}(x);$$

(c) the stochastic gradients of f_i and g_i are bounded in expectation, that is

$$\mathbb{E}\left[\left\|\nabla G_i(x;\phi_i)\right\|^2 |\zeta_i\right] \le C_g, \quad \mathbb{E}\left[\left\|\nabla F_i(y;\zeta_i)\right\|^2\right] \le C_f;$$

(d) function $G(\cdot; \phi_i)$ has bounded variance, i.e., $\mathbb{E}\left[\|G_i(x; \phi_i) - g_i(x)\|^2\right] \leq V_g;$

Assumption 1 is standard assumption for stochastic compositional optimization problem [2, 10, 26]. Conditions (b) and (d) in Assumption 1 are analogous to the unbiasedness and bounded variance assumptions for non-compositional stochastic optimization problems.

Assumption 2. [weight matrices and networks] Let $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}_A)$ and $\mathcal{G}_{B^{\intercal}} = (\mathcal{V}, \mathcal{E}_{B^{\intercal}})$ be subgraphs of \mathcal{G} induced by matrices A and B^{\intercal} respectively.

- (a) The matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is nonnegative row stochastic and $\mathbf{B} \in \mathbb{R}^{n \times n}$ is nonnegative column stochastic, i.e., $\mathbf{A}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^{\mathsf{T}}\mathbf{B} = \mathbf{1}^{\mathsf{T}}$. In addition, the diagonal entries of \mathbf{A} and \mathbf{B} are positive, i.e., $\mathbf{A}_{ii} > 0$ and $\mathbf{B}_{ii} > 0$ for all $i \in \mathcal{V}$.
- (b) The graphs \mathcal{G}_A and $\mathcal{G}_{B^{\intercal}}$ each contain at least one spanning tree. Moreover, there exists at least one node that is a root of spanning trees for both \mathcal{G}_A and $\mathcal{G}_{B^{\intercal}}$, i.e. $\mathcal{R}_A \cap \mathcal{R}_{B^{\intercal}} \neq \emptyset$, where $\mathcal{R}_A (\mathcal{R}_{B^{\intercal}})$ is the set of roots of all possible spanning trees in the graph $\mathcal{G}_A (\mathcal{G}_{B^{\intercal}})$.

It is worth noting that Assumption 2 (b) is weaker than requiring that both \mathcal{G}_A and $\mathcal{G}_{B^{\mathsf{T}}}$ are strongly connected, which offers greater flexibility in the design of \mathcal{G}_A and \mathcal{G}_B [20]. Under Assumption 2, the matrix **A** has a nonnegative left eigenvector \mathbf{u}^{T} (w.r.t. eigenvalue 1) with $\mathbf{u}^{\mathsf{T}}\mathbf{1} = n$, and the matrix **B** has a nonnegative right eigenvector \mathbf{v} (w.r.t. eigenvalue 1) with $\mathbf{1}^T\mathbf{v} = n$. Moreover, $\mathbf{u}^{\mathsf{T}}\mathbf{v} > 0$.

For easy of presentation, we rewrite AB-DSCSC ((7)-(8)) in a compact form:

$$\mathbf{x}_{k+1} = \tilde{\mathbf{A}} \left(\mathbf{x}_k - \alpha_k \mathbf{y}_k \right),$$

$$\mathbf{z}_{k+1} = (1 - \beta_k) \left(\mathbf{z}_k + \mathbf{G}_{k+1}^{(1)} - \mathbf{G}_{k+1}^{(2)} \right) + \beta_k \mathbf{G}_{k+1}^{(1)},$$

$$\mathbf{y}_{k+1} = \tilde{\mathbf{B}} \mathbf{y}_k + \mathbf{H}_{k+1} - \mathbf{H}_k,$$
(10)

where $\tilde{\mathbf{A}} := \mathbf{A} \otimes \mathbf{I}_d$, $\tilde{\mathbf{B}} := \mathbf{B} \otimes \mathbf{I}_d$, the vectors \mathbf{x}_k , \mathbf{y}_k , \mathbf{z}_k , $\mathbf{G}_{k+1}^{(1)}$, $\mathbf{G}_{k+1}^{(2)}$ and \mathbf{H}_k concatenate all $x_{i,k}$'s, $y_{i,k}$'s, $z_{i,k}$'s, $G_i(x_{i,k+1}, \phi'_{i,k+1})$'s, $G_i(x_{i,k}, \phi'_{i,k+1})$'s and $\nabla G_i(x_{i,k}; \phi_{i,k}) \nabla F_i(z_{i,k}; \zeta_{i,k})$'s respectively.

3 Convergence analysis

In this section, we derive convergence rates of AB-DSCSC. We also investigate the asymptotic normality of AB-DSCSC when the objective function is strongly convex. We first present a technical lemma which provides some norms for studying the consensus of AB-DSCSC.

Lemma 1. Under Assumption 2, there exist vector norms, denoted as $\|\cdot\|_{\mathbf{A}}$, $\|\cdot\|_{\mathbf{B}}$, on \mathbb{R}^{nd} such that the corresponding induced matrix norms $\|\|\mathbf{W}\|\|_{\mathbf{A}} := \sup_{\mathbf{x}\neq 0} \frac{\|\hat{\mathbf{W}}\mathbf{x}\|_{\mathbf{A}}}{\|\|\mathbf{x}\|_{\mathbf{A}}}$, $\|\|\mathbf{W}\|\|_{\mathbf{B}} := \sup_{\mathbf{x}\neq 0} \frac{\|\hat{\mathbf{W}}\mathbf{x}\|_{\mathbf{A}}}{\|\|\mathbf{x}\|_{\mathbf{B}}}$ for $\mathbf{W} \in \mathbb{R}^{nd \times nd}$ satisfy:

$$\left\| \tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{A}} < 1, \quad \left\| \tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}} < 1.$$
(11)

Additionally, let $\|\cdot\|_*$ and $\|\cdot\|_{**}$ be any two vector norms of $\|\cdot\|$, $\|\cdot\|_{\mathbf{A}}$ or $\|\cdot\|_{\mathbf{B}}$. There exists a constant $\overline{c} > 1$ such that

$$\|\mathbf{x}\|_* \le \overline{c} \|\mathbf{x}\|_{**}, \quad \forall \mathbf{x} \in \mathbb{R}^{nd}.$$
 (12)

Proof. Under Assumption 2, the conditions of [25, Lemma 3] hold and then there exists an invertible matrix $\mathbf{A}_* \in \mathbb{R}^{d \times d}$ such that

$$\left\| \left\| \mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right\|_{*} = \left\| \left\| \mathbf{A}_{*} \left(\mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right) \mathbf{A}_{*}^{-1} \right\| \right\| < 1,$$

where $\|\|\cdot\|\|_*$ and $\|\|\cdot\|\|$ are matrix norms induced by vector norms $\|x\|_* := \|\mathbf{A}_*x\|$ and 2-norm respectively. Let $\hat{\mathbf{A}} = \mathbf{A}_* \otimes \mathbf{I}_d$. Noting that $(\mathbf{W}_1 \otimes \mathbf{W}_2)^{-1} = \mathbf{W}_1^{-1} \otimes \mathbf{W}_2^{-1}$ for any invertible matrices $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{R}^{nd \times nd}$, $\hat{\mathbf{A}}^{-1} = \mathbf{A}_*^{-1} \otimes \mathbf{I}_d$. Therefore, vector matrix $\|\mathbf{x}\|_{\mathbf{A}} := \|\hat{\mathbf{A}}\mathbf{x}\|$ is well defined and the corresponding induced matrix norm $\|\|\cdot\|\|_{\mathbf{A}}$ satisfies

$$\begin{split} \left\| \left\| \tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\| \right\|_{\mathbf{A}} &= \left\| \left\| \hat{\mathbf{A}} \left(\tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right) \hat{\mathbf{A}}^{-1} \right\| \\ &= \left\| \left\| \left[\mathbf{A}_{*} \left(\mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right) \mathbf{A}_{*}^{-1} \right] \otimes \mathbf{I}_{d} \right\| \\ &= \left\| \left\| \mathbf{A}_{*} \left(\mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right) \mathbf{A}_{*}^{-1} \right\| \right\| < 1. \end{split}$$

By the similar analysis, there exists $\hat{\mathbf{B}}$ such that

$$\left\| \tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}} = \left\| \hat{\mathbf{B}} \left(\tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right) \hat{\mathbf{B}}^{-1} \right\| < 1.$$

The inequality (12) follows from the equivalence relation of all norms on \mathbb{R}^d . The proof is complete.

The next lemma studies the asymptotic consensus of AB-DSCSC.

Lemma 2. Suppose Assumptions 1-2 hold. Stepsize α_k is nonincreasing and $\lim_{k\to\infty} \frac{\alpha_k}{\alpha_{k+1}} = 1$. Define auxiliary sequence $\{\mathbf{y}'_k\}$ as

$$\mathbf{y}_{k+1}^{'} = \tilde{\mathbf{B}}\mathbf{y}_{k}^{'} + \mathbf{J}_{k+1} - \mathbf{J}_{k}, \tag{13}$$

where vectors \mathbf{J}_k and \mathbf{y}'_1 concatenate all $\nabla g_i(x_{i,k}) \nabla f_i(z_{i,k})$'s and $\nabla g_i(x_{i,1}) \nabla f_i(z_{i,1})$'s respectively. Then

$$\mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{1} \otimes \bar{x}_{k+1}\|_{\mathbf{A}}^{2}\right] + c_{4}\mathbb{E}\left[\|\mathbf{y}_{k+1}^{'} - \mathbf{v} \otimes \bar{y}_{k+1}^{'}\|_{\mathbf{B}}^{2}\right]$$

$$\leq \rho^{k} \left(\mathbb{E}\left[\|\mathbf{x}_{1} - \mathbf{1} \otimes \bar{x}_{1}\|_{\mathbf{A}}^{2}\right] + c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{1}^{'} - \mathbf{v} \otimes \bar{y}_{1}^{'}\right\|_{\mathbf{B}}^{2}\right]\right) + (c_{1} + c_{3}c_{4})\sum_{t=1}^{k} \rho^{k-t}\alpha_{t}^{2}, \quad (14)$$

where $\bar{x}_k := \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \mathbf{x}_k, \ \bar{y}'_k := \left(\frac{\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \mathbf{y}'_k \ and \ \rho = \max\left\{\frac{1+\tau_{\mathbf{B}}^2}{2}, \ \frac{3+\tau_{\mathbf{A}}^2}{4}\right\}, \ c_b = \max\left\{\bar{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\bar{c}\right\},$

$$c_{1} = \frac{1 + \tau_{\mathbf{A}}^{2}}{1 - \tau_{\mathbf{A}}^{2}} \left\| \mathbf{I}_{n} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right\|_{\mathbf{A}}^{2} \overline{c}^{2} \frac{c_{b}^{2} n C_{g} C_{f}}{(1 - \tau_{\mathbf{B}})^{2}}$$

$$c_{2} = 8 \frac{1 + \tau_{\mathbf{B}}^{2}}{1 - \tau_{\mathbf{B}}^{2}} \left\| \mathbf{I}_{nd} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}}^{2} \overline{c}^{4} \left(C_{f} L_{g}^{2} + C_{g} L_{f}^{2} \right) \left\| \mathbf{A} - \mathbf{I}_{nd} \right\|_{2}^{2},$$

$$c_{3} = 8 \frac{1 + \tau_{\mathbf{B}}^{2}}{1 - \tau_{\mathbf{B}}^{2}} \left\| \mathbf{I}_{nd} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}}^{2} \overline{c}^{2} \left(C_{f} L_{g}^{2} + C_{g} L_{f}^{2} \right) \frac{c_{b}^{2} n C_{g} C_{f}}{(1 - \tau_{\mathbf{B}})^{2}}, \quad c_{4} = \frac{1 - \tau_{\mathbf{A}}^{2}}{4c_{2}},$$

$$\tau_{\mathbf{A}} := \left\| \mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{A}}, \quad \tau_{\mathbf{B}} := \left\| \mathbf{B} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}}, \quad (15)$$

u and **v** are the left eigenvector of **A** and the right eigenvector of **B** respectively, vector norms $\|\cdot\|_{\mathbf{A}}$, $\|\cdot\|_{\mathbf{B}}$ and matrix norms $\|\cdot\|_{\mathbf{A}}$, $\|\cdot\|_{\mathbf{B}}$ are introduced in Lemma 1.

Proof. We first provide the upper bound of consensus error $\mathbf{x}_{k+1} - \mathbf{1} \otimes \bar{x}_{k+1}$ in the mean square sense. Note that for any random vectors θ , θ' and positive scalar τ ,

$$\mathbb{E}\left[\left\|\theta + \theta'\right\|_{*}^{2}\right] \leq (1+\tau)\mathbb{E}\left[\left\|\theta\right\|_{*}^{2}\right] + \left(1 + \frac{1}{\tau}\right)\mathbb{E}\left[\left\|\theta'\right\|_{*}^{2}\right],\tag{16}$$

where the norm $\|\cdot\|_*$ may be $\|\cdot\|_{\mathbf{A}}$ or $\|\cdot\|_{\mathbf{B}}$. Choosing

$$\theta = \left(\tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \left(\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k}\right), \quad \theta' = -\alpha_{k} \left(\tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \mathbf{y}_{k},$$

we have $\mathbf{x}_{k+1} - \mathbf{1} \otimes \bar{x}_{k+1} = \theta + \theta'$ and

$$\mathbb{E}\left[\left\|\mathbf{x}_{k+1} - \mathbf{1} \otimes \bar{x}_{k+1}\right\|_{\mathbf{A}}^{2}\right] \\
\leq (1+\tau)\mathbb{E}\left[\left\|\left(\tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right)(\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k})\right\|_{\mathbf{A}}^{2}\right] + \left(1 + \frac{1}{\tau}\right)\mathbb{E}\left[\left\|\alpha_{k}\left(\tilde{\mathbf{A}} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right)\mathbf{y}_{k}\right\|_{\mathbf{A}}^{2}\right] \\
\leq \frac{1+\tau_{\mathbf{A}}^{2}}{2}\mathbb{E}\left[\left\|\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k}\right\|_{\mathbf{A}}^{2}\right] + \alpha_{k}^{2}\frac{1+\tau_{\mathbf{A}}^{2}}{1-\tau_{\mathbf{A}}^{2}}\left\|\left\|\mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n}\right\|_{\mathbf{A}}^{2}\bar{c}^{2}\mathbb{E}\left[\left\|\mathbf{y}_{k}\right\|^{2}\right], \tag{17}$$

where $\tau_{\mathbf{A}}$ is defined in (15), $\tau = (1 - \tau_{\mathbf{A}}^2)/(2\tau_{\mathbf{A}}^2)$ and the last inequality follows from the fact (12). By the definition of \mathbf{y}_k in (10),

$$\mathbb{E}\left[\left\|\mathbf{y}_{k}\right\|^{2}\right] = \mathbb{E}\left[\left\|\sum_{t=1}^{k-1} \tilde{\mathbf{B}}^{k-1-t}(\tilde{\mathbf{B}} - \mathbf{I}_{nd})\mathbf{H}_{t} + \mathbf{H}_{k}\right\|^{2}\right] \leq \sum_{t_{1}=1}^{k} \sum_{t_{2}=1}^{k} \left\|\left\|\tilde{\mathbf{B}}(k, t_{1})\right\|\right\|\left\|\tilde{\mathbf{B}}(k, t_{2})\right\|\right\| \mathbb{E}\left[\left\|\mathbf{H}_{t_{1}}\right\|\left\|\mathbf{H}_{t_{2}}\right\|\right],$$
where

where

$$\tilde{\mathbf{B}}(k,t) := \tilde{\mathbf{B}}^{k-1-t} (\tilde{\mathbf{B}} - \mathbf{I}_{nd}) \ (t \le k-1), \quad \tilde{\mathbf{B}}(k,k) := \mathbf{I}_{nd}.$$
(18)

Obviously, $\left\| \left\| \tilde{\mathbf{B}}(k,k) \right\| = 1$, $\left\| \left\| \tilde{\mathbf{B}}(k,k-1) \right\| \le \bar{c} \right\| \left\| \tilde{\mathbf{B}} - \mathbf{I}_{nd} \right\|_{\mathbf{B}}$ and for t < k-1, $\left\| \left\| \tilde{\mathbf{B}}(k,t) \right\| \le \bar{c} \right\| \left\| \tilde{\mathbf{B}}^{k-1-t} \left(\tilde{\mathbf{B}} - \mathbf{I}_{-1} \right) \right\|_{\mathbf{B}} = \bar{c} \left\| \left(\tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{\mathbf{v}} \le \mathbf{I}_{+} \right) \tilde{\mathbf{B}}^{k-2-t} \left(\tilde{\mathbf{B}} - \mathbf{I}_{-1} \right) \right\|_{\mathbf{B}}$

$$\begin{split} \left\| \left\| \tilde{\mathbf{B}}(k,t) \right\| &\leq \overline{c} \left\| \left\| \tilde{\mathbf{B}}^{k-1-t} \left(\tilde{\mathbf{B}} - \mathbf{I}_{nd} \right) \right\| \right\|_{\mathbf{B}} = \overline{c} \left\| \left\| \left(\tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right) \tilde{\mathbf{B}}^{k-2-t} \left(\tilde{\mathbf{B}} - \mathbf{I}_{n} \right) \right\| \right\|_{\mathbf{B}} \\ &\leq \overline{c}\tau_{\mathbf{B}} \left\| \left\| \tilde{\mathbf{B}}^{k-2-t} \left(\tilde{\mathbf{B}} - \mathbf{I}_{nd} \right) \right\| \right\|_{\mathbf{B}} \\ &\leq \cdots \leq \overline{c}\tau_{\mathbf{B}}^{k-1-t} \left\| \left\| \tilde{\mathbf{B}} - \mathbf{I}_{nd} \right\| \right\|_{\mathbf{B}}. \end{split}$$

Denoting $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}$, we have

$$\left\| \left\| \tilde{\mathbf{B}}(k,t) \right\| \le c_b \tau_{\mathbf{B}}^{k-t}$$
(19)

and

$$\mathbb{E}\left[\left\|\mathbf{y}_{k}\right\|^{2}\right] \leq c_{b}^{2} \sum_{t_{1}=1}^{k} \sum_{t_{2}=1}^{k} \tau_{\mathbf{B}}^{2k-t_{1}-t_{2}} \mathbb{E}\left[\left\|\mathbf{H}_{t_{1}}\right\|\left\|\mathbf{H}_{t_{2}}\right\|\right]$$

$$\leq c_{b}^{2} \sum_{t_{1}=1}^{k} \sum_{t_{2}=1}^{k} \tau_{\mathbf{B}}^{2k-t_{1}-t_{2}} \frac{\mathbb{E}\left[\left\|\mathbf{H}_{t_{1}}\right\|^{2}\right] + \mathbb{E}\left[\left\|\mathbf{H}_{t_{2}}\right\|^{2}\right]}{2}$$

$$\leq c_{b}^{2} \sum_{t_{1}=1}^{k} \sum_{t_{2}=1}^{k} \tau_{\mathbf{B}}^{2k-t_{1}-t_{2}} nC_{g}C_{f} \leq \frac{c_{b}^{2}nC_{g}C_{f}}{(1-\tau_{\mathbf{B}})^{2}},$$
(20)

where the third inequality follows from Assumption 1 (c). Substitute (20) into (17),

$$\mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{1} \otimes \bar{x}_{k+1}\|_{\mathbf{A}}^{2}\right] \leq \frac{1 + \tau_{\mathbf{A}}^{2}}{2} \mathbb{E}\left[\|\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k}\|_{\mathbf{A}}^{2}\right] + c_{1}\alpha_{k}^{2},$$
(21)
$$\frac{2}{\mathbf{A}} \left[\|\mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{2}\|_{\mathbf{A}}^{2} \bar{c}^{2} \frac{c_{b}^{2}nC_{g}C_{f}}{(1 - \varepsilon)^{2}}\right].$$

where $c_1 = \frac{1+\tau_{\mathbf{A}}^2}{1-\tau_{\mathbf{A}}^2} \left\| \left\| \mathbf{A} - \frac{\mathbf{1}\mathbf{u}^{\mathsf{T}}}{n} \right\| \right\|_{\mathbf{A}}^2 \overline{c}^2 \frac{c_b^2 n C_g C_f}{(1-\tau_{\mathbf{B}})^2}$

Next, we estimate the upper bound of consensus error $\|\mathbf{y}_k' - \mathbf{v} \otimes \bar{y}_k'\|^2$ in the mean sense. Set

$$\theta = \left(\tilde{\mathbf{B}} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \left(\mathbf{y}_{k}^{'} - \mathbf{v} \otimes \bar{y}_{k}^{'}\right), \quad \theta^{'} = \left(\mathbf{I}_{nd} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \left(\mathbf{J}_{k+1} - \mathbf{J}_{k}\right)$$

in (16). By the definitions of $\mathbf{v}_{k+1}^{'}$ and $\bar{y}_{k+1}^{'}$, we have $\mathbf{v}_{k+1}^{'} - \mathbf{1} \otimes \bar{y}_{k+1}^{'} = \theta + \theta^{'}$ and

$$\mathbb{E}\left[\left\|\mathbf{y}_{k+1}^{'}-\mathbf{v}\otimes\bar{y}_{k+1}^{'}\right\|_{\mathbf{B}}^{2}\right]$$

$$\leq (1+\tau)\mathbb{E}\left[\left\|\left(\tilde{\mathbf{B}}-\frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\left(\mathbf{y}_{k}^{'}-\mathbf{v}\otimes\bar{y}_{k}^{'}\right)\right\|_{\mathbf{B}}^{2}\right]+\left(1+\frac{1}{\tau}\right)\mathbb{E}\left[\left\|\left(\mathbf{I}_{nd}-\frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\left(\mathbf{J}_{k+1}-\mathbf{J}_{k}\right)\right\|_{\mathbf{B}}^{2}\right]$$

$$\leq \frac{1+\tau_{\mathbf{B}}^{2}}{2}\mathbb{E}\left[\left\|\mathbf{y}_{k}^{'}-\mathbf{v}\otimes\bar{y}_{k}^{'}\right\|_{\mathbf{B}}^{2}\right]+2\frac{1+\tau_{\mathbf{B}}^{2}}{1-\tau_{\mathbf{B}}^{2}}\left\|\left\|\mathbf{I}_{nd}-\frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right\|_{\mathbf{B}}^{2}\bar{c}^{2}\mathbb{E}\left[\left\|\mathbf{J}_{k+1}-\mathbf{J}_{k}\right\|^{2}\right],$$
(22)

where $\tau_{\mathbf{B}}$ is defined in (15), the second inequality follows from the setting $\tau = (1 - \tau_{\mathbf{B}}^2)/(2\tau_{\mathbf{B}}^2)$ and (12). For the term $\mathbb{E}\left[\|\mathbf{J}_{k+1} - \mathbf{J}_k\|^2\right]$,

$$\mathbb{E}\left[\left\|\mathbf{J}_{k+1} - \mathbf{J}_{k}\right\|^{2}\right] = \mathbb{E}\left[\left\|\left(\nabla \mathbf{g}_{k+1} - \nabla \mathbf{g}_{k}\right)\nabla \mathbf{f}_{k+1} + \nabla \mathbf{g}_{k}\left(\nabla \mathbf{f}_{k+1} - \nabla \mathbf{f}_{k}\right)\right\|^{2}\right]$$

$$\leq 2\left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2}\right)\mathbb{E}\left[\left\|\mathbf{x}_{k+1} - \mathbf{x}_{k}\right\|^{2}\right]$$

$$= 2\left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2}\right)\mathbb{E}\left[\left\|\left(\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right)\left(\mathbf{x}_{k} - \mathbf{1}\otimes\bar{x}_{k}\right) - \alpha_{k}\tilde{\mathbf{A}}\mathbf{y}_{k}\right\|^{2}\right]$$

$$\leq 4\left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2}\right)\left\|\left\|\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right\|^{2}\bar{c}^{2}\mathbb{E}\left[\left\|\mathbf{x}_{k} - \mathbf{1}\otimes\bar{x}_{k}\right\|^{2}\right]$$

$$+ 4\left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2}\right)^{2}\alpha_{k}^{2}\mathbb{E}\left[\left\|\mathbf{y}_{k}\right\|^{2}\right],$$
(23)

where $\nabla \mathbf{g}_k = [\nabla g_1(x_{1,k})^{\mathsf{T}}, \cdots, \nabla g_n(x_{n,k})^{\mathsf{T}}]^{\mathsf{T}}$ and $\nabla \mathbf{f}_k = [\nabla f_1(x_{1,k})^{\mathsf{T}}, \cdots, \nabla f_n(x_{n,k})^{\mathsf{T}}]^{\mathsf{T}}$, the first inequality follows from Assumption 1 (a) and (c), the second equality follows from the fact $\left(\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right) (\mathbf{1} \otimes \bar{x}_k) = \mathbf{0}$ as \mathbf{A} is a row stochastic matrix. Substitute (20) and (23) into (22),

$$\mathbb{E}\left[\left\|\mathbf{y}_{k+1}^{'}-\mathbf{v}\otimes\bar{y}_{k+1}^{'}\right\|_{\mathbf{B}}^{2}\right] \leq \frac{1+\tau_{\mathbf{B}}^{2}}{2}\mathbb{E}\left[\left\|\mathbf{y}_{k}^{'}-\mathbf{v}\otimes\bar{y}_{k}^{'}\right\|_{\mathbf{B}}^{2}\right]+c_{2}\mathbb{E}\left[\left\|\mathbf{x}_{k}-\mathbf{1}\otimes\bar{x}_{k}\right\|_{\mathbf{A}}^{2}\right]+c_{3}\alpha_{k}^{2},\tag{24}$$

where the constants

$$c_{2} = 8 \frac{1 + \tau_{\mathbf{B}}^{2}}{1 - \tau_{\mathbf{B}}^{2}} \left\| \mathbf{I}_{nd} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}}^{2} \overline{c}^{4} \left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2} \right) \left\| \left\| \tilde{\mathbf{A}} - \mathbf{I}_{nd} \right\| \right\|^{2},$$

$$c_{3} = 8 \frac{1 + \tau_{\mathbf{B}}^{2}}{1 - \tau_{\mathbf{B}}^{2}} \left\| \mathbf{I}_{nd} - \frac{\mathbf{v}\mathbf{1}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d} \right\|_{\mathbf{B}}^{2} \overline{c}^{2} \left(C_{f}L_{g}^{2} + C_{g}L_{f}^{2} \right) \left\| \left\| \mathbf{A} \right\| \right\|^{2} \frac{c_{b}^{2}nC_{g}C_{f}}{(1 - \tau_{\mathbf{B}})^{2}}.$$

Lastly, we show (14) through combining (21) with (24). Multiplying $c_4 = \frac{1-\tau_A^2}{4c_2}$ on both sides of inequality (24),

$$c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k+1}^{'}-\mathbf{v}\otimes\bar{y}_{k+1}^{'}\right\|_{\mathbf{B}}^{2}\right]$$

$$\leq\frac{1+\tau_{\mathbf{B}}^{2}}{2}c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k}^{'}-\mathbf{v}\otimes\bar{y}_{k}^{'}\right\|_{\mathbf{B}}^{2}\right]+\frac{1-\tau_{\mathbf{A}}^{2}}{4}\mathbb{E}\left[\left\|\mathbf{x}_{k}-\mathbf{1}\otimes\bar{x}_{k}\right\|_{\mathbf{A}}^{2}\right]+c_{3}c_{4}\alpha_{k}^{2}.$$

Substituting above inequality into (21), we have

$$\mathbb{E}\left[\left\|\mathbf{x}_{k+1}-\mathbf{1}\otimes\bar{x}_{k+1}\right\|_{\mathbf{A}}^{2}\right]+c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k+1}'-\mathbf{v}\otimes\bar{y}_{k+1}'\right\|_{\mathbf{B}}^{2}\right]$$

$$\leq\frac{3+\tau_{\mathbf{A}}^{2}}{4}\mathbb{E}\left[\left\|\mathbf{x}_{k}-\mathbf{1}\otimes\bar{x}_{k}\right\|_{\mathbf{A}}^{2}\right]+\frac{1+\tau_{\mathbf{B}}^{2}}{2}c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k}'-\mathbf{v}\otimes\bar{y}_{k}'\right\|_{\mathbf{B}}^{2}\right]+(c_{1}+c_{3}c_{4})\alpha_{k}^{2}$$

$$\leq\rho\left(\mathbb{E}\left[\left\|\mathbf{x}_{k}-\mathbf{1}\otimes\bar{x}_{k}\right\|_{\mathbf{A}}^{2}\right]+c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k}'-\mathbf{v}\otimes\bar{y}_{k}'\right\|_{\mathbf{B}}^{2}\right]\right)+(c_{1}+c_{3}c_{4})\alpha_{k}^{2}$$

$$\cdots$$

$$\leq\rho^{k}\left(\mathbb{E}\left[\left\|\mathbf{x}_{1}-\mathbf{1}\otimes\bar{x}_{1}\right\|_{\mathbf{A}}^{2}\right]+c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{1}'-\mathbf{v}\otimes\bar{y}_{1}'\right\|_{\mathbf{B}}^{2}\right]\right)+(c_{1}+c_{3}c_{4})\sum_{t=1}^{k}\rho^{k-t}\alpha_{t}^{2},$$

$$=\max\left\{\frac{1+\tau_{\mathbf{B}}^{2}}{2},\frac{3+\tau_{\mathbf{A}}^{2}}{4}\right\}.$$
 The proof is complete.

where $\rho = \max\left\{\frac{1+\tau_{\mathbf{B}}^2}{2}, \frac{3+\tau_{\mathbf{A}}^2}{4}\right\}$. The proof is complete.

Lemma 2 indicates that the consensus errors can be explicitly decomposed into "bias" and "variance" terms. The bias term characterizes how fast initial conditions are forgotten and is related to condition numbers $\tau_{\mathbf{A}}$ and $\tau_{\mathbf{B}}$ of network topology. The variance term characterizes the effect of new stochastic gradient, which is independent of the starting point and increases with the gradient upper bounds C_f, C_g and Lipschitz parameters L_f, L_g .

The following lemma is a technical result.

Lemma 3. Suppose that stepsize α_k is nonincreasing and $\lim_{k\to\infty} \frac{\alpha_k}{\alpha_{k+1}} = 1$. Then there exists a constant c such that

$$\sum_{t=1}^{k} \rho^{k-t} \alpha_t \le c \alpha_k,$$

where scalar $\rho = \max\left\{\frac{1+\tau_{\mathbf{B}}^2}{2}, \frac{3+\tau_{\mathbf{A}}^2}{4}\right\}.$

Proof. Let $\beta_k = \sum_{t=1}^k \rho^{k-t} \alpha_t$, then $\beta_k = \rho \sum_{t=1}^{k-1} \rho^{k-1-t} \alpha_t + \alpha_k = \rho \beta_{k-1} + \alpha_k$. Denoting $b_k = \beta_k / \alpha_k$, then $b_k = \rho \frac{\alpha_{k-1}}{\alpha_k} b_{k-1} + 1$. Noting that $\lim_{k \to \infty} \frac{\alpha_{k-1}}{\alpha_k} = 1$ and $\rho < 1$, there exists an integer $k_0 > 0$ such that $\frac{\alpha_{k-1}}{\alpha_k} \leq \frac{2}{\rho+1}$ for $k > k_0$. Taking $c = \max\left\{\sup_{1 \leq k \leq k_0} b_k, \frac{\rho+1}{1-\rho}\right\}$, we have $b_k \leq c$ for $k \leq k_0$. Suppose that the claim holds for k-1 $(k-1 \geq k_0)$, that is $b_{k-1} \leq c$, then

$$b_k = \rho \frac{\alpha_{k-1}}{\alpha_k} b_{k-1} + 1 \le \frac{2\rho}{\rho+1} c + 1 \le \frac{2\rho}{\rho+1} c + \frac{1-\rho}{\rho+1} c = c.$$

The proof is complete.

By the fact $\lim_{k\to\infty} \frac{\alpha_k^2}{\alpha_{k+1}^2} = 1$ and Lemma 3, (14) can be rewritten as

$$\mathbb{E}\left[\left\|\mathbf{x}_{k+1}-\mathbf{1}\otimes\bar{x}_{k+1}\right\|_{\mathbf{A}}^{2}\right]+c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{k+1}^{'}-\mathbf{v}\otimes\bar{y}_{k+1}^{'}\right\|_{\mathbf{B}}^{2}\right]$$

$$\leq\rho^{k}\left(c_{4}\mathbb{E}\left[\left\|\mathbf{y}_{1}^{'}-\mathbf{v}\otimes\bar{y}_{1}^{'}\right\|_{\mathbf{B}}^{2}\right]+\mathbb{E}\left[\left\|\mathbf{x}_{1}-\mathbf{1}\otimes\bar{x}_{1}\right\|_{\mathbf{A}}^{2}\right]\right)+(c_{1}+c_{3}c_{4})c\alpha_{k}^{2}.$$

Assuming $\rho^k = o(\alpha_k^2)^1$, the consensus errors have a rough upper bounds

$$\mathbb{E}\left[\|\mathbf{x}_{k}-\mathbf{1}\otimes\bar{x}_{k}\|_{\mathbf{A}}^{2}\right] \leq U_{1}\alpha_{k}^{2}, \quad \mathbb{E}\left[\|\mathbf{y}_{k}^{'}-\mathbf{v}\otimes\bar{y}_{k}^{'}\|^{2}\right] \leq U_{1}\alpha_{k}^{2}, \quad (25)$$

where the constant U_1 depends on parameters $\tau_{\mathbf{A}}, \tau_{\mathbf{B}}, C_f, C_g, L_f$ and L_g .

The next lemma quantifies the error of estimating $g_i(x_{i,k})$ by $z_{i,k}$.

Lemma 4. Suppose that stepsizes α_k and β_k are nonincreasing and $\lim_{k\to\infty} \frac{\alpha_k}{\alpha_{k+1}} = 1$, $\beta_k \leq 1$. Then under Assumptions 1-2,

$$\mathbb{E}\left[\left\|\mathbf{z}_{k+1} - \mathbf{g}_{k+1}\right\|^{2}\right] \leq (1 - \beta_{k})^{2} \mathbb{E}\left[\left\|\mathbf{z}_{k} - \mathbf{g}_{k}\right\|^{2}\right] + \left(12C_{g}\bar{c}^{2}\left\|\left\|\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right\|\right\|^{2}U_{1} + \frac{12c_{b}^{2}nC_{g}^{2}C_{f}\left\|\left\|\mathbf{A}\right\|\right\|^{2}}{(1 - \tau_{\mathbf{B}})^{2}}\right)\alpha_{k}^{2} + 3V_{g}\beta_{k}^{2},$$
(26)

where $\mathbf{g}_k = [g_1(x_{1,k})^{\mathsf{T}}, \cdots, g_n(x_{n,k})^{\mathsf{T}}]^{\mathsf{T}}$ and \overline{c} is defined in (12).

¹In this paper, we use constant stepsize and sublinear diminishing stepsize. Then $\rho^k = o(\alpha_k^2)$ holds.

Proof. By the definitions of \mathbf{z}_{k+1} and \mathbf{g}_{k+1} ,

$$\mathbf{z}_{k+1} - \mathbf{g}_{k+1} = (1 - \beta_k) \left(\mathbf{z}_k - \mathbf{g}_k \right) + \left(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1} \right) + (1 - \beta_k) \left(\mathbf{g}_k - \mathbf{G}_{k+1}^{(2)} \right).$$
(27)

Then

$$\mathbb{E}\left[\|\mathbf{z}_{k+1} - \mathbf{g}_{k+1}\|^{2}\right] = (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + \mathbb{E}\left[\|(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_{k})(\mathbf{g}_{k} - \mathbf{G}_{k+1}^{(2)})\|^{2}\right] + 2\mathbb{E}\left[\left\langle (1 - \beta_{k})(\mathbf{z}_{k} - \mathbf{g}_{k}), (\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_{k})(\mathbf{g}_{k} - \mathbf{G}_{k+1}^{(2)})\right\rangle\right] = (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + \mathbb{E}\left[\|(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_{k})(\mathbf{g}_{k} - \mathbf{G}_{k+1}^{(2)})\|^{2}\right], \quad (28)$$

where the second equality follows from the fact

$$\mathbb{E}\left[(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_k)(\mathbf{g}_k - \mathbf{G}_{k+1}^{(2)}) \right] = \mathbb{E}\left[\mathbb{E}\left[(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_k)(\mathbf{g}_k - \mathbf{G}_{k+1}^{(2)}) \middle| \mathcal{F}_k' \right] \right] = \mathbf{0}$$

with

$$\mathcal{F}_{1}' = \sigma \left(x_{i,1}, z_{i,1}, \phi_{i,1}, \zeta_{i,1} : i \in \mathcal{V} \right), \mathcal{F}_{k}' = \sigma \left(\left\{ x_{i,1}, z_{i,1}, \phi_{i,t}, \zeta_{i,t} : i \in \mathcal{V}, 1 \le t \le k \right\} \cup \left\{ \phi_{i,t}' : i \in \mathcal{V}, 2 \le t \le k \right\} \right) (k \ge 2).$$
(29)

For the second term on the right hand side of (28),

$$\begin{split} & \mathbb{E}\left[\|(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_k)(\mathbf{g}_k - \mathbf{G}_{k+1}^{(2)})\|^2\right] \\ &= \mathbb{E}\left[\|(1 - \beta_k)(\mathbf{G}_{k+1}^{(1)} - \mathbf{G}_{k+1}^{(2)}) + \beta_k(\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}) + (1 - \beta_k)(\mathbf{g}_k - \mathbf{g}_{k+1})\|^2\right] \\ &\leq 3(1 - \beta_k)^2 \mathbb{E}\left[\|\mathbf{G}_{k+1}^{(1)} - \mathbf{G}_{k+1}^{(2)}\|^2\right] + 3\beta_k^2 \mathbb{E}\left[\|\mathbf{G}_{k+1}^{(1)} - \mathbf{g}_{k+1}\|^2\right] + 3(1 - \beta_k)^2 \mathbb{E}\left[\|\mathbf{g}_k - \mathbf{g}_{k+1}\|^2\right] \\ &\leq 6(1 - \beta_k)^2 C_g \mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2\right] + 3\beta_k^2 V_g, \end{split}$$

where the second inequality follows from the conditions (c) and (d) in Assumption 1. Substitute above inequality into (28),

$$\begin{split} & \mathbb{E}\left[\|\mathbf{z}_{k+1} - \mathbf{g}_{k+1}\|^{2}\right] \\ & \leq (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + 6(1 - \beta_{k})^{2} C_{g} \mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{x}_{k}\|^{2}\right] + 3\beta_{k}^{2} V_{g} \\ & = (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + 6(1 - \beta_{k})^{2} C_{g} \mathbb{E}\left[\left\|\left(\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right)\left(\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k}\right) - \alpha_{k} \tilde{\mathbf{A}} \mathbf{y}_{k}\right\|^{2}\right] + 3\beta_{k}^{2} V_{g} \\ & \leq (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + 12(1 - \beta_{k})^{2} C_{g} \bar{c}^{2}\left\|\left|\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right\|\right|^{2} \mathbb{E}\left[\|\mathbf{x}_{k} - \mathbf{1} \otimes \bar{x}_{k}\|_{\mathbf{A}}^{2}\right] \\ & \quad + 12(1 - \beta_{k})^{2} C_{g} \alpha_{k}^{2} \|\|\mathbf{A}\|\|^{2} \mathbb{E}\left[\|\mathbf{y}_{k}\|^{2}\right] + 3\beta_{k}^{2} V_{g} \\ & \leq (1 - \beta_{k})^{2} \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] + \left(12 C_{g} \bar{c}^{2} \left\|\left|\tilde{\mathbf{A}} - \mathbf{I}_{nd}\right\|\right|^{2} U_{1} + \frac{12 c_{b}^{2} n C_{g}^{2} C_{f} \|\|\mathbf{A}\|\|^{2}}{(1 - \tau_{\mathbf{B}})^{2}}\right) \alpha_{k}^{2} + 3 V_{g} \beta_{k}^{2}, \end{split}$$

where \bar{c} is defined in (12), the equality follows from the fact $(\tilde{\mathbf{A}} - \mathbf{I}_{nd})(\mathbf{1} \otimes \bar{x}_k) = \mathbf{0}$ by the row stochasticity of \mathbf{A} , the last inequality follows from (20), (25) and the definition of β_k . The proof is complete.

The following lemma studies the boundness of stochastic noise accumulated in gradient tracking process.

Lemma 5. Define

$$\xi_k := y_k - y'_k. \tag{30}$$

Under the conditions of Lemma 2,

(i)
$$\mathbb{E}\left[\left\|\xi_{k}\right\|^{2}\right] \leq \frac{c_{b}^{2}4nC_{f}C_{g}}{(1-\tau_{\mathbf{B}})^{2}}, \text{ where } c_{b} = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_{n}\|\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\};$$

(ii) there exists constant $U_3 > 0$ such that

$$\left| \mathbb{E}\left[\left\langle \nabla h(\bar{x}_k), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \xi_k \right\rangle \right] \right| \le U_3 \alpha_k.$$

Proof. We first show part (i). By the definition of ξ_k ,

$$\xi_k = \sum_{t=1}^{k-1} \tilde{\mathbf{B}}^{k-1-t} (\tilde{\mathbf{B}} - \mathbf{I}_{nd}) \epsilon_t + \epsilon_k = \sum_{t=1}^k \tilde{\mathbf{B}}(k, t) \epsilon_t,$$
(31)

where $\epsilon_t := \mathbf{H}_t - \mathbf{J}_t$, \mathbf{H}_t and \mathbf{J}_t present in (10) and Lemma 2 respectively, $\tilde{\mathbf{B}}(k, t)$ is defined in (18). Then we have

$$\mathbb{E}[\|\xi_k\|^2] \leq \sum_{t_1=1}^k \sum_{t_2=1}^k \left\| \|\tilde{\mathbf{B}}(k,t_1) \| \| \|\tilde{\mathbf{B}}(k,t_2) \| \|\mathbb{E}[\|\epsilon_{t_1}\| \|\epsilon_{t_2}\|] \right\|$$

$$\leq c_b^2 \sum_{t_1=1}^k \sum_{t_2=1}^k \tau_{\mathbf{B}}^{2k-t_1-t_2} \mathbb{E}[\|\epsilon_{t_1}\| \|\epsilon_{t_2}\|]$$

$$\leq c_b^2 \sum_{t_1=1}^k \sum_{t_2=1}^k \tau_{\mathbf{B}}^{2k-t_1-t_2} \frac{\mathbb{E}[\|\epsilon_{t_1}\|^2 + \|\epsilon_{t_2}\|^2]}{2}, \qquad (32)$$

where $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}$, the second inequality follows from (19). By the definition of ϵ_k ,

$$\mathbb{E}\left[\|\epsilon_{k}\|^{2}\right] = \sum_{j=1}^{n} \mathbb{E}\left[\|\nabla G_{j}(x_{j,k};\phi_{j,k})\nabla F_{j}(z_{j,k};\zeta_{j,k}) - \nabla g_{j}(x_{j,k})\nabla f_{j}(z_{j,k})\|^{2}\right]$$

$$\leq 2\sum_{j=1}^{n} \left(\mathbb{E}\left[\|\nabla G_{j}(x_{j,k};\phi_{j,k})\|^{2}\|\nabla F_{j}(z_{j,k};\zeta_{j,k})\|^{2}\right] + C_{f}C_{g}\right)$$

$$= 2\sum_{j=1}^{n} \left(\mathbb{E}\left[\mathbb{E}\left[\|\nabla G_{j}(x_{j,k};\phi_{j,k})\|^{2}\|\nabla F_{j}(z_{j,k};\zeta_{j,k})\|^{2}\right]\mathcal{F}_{k},\zeta_{j,k}\right]\right] + C_{f}C_{g}\right)$$

$$\leq 2\sum_{j=1}^{n} \left(C_{g}\mathbb{E}\left[\|\nabla F_{j}(z_{j,k};\zeta_{j,k})\|^{2}\right] + C_{f}C_{g}\right) \leq 4nC_{f}C_{g},$$
(33)

where

$$\mathcal{F}_{1} = \sigma\{x_{i,1}, z_{i,1} : i \in \mathcal{V}\},\$$

$$\mathcal{F}_{k} = \sigma\left(\{x_{i,1}, z_{i,1}, \phi_{i,t}, \zeta_{i,t} : i \in \mathcal{V}, 1 \le t \le k - 1\} \cup \{\phi_{i,t}' : i \in \mathcal{V}, 2 \le t \le k\}\right)(k \ge 2).$$
(34)

Substitute (33) into (32), $\mathbb{E}[\|\xi_k\|^2] \leq c_b^2 4nC_f C_g \sum_{t_1=1}^k \sum_{t_2=1}^k \tau_{\mathbf{B}}^{2k-t_1-t_2} \leq \frac{c_b^2 4nC_f C_g}{(1-\tau_{\mathbf{B}})^2}$. Part (i) is obtained.

By (31),

$$\begin{split} & \left| \mathbb{E} \left[\left\langle \nabla h(\bar{x}_k), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \xi_k \right\rangle \right] \right| \\ &= \left| \sum_{t=1}^k \mathbb{E} \left[\left\langle \nabla h(\bar{x}_k), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \tilde{\mathbf{B}}(k, t) \epsilon_t \right\rangle \right] \right| \\ &= \left| \sum_{t=1}^{k-1} \mathbb{E} \left[\mathbb{E} \left[\left\langle \sum_{l=t+1}^k \left(\nabla h(\bar{x}_l) - \nabla h(\bar{x}_{l-1}) \right) + \nabla h(\bar{x}_t), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \tilde{\mathbf{B}}(k, t) \epsilon_t \right\rangle \right| \mathcal{F}_t \right] \right] \\ &+ \mathbb{E} \left[\mathbb{E} \left[\left\langle \nabla h(\bar{x}_k), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \tilde{\mathbf{B}}(k, t) \epsilon_k \right\rangle \right| \mathcal{F}_k \right] \right] \right| \\ &= \left| \sum_{t=1}^{k-1} \mathbb{E} \left[\left\langle \sum_{l=t+1}^k \left(\nabla h(\bar{x}_l) - \nabla h(\bar{x}_{l-1}) \right), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \tilde{\mathbf{B}}(k, t) \epsilon_t \right\rangle \right] \right| \\ &\leq \frac{\|\mathbf{u}\| L c_b}{n} \sum_{t=1}^{k-1} \tau_{\mathbf{B}}^{k-t} \sum_{l=t+1}^k \mathbb{E} \left[\|\bar{x}_l - \bar{x}_{l-1}\| \| \epsilon_t \| \right] \\ &\leq \frac{\|\mathbf{u}\|^2 L c_b}{n^2} \sum_{t=1}^{k-1} \tau_{\mathbf{B}}^{k-t} \sum_{l=t+1}^k \alpha_l \mathbb{E} \left[\|\mathbf{y}_l\| \| \epsilon_t \| \right], \end{split}$$

where the third equality holds as $\{\epsilon_t\}$ is a martingale difference sequence, the first inequality follows from (19) and the last inequality follows from the fact $\bar{x}_{k+1} = \bar{x}_k - \alpha_k \left(\frac{\mathbf{u}^{\intercal}}{n} \otimes \mathbf{I}_d\right) \mathbf{y}_k$. By (20) and (33),

$$\mathbb{E}\left[\left\|\mathbf{y}_{l}\right\|\left\|\epsilon_{t}\right\|\right] \leq \frac{\mathbb{E}\left[\left\|\mathbf{y}_{l}\right\|^{2}\right] + \mathbb{E}\left[\left\|\epsilon_{k}\right\|^{2}\right]}{2} \leq \frac{c_{b}^{2}n \ C_{f}C_{g}}{2(1-\tau_{\mathbf{B}})^{2}} + 2nC_{f}C_{g}.$$
(35)

Let $U = \frac{\|\mathbf{u}\|^2 L c_b}{n} \left(\frac{c_b^2 C_f C_g}{2(1-\tau_{\mathbf{B}})^2} + 2C_f C_g \right),$ $\mathbb{E}\left[\left\langle \nabla h(\bar{x}_k), \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \xi_k \right\rangle \right] \leq (1-\tau_{\mathbf{B}}) U \sum_{t=1}^{k-1} \tau_{\mathbf{B}}^{k-t} \sum_{l=t+1}^k \alpha_l = (1-\tau_{\mathbf{B}}) U \sum_{t=2}^k \alpha_t \tau_{\mathbf{B}}^{k-t} \left(\sum_{l=1}^{t-1} \tau_{\mathbf{B}}^l \right) \leq U c \alpha_k,$

where the last inequality follows from the fact $(1 - \tau_{\mathbf{B}}) \left(\sum_{l=1}^{t-1} \tau_{\mathbf{B}}^{l} \right) \leq 1$ and Lemma 3. Part (ii) holds. The proof is complete.

With Lemmas 1-5 at hand, we are ready to present the convergence rate of AB-DSCSC.

Theorem 1. Let $\alpha_k = \frac{a}{\sqrt{K}}$, $\beta_k = \frac{\alpha_k C_g L_f^2}{n}$ and $a < \frac{n}{C_g L_f^2}$. Then under Assumptions 1-2,

$$\frac{1}{K}\sum_{k=1}^{K} \mathbb{E}\left[\|\nabla h(x_{i,k})\|^2\right] \le \frac{8\left(\mathbb{E}\left[h(\bar{x}_1)\right] + \mathbb{E}\left[\|\mathbf{z}_1 - \mathbf{g}_1\|^2\right]\right)/a + 8aU_4}{\sqrt{K}} + \frac{2\left(\frac{4L^2U_1}{n} + \frac{4\|\mathbf{u}\|^2U_1}{n^2} + L^2U_1\right)a^2}{K},$$

where constants U_1 is defined in (25), U_3 presents in Lemma 5 and

$$U_4 = \frac{L \|\mathbf{u}\|^2 c_b^2 C_f C_g}{2n^2 (1 - \tau_{\mathbf{B}})^2} + U_3 + 12C_g \overline{c}^2 \left\| \tilde{\mathbf{A}} - \mathbf{I}_{nd} \right\|^2 U_1 + \frac{12c_b^2 n C_g^2 C_f \|\|\mathbf{A}\|\|^2}{(1 - \tau_{\mathbf{B}})^2} + 3V_g \frac{C_g^2 L_f^4}{n^2}$$

Proof. We first estimate the upper bound of $\nabla h(\bar{x}_k)$ in expectation. Noting that $\nabla h(x)$ is $L(:= C_g^2 L_f + C_f L_g)$ -smooth [31],

$$\begin{split} h(\bar{x}_{k+1}) &\leq h(\bar{x}_k) + \langle \nabla h(\bar{x}_k), \bar{x}_{k+1} - \bar{x}_k \rangle + \frac{L}{2} \| \bar{x}_{k+1} - \bar{x}_k \|^2 \\ &= h(\bar{x}_k) - \left\langle \nabla h(\bar{x}_k), \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \left(\mathbf{y}'_k + \xi_k \right) \right\rangle + \frac{L}{2} \left\| \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \mathbf{y}_k \right\|^2 \\ &= h(\bar{x}_k) - \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_k}{n} \| \nabla h(\bar{x}_k) \|^2 + \frac{L}{2} \left\| \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \mathbf{y}_k \right\|^2 \\ &+ \left\langle \nabla h(\bar{x}_k), \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v}}{n} \nabla h(\bar{x}_k) - \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \left(\mathbf{y}'_k + \xi_k \right) \right) \right\rangle, \end{split}$$

where the second equality follows from the fact that

$$\bar{x}_{k+1} = \bar{x}_k - \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \mathbf{y}_k = \bar{x}_k - \alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \left(\mathbf{y}_k' + \xi_k\right).$$

Take expectation on both sides of above inequality,

$$\mathbb{E}\left[h(\bar{x}_{k+1})\right] \leq \mathbb{E}\left[h(\bar{x}_{k})\right] - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\mathbb{E}\left[\|\nabla h(\bar{x}_{k})\|^{2}\right] + \frac{L}{2}\mathbb{E}\left[\left\|\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\mathbf{y}_{k}\right\|^{2}\right] + \mathbb{E}\left[\left\langle\nabla h(\bar{x}_{k}),\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\nabla h(\bar{x}_{k}) - \left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\left(\mathbf{y}_{k}'+\xi_{k}\right)\right)\right\rangle\right].$$
(36)

For the third term on the right hand of (36),

$$\frac{L}{2}\mathbb{E}\left[\left\|\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\mathbf{y}_{k}\right\|^{2}\right] \leq \frac{L\alpha_{k}^{2}\|\mathbf{u}\|^{2}}{2n^{2}}\mathbb{E}\left[\left\|\mathbf{y}_{k}\right\|^{2}\right] \leq \frac{L\|\mathbf{u}\|^{2}c_{b}^{2}C_{f}C_{g}}{2n^{2}(1-\tau_{\mathbf{B}})^{2}}\alpha_{k}^{2},\tag{37}$$

where the second inequalities follows from (20).

For the fourth term on the right hand of (36),

$$\mathbb{E}\left[\left\langle \nabla h(\bar{x}_{k}), \alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\nabla h(\bar{x}_{k}) - \left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\left(\mathbf{y}_{k}^{'} + \xi_{k}\right)\right)\right\rangle\right] \\
\leq \frac{\alpha_{k}^{2}}{2\tau}\mathbb{E}\left[\left\|\nabla h(\bar{x}_{k})\right\|^{2}\right] + \frac{3\tau}{2}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\right)^{2}\mathbb{E}\left[\left\|P_{1}\right\|^{2}\right] + \frac{3\tau}{2}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\right)^{2}\mathbb{E}\left[\left\|P_{2}\right\|^{2}\right] \\
+ \frac{3\tau\|\mathbf{u}\|^{2}}{2n^{2}}\mathbb{E}\left[\left\|\mathbf{v}\otimes\bar{y}_{k}^{'} - \mathbf{y}_{k}^{'}\right\|^{2}\right] + \frac{\alpha_{k}\|\mathbf{u}\|}{n}\left|\mathbb{E}\left[\left\langle\nabla h(\bar{x}_{k}), -\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\xi_{k}\right\rangle\right]\right| \\
\leq \frac{\alpha_{k}^{2}}{2\tau}\mathbb{E}\left[\left\|\nabla h(\bar{x}_{k})\right\|^{2}\right] + \frac{3\tau L^{2}}{2n}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\right)^{2}\mathbb{E}\left[\left\|\mathbf{x}_{k} - \mathbf{1}\otimes\bar{x}_{k}\right\|^{2}\right] + \frac{3\tau C_{g}L_{f}^{2}}{2n}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n}\right)^{2}\mathbb{E}\left[\left\|\mathbf{g}_{k} - \mathbf{z}_{k}\right\|^{2}\right] \\
+ \frac{3\tau\|\mathbf{u}\|^{2}}{2n^{2}}\mathbb{E}\left[\left\|\mathbf{v}\otimes\bar{y}_{k}^{'} - \mathbf{y}_{k}^{'}\right\|^{2}\right] + \frac{\alpha_{k}\|\mathbf{u}\|}{n}\left|\mathbb{E}\left[\left\langle\nabla h(\bar{x}_{k}), -\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\xi_{k}\right\rangle\right]\right| \\
\leq \frac{\alpha_{k}^{2}}{2\tau}\mathbb{E}\left[\left\|\nabla h(\bar{x}_{k})\right\|^{2}\right] + \frac{3\tau L^{2}U_{1}}{2n}\alpha_{k}^{2} + \frac{3\tau C_{g}L_{f}^{2}}{2n}\mathbb{E}\left[\left\|\mathbf{g}_{k} - \mathbf{z}_{k}\right\|^{2}\right] + \frac{3\tau\|\mathbf{u}\|^{2}U_{1}}{2n^{2}}\alpha_{k}^{2} + U_{3}\alpha_{k}^{2}, \quad (38)$$

where $P_1 = \nabla h(\bar{x}_k) - \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \nabla f_j(g_j(x_{j,k})), P_2 = \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \nabla f_j(g_j(x_{j,k})) - \bar{y}'_k$ and τ can be any positive scalar, the first inequality follows from Cauchy-Schwartz inequality and the fact $ab \leq \frac{1}{2\tau}a^2 + \frac{\tau}{2}b^2$, the second inequality follows from the Lipschitz continuity of $\nabla g_j(\cdot) \nabla f_j(g_j(\cdot))$, Assumption 1 and the fact $\bar{y}'_k = \frac{1}{n} \sum_{j=1}^n \nabla g_j(x_{j,k}) \nabla f_j(z_{j,k})$, the third inequality follows from (25), the fact $\mathbf{u}^{\mathsf{T}} \mathbf{v} \leq n$ and Lemma 5 (ii).

Plug (37)-(38) into (36) and set $\tau = \frac{2\alpha_k}{3}$,

$$\mathbb{E}\left[h(\bar{x}_{k+1})\right] \leq \mathbb{E}\left[h(\bar{x}_{k})\right] - \alpha_{k} \left(1 - \frac{\alpha_{k}}{2\tau}\right) \mathbb{E}\left[\|\nabla h(\bar{x}_{k})\|^{2}\right] + \frac{L\|\mathbf{u}\|^{2}c_{b}^{2}C_{f}C_{g}}{2n^{2}(1 - \tau_{\mathbf{B}})^{2}}\alpha_{k}^{2} + \frac{3\tau L^{2}U_{1}}{2n}\alpha_{k}^{2}
+ \frac{3\tau C_{g}L_{f}^{2}}{2n} \mathbb{E}\left[\|\mathbf{g}_{k} - \mathbf{z}_{k}\|^{2}\right] + \frac{3\tau \|\mathbf{u}\|^{2}U_{1}}{2n^{2}}\alpha_{k}^{2} + U_{3}\alpha_{k}^{2}
\leq \mathbb{E}\left[h(\bar{x}_{k})\right] - \frac{\alpha_{k}}{4} \mathbb{E}\left[\|\nabla h(\bar{x}_{k})\|^{2}\right] + \left(\frac{L\|\mathbf{u}\|^{2}c_{b}^{2}C_{f}C_{g}}{2n^{2}(1 - \tau_{\mathbf{B}})^{2}} + U_{3}\right)\alpha_{k}^{2}
+ \left(\frac{L^{2}U_{1}}{n} + \frac{\|\mathbf{u}\|^{2}U_{1}}{n^{2}}\right)\alpha_{k}^{3} + \beta_{k}\mathbb{E}\left[\|\mathbf{g}_{k} - \mathbf{z}_{k}\|^{2}\right].$$
(39)

Combining (26) with (39),

$$\mathbb{E}\left[h(\bar{x}_{k+1})\right] + \mathbb{E}\left[\|\mathbf{z}_{k+1} - \mathbf{g}_{k+1}\|^{2}\right] \\
\leq \mathbb{E}\left[h(\bar{x}_{k})\right] + \left[(1 - \beta_{k})^{2} + \beta_{k}\right] \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] - \frac{\alpha_{k}}{4} \mathbb{E}\left[\|\nabla h(\bar{x}_{k})\|^{2}\right] + \left(\frac{L\|\mathbf{u}\|^{2}c_{b}^{2}C_{f}C_{g}}{2n^{2}(1 - \tau_{\mathbf{B}})^{2}} + U_{3}\right)\alpha_{k}^{2} \\
+ \left(\frac{L^{2}U_{1}}{n} + \frac{\|\mathbf{u}\|^{2}U_{1}}{n^{2}}\right)\alpha_{k}^{3} + \left(12C_{g}\bar{c}^{2}\|\|\tilde{\mathbf{A}} - \mathbf{I}_{nd}\|\|^{2}U_{1} + \frac{12c_{b}^{2}nC_{g}^{2}C_{f}\|\|\mathbf{A}\|\|^{2}}{(1 - \tau_{\mathbf{B}})^{2}}\right)\alpha_{k}^{2} + 3V_{g}\beta_{k}^{2} \\
\leq \mathbb{E}\left[h(\bar{x}_{k})\right] + \mathbb{E}\left[\|\mathbf{z}_{k} - \mathbf{g}_{k}\|^{2}\right] - \frac{\alpha_{k}}{4}\mathbb{E}\left[\|\nabla h(\bar{x}_{k})\|^{2}\right] + U_{4}\alpha_{k}^{2} + \left(\frac{L^{2}U_{1}}{n} + \frac{\|\mathbf{u}\|^{2}U_{1}}{n^{2}}\right)\alpha_{k}^{3},$$
(40)

where

$$U_4 = \frac{L \|\mathbf{u}\|^2 c_b^2 C_f C_g}{2n^2 (1 - \tau_{\mathbf{B}})^2} + U_3 + 12C_g \bar{c}^2 \left\| \left\| \tilde{\mathbf{A}} - \mathbf{I}_{nd} \right\| \right\|^2 U_1 + \frac{12c_b^2 n C_g^2 C_f \|\|\mathbf{A}\|\|^2}{(1 - \tau_{\mathbf{B}})^2} + 3V_g \frac{C_g^2 L_f^4}{n^2}.$$

Reordering the terms of (40) and summing over k from 1 to K,

$$\sum_{k=1}^{K} \frac{\alpha_{k}}{4} \mathbb{E} \left[\|\nabla h(\bar{x}_{k})\|^{2} \right] \leq \mathbb{E} \left[h(\bar{x}_{1}) \right] + \mathbb{E} \left[\|\mathbf{z}_{1} - \mathbf{g}_{1}\|^{2} \right] - \left(\mathbb{E} \left[h(\bar{x}_{K+1}) \right] + \mathbb{E} \left[\|\mathbf{z}_{K+1} - \mathbf{g}_{K+1}\|^{2} \right] \right) \\ + U_{4} \sum_{k=1}^{K} \alpha_{k}^{2} + \left(\frac{L^{2}U_{1}}{n} + \frac{\|\mathbf{u}\|^{2}U_{1}}{n^{2}} \right) \sum_{k=1}^{K} \alpha_{k}^{3} \\ \leq \mathbb{E} \left[h(\bar{x}_{1}) \right] + \mathbb{E} \left[\|\mathbf{z}_{1} - \mathbf{g}_{1}\|^{2} \right] + U_{4} \sum_{k=1}^{K} \alpha_{k}^{2} + \left(\frac{L^{2}U_{1}}{n} + \frac{\|\mathbf{u}\|^{2}U_{1}}{n^{2}} \right) \sum_{k=1}^{K} \alpha_{k}^{3}.$$

Multiplying both sides of the above inequality by $\frac{4}{a\sqrt{K}}$,

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\|\nabla h(\bar{x}_k)\|^2 \right] \le \frac{4\left(\mathbb{E}\left[h(\bar{x}_1)\right] + \mathbb{E}\left[\|\mathbf{z}_1 - \mathbf{g}_1\|^2\right]\right)/a + 4aU_4}{\sqrt{K}} + \frac{4\left(\frac{L^2U_1}{n} + \frac{\|\mathbf{u}\|^2U_1}{n^2}\right)a^2}{K}.$$

By the Lipschitz continuity of $\nabla h(\cdot)$, we have

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{E} \left[\|\nabla h(x_{i,k})\|^2 \right] \leq \frac{2}{K} \sum_{k=1}^{K} \mathbb{E} \left[\|\nabla h(\bar{x}_k)\|^2 \right] + \frac{2L^2}{K} \sum_{k=1}^{K} \mathbb{E} \left[\|x_{i,k} - \bar{x}_k\|^2 \right] \\
\leq \frac{8 \left(\mathbb{E} \left[h(\bar{x}_1) \right] + \mathbb{E} \left[\|\mathbf{z}_1 - \mathbf{g}_1\|^2 \right] \right) / a + 8aU_4}{\sqrt{K}} + \frac{2 \left(\frac{4L^2 U_1}{n} + \frac{4\|\mathbf{u}\|^2 U_1}{n^2} + L^2 U_1 \right) a^2}{K},$$

where the last inequality follows from (25). The proof is complete.

Theorem 1 presents that the AB-DSCSC achieves the convergence rate $\mathcal{O}(K^{-1/2})$ finding the (ϵ)-stationary point, which is same as the convergence rate of stochastic gradient descent for non-compositional problems. On the other hand, the sample complexity for finding the (ϵ)stationary point is $\mathcal{O}\left(\frac{1}{\epsilon^2}\right)$ as AB-DSCSC does not need the increasing batch size strategy in each iteration.

Next, we study the convergence rate of AB-DSCSC for the strongly convex objective under diminishing stepsize strategy.

Theorem 2. Let $\alpha_k = a/(k+b)^{\alpha}$, $\beta_k = \beta \alpha_k$, where $a > 0, b \ge 0$, $\alpha \in (1/2, 1)$, $\beta \in (0, 1/a)$ and $a/(1+b)^{\alpha} \le \frac{n}{\mathbf{u}^{\mathsf{T}} \mathbf{v} \mu} \min\{1, 2/(C_g^2 L_f + C_f L_g)\}$. Under Assumptions 1-2 and the condition that objective function h(x) is μ -strongly convex,

$$\mathbb{E}\left[\|\bar{x}_k - x^*\|^2\right] = \mathcal{O}\left(\alpha_k\right).$$

Moreover, if $\alpha_k = a/(k+b)$, $\beta_k = \beta \alpha_k$, $\frac{2n}{\mathbf{u}^{\mathsf{T}} \mathbf{v} \mu} < a \leq \frac{n(b+1)}{\mathbf{u}^{\mathsf{T}} \mathbf{v} \mu} \min\left\{1, 2/(C_g^2 L_f + C_f L_g)\right\}$ and $1 < \beta a \leq 1+b$, $\mathbb{E}\left[\|\bar{x}_k - x^*\|^2\right] = \mathcal{O}\left(\frac{1}{k}\right).$

Proof. Recall the definition $\bar{x}_{k+1} = \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \mathbf{x}_{k+1}$ in Lemma 2,

$$\bar{x}_{k+1} = \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \tilde{\mathbf{A}} \left(\mathbf{x}_{k} - \alpha_{k} \mathbf{y}_{k}\right) \\
= \bar{x}_{k} - \alpha_{k} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \left(\mathbf{y}_{k}^{'} + \xi_{k}\right) \\
= \bar{x}_{k} - \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \nabla h(\bar{x}_{k}) + \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \left(\underbrace{\nabla h(\bar{x}_{k}) - \frac{1}{n} \sum_{j=1}^{n} \nabla g_{j}(x_{j,k}) \nabla f_{j}(g_{j}(x_{j,k}))}_{P_{k}^{(1)}} + \underbrace{\frac{1}{n} \sum_{j=1}^{n} \nabla g_{j}(x_{j,k}) \nabla f_{j}(g_{j}(x_{j,k})) - \bar{y}_{k}^{'}}_{P_{k}^{(2)}} + \underbrace{\frac{n}{\mathbf{u}^{\mathsf{T}} \mathbf{v}} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \left(\mathbf{v} \otimes \bar{y}_{k}^{'} - \mathbf{y}_{k}^{'}\right)}_{P_{k}^{(3)}} \\
+ \underbrace{\left(-\frac{n}{\mathbf{u}^{\mathsf{T}} \mathbf{v}}\right) \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \xi_{k}}_{P_{k}^{(4)}}\right),$$
(41)

where $\mathbf{y}_{k}^{'}$ and ξ_{k+1} are defined in (13) and (30), the second equality follows from the fact $\mathbf{u}^{\mathsf{T}}\mathbf{A} = \mathbf{1}$. Subsequently,

$$\mathbb{E}\left[\left\|\bar{x}_{k+1} - x^*\right\|^2\right] = \mathbb{E}\left[\left\|\bar{x}_k - x^* - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\nabla h(\bar{x}_k)\right\|^2\right] + \left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right)^2 \mathbb{E}\left[\left\|P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\|^2\right] \\
+ 2\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right) \mathbb{E}\left[\left\langle\bar{x}_k - x^* - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\nabla h(\bar{x}_k), P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\rangle\right] \\
\leq \left(1 - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu\alpha_k}{n}\right)^2 \mathbb{E}\left[\left\|\bar{x}_k - x^*\right\|^2\right] + \left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right)^2 \mathbb{E}\left[\left\|P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\|^2\right] \\
+ 2\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right) \mathbb{E}\left[\left\langle\bar{x}_k - x^* - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\nabla h(\bar{x}_k), P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\rangle\right] \\
\leq \left(\left(1 - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu\alpha_k}{n}\right)^2 + \frac{\tau}{2}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right)^2 L^2\right) \mathbb{E}\left[\left\|\bar{x}_k - x^*\right\|^2\right] \\
+ \left(1 + \frac{1}{2\tau}\right)\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right)^2 \mathbb{E}\left[\left\|P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\|^2\right] \\
+ 2\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_k}{n}\right) \mathbb{E}\left[\left\langle\bar{x}_k - x^*, P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right\rangle\right],$$
(42)

where τ is any positive scalar, the first inequality follows from [22, Lemm 10], the second inequalities follows from the inequality $ab \leq \frac{\tau a^2}{2} + \frac{b^2}{2\tau}$ and the fact that $\nabla h(x)$ is $L(:= C_g^2 L_f + C_f L_g)$ -smooth.

For the second term on the right hand side of (42),

$$\begin{pmatrix} 1 + \frac{1}{2\tau} \end{pmatrix} \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \right)^{2} \mathbb{E} \left[\left\| P_{k}^{(1)} + P_{k}^{(2)} + P_{k}^{(3)} + P_{k}^{(4)} \right\|^{2} \right]$$

$$\leq \left(1 + \frac{1}{2\tau} \right) \left(4 \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \right)^{2} \frac{L^{2} \overline{c}^{2}}{n} \mathbb{E} \left[\left\| x_{k} - \mathbf{1} \otimes \overline{x}_{k} \right\|_{\mathbf{A}}^{2} \right] + 4 \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \right)^{2} \frac{C_{g}^{2} L_{f}^{2}}{n} \mathbb{E} \left[\left\| \mathbf{g}_{k} - \mathbf{z}_{k} \right\|^{2} \right] \right]$$

$$+ 4\alpha_{k}^{2} \frac{\|\mathbf{u}\|^{2}}{n^{2}} \overline{c}^{2} \mathbb{E} \left[\left\| \mathbf{y}_{k}' - \mathbf{v} \otimes \overline{y}_{k}' \right\|_{\mathbf{B}}^{2} \right] + 4 \frac{\|\mathbf{u}\|^{2}}{n^{2}} \alpha_{k}^{2} \mathbb{E} \left[\left\| \xi_{k} \right\|^{2} \right] \right)$$

$$\leq \left(1 + \frac{1}{2\tau} \right) \left(4 \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \right)^{2} \frac{L^{2} \overline{c}^{2} U_{1} \alpha_{k}^{2}}{n} + 4 \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n} \right)^{2} \frac{C_{g}^{2} L_{f}^{2}}{n} \mathbb{E} \left[\left\| \mathbf{g}_{k} - \mathbf{z}_{k} \right\|^{2} \right] + 4 \frac{\|\mathbf{u}\|^{2}}{n^{2}} \overline{c}^{2} U_{1} \alpha_{k}^{3} \right)$$

$$+ \left(1 + \frac{1}{2\tau} \right) 4 \frac{\|\mathbf{u}\|^{2}}{n^{2}} \frac{c_{b}^{2} 4n C_{f} C_{g}}{(1 - \tau_{\mathbf{B}})^{2}} \alpha_{k}^{2},$$

$$\tag{43}$$

where $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}$, the first inequality follows from Assumption 1(c) and the Lipschitz continuity of $\nabla f_j(\cdot)$, the second inequality follows from (25) and Lemma 5. In addition, by Lemma 4 and [17, Lemmas 4-5 in Chapter 2], there exists a constant U_2 such that

$$\mathbb{E}\left[\|\mathbf{g}_k - \mathbf{z}_k\|^2\right] \le U_2 \beta_k = U_2 \beta \alpha_k.$$
(44)

Combining (43) with (44), we have

$$\left(1 + \frac{1}{2\tau}\right) \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n}\right)^{2} \mathbb{E}\left[\left\|P_{k}^{(1)} + P_{k}^{(2)} + P_{k}^{(3)} + P_{k}^{(4)}\right\|^{2}\right]$$

$$\leq \left(1 + \frac{1}{2\tau}\right) \left(4\left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n}\right)^{2} \frac{L^{2} \overline{c}^{2} U_{1} \alpha_{k}^{2}}{n} + 4\left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_{k}}{n}\right)^{2} \frac{C_{g}^{2} L_{f}^{2}}{n} U_{2} \beta \alpha_{k} + 4 \frac{\|\mathbf{u}\|^{2}}{n^{2}} \overline{c}^{2} U_{1} \alpha_{k}^{3}\right)$$

$$+ \left(1 + \frac{1}{2\tau}\right) 4 \frac{\|\mathbf{u}\|^{2}}{n} \frac{c_{b}^{2} 4n C_{f} C_{g}}{n^{2}} \alpha_{k}^{2}$$

$$(45)$$

$$\leq 16 \left(1 + \frac{(C_g^2 L_f + C_f L_g)}{\mu^2} \right) \frac{\|\mathbf{u}\|^2}{n} \frac{c_b^2 C_f C_g}{(1 - \tau_{\mathbf{B}})^2} \alpha_k^2 + o(\alpha_k^2),$$
(46)

where

$$\tau = \frac{\mu^2}{2\left(C_g^2 L_f + C_f L_g\right)}.$$
(47)

For the third term on the right hand side of (42),

$$2\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\right)\mathbb{E}\left[\left\langle\bar{x}_{k}-x^{*},P_{k}^{(1)}+P_{k}^{(2)}+P_{k}^{(3)}+P_{k}^{(4)}\right\rangle\right]$$

$$\leq\tau_{1}\mathbb{E}\left[\left\|\bar{x}_{k}-x^{*}\right\|^{2}\right]+\frac{1}{\tau_{1}}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\right)^{2}\mathbb{E}\left[\left\|P_{k}^{(1)}+P_{k}^{(2)}+P_{k}^{(3)}\right\|^{2}\right]+2\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\right)\mathbb{E}\left[\left\langle\bar{x}_{k}-x^{*},P_{k}^{(4)}\right\rangle\right]$$

$$\leq\tau_{1}\mathbb{E}\left[\left\|\bar{x}_{k}-x^{*}\right\|^{2}\right]+\frac{12}{\tau_{1}}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\right)^{2}\frac{L^{2}\bar{c}^{2}U_{1}\alpha_{k}^{2}}{n}+\frac{12}{\tau_{1}}\left(\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\alpha_{k}}{n}\right)^{2}\frac{C_{g}^{2}L_{f}^{2}}{n}U_{2}\beta\alpha_{k}+\frac{12}{\tau_{1}}\frac{\|\mathbf{u}\|^{2}}{n^{2}}\bar{c}^{2}U_{1}\alpha_{k}^{3}$$

$$+\frac{\|\mathbf{u}\|^{2}c_{b}c_{0}}{n^{2}(1-\tau_{\mathbf{B}})}\left(\frac{c_{b}^{2}nC_{g}C_{f}}{(1-\tau_{\mathbf{B}})^{2}}+4nC_{g}C_{f}\right)\alpha_{k}^{3}$$

$$\leq\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu\alpha_{k}}{4n}\mathbb{E}\left[\left\|\bar{x}_{k}-x^{*}\right\|^{2}\right]+\left(\frac{48\mathbf{u}^{\mathsf{T}}\mathbf{v}C_{g}^{2}L_{f}^{2}\mu U_{2}\beta}{n^{2}}+\frac{48\|\mathbf{u}\|^{2}\bar{c}^{2}U_{1}}{n\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu}\right)\alpha_{k}^{2}+o(\alpha_{k}^{2}),\qquad(48)$$

where c_0 is some constant scalar,

$$\tau_1 = \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \mu}{4n} \alpha_k,\tag{49}$$

the first inequality follows from the fact $ab \leq \frac{\tau_1 a^2}{2} + \frac{b^2}{2\tau_1}$ for any positive scalar τ_1 , the second inequality follows from (45) and Lemma 6 in Appendix.

Substitute (46)-(49) into (42),

$$\mathbb{E}\left[\|\bar{x}_{k+1} - x^*\|^2\right] \le \left(1 - \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu\alpha_k}{2n}\right) \mathbb{E}\left[\|\bar{x}_k - x^*\|^2\right] + o\left(\alpha_k^2\right) + \left[\frac{48\mathbf{u}^{\mathsf{T}}\mathbf{v}C_g^2 L_f^2 \mu U_2 \beta}{n^2} + \frac{48\|\mathbf{u}\|^2 \overline{c}^2 U_1}{n\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu} + 16\left(1 + \frac{\left(C_g^2 L_f + C_f L_g\right)}{\mu^2}\right) \frac{\|\mathbf{u}\|^2}{n} \frac{c_b^2 C_f C_g}{(1 - \tau_{\mathbf{B}})^2}\right] \alpha_k^2.$$

Then by [17, Lemmas 4-5 in Chapter 2],

$$\mathbb{E}\left[\|\bar{x}_{k+1} - x^*\|^2\right] = \mathcal{O}\left(\alpha_k\right) \text{ if } \alpha_k = a/(k+b)^{\alpha}, \alpha \in (1/2, 1),$$

and

$$\mathbb{E}\left[\|\bar{x}_k - x^*\|^2\right] = \mathcal{O}\left(\frac{1}{k}\right) \text{ if } \alpha_k = a/(k+b), a > \frac{2n}{\mathbf{u}^{\mathsf{T}}\mathbf{v}\mu}.$$

The proof is complete.

Theorem 2 shows that AB-DSCSC achieves the convergence rate $\mathcal{O}\left(\frac{1}{k}\right)$ for finding the optimal solution, which is also the optimal convergence rate of stochastic gradient descent for non-compositional stochastic strongly convex optimization [23].

The next theorem studies the asymptotic normality of AB-DSCSC.

Theorem 3. Let stepsizes $\alpha_k = a/(k+b)^{\alpha}$, $\beta_k = \beta \alpha_k$, where $a > 0, b \ge 0$, $\alpha \in (1/2, 1)$, $\beta \in (0, 1/a)$ and $a/(1+b)^{\alpha} \le \frac{n}{\mathbf{u}^{\mathsf{T}} \mathbf{v} \mu} \min\{1, 2/(C_g^2 L_f + C_f L_g)\}$. Suppose

- (a) Assumptions 1-2 hold;
- (b) h(x) is μ -strongly convex;
- (c) there exist scalar C and matrix \mathbf{H} such that

$$\left\|\nabla h(x) - \frac{1}{n}\mathbf{H}(x - x^*)\right\| \le C \|x - x^*\|^{1+\gamma}, \quad \forall x \in \mathbb{R}^d,$$

where $\gamma \in (0,1]$ satisfies that $\sum_{k=1}^{\infty} \frac{\alpha_k^{(1+\gamma)/2}}{\sqrt{k}} < \infty;$

(d) for any $i \in \mathcal{V}$, there exist scalar C_i and matrix \mathbf{T}_i such that

$$\left\|\nabla f_{i}(y) - \nabla f_{i}(y') - \mathbf{T}_{i}\left(y - y'\right)\right\| \leq C_{i}\|y - y'\|^{1+\gamma}, \quad \forall y, y' \in \mathbb{R}^{p},$$

(e) for any $i \in \mathcal{V}$, $G_i(\cdot; \phi)$ is Lipschitz continuous with coefficient L'_g , that is

$$\left\|G_{i}(x;\phi) - G_{i}(x';\phi)\right\| \leq L'_{g}\|x - x'\|, \quad \forall y, y' \in \mathbb{R}^{p}.$$

Then for any $i \in \mathcal{V}$,

$$\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \begin{pmatrix} x_{i,t} - x^* \\ \sum_{j=1}^{n} \nabla g_j(x^*) \mathbf{T}_j(z_{j,t} - g_j(x_{j,t})) \\ n \end{pmatrix} \xrightarrow{d} N \begin{pmatrix} \mathbf{0}, \begin{pmatrix} \mathbf{H}^{-1} \left(\mathbf{S}_1 + \mathbf{S}_2\right) \left(\mathbf{H}^{-1}\right)^{\mathsf{T}} & -\frac{1}{n} \mathbf{H}^{-1} \mathbf{S}_2 \\ -\frac{1}{n} \mathbf{S}_2 \left(\mathbf{H}^{-1}\right)^{\mathsf{T}} & \frac{1}{n^2} \mathbf{S}_2 \end{pmatrix} \end{pmatrix}, \quad (50)$$

where $\mathbf{S}_1 = \operatorname{Cov}\left(\nabla G_j(x^*;\phi_j)\nabla F_j(g(x^*);\zeta_j)\right), \ \mathbf{S}_2 = \operatorname{Cov}\left(\sum_{j=1}^n \nabla g_j(x^*)\mathbf{T}_jG_j(x^*;\phi_j)\right).$

Proof. By (25),

$$\mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=0}^{k-1} (\bar{x}_t - x^*) - \frac{1}{\sqrt{k}}\sum_{t=0}^{k-1} (x_{i,t} - x^*)\right\|\right] \\ \leq \frac{1}{\sqrt{k}}\sum_{t=0}^{k-1} \sqrt{\mathbb{E}\left[\|x_t - \mathbf{1} \otimes \bar{x}_t\|^2\right]} \leq \frac{\sqrt{U_1}}{\sqrt{k}}\sum_{t=0}^{k-1} \alpha_t \to 0$$

Then by Slutsky's theorem, it is sufficient to show

$$\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \left(\begin{array}{c} \bar{x}_{t} - x^{*} \\ \underline{\sum_{j=1}^{n} \nabla g_{j}(x^{*}) \mathbf{T}_{j}(z_{j,t} - g_{j}(x_{j,t}))}{n} \end{array} \right) \xrightarrow{d} N \left(\mathbf{0}, \left(\begin{array}{c} \mathbf{H}^{-1} \left(\mathbf{S}_{1} + \mathbf{S}_{2} \right) (\mathbf{H}^{-1})^{\mathsf{T}} & -\frac{1}{n} \mathbf{H}^{-1} \mathbf{S}_{2} \\ -\frac{1}{n} \mathbf{S}_{2} (\mathbf{H}^{-1})^{\mathsf{T}} & \frac{1}{n^{2}} \mathbf{S}_{2} \end{array} \right) \right)$$

Subtract x^* from both sides of (41),

$$\bar{x}_{k+1} - x^* = \bar{x}_k - x^* - \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_k}{n} \nabla h(\bar{x}_k) + \left(\frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_k}{n}\right) \left(P_k^{(1)} + P_k^{(2)} + P_k^{(3)} + P_k^{(4)}\right) = \left(\mathbf{I}_d - \tilde{\alpha}_k \frac{1}{n} \mathbf{H}\right) (\bar{x}_k - x^*) - \tilde{\alpha}_k \frac{1}{n} \sum_{j=1}^n \nabla g_j(x^*) \mathbf{T}_j \left(z_{j,k} - g_j\left(x_{j,k}\right)\right) + \tilde{\alpha}_k \left(P_k^{(0)} + P_k^{(1)} + P_k^{(3)} + P_k^{(4)}\right),$$
(51)

where $\tilde{\alpha}_k = \frac{\mathbf{u}^{\mathsf{T}} \mathbf{v} \alpha_k}{n}$,

$$P_k^{(0)} = -\left(\nabla h(\bar{x}_k) - \frac{1}{n}\mathbf{H}(\bar{x}_k - x^*)\right) + \frac{1}{n}\sum_{j=1}^n \nabla g_j(x^*)\mathbf{T}_j\left(z_{j,k} - g_j\left(x_{j,k}\right)\right) + P_k^{(2)}.$$
 (52)

According to recursion (27) and the definition of β_k ,

$$z_{i,k+1} - g_i\left(x_{i,k+1}\right) = \left(1 - \frac{n\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\tilde{\alpha}_k\right)\left(z_{i,k} - g_i\left(x_{i,k}\right)\right) + G_{i,k+1}^{(1)} - g_i(x_{i,k+1}) + (1 - \beta_k)\left(g_i(x_{i,k}) - G_{i,k+1}^{(2)}\right) + G_{i,k+1}^{(2)}\right)$$

where $G_{i,k+1}^{(1)} = G_i(x_{i,k+1}; \phi'_{i,k+1}), \ G_{i,k+1}^{(2)} = G_i(x_{i,k}; \phi'_{i,k+1})$. Combining above equation with (51), Δ

$$\Delta_{k+1} = \left(\mathbf{I}_{2d} - \tilde{\alpha}_k \mathbf{H}_{\theta}\right) \Delta_k + \tilde{\alpha}_k \eta_k^{(1)} + \tilde{\alpha}_k \left(\eta_k^{(2)} + \eta_k^{(3)}\right),\tag{53}$$

where

here

$$\Delta_{k} = \begin{pmatrix} \bar{x}_{k} - x^{*} \\ \frac{\sum_{j=1}^{n} \nabla g_{j}(x^{*}) \mathbf{T}_{j}(z_{j,k} - g_{j}(x_{j,k}))}{n} \end{pmatrix}, \quad \mathbf{H}_{\theta} = \begin{pmatrix} \frac{1}{n} \mathbf{H} & \mathbf{I}_{d} \\ \mathbf{0} & \frac{n\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}} \mathbf{I}_{d} \end{pmatrix},$$

$$\eta_{k}^{(1)} = \begin{pmatrix} P_{k}^{(4)} \\ \frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}} \sum_{j=1}^{n} \nabla g_{j}(x^{*}) \mathbf{T}_{j} \left(G_{j}(x^{*}; \phi_{j,k+1}^{'}) - g_{j}(x^{*}) \right) \end{pmatrix}, \quad \eta_{k}^{(2)} = \begin{pmatrix} P_{k}^{(0)} + P_{k}^{(1)} + P_{k}^{(3)} \\ \mathbf{0} \end{pmatrix}, \quad (54)$$

and

$$\eta_{k}^{(3)} = \left(\begin{array}{c} \mathbf{0} \\ \sum_{j=1}^{n} \nabla g_{j}(x^{*}) \mathbf{T}_{j} \left(\frac{G_{j,k+1}^{(1)} - g_{j}(x_{j,k+1}) + (1-\beta_{k}) \left(g_{j}(x_{j,k}) - G_{j,k+1}^{(2)} \right)}{n \tilde{\alpha}_{k}} - \frac{\beta}{\mathbf{u}^{\mathsf{T}} \mathbf{v}} \left(G_{j}(x^{*}; \phi_{j,k+1}') - g_{j}(x^{*}) \right) \right) \right).$$

Denote $\mathbf{M}(k,t) = \tilde{\alpha}_t \sum_{l_1=t}^k \prod_{l_2=t+1}^{l_1} \left(\mathbf{I}_{2d} - \tilde{\alpha}_k \mathbf{H}_{\theta} \right), \quad \mathbf{N}(k,t) = \mathbf{M}(k,t) - \mathbf{H}_{\theta}^{-1}.$ Then by the recursion (53),

$$\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \Delta_{t} = \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbf{H}_{\theta}^{-1} \eta_{t}^{(1)} + \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbf{N}(k,t) \eta_{t}^{(1)} + \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbf{M}(k,t) \eta_{t}^{(2)} + \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbf{M}(k,t) \eta_{t}^{(3)} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$
(55)

It is easy to show that the second term on the right hand side of (55) converge to 0 in probability, see Lemma 7 in Appendix for details. For the third term on the right hand side of (55),

$$\begin{split} & \mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{M}(k,t)\eta_{t}^{(2)}\right\|\right] \\ & \leq \frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\mathbf{M}(k,t)\|\left(\mathbb{E}\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\nabla g_{j}(x^{*})\left(\nabla f_{j}(g_{j}(x_{j,t}))-\nabla f_{j}(z_{j,t})-\mathbf{T}_{j}\left(z_{j,t}-g_{j}\left(x_{j,t}\right)\right)\right)\right\|\right] \\ & +\mathbb{E}\left[\left\|\nabla h(\bar{x}_{t})-\frac{1}{n}\mathbf{H}(\bar{x}_{t}-x^{*})\right\|\right]+\mathbb{E}\left[\left\|\frac{1}{n}\sum_{j=1}^{n}\left(\nabla g_{j}(x_{j,t})-\nabla g_{j}(x^{*})\right)\left(\nabla f_{j}(g_{j}(x_{j,t}))-\nabla f_{j}(z_{j,t})\right)\right\|\right]\right) \\ & +\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\|\mathbf{M}(k,t)\|\|\mathbb{E}\left[\left\|P_{t}^{(1)}+P_{t}^{(3)}\right\|\right] \\ & \leq \frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\|\mathbf{M}(k,t)\|\|\left(\frac{1}{n}\sum_{j=1}^{n}\|\nabla g_{j}(x^{*})\|\|\mathbb{E}\left[\|z_{j,t}-g_{j}\left(x_{j,t}\right)\right\|^{1+\gamma}\right]+\mathbb{E}\left[\|\bar{x}_{t}-x^{*}\|^{1+\gamma}\right] \\ & +\frac{1}{n}\sum_{j=1}^{n}L_{g}L_{f}\sqrt{\mathbb{E}\left[\|x_{j,t}-x^{*}\|^{2}\right]\mathbb{E}\left[\|g_{j}(x_{j,t})-z_{j,t}\|^{2}\right]}\right)+\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\|\mathbf{M}(k,t)\|\|\mathbb{E}\left[\left\|P_{t}^{(1)}+P_{t}^{(3)}\right\|\right] \\ & = \frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\|\mathbf{M}(k,t)\|\|\mathcal{O}\left(\alpha_{t}^{(1+\gamma)/2}+\alpha_{t}\right), \end{split}$$

where the first inequality follows from the definitions of $\eta_t^{(2)}$, $P_t^{(0)}$ and $P_t^{(2)}$ in (54), (52) and (41), the second inequality follows from condition (d), Assumption 1 (a) and the Hölder inequality, the equality follows from (25), Lemma 4, Theorem 2 and (43). Then by the boundedness of $\mathbf{M}(k,t)$ [18, Lemma 1 (ii)], the fact $\sum_{k=1}^{\infty} \frac{\alpha_k^{(1+\gamma)/2}}{\sqrt{k}} < \infty$ and Kronecker Lemma, we have

$$\mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{M}(k,t)\eta_{t}^{(2)}\right\|\right] \leq \frac{1}{\sqrt{k}}\sum_{t=1}^{k}\|\|\mathbf{M}(k,t)\|\|\mathcal{O}\left(\alpha_{t}^{(1+\gamma)/2}+\alpha_{t}\right)\longrightarrow 0.$$

Noting that $\eta_k^{(3)}$ is a martingale difference sequence adapted to the filtration \mathcal{F}_k (34), the fourth term on the right hand side of (55)

$$\begin{split} & \mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{M}(k,t)\eta_{t}^{(3)}\right\|^{2}\right] \\ &= \frac{1}{k}\sum_{t=1}^{k}\mathbb{E}\left[\left\|\mathbf{M}(k,t)\sum_{j=1}^{n}\nabla g_{j}(x^{*})\mathbf{T}_{j}\left(\frac{G_{j,t+1}^{(1)} - g_{j}(x_{j,t+1}) - \left(G_{j,t+1}^{(2)} - g_{j}(x_{j,t})\right)\right)}{n\tilde{\alpha}_{t}} \\ &+ \frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\left(G_{j,t+1}^{(2)} - g_{j}(x_{j,t}) - \left(G_{j}(x^{*};\phi_{j,t+1}') - g_{j}(x^{*})\right)\right)\right)\right\|^{2}\right] \\ &\leq \frac{1}{k}\sum_{t=1}^{k}\frac{1}{n}\sum_{j=1}^{n}\|\|\mathbf{M}(k,t)\|^{2}\|\nabla g_{j}(x^{*})\|^{2}\|\mathbf{T}_{j}\|^{2}4\left(\left(\frac{L_{g}'}{\tilde{\alpha}_{t}}\right)^{2}\mathbb{E}\left[\|x_{j,t+1} - x_{j,t}\|^{2}\right] + \left(\frac{n\beta L_{g}'}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right)^{2}\mathbb{E}\left[\|x_{j,t} - x^{*}\|^{2}\right]\right) \\ &= \frac{1}{k}\sum_{t=1}^{k}\mathcal{O}\left(\alpha_{t}\right), \end{split}$$

where the inequality follows from the Lipschitz continuity of $G_j(\cdot; \phi)$, the second equality follows from (25), Theorem 2 and the fact

$$\mathbb{E}\left[\left\|x_{j,t+1} - x_{j,t}\right\|^{2}\right] \leq 3\left(\mathbb{E}\left[\left\|x_{j,t+1} - \bar{x}_{t+1}\right\|^{2}\right] + \mathbb{E}\left[\left\|x_{j,t} - \bar{x}_{t}\right\|^{2}\right] + \mathbb{E}\left[\left\|\bar{x}_{t+1} - \bar{x}_{t}\right\|^{2}\right]\right) = \mathcal{O}\left(\alpha_{t}^{2}\right).$$

Then by Kronecker Lemma, $\mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{M}(k,t)\eta_{t}^{(3)}\right\|^{2}\right] = \frac{1}{k}\sum_{t=1}^{k}\mathcal{O}\left(\alpha_{t}\right) \longrightarrow 0.$

It is left to show the asymptotic normality of the first term on the right hand side of (55). Indeed, by the similar way to [32, Lemma 6 in Appendix B], we may obtain that

$$\mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=0}^{k-1}P_{k}^{(4)}-\frac{1}{\sqrt{k}}\sum_{t=0}^{k-1}\left(\frac{\mathbf{1}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\epsilon_{t}^{*}\right\|^{2}\right]\longrightarrow0,\quad\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\left(\frac{\mathbf{1}^{\mathsf{T}}}{n}\otimes\mathbf{I}_{d}\right)\epsilon_{t}^{*}\stackrel{d}{\to}N\left(\mathbf{0},\frac{1}{n^{2}}\mathbf{S}_{1}\right)$$

and

$$\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\sum_{j=1}^{n}\nabla g_{j}(x^{*})\mathbf{T}_{j}\left(G_{j}(x^{*};\phi_{j,k+1}^{'})-g_{j}(x^{*})\right)\overset{d}{\rightarrow}N\left(\mathbf{0},\left(\frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right)^{2}\mathbf{S}_{2}\right),$$

where

$$\epsilon_t^* = [(\nabla G_1(x^*; \phi_{1,t}) \nabla F_1(g(x^*); \zeta_{1,t}) - \nabla g_1(x^*; \phi_{1,t}) \nabla f_1(g(x^*)))^{\mathsf{T}}, \cdots, \\ (\nabla G_n(x^*; \phi_{n,t}) \nabla F_n(g(x^*); \zeta_{n,t}) - \nabla g_n(x^*; \phi_{n,t}) \nabla f_n(g(x^*)))^{\mathsf{T}}]^{\mathsf{T}}.$$

Note that
$$\mathbf{H}_{\theta}^{-1} = \begin{pmatrix} n\mathbf{H}^{-1} & -\frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{\beta}\mathbf{H}^{-1} \\ \mathbf{0} & \frac{\mathbf{u}^{\mathsf{T}}\mathbf{v}}{n\beta}\mathbf{I}_{d} \end{pmatrix}$$
 and $\phi_{i,k}$ is independent of $\phi'_{i,k}$. Then
$$\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{H}_{\theta}^{-1}\eta_{t}^{(1)} \stackrel{d}{\longrightarrow} N\left(\mathbf{0}, \begin{pmatrix} \mathbf{H}^{-1}\left(\mathbf{S}_{1}+\mathbf{S}_{2}\right)\left(\mathbf{H}^{-1}\right)^{\mathsf{T}} & -\frac{1}{n}\mathbf{H}^{-1}\mathbf{S}_{2} \\ -\frac{1}{n}\mathbf{S}_{2}(\mathbf{H}^{-1})^{\mathsf{T}} & \frac{1}{n^{2}}\mathbf{S}_{2} \end{pmatrix}\right)\right).$$
The proof is complete.

The proof is complete.

Theorem 3 shows that Polyak-Ruppert averaged iterates of the proposed method converge in distribution to a normal random vector for any agent. Different from the traditional asymptotic normality results on SA based methods [3, 8], the asymptotic covariance matrix in (50) has two parts, $\mathbf{H}^{-1}\mathbf{S}_1(\mathbf{H}^{-1})^{\mathsf{T}}$ and $\mathbf{H}^{-1}\mathbf{S}_2(\mathbf{H}^{-1})^{\mathsf{T}}$, where the first one is induced by the randomness of gradient and the second one is induced by the randomness of the inner function. Indeed, the asymptotic normality on the SAA scheme for stochastic compositional optimization has been studied by Dentcheva et al. [5]. To the best of our knowledge, Theorem 3 is the first asymptotic normality result for the SA based method on distributed stochastic compositional optimization problem.

Experimental Results 4

We test the proposed method for two applications, i.e., model-agnostic meta learning problem and logistic regression problem.



Figure 1: Meta-training and meta-test.

4.1 Model-agnostic meta learning

Model-agnostic meta learning (MAML) is a powerful tool for learning a new task by using the prior experience from related tasks [9]. It is to find a good initialization parameter from similar learning tasks such that taking several gradient steps would produce good results on new tasks, and the optimizations model is

$$\min_{x \in \mathbb{R}^d} \frac{1}{M} \sum_{m=1}^M f_m \left(x - \alpha \nabla f_m(x) \right), \tag{56}$$

where $m = 1, 2, \dots, M$ is the index of training tasks, α is the adaptation stepsize, $f_m(x) = \mathbb{E}[F_m(x;\zeta_m)]$ is the loss function of task m. We illustrate the empirical performance of AB-DSCSC to solve MAML problem (56) and compare it with GP-DSCGD and GT-DSCGD [10].

The setting of MAML is as follows [2, 10]. Each task $m \in \mathcal{M} = \{1, 2, \dots, M\}$ maps the input b to a sine wave $s(b; a_m, \phi_m) := a_m \sin(b + \phi_m)$ where the amplitude a_m and phase ϕ_m of the sinusoid vary across tasks. The tasks' parameters a_m and ϕ_m are sampled uniformly from [0.1, 5] and $[0, 2\pi]$ respectively, input domain of b is uniform on [5, -5]. The regressor of $s(b; a_m, \phi_m)$ is a fully-connected neural network $\hat{s}(b; x)$, which consists of two hidden layers with 40 ReLU nodes. The loss function $f_m(z) = \mathbb{E}_b \left[\|\hat{s}(b; z) - a_m \sin(b + \phi_m)\|^2 \right]$ and the one-step adaptation stepsize $\alpha = 0.01$.

In this experiment, we generate a directed graph \mathcal{G} of 5 agents by adding random links to a ring network. Each agent is assigned with 200 training tasks, i.e. M = 1000 in problem (56). We utilize 2500 new tasks of sinusoidal regression to test the obtained parameters. For AB-DSCSC, stepsize $\alpha_k = 0.01, \beta_k = 0.8$ and communication graphs $\mathcal{G}_{\mathbf{A}} = \mathcal{G}_{\mathbf{BT}} = \mathcal{G}$. For GP-DSCGD and GT-DSCGD, stepsize $\eta = 0.03, \gamma = 3, \beta_k = 0.33$, and set the underlying graph² of \mathcal{G} as the communication graph. In each task, we use 10 samples for training and testing.

²The underlying graph of a directed graph \mathcal{G}' is an undirected graph obtained by replacing all directed edges of \mathcal{G}' with undirected edges.

We run AB-DSCSC, GP-DCSGD and GT-DCSGD for 5000 iterations and record their performance on the training loss and test loss in Figure 1, where the solid curve, dash-dot curve and dashed curve display the averaged training loss of AB-DSCSC, GT-DCSGD and GP-DCSGD over different agents respectively. We can observe from Figure 1 (left) that the three methods achieve similar performance. Figure 1 (right) depicts the test loss on new tasks after 10 gradient descent steps with the learned model parameters as the initialization. Again, the three methods achieve similar performance on the new tasks and they are well adaptable to new tasks as test loss decreasing quickly.

4.2 Conditional stochastic optimization

We consider a modified logistic regression problem, in which the inner and outer randomness are independent of each other [13],

$$\min_{x \in \mathbb{R}^d} h(x) = \frac{1}{n} \sum_{i=1}^n \frac{1}{m_i - m_{i-1}} \sum_{j=m_{i-1}+1}^{m_i} \log\left(1 + \exp\left(-b_j \left(\frac{1}{l} \sum_{s=1}^l \phi_s + a_j\right)^{\mathsf{T}} x\right)\right), \quad (57)$$

where n = 50, $m_i = 20i$, l = 10000, $a_j \sim N(\mathbf{0}, \mathbf{I}_d)$, $b_j \in \{1, -1\}$, $\phi_s \sim N(\mathbf{0}, \mathbf{I}_d)$. Obviously, problem (57) falls in the form of DSCO with inner function

$$g_i(x) = \left[-b_{m_{i-1}+1} \left(\frac{1}{l} \sum_{s=1}^l \phi_s + a_{m_{i-1}+1} \right)^{\mathsf{T}} x, \cdots, -b_{m_i} \left(\frac{1}{l} \sum_{s=1}^l \phi_s + a_{m_i} \right)^{\mathsf{T}} x \right]^{\mathsf{T}},$$

outer function $f_i(z) = \frac{1}{m_i - m_{i-1}} \sum_{j=m_{i-1}+1}^{m_i} \log(1 + \exp(z_j)), z_j$ is the *j*-th component of vector $z \in \mathbb{R}^{m_i}$.



Figure 2: Optimality gap and residual.

Similarly, we generate a directed graph \mathcal{G} of 50 agents by adding random links to a ring network, and set communication graphs $\mathcal{G}_{\mathbf{A}} = \mathcal{G}_{\mathbf{B}^{\intercal}} = \mathcal{G}$ for AB-DSCSC. The communication graph of GP-DCSGD and GT-DSCGD is also set as the underlying graph of \mathcal{G} . The stepsize

 $\alpha_k = 0.01/k^{0.55}$, $\beta_k = 0.8/k^{0.6}$ for AB-DSCSC and $\eta = 0.03, \gamma = 3, \beta_k = 0.33/k^{0.6}$ for GP-DCSGD and GT-DCSGD.

Note that problem (57) is a convex optimization problem, we solve it by centralized gradient descent and denote the optimal solution as x^* . Then, we run AB-DSCSC, GP-DCSGD and GT-DCSGD for 10000 iterations and record their performance on the averaged optimality gap $\frac{1}{n} \sum_{i=1}^{n} ||x_{i,k} - x^*||^2$ and average residual $\frac{1}{n} \sum_{i=1}^{n} (h(x_{i,k}) - h(x^*))$ in Figure 2. Obviously, the three methods can solve the problem efficiently and achieve similar performance.

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Appendix

Lemma 6. Let $\alpha_k = a/(k+b)^{\alpha}$, $a > 0, b \ge 0$, $\alpha \in (1/2, 1]$. Under Assumptions 1-2 and the condition that objective function h(x) is μ -strongly convex,

$$\mathbb{E}\left[\left\langle \bar{x}_{k} - x^{*}, -\alpha_{k} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \xi_{k} \right\rangle\right] \leq \frac{\|\mathbf{u}\| c_{b} c_{0}}{2n(1-\tau_{\mathbf{B}})} \left(\frac{c_{b}^{2} n C_{g} C_{f}}{(1-\tau_{\mathbf{B}})^{2}} + 4n C_{g} C_{f}\right) \alpha_{k}^{2},$$

where $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}, c_0 \text{ is some constant scalar.}$

Proof. Recall the definition $\bar{x}_k = \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \mathbf{x}_k$ in Lemma 2,

$$\bar{x}_k - x^* = \bar{x}_{k-1} - x^* - \alpha_{k-1} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \mathbf{y}_{k-1} = \bar{x}_1 - x^* - \sum_{t=1}^{k-1} \alpha_t \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \mathbf{y}_t,$$

and then

$$\mathbb{E}\left[\left\langle \bar{x}_{k} - x^{*}, -\alpha_{k} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \xi_{k} \right\rangle\right]$$

= $\mathbb{E}\left[\left\langle \bar{x}_{1} - x^{*} - \sum_{t=1}^{k-1} \alpha_{t} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \mathbf{y}_{t}, -\alpha_{k} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \xi_{k} \right\rangle\right]$
= $-\alpha_{k} \mathbb{E}\left[\left\langle \bar{x}_{1} - x^{*} - \sum_{t=1}^{k-1} \alpha_{t} \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \mathbf{y}_{t}, \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \sum_{t=1}^{k} \tilde{\mathbf{B}}(k, t) \epsilon_{t} \right\rangle\right],$

where $\epsilon_t = \mathbf{H}_t - \mathbf{J}_t$, \mathbf{H}_t and \mathbf{J}_t are defined in (10) and Lemma 2 respectively, the second equality follows from (31). Note that $\mathbb{E}\left[\left\langle \bar{x}_0 - x^*, \left(\frac{\mathbf{u}^{\intercal}}{n} \otimes \mathbf{I}_d\right) \tilde{\mathbf{B}}(k,t) \epsilon_t \right\rangle \middle| \mathcal{F}_t \right] = 0$ and

$$\mathbb{E}\left[\left\langle \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \mathbf{y}_{t_{1}}, \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right) \tilde{\mathbf{B}}(k, t_{2}) \epsilon_{t_{2}} \right\rangle \middle| \mathcal{F}_{t_{2}} \right] = 0 \ (t_{1} < t_{2}),$$

where \mathcal{F}_k is defined in (34). Then

$$\mathbb{E}\left[\left\langle \bar{x}_{k} - x^{*}, -\alpha_{k}\left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_{d}\right)\xi_{k}\right\rangle\right] \leq \alpha_{k}\sum_{t_{1}=1}^{t_{1}}\sum_{t_{2}=1}^{t_{1}}\alpha_{t_{1}}\frac{\|\mathbf{u}\|^{2}}{2n^{2}}\left\|\left|\tilde{\mathbf{B}}(k, t_{2})\right|\right\|\left(\mathbb{E}\left[\|\mathbf{y}_{t_{1}}\|^{2}\right] + \mathbb{E}\left[\|\epsilon_{t_{2}}\|^{2}\right]\right)\right)$$
$$\leq \alpha_{k}\sum_{t_{1}=1}^{k-1}\sum_{t_{2}=1}^{t_{1}}\alpha_{t_{1}}\frac{\|\mathbf{u}\|^{2}c_{b}}{2n^{2}}\tau_{\mathbf{B}}^{k-t_{2}}\left(\frac{c_{b}^{2}nC_{g}C_{f}}{(1-\tau_{\mathbf{B}})^{2}} + 4nC_{g}C_{f}\right)$$
$$\leq \frac{\|\mathbf{u}\|^{2}c_{b}c}{2n^{2}(1-\tau_{\mathbf{B}})}\left(\frac{c_{b}^{2}nC_{g}C_{f}}{(1-\tau_{\mathbf{B}})^{2}} + 4nC_{g}C_{f}\right)\alpha_{k}\alpha_{k-1},$$

where $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}$ and c is some constant scalar, the second inequality follows from (19), (20) and (33), the third inequality follows from Lemma 3. Noting that $\lim_{k\to\infty} \frac{\alpha_{k-1}}{\alpha_k} = 1$, there exists constant $c_0 > c$ such that

$$\mathbb{E}\left[\left\langle \bar{x}_k - x^*, -\alpha_k \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \xi_k \right\rangle\right] \le \frac{\|\mathbf{u}\|^2 c_b c_0}{2n^2 (1 - \tau_{\mathbf{B}})} \left(\frac{c_b^2 n C_g C_f}{(1 - \tau_{\mathbf{B}})^2} + 4n C_g C_f\right) \alpha_k^2$$

The proof is complete.

Lemma 7. Let $\alpha_k = a/(k+b)^{\alpha}$, a > 0, $b \ge 0$, $\alpha \in (1/2, 1)$. Suppose that

- (a) Assumptions 1-2 hold;
- (b) for any $i \in \mathcal{V}$, there exist scalar C_i and matrix \mathbf{T}_i such that

$$\begin{split} \left\|\nabla f_i(y) - \nabla f_i(y') - \mathbf{T}_i\left(y - y'\right)\right\| &\leq C_i \|y - y'\|^{1+\gamma}, \quad \forall y, y' \in \mathbb{R}^p, \\ where \ \gamma \in (0, 1] \ satisfies \ that \ \sum_{k=1}^{\infty} \frac{\alpha_k^{(1+\gamma)/2}}{\sqrt{k}} < \infty. \end{split}$$

Denote $\mathbf{M}(k,t) = \tilde{\alpha}_t \sum_{l_1=t}^k \prod_{l_2=t+1}^{l_1} (\mathbf{I}_{2d} - \tilde{\alpha}_k \mathbf{H}_{\theta}), \ \mathbf{N}(k,t) = \mathbf{M}(k,t) - \mathbf{H}_{\theta}^{-1}$ and

$$\eta_t^{(1)} = \begin{pmatrix} \left(-\frac{n}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right) \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d\right) \xi_t \\ \frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}} \sum_{j=1}^n \nabla g_j(x^*) \mathbf{T}_j \left(G_j(x^*; \phi'_{i,t+1}) - g_j(x^*)\right) \end{pmatrix}.$$

We have

$$\lim_{k \to \infty} \mathbb{E}\left[\left\| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbf{N}(k, t) \eta_t^{(1)} \right\|^2 \right] = 0.$$

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Proof. Note that

$$\eta_t^{(1)} = \begin{pmatrix} \left(-\frac{n}{\mathbf{u}^{\mathsf{T}} \mathbf{v}} \right) \left(\frac{\mathbf{u}^{\mathsf{T}}}{n} \otimes \mathbf{I}_d \right) \xi_t \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \frac{\beta}{\mathbf{u}^{\mathsf{T}} \mathbf{v}} \sum_{j=1}^n \nabla g_j(x^*) \mathbf{T}_j \left(G_j(x^*; \phi'_{i,t+1}) - g_j(x^*) \right) \end{pmatrix}$$

and

$$\mathbb{E}\left[\left<\xi_{t_{1}},\xi_{t_{2}}\right>\right] = \mathbb{E}\left[\mathbb{E}\left[\left<\xi_{t_{1}},\xi_{t_{2}}\right>\left|\mathcal{F}_{\min\{t_{1},t_{2}\}}\right]\right] = \mathbb{E}\left[\left<\xi_{\min\{t_{1},t_{2}\}},\sum_{l=1}^{\min\{t_{1},t_{2}\}}\tilde{\mathbf{B}}(\max\{t_{1},t_{2}\},l)\epsilon_{l}\right>\right]\right] \quad (t_{1} \le t_{2}),$$
$$\mathbb{E}\left[\left\left|\mathcal{F}_{\min\{t_{1},t_{2}\}}\right] = 0 \quad (t_{1} \ne t_{2}),$$

where \mathcal{F}_t is defined in (34). Then

$$\begin{split} \mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{N}(k,t)\eta_{t}^{(1)}\right\|^{2}\right] &= \mathbb{E}\left[\mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{N}(k,t)\eta_{t}^{(1)}\right\|^{2}\left|\mathcal{F}_{\min\{t_{1},t_{2}\}}\right]\right] \\ &\leq \left(\frac{\|\mathbf{u}\|}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right)^{2}\frac{4}{k}\sum_{t_{1}=1}^{k}\sum_{t_{2}=t_{1}}^{k}\|\|\mathbf{N}(k,t_{1})\|\|\|\mathbf{N}(k,t_{2})\|\|\sum_{l=1}^{t_{1}}\left\|\left\|\tilde{\mathbf{B}}(t_{2},l)\right\|\|\mathbb{E}\left[\|\xi_{t_{1}}\|\|\epsilon_{l}\|\right] \\ &+ \frac{2}{k}\sum_{t=1}^{k}\|\|\mathbf{N}(k,t)\|^{2}\mathbb{E}\left[\left\|\frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\sum_{j=1}^{n}\nabla g_{j}(x^{*})\mathbf{T}_{j}\left(G_{j}(x^{*};\phi_{i,t}') - g_{j}(x^{*})\right)\right\|^{2}\right] \\ &\leq c_{b}c_{N}(c_{b}^{2}+1)nC_{f}C_{g}\left(\frac{\|\mathbf{u}\|}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right)^{2}\frac{1}{(1-\tau_{\mathbf{B}})^{4}}\frac{8}{k}\sum_{t_{1}=1}^{k}\|\|\mathbf{N}(k,t_{1})\|\| \\ &+ n\left(\frac{\beta}{\mathbf{u}^{\mathsf{T}}\mathbf{v}}\right)^{2}\left(\sum_{j=1}^{n}\|\nabla g_{j}(x^{*})\|^{2}\|\mathbf{T}_{j}\|^{2}\right)V_{g}c_{N}\frac{2}{k}\sum_{t=1}^{k}\|\|\mathbf{N}(k,t)\|\|, \end{split}$$

where $c_b = \max\left\{\overline{c}, \frac{\|\mathbf{B}-\mathbf{I}_n\|_{\mathbf{B}}}{\tau_{\mathbf{B}}}\overline{c}\right\}, c_N = \sup_{k,t} \|\|\mathbf{N}(k,t)\|\|$, the second inequality follows from the fact $\sup_{k,t} \|\|\mathbf{N}(k,t)\|\| < \infty$ [18, Lemma 1 (ii)], (19), (33), Lemma 5 (i) and Assumption 1 (c). By [18, Lemma 1 (ii)],

$$\lim_{k \to \infty} \frac{1}{k} \sum_{t=1}^{k} ||| \mathbf{N}(k, t) ||| = 0,$$

which implies $\lim_{k\to\infty} \mathbb{E}\left[\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}\mathbf{N}(k,t)\eta_{t}^{(1)}\right\|^{2}\right] = 0$. The proof is complete.