

ABACUS: frequent pAttern mining-BAsed Community discovery in mUltidimensional networkS

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Abstract Community Discovery in complex networks is the problem of detecting, for each node of the network, its membership to one of more groups of nodes, the *communities*, that are densely connected, or highly interactive, or, more in general, *similar*, according to a similarity function. So far, the problem has been widely studied in monodimensional networks, i.e. networks where only one connection between two entities may exist. However, real networks are often multidimensional, i.e., multiple connections between any two nodes may exist, either reflecting different kinds of relationships, or representing different values of the same type of tie. In this context, the problem of Community Discovery has to be redefined, taking into account multidimensional structure of the graph. We define a new concept of community that groups together nodes sharing memberships to the same monodimensional communities in the different single dimensions. As we show, such communities are meaningful and able to group nodes even if they might not be connected in any of the monodimensional networks. We devise ABACUS (frequent pAttern mining-BAsed Community discoverer in mUltidimensional networkS), an algorithm that is able to extract multidimensional communities based on the extraction of frequent closed itemsets from monodimensional community memberships. Experiments on two different real multidimensional networks confirm the meaningfulness of the introduced concepts, and open the way for a new class of algorithms for community discovery that do not rely on the dense connections among nodes.

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1 Introduction

Inspired by real-world scenarios such as social networks, technology networks, the Web, biological networks, and so on, in the last years, wide, multidisciplinary, and extensive research has been devoted to the extraction of non trivial knowledge from networks. Predicting future links among the nodes or actors of a network ([13]), detecting and studying the diffusion of information among them ([23,29]), mining frequent patterns of nodes' behaviors ([4, 47,20]), are only a few examples of tasks in the field of Complex Network Analysis, that includes, among all, physicians, mathematicians, computer scientists, sociologists, economists and biologists. The data at the basis of this field of research is huge, heterogeneous, and semantically rich, and this allows to identify many properties and behaviors of the actors involved in a network. One crucial task at the basis of Complex Network Analysis is Community Discovery, i.e., the discovery of a group of nodes densely connected, or highly related. Many techniques have been proposed to identify communities in networks ([28,21]), allowing the detection of hierarchical connections, influential nodes in communities, or just groups of nodes that share some properties or behaviors. In order to do so, the connections among the nodes of a network were so far posed at the center of investigation, since they play a key role in the study of the network structure, evolution, and behavior.

Nowadays, most of the work done in the literature is limited to a very simplified perspective of such relations, focusing only on whether two nodes are connected or not, and possibly assigning a strength to this connection. In the real world, however, this is not always enough to model all the available information about the interactions between actors, including their multiple preferences, their multifaceted behaviors, and their complex interactions. While multiple types of connections among actors could still be represented into a monodimensional network, by collapsing all connections to one type and potentially affecting a measure of tie strength, a more sophisticated analysis of the network structure, which could maintain information on the semantic differences in how actors are connected, would help all the techniques to provide more meaningful communities.

To this aim, in this paper we deal with *multidimensional networks*, i.e. networks in which multiple connections may exist between a pair of nodes, reflecting various interactions (i.e., dimensions) between them. Multidimensionality in real networks may be expressed by either different types of connections (two persons may be connected because they are friends, colleagues, they play together in a team, and so on), or different quantitative values of one specific relationship (co-authorship between two authors may occur in several different years, for example).

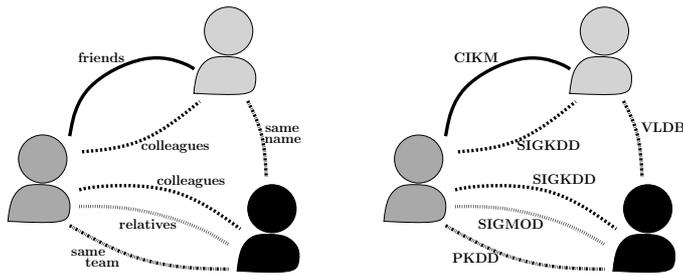


Fig. 1 Example of multidimensional networks

This distinction is reported in Figure 1, where on the left we have different types of links, while on the right we have different values (conferences) for one relationship (for example, co-authorship). We can also distinguish between *explicit* or *implicit* dimensions, the former being relationships explicitly set by the nodes (friendship, for example), while the latter being relationships inferred by the analyst, that may link two nodes according to their similarity or other principles (two users may be passively linked if they wrote a post on the same topic).

In this scenario, we deal with the problem of *Multidimensional Community Discovery*, i.e. the problem of detecting communities of actors in a multidimensional network. We define a new concept of multidimensional community that groups nodes sharing their membership to the same monodimensional communities in the same single dimensions. This concept gives us the possibility to leverage traditional monodimensional community discovery algorithms. It then allows us to define the lattice of multidimensional communities as function of the subset of dimensions for which the monodimensional community memberships of nodes are shared. Each multidimensional community can then be represented by the associated subset of dimensions, providing a semantic meaning to the community. Note that while the problem of finding cross-dimensional or cross-network structures is not new [15, 5, 46], our definition of multidimensional community differs from the previous ones. In fact, using this definition, a multidimensional community could be unconnected, i.e. composed of nodes which are not directly connected in any of the dimensions. This represents a complex phenomenon that can be seen in the real world: not all the people in a social community are necessarily connected directly, and, if they share their memberships in more than one dimension, they can be seen as a (potentially unconnected) group of highly related (both positively and negatively) people.

We devise ABACUS (frequent pAttern mining-BAsed Community discoverer in mUltidimensional networkS), an algorithm that extracts multidimensional communities such as the one in Figure 2 working in four steps:

1. Each dimension is treated separately and monodimensional communities are extracted

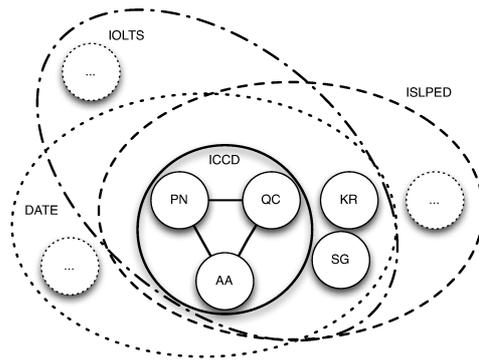


Fig. 2 An example of a real multidimensional community found in DBLP by our algorithm ABACUS, that other methods are not able to detect. Nodes in the community: Amit Agarwal, Qikai Chen, Swaroop Ghosh, Patrick Ndai, Kaushik Roy.

2. Each node is labeled with a list of pairs (dimension, community the node belongs to in that dimension)
3. Each pair is treated as an item and a frequent closed itemset mining algorithm is applied
4. Frequent closed itemsets represent multidimensional communities described by the itemsets

ABACUS is based on existing monodimensional algorithms for community discovery (used as a parameter), and on the extraction of frequent closed itemsets, that, in our scenario, represent the multidimensional description of the communities.

Our main contribution can be then summarized as follows: we introduce the new concept of multidimensional communities, and the ABACUS algorithm to extract them (Section 4); we show the applicability of ABACUS to real world multidimensional networks (Section 5), together with a comparison with previous approaches to the problem of community discovery in multidimensional network.

2 Related work

Detecting communities in networks has been studied from many angles. Two comprehensive surveys on the topic can be found in [21,28]. From one side, a community has been defined as a set of nodes with a high density of links among them, and sparse connections with nodes outside the community. The papers working with this quantitative definition rely on information theory principles [35] or on the notion of modularity [19], which is a function defined to detect the ratio between intra- and inter-community number of edges. Modularity is widely used in many works, and several algorithms have been proposed to extract high modularity partitioning of a network: one of them is a greedy optimization able to scale up to networks with billions of edges [9].

From another side, communities have been approached looking at the statistical properties of the graph. In [24], a framework for the detection of overlapping communities, i.e. communities allowing the vertices to be in more than one community, is presented. The framework is based on the “split betweenness” concept: vertices and edges are ranked by their betweenness centrality (the portion of shortest path in which they appear) and then split in order to form a transformed network, where classical algorithms can be used to detect communities. The resulting communities are then merged in order to find overlaps. Another class of approaches relies on the propagation in the network of a label [39] or a particular definition of structure (usually a clique [34]). The first approach is known for being a quasi linear solution for the problem, the second one allows to find overlapping communities. One algorithm that maximizes quality and quantity measures on its results is InfoMap [41], a random walk-based algorithm. An emerging novel problem definition can be found in [1], in which the authors state that community discovery algorithms should not group nodes but edges, emphasizing the role of the relation residing in a community. Previously described methods have focused on both unweighted or weighted graphs, but still considering the network as a monodimensional entity. Only since recently, multidimensionality has started to be taken into account in network analysis. A few examples of studies are: link prediction in networks with positive and negative links [27] or in multidimensional networks [40]; a statistical analysis over different kinds of relations in the same network in an online game community [43]; analysis of structural properties of multidimensional networks [6, 8] and its applications to multidimensional hub analysis [7].

From a community discovery point of view, to the best of our knowledge, the main approaches to take into account multiple dimensions are three. In [31] the authors extend the definition of modularity to fit the multidimensional case, which they call “multislice”. However, no definition of “multidimensional community” is provided, nor the approach characterizes and analyzes the communities found. Instead, the authors use the multidimensional information to extract monodimensional communities. In [44] the authors create a machine learning procedure which detects the possible different latent dimensions among the entities in the network and uses them as features for the node classification algorithm. In other words, they use the multidimensional labels of some nodes to infer labels for other nodes, by means of edge clustering. Hence, multidimensionality is only present here in terms of node labels, but the input network is not multidimensional according to our definition (see Section 3), and the output is not in the form of multidimensional communities. In [5], a possible formulation of community discovery and characterization in multidimensional networks was given. A new measure was introduced to capture the interplay among the dimensions, that makes multidimensional communities emerge even where the connections among nodes reside in different dimensions. In this paper, we approach the problem from a similar angle, but focus on extracting communities using frequent itemset mining, and giving a semantic description to each multidimensional community as the subset of

dimensions used to characterize it. Resulting multidimensional communities may be different from the ones extracted in [5] and are navigable using the lattice extracted in the frequent itemset mining process.

Another work that deals with networks containing heterogenous information, but not multiple dimensions, is presented in [42], where the authors propose a method to generate net-clusters using links across multi-typed objects. This approach works on heterogeneous networks, i.e. networks where nodes may have different types (e.g. papers or authors), and does not deal with multidimensional networks, i.e. networks where edges may be of different types and two nodes may be connected by multiple edges.

The authors of [15] studied the problem of community mining in multi-relational networks. The problem setting, however, is different: the authors exploit the multi-relational links to evaluate the importance of the relations based on labeled examples, provided by a user as queries. Hence, they do not perform community detection, but rather extract the importance of each dimension for a given node, in the form of a weight.

The idea of applying closed frequent pattern mining to multi-relational data is not new. In [17], the authors extract all closed n -sets satisfying given piecewise (anti-)monotonic constraints, from n -ary relations. In [33], the authors presented a framework for constraint-based pattern mining in multi-relational databases, finding patterns not under (anti-)monotonic and closedness constraints, expressed over complex aggregates over multiple relations. However, both the works solve the technical problem of finding the frequent closed patterns, but do not apply this technique to the setting of multidimensional network analysis.

There are other works in the literature that deal with the extraction of knowledge across networks. In [46] and [48], for example, the authors deal with the problem of finding cross-graph quasi-cliques. This problem can be seen as a sub-problem of the one we deal with in this paper. However, our concept of community is independent from the density of the connections among the nodes. Other two papers [18,30] deal with the extraction of cliques with particular constraints: in the first work, the authors search cliques that remain cliques over time; in the second, cliques with homogeneous node attributes are found. They however do not deal with the community discovery problem.

Based on all the above, we believe that two approaches may be considered really related to our problem formulation, namely [31] and [5], thus we use these as baselines for comparison in Section 5.

3 Multidimensional networks

In the world as we know it we can see a large number of interactions and connections among information sources, events, people, or items, giving birth to complex networks. Enumerating all the possible networks detectable within our world, or their properties, would be difficult due to their number and heterogeneity, and it is not the scope of this paper. An excellent survey on complex

networks can be found in [32], where the author gives a good classification of networks into *social* (where, for example, we find on-line social network such as Facebook), *information* (such as for example citation networks), *technological* (among which we mention the power grid, the train routes, or the Internet), and *biological* (e.g., protein interaction networks) networks.

While all the example networks presented in [32] are monodimensional, in the real world it is possible to find many multidimensional networks: transportation networks (transport means are different dimensions), social networks (different online services may be seen as different dimensions connecting the same users), co-authorship networks (different venues as dimensions), constitute a short, non-exhaustive list of possible real-world examples.

3.1 A model for multidimensional networks

In its classical definition, a network is defined as a structure that is made up of a set of entities and connections among them. We want to extend this definition by allowing connections of different kinds, that we call *dimensions*.

We use a *multigraph* to model a multidimensional network and its properties. For the sake of simplicity, in our model we only consider undirected multigraphs and since we do not consider node labels, hereafter we use *edge-labeled undirected multigraphs*, denoted by a triple $\mathcal{G} = (V, E, L)$ where: V is a set of nodes; L is a set of labels; E is a set of labeled edges, i.e. the set of triples (u, v, d) where $u, v \in V$ are nodes and $d \in L$ is a label. Also, we use the term *dimension* to indicate *label*, and we say that a node *belongs to* or *appears in* a given dimension d if there is at least one edge labeled with d adjacent to it. We also say that an edge *belongs to* or *appears in* a dimension d if its label is d . We assume that given a pair of nodes $u, v \in V$ and a label $d \in L$ only one edge (u, v, d) may exist. Thus, each pair of nodes in \mathcal{G} can be connected by at most $|L|$ possible edges.

3.2 Real world dataset

We created two multidimensional networks from the well known digital bibliography database DBLP¹ and from a search engine query log².

- **DBLP** We extracted author-author relationships if two authors collaborated in writing at least one paper. The dimensions of this network are defined as the venues in which the paper was published, resulting in 2,536 conferences that took place in years 2000-2010 (all the editions of a conference are considered as one dimension). As the network was created in year 2012, we consider our temporal subset to be complete for years 2000-2010. We weighted each edge by the number of papers published by the

¹ <http://dblp.uni-trier.de/xml>

² <http://www.gregsadetsky.com/aol-data>

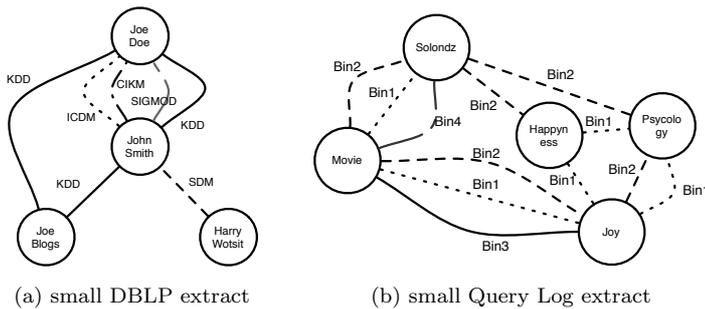


Fig. 3 Small extracts from the multidimensional DBLP and Query Log networks. Edge line styles correspond to dimensions in the network, i.e. distinct conferences for DBLP, and binned ranks for Query Log.

two connected authors in the same conference (dimension). The final network consisted of 558,800 nodes, connected by 2,668,497 edges in 2,536 dimensions. A small extract of this network is represented in Figure 3(a). Figure 4(a) reports the distribution of the number of edges per dimension (the dimensions are sorted by the values of the y axis). High number of edges corresponds to high number of editions of a conference and/or high number of published papers and/or high co-authorship number per paper.

- **QueryLog.** This network was constructed from a query log of approximately 20 millions web-search queries submitted by 650,000 users, as described in [36]. We extracted a word-word network of query terms (nodes), connecting two words if they appeared together in a query. The dimensions are defined as the rank positions of the clicked results, grouped into six almost equi-populated bins: “Bin1” for rank 1, “Bin2” for ranks 2-3, “Bin3” for ranks 4-6, “Bin4” for ranks 7-10, “Bin5” for ranks 11-500. Hence two words appeared together in a query for which the user clicked on a resulting url ranked #4 produce a link in dimension “Bin3” between the two words. We weighted each edge by the number of queries in which the two connected words appeared together in the same dimension. The final network consisted of 131,268 nodes, connected by 2,313,224 edges in 5 dimensions. A small extract of this network is represented in Figure 3(b). Figure 4(b) reports the distribution of the number of edges per dimension. This network was used in [7] for tasks such as query term disambiguation. More in general, word-word networks from query logs have been used in the information retrieval literature to mine the semantics of web search queries [26,22]

Note that we have weighted the edges of both DBLP and QueryLog to preserve as much information as possible from the original data. These edge weights may be taken into account by the monodimensional community discovery algorithms to drive their search in a more meaningful way, as they reflect the strength of the connections between nodes.

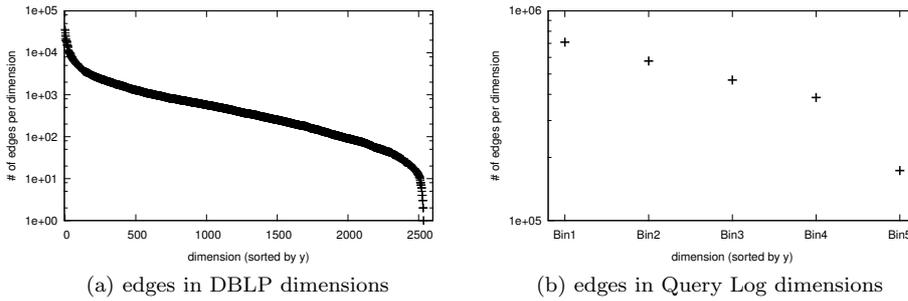


Fig. 4 Distribution of the number of edges in each dimension in DBLP and Query Log. Rank of the dimensions based on number of edges.

Following the classification in [32], we took one social and one information network, with different features (semantic: social vs information network; number of dimensions: thousands vs five; number of nodes: $\sim 600,000$ vs $\sim 100,000$, types of dimensions: categorical vs numerical attributes). Although they do not cover the entire space of possible networks, according to the classification in [32], our networks partially cover, with their structural characteristics and semantic, the spaces of social networks, information networks, and technological networks.

4 The ABACUS framework

In this section we present the core theoretical concepts of our problem. After defining the types of communities we are seeking, we show how to map the problem of finding multidimensional communities to the problem of extracting frequent closed itemsets from community memberships, and then finally present ABACUS, the algorithm proposed to solve our definition of the problem.

4.1 A new concept

As said above, most of the existing approaches to the problem of community discovery rely on a concept of community which is structure-based. That is, nodes with dense connections (or high interaction) are grouped together (in some cases, overlapping communities are also discovered). In this paper, we change this perspective. Let us start with a real-world example. In the WWW context, nowadays it is very popular to be connected in services like Facebook, Twitter, Google+ and, possibly, all of them. Each of these services sees different communities that can be spotted within their sets of nodes. As today many of the users have their online identities replicated across the different social networks, it is very likely that people sharing their membership in community k in service s , are also sharing their membership in community k' in service

s' . Extending this, we can easily imagine that many communities (especially small ones) would be exactly replicated across different dimensions.

In addition to this, there is another effect that can be detected in the real world. Even within close circles of friends, it usually happens to see pairs of people which are not directly connected. There can be many reasons for this: they can be enemies, or potential friends not yet connected, or there can be obstacles for their connection to setup (in some example networks, spatial constraints may inhibit people living too far away from connecting to each other). Yet, in these cases, two or more persons can share their memberships to communities in different contexts, or social networks, or, more in general, dimensions.

Two nodes A and B can then end up being logically connected by their shared memberships (say, to community 3 in dimension Google+ and to community 4 in dimension Facebook), but never actually connected in any dimensions in which they appear. This concept of logical connection here is crucial. While in previous community discovery algorithms, monodimensional approaches have a limited view of the rich set of connections residing within nodes, disregarding the additional information provided by multiple dimensions would be restrictive. Let us consider a co-authorship graph in DBLP, where each conference is a different dimension. Two persons in such network can be easily spotted to have connections in conferences such as KDD, VLDB, and SDM, while they are not connected, or not even present, in other dimensions such as AAAI, or SIGGRAPH, and so on. This piece of information is usually lost in traditional algorithms working on monodimensional networks, and, unfortunately, weights do not help in conveying entirely this additional knowledge.

On the other hand, if we use the shared memberships as key concept for connecting people (thus, not necessarily directly connected), we are linking them logically, using the semantic residing in the dimensions.

4.2 From communities to itemsets

Following the above idea, we can proceed as follows. First, we can split a multidimensional network into several monodimensional ones. We can then perform any existing technique for monodimensional community discovery, obtaining, for each node of the original network, a set of memberships to communities in each single dimension. We are now using the nodes as *transactions* of items, where an item is a pair (*dimension, community*) expressing the membership of the node in the various dimensions. At this point, applying frequent pattern mining to find *frequent closed itemsets* [3] appears to be natural. There is, in fact, a natural mapping of almost all the concepts in the frequent closed itemset mining (FCIM) paradigm in our problem: nodes are transactions; memberships are items; multidimensional communities are itemsets; the support of an itemset is the number of nodes sharing that set of memberships, and so on. Even the constraint-based paradigm [10] has a role in our problem: one

can, in fact, use constraints on the itemsets (eg. excluding/including specific items, computing any monotonic or convertible measure on itemsets, and so on). For the sake of simplicity, we reserve for future work this part of the problem, and we focus only on the extraction of frequent closed itemsets. In this new domain, it is also necessary to define concepts for a common understanding. With the term *support*, we intend the number of nodes that are members of a given multidimensional community. For instance, in the case of a co-authorship multidimensional network, two is the support of a multidimensional community formed by two authors as members. Moreover, the *size* represents the number of different dimensions involved in a multidimensional community. Again in the multidimensional co-authorship network, two is the size of a multidimensional community composed of two dimensions such as two conferences.

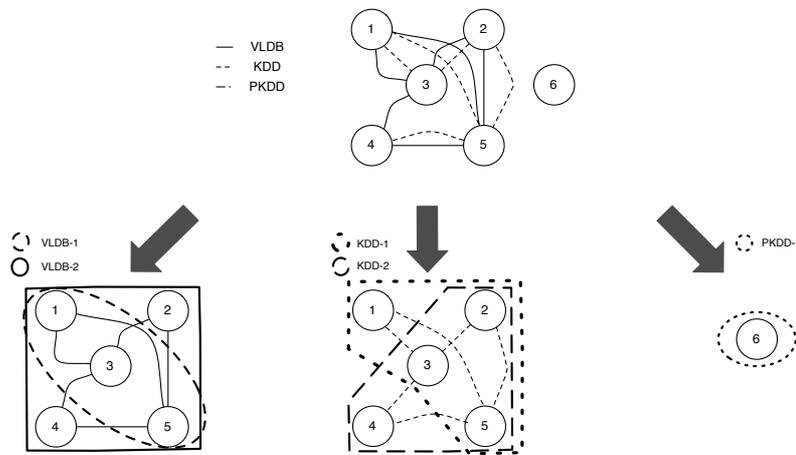


Fig. 5 Run-through example: co-authorship network with two dimensions: KDD and VLDB (top), monodimensional overlapping communities (bottom)

Let us follow a run-through example of our search strategy. Figure 5 describes our toy input network (top), consisting of six nodes, connected in three different dimensions (KDD, VLDB and PKDD). From the top image to the ones below, we perform two steps: first, we split the multidimensional network into three monodimensional ones; then, we perform the community discovery on each of them. The algorithm finds two different communities (highlighted by different line styles) in the VLDB and KDD dimensions, and one formed by a single node in the PKDD dimension. Note that, as this is just meant to provide an example to guide the reader through the steps of our methodology, we did not run a real community discovery algorithm here, and instead built the communities such that the resulting output would contain all the features we want to explain by means of this example. In particular, we imagined to

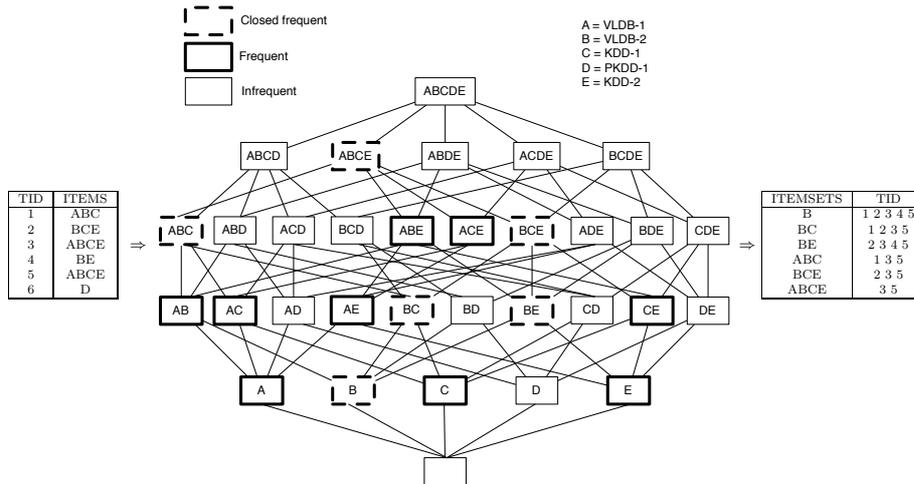


Fig. 6 Run-through example: set of monodimensional communities (items) associated to each node (transaction), on the left; lattice of multidimensional communities extracted using an algorithm for mining frequent closed itemsets, in the middle; resulting frequent closed itemsets (multidimensional communities) and supporting transactions (nodes) on the right.

be running an overlapping monodimensional community discoverer, assigning communities also to single nodes (node 6). The output of this process is represented on the left of Figure 6 that shows the list of transactions that it is possible to build from the memberships of the five nodes. The central part of Figure 6 shows then how the lattice of multidimensional communities is created. We see how the community found in the PKDD dimension gets cut due to a minimum support threshold $\sigma = 2$. In bold black we have the frequent itemsets, while with bold dashed line we highlighted the closed frequent ones. Finally, we see that the closed frequent itemsets clearly summarize the entire set of frequent itemsets found, so it would be redundant to return also non-closed items. In the right part of the figure we show the final output. The first community is found with a single membership to B, i.e. VLDB-2. This is clearly a monodimensional community, that our algorithm is still able to extract. The last community, formed by nodes 3 and 5, shows how we are able to extract communities of nodes that were not necessarily connected in the initial input. Indeed, nodes 3 and 5 are unconnected in all the dimensions of the example. We want to emphasize here that the ability to find such communities is not given by the type of monodimensional community discoverer, but it is due to the mapping to the frequent pattern mining paradigm, and the fact that our concept of multidimensional communities groups together nodes sharing memberships to the same monodimensional communities in the same dimensions. Nodes 3 and 5, in fact, share their memberships to the communities VLDB-1, VLDB-2, KDD-1 and KDD-2.

4.3 The ABACUS algorithm.

Algorithm 1 is the core of our approach. It takes as input three parameters: the multidimensional network \mathcal{G} , a monodimensional algorithm for community discovery CD , and a minimum support threshold σ . The algorithm works by building a set of transactions *memberships* that, for each node n , record a set of pairs (i, j) representing memberships of node n to community j in dimension i . Note that if CD is able to find overlapping communities, one node may have more than one pair associated to a specific dimension. This would result in more possible combinations, i.e. more different items, thus an higher number of resulting communities. However, this does not change the type of communities that ABACUS may find, namely groups of nodes sharing memberships to the same monodimensional communities in the same dimensions. Thus, for the sake of simplicity, and without lack of generality, in the rest of the paper we show experiments conducted with a non-overlapping community discovery algorithm.

Note also that CD may or may not take into account edge weights to drive the search for communities in a more meaningful way. As we have weighted our networks presented in Section 3, in our experimental evaluation in Section 5 we use an algorithm that takes into account edge weights.

In line 4 the function ϕ is called to split the multidimensional network into a set of monodimensional ones, by replicating each node into each of the dimensions in which it has at least one edge, and adding to it all of its adjacent edges in their corresponding dimensions. Each dimension is then processed as a separate network G_i by CD in a for loop, returning a different set of communities per dimension. In lines 6 – 8, for each node in each community, its memberships are updated with the pair $(dimension, community)$, building a set of transactions (one per node). The function *map* returns a unique item code for its argument. Such set is then passed to the frequent closed itemset miner ($FCIM$) in line 11, together with a threshold of minimum support, and the resulting set of frequent closed itemsets are returned, constituting the multidimensional description of each community. In Section 5 we show how, by using an implementation of $FCIM$ returning also the transaction ids of each itemset, we also get the set of nodes contained in each community (i.e., the ids of the transactions supporting the frequent closed itemset).

The complexity of ABACUS is directly inherited by the complexity of the algorithm used for $FCIM$, and by that of the method for monodimensional community discovery. The additional complexity introduced by ABACUS, in fact, resides only in the problem-mapping phase, where we perform a linear scan of the list of communities found and we prepare the input for $FCIM$. We then refer to the corresponding papers for discussion on the complexity, although in Section 5.3.3 we present an empirical evaluation of the complexity of ABACUS.

Algorithm 1 ABACUS**Require:** \mathcal{G}, CD, σ

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1: for all  $n \in nodes(\mathcal{G})$  do
2:    $memberships[n] = \emptyset$ 
3: end for
4: for all  $G_i \in \phi(\mathcal{G})$  do
5:   for all  $c_j \in CD(G_i)$  do
6:     for all  $n \in nodes(c_j)$  do
7:        $memberships[n] \leftarrow memberships[n] \cup map((i, j))$ 
8:     end for
9:   end for
10: end for
11:  $\mathcal{I} \leftarrow FCIM(memberships, \sigma)$ 
12: return  $\mathcal{I}$ 

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5 Case study on DBLP and Query Log

5.1 Tools

We have implemented ABACUS in c++, making use of the igraph³ library.

As CD parameter, we use the community discovery algorithm based on label propagation [39], that takes into account edge weights. This algorithm is well known to be scalable, and, as a result, our running times to process the network were considerably low (a few seconds up to the creation of the transaction file, plus a few minutes to perform frequent closed itemset mining, see Section 5.3 for running times). In all the experiments we set the minimum support threshold to 2, in order to capture all the possible connections among nodes.

We chose an efficient implementation of Eclat [12] as frequent itemset miner, with options to return both frequent closed itemsets and list of supporting transactions for every itemset.

Note that many other choices are possible for the CD and the frequent itemset mining steps and that, for the sake of simplicity and presentation, we only report the results obtained by the above choice. Note also that while the choice for the frequent closed itemset mining implementation is usually mainly driven by scalability issues, selecting a different algorithm for community discovery may lead to very different communities. The debate on which algorithm to choose is however out of scope in this paper, and we refer to Section 2 and to the surveys on community discovery for driving the reader to the best choice for this step, which is mainly driven by the final application [21,28]. Moreover, despite the possibility of returning different types of communities, we want to emphasize that the ability to return potentially unconnected communities is given by the mapping of the problem as described in Section 4, and *not* to the choice of CD . In fact, as said above, our multidimensional communities represent nodes that share memberships to the same monodimensional communities in the same dimensions. This concept is not tied to the fact that the nodes

³ <http://igraph.sourceforge.net>

must be directly connected in all the dimensions found in the multidimensional community.

All the experiments were performed on a laptop equipped with an Intel i7 processor at 2.2GHz, with 4GB of RAM.

5.2 Experiments

We performed our experiments following three questions related to our problem:

- Q1.** Quantitative evaluation: given the high number of resulting communities, how can we easily reduce the patterns to select only a set of meaningful ones?
- Q2.** Are there relational dependencies between our concept of communities and structural properties of them?
- Q3.** Qualitative evaluation: among the communities found, are there any relevant ones? Can we reason on the multidimensional density of the connections within the communities?

In order to answer the above questions, we define a simple and easy to compute measure of connectedness within communities. The Multidimensional Community Density (MCD) is then the number of edges in a community normalized by the maximum possible for that community, or, in formula:

$$\frac{\#edges}{ndim \times \frac{\#nodes \times (\#nodes - 1)}{2}} \quad (1)$$

where $ndim$ is the number of different dimensions found in the community.

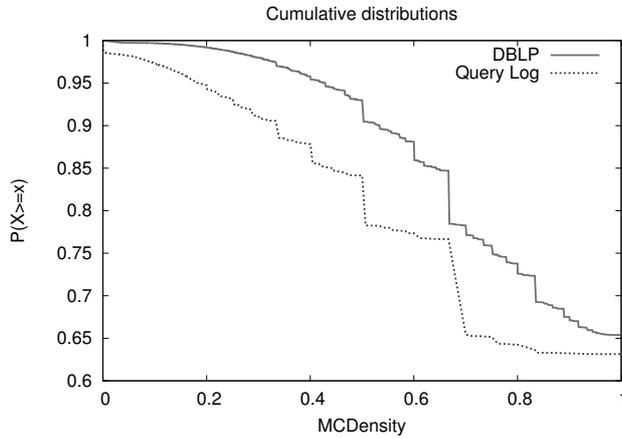


Fig. 7 Cumulative distributions of Multidimensional Community Density (MCD) for the two networks

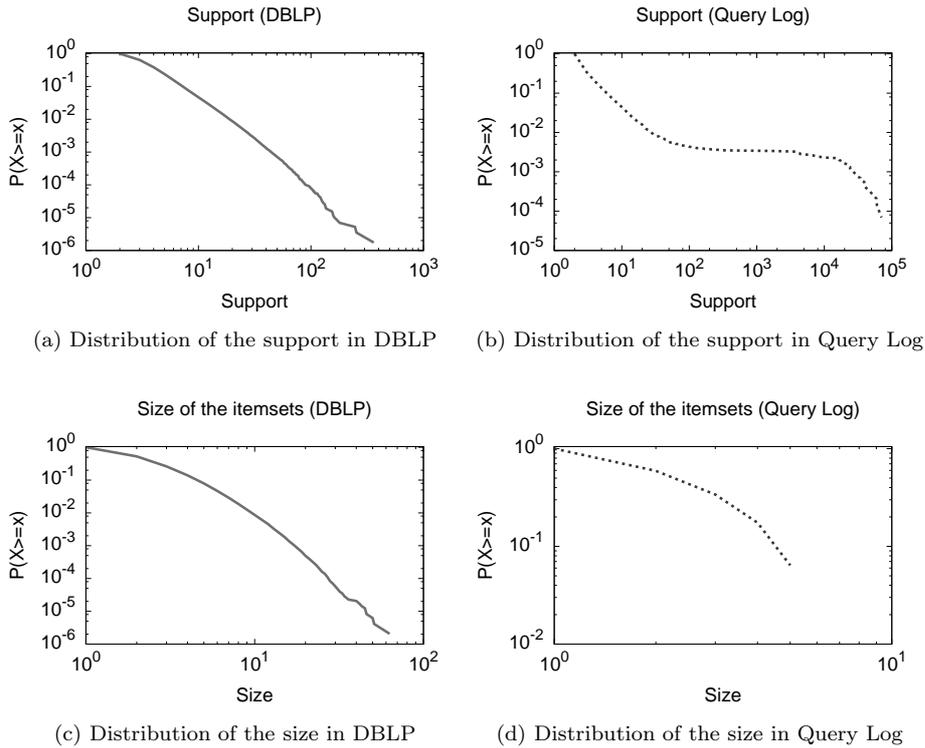


Fig. 8 Cumulative distribution of support (top row) and size of the itemsets (bottom row), for DBLP (left column) and Query Log (right column)

Let us answer **Q1**. The frequent pattern mining literature reports that the problem of finding few relevant patterns to be interpreted, among the many returned, is hard [14, 25]. We can overcome this problem in three different ways. First, we can look at the distributions of the MCD (defined above), the support of the patterns, and the size of the itemsets to focus our search towards the communities that we consider relevant, depending on the final application. Figures 7, 8 report the mentioned distributions (we report the cumulative versions, to be able to use the three measures as straightforward filters). For better comparison, we reported on the y -axes the percentage of communities with values of the measures greater than a certain thresholds. However, the absolute number of communities can be used to choose, depending on the application, the best support, size of the itemset, and MCD to select only the relevant communities. Second, more generally speaking, the entire Constraint-Based Frequent Pattern Mining literature can be applied in our scenario at running stage, to drive the search to fewer, more focused, patterns [38, 37, 11]. For example, we may want only patterns including or excluding a specific dimension, or patterns including dimensions with specific properties (e.g., at least 1000 authors). To this extent, it is worth noting that MCD is neither

(anti-)monotone, nor convertible, nor loose-antimonotone. We leave for further research the definition of meaningful, application-driven constraints, and their effect to the results. Lastly, the authors of [14] present another methodology for selecting few interesting patterns among many, which is not based on constraints. We believe that this technique may be also used, and we plan to investigate this opportunity in the future.

To answer **Q2** we check whether MCD is correlated to other structural properties of the nodes of the communities. For example, one possible intuition is that communities with low density may group together nodes that were at the borders of the monodimensional communities. To study this, we computed the closeness centrality for each node and for each dimension, and checked the correlation between the centrality and the density. We did not find any clear sign of direct correlation. We checked also for correlation with PageRank, the degree centrality and the betweenness centrality, for which again we did not have signs of correlation. Based on these results, we believe that MCD is yet another measure to be used to filter the results towards more focused results.

Lastly, in order to answer **Q3**, we extracted a few communities either minimizing or maximising MCD. In the remainder, we call MCS the number of nodes in a community. As we have stated above, we can use the distributions of size, support and MCD to post-process the results to get only the few interesting ones. We have extracted a few (i.e., 200) communities for each network, and we report in Figure 9 four of them. Besides the first example, that was found by searching for one of the co-authors of this paper, the other ones were found by examining the results filtered by means of the above mentioned three measures. In particular: Figure 9(b) was found within 260 communities obtained by constraining $MCD < 0.1$, $MCS \geq 2$ and $size \geq 3$; Figure 9(c) was found among 287 communities obtained by constraining $MCD = 1$, $MCS \geq 3$ and $size \geq 2$; Figure 9(d) was found within 286 communities obtained by constraining $MCD < 0.5$, $MCS \geq 4$ and $size \geq 4$. These thresholds were obtained by looking at the distributions reported above.

Consider the one in Figure 9(a). We discovered a size-4 community connecting FP, FG, MN and DP with dimensions set $\{KDD, GIS, SAC, SEBD\}$. It is interesting to observe that, given its very dense connections, this multidimensional community would have been found also by using the methods proposed in [5,31].

However the method proposed in this paper has the possibility to discover more complex interactions between dimensions. Indeed, the lattice can be used to browse the multidimensional communities by selecting different dimensions sets. To give an example we extracted a size-3 community composed of authors AA, PN, QC, KR and SG with connections in dimensions set $\{IOLTS, DATE, ISLPED\}$, see Figure 9(b) where the different multidimensional memberships are shown. These authors are part of three monodimensional communities, but have not co-authored papers at these three conferences (there are no links connecting them). By adding the dimension *ICCD* (solid line circle), we are able to extract a size-4 community composed of the first three authors. This fourth dimension includes a paper co-authored by the three

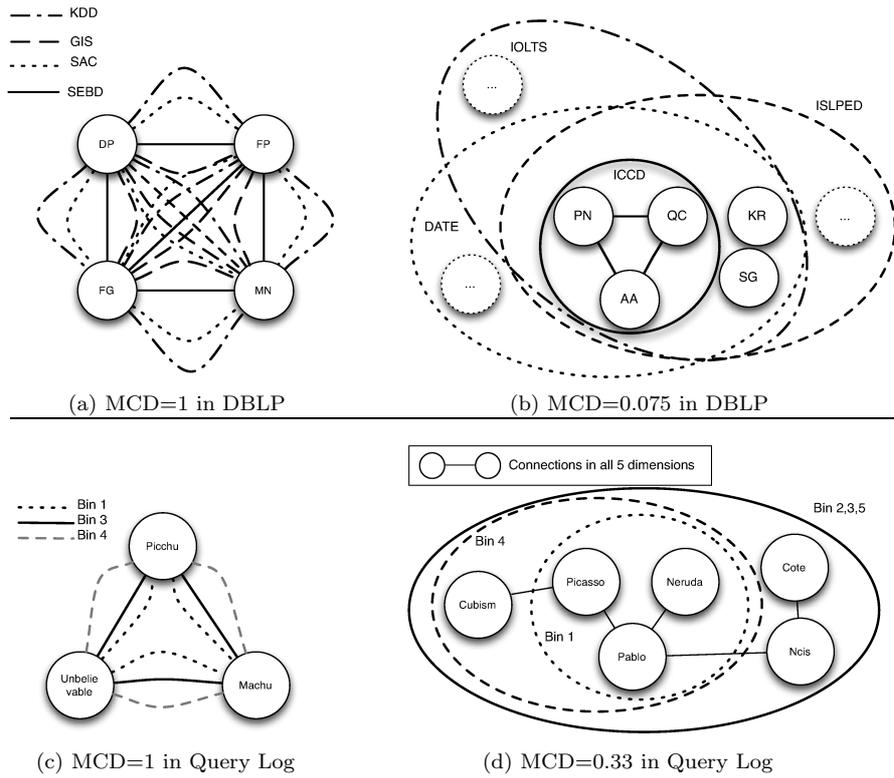


Fig. 9 Four communities with high and low MCD extracted from the two networks. Nodes in (a): Fosca Giannotti, Mirco Nanni, Dino Pedreschi, Fabio Pinelli. Nodes in (b): Amit Agarwal, Qikai Chen, Swaroop Ghosh, Patrick Ndai, Kaushik Roy. The dashed ovals represent the shared memberships to the same community in the corresponding dimensions (see circle labels). The dashed anonymous nodes in (b) represent several nodes belonging to the communities in dimensions IOLTS, ISLPED and DATE and are not visualized to simplify the readability. In (d), we report with a single solid line the point to point connections found in all the five dimensions.

authors, which resulted in a *ICCD*-monodimensional community formed by the three nodes. Interestingly, through this dimension we are able to specialize the previously discovered 5 authors community. Note that by using the methods proposed in [5,31] it would not be possible to discover how the *ICCD* dimension could specialize the community, and so its semantic meaning. This is due to the fact that more information is included in the results w.r.t. the mentioned works.

Similar results can be obtained by applying ABACUS to the Query Log dataset. Finding a community of words in this network means finding a set of words typically used together in queries that lead to good or bad results. A set of words found together only in dimension 1 is a set of words that, used together in a query, lead to very specific results (users clicked on the first result). Words found together only in dimension 5, on the other hand, lead to lower

ranked results. If we find words together in different dimensions, it may mean that either the concept the users were looking for is only representable by words used in conjunction, or that they need more terms to be disambiguated. An example of the first kind is shown in Figure 9(c), where, maximising the MCD, we were able to detect a highly connected multidimensional community where the words *Machu*, *Picchu*, and *unbelievable* are connected in three different dimensions of the dataset. An example of the second kind, on the other hand, is shown in Figure 9(d). In this example, we can observe that, if we consider all the dimensions, we obtain a set of words belonging to the same multidimensional community with a strong intrinsic semantic correlation (i.e. Pablo, Picasso, Neruda -besides sharing their first name, there exists an edition of a book from Neruda with a Picasso painting on the cover), removing, then, the most specific dimension (Bin 1 – i.e. click on the first returned result) we include words that make the concept broader. Also in this case, the methods proposed in [5,31] do not allow to investigate the effect of the different dimensions on the specialization of the communities and, thus, the intrinsic semantic correlation among different words.

5.3 Comparison with previous approaches

As reported in Section 2, in [5], the authors proposed another way to extract multidimensional communities. Their approach is based however on a different concept of communities: a multidimensional community groups nodes that are highly multidimensionally connected. How this multidimensional connectedness is evaluated is left at the end of the process, by post-processing the resulting communities. Their approach is composed of the following steps: first, the multidimensional network is collapsed to a monodimensional one (i.e., they follow exactly the opposite of our first step), by weighing the edges in different ways; second, monodimensional community discovery is performed on the resulting network; on the resulting communities, multidimensional connections are restored from the original networks; the communities are then evaluated by means of multidimensional measures.

The approach described in [31] works in a similar way, although it presents some differences. The approach works in two phases: in the first phase, the adjacency matrices corresponding to each different dimensions are coupled by connecting, for each entity, its node representation i in dimension k' to its node representation j in k'' . This step is driven by a coupling parameter ω which controls the weight of this inter-dimension connection. This is basically a node-centric monodimensional collapsing pre-process on the multidimensional information, as opposed to edge-centric as done in [5]. In the second phase, the authors apply a modularity-driven monodimensional community discovery to extract the communities. There are then two main differences between this baseline and the one presented in [5]: first, the pre-process step in which the multidimensional information is collapsed is done at the node level, rather than on the edges; second, instead of being parametric in the monodimensional

Net subset	Dim.	Nodes	Edges	Net subset	Dim.	Nodes	Edges
2000-2010	2,536	558,800	2,668,497	1980-1990	390	37,440	71,535
2001-2010	2,477	544,608	2,585,251	1981-1990	380	36,735	69,766
2002-2010	2,401	528,958	2,489,462	1982-1990	376	35,912	67,733
2003-2010	2,317	511,178	2,365,979	1983-1990	368	34,853	65,048
2004-2010	2,244	489,228	2,216,735	1984-1990	359	33,688	61,955
2005-2010	2,126	458,763	2,009,903	1985-1990	344	31,797	57,432
2006-2010	1,977	423,755	1,772,646	1986-1990	335	29,770	52,271
2007-2010	1,848	379,182	1,496,042	1987-1990	321	26,506	44,631
2008-2010	1,707	325,246	1,182,161	1988-1990	303	23,031	36,935
2009-2010	1,530	260,248	840,916	1989-1990	265	18,275	27,065
2010-2010	1,172	163,374	431,296	1990-1990	188	11,780	15,927

(a) Large nets

(a) Small nets

Table 1 Statistics of the large and small nets used in the comparisons

community discovery algorithm, the authors apply a strategy that aims at maximizing a multidimensional version of the modularity function.

We then compared against both these approaches, using a c++ implementation⁴ of the method presented in [31], and a c++ implementation of the method presented in [5].

We wanted to compare the three approaches at different levels. In particular we wanted to answer the following:

- Q4.** Quantitative evaluation: how do the sets of returned communities found compare? Can we measure their intersection and the number of communities that only our method or a given baseline may find?
- Q5.** Qualitative evaluation: what do the different concepts of community look like?
- Q6.** Scalability: how do the methods perform on networks of different size?

In order to address the above, we ran ABACUS and the two different baselines, on several subsets of the DBLP dataset. Hereafter, we refer to MD for the method proposed in [5] and to GL for the method proposed in [31].

We created two additional (w.r.t. the networks presented in Section 3) sets of networks by taking incrementally large subsets of DBLP, by taking all the nodes, edges and dimensions contained in different temporal windows. This was needed to be able to compare against the two different baselines, which present different scalability in terms of both running times and memory occupation, as we see in Section 5.3.3 (in particular, we were not able to run GL on large networks). The first set, called “large nets” hereafter, consists of 11 networks corresponding to the single year 2010, the years 2009 and 2010, the years between 2008 and 2010, and so on, up to the years from 2000 to 2010. The second set, called “small nets” hereafter, consists of 11 networks corresponding to the single year 1990, the years 1989 and 1990, and so on, up to the years from 1980 to 1990. Table 1 reports the basic statistics of the two

⁴ <https://code.launchpad.net/louvain>

sets of networks. As we see, the small nets are much smaller than the large ones, in terms of nodes, edges and number of dimensions.

5.3.1 Quantitative evaluation

Figure 10(a) reports the number of communities found by ABACUS and MD in the large networks, while Figure 10(b) reports the number of communities found by ABACUS, GL with different values of ω and MD in the small networks.

In the large networks, as we see, due to the strategy of collapsing the multidimensional network to a monodimensional one, the number of communities found by MD becomes nearly stable after adding four years. In fact, after the first step, each additional year included into the subset is only changing the weight of existing edges, instead of creating new ones (and bringing new nodes). On the other hand, the search space of ABACUS grows consistently up to the last two or three steps, where the growth slows down. By keeping the dimensions separated, in fact, each additional year is able to provide a significant number of new combinations to the previous ones. Although the number of results returned by ABACUS is high, we have discussed in **Q1** how to deal with it.

In the small networks, the above trend is followed as well, but we can make further considerations regarding different baselines. First of all, we see how, due to the definition of the GL approach, setting $\omega = 0$ leads to a larger number of communities when comparing to other values of ω . This value of the parameter actually forces each node representing the same entity in different dimensions to be grouped separately. In other words, the dimensions are treated in a disjoint way, i.e. the algorithm performs community discovery in each dimension separately.

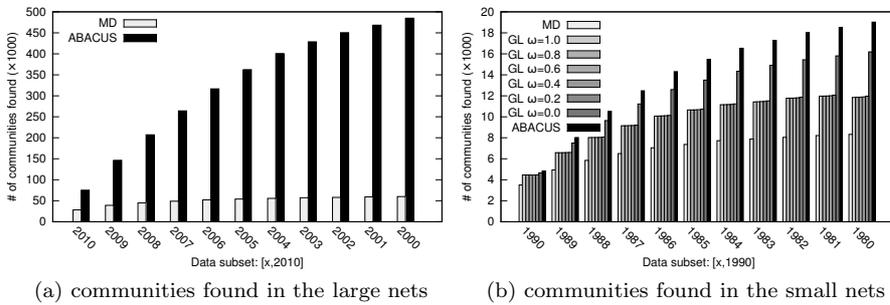


Fig. 10 Quantitative comparisons between ABACUS and the baselines (each x value corresponds to an additional year included in the subset, from 2000 to 2010 in (a), and from 1980 to 1990 in (b)). Number of communities found by ABACUS and MD on the large net in (a), and by ABACUS and all the baselines on the small net in (b). In these plots, MD is always the leftmost bar within a stack, and ABACUS is always the rightmost one.

The plots also shows the difference in the number of returned communities by values of ω greater than zero, although there appear to be no much dif-

ferences within the experiments ran with values greater than zero. To better explore the sensitivity of these experiments to the ω parameter we also ran GL with values up to 10000. Figure 11 shows that there is no substantial difference in running GL with values larger than zero and up to 10000. Because of this, and for sake of simplicity, in the following we only show the results obtained with a few values of ω .

Looking at the plots, a few clear questions arise: are the three methods finding the same communities? Is one method returning communities found also by the competitors? Can we identify (classes of) communities that can be found only by one of the three methods? Figure 12 partially answers these questions from a quantitative point of view. Calling A the set of communities found by ABACUS and B the set returned a given baseline, the light gray bar (always the leftmost bar in a stack) shows $|A \cap B|/|A \cup B|$ -i.e. the portion of communities found by both, the dark gray bar (always the bar in the middle of a stack) shows $|B \setminus A|/|A \cup B|$ -i.e. the portion of communities found only by the baseline-, and the black bar (always the rightmost bar in a stack) shows $|A \setminus B|/|A \cup B|$ -i.e. the portion of communities found only by ABACUS. Note that in order to compare the communities found we had to remove the multidimensional information contained in those found by ABACUS. This step is however correct, i.e. there cannot be two instances of the same set of nodes tied to two different sets of dimensions (itemsets) as this would violate the theory behind the *closed* itemsets. Note also that, in analogy with the majority of the works on community discovery, and on frequent pattern mining, we perform *exact* matching here, thus we are only counting the identical communities in this comparison.

As we see, since the bars report relative numbers, the ratio of communities that can be found only by the baselines decreases as the subset of years grows. Put in other words, even if we know that ABACUS is meant to find communities of a different type than the ones found by the baselines, we see how, for large datasets, the set of communities found only by one of the baselines becomes

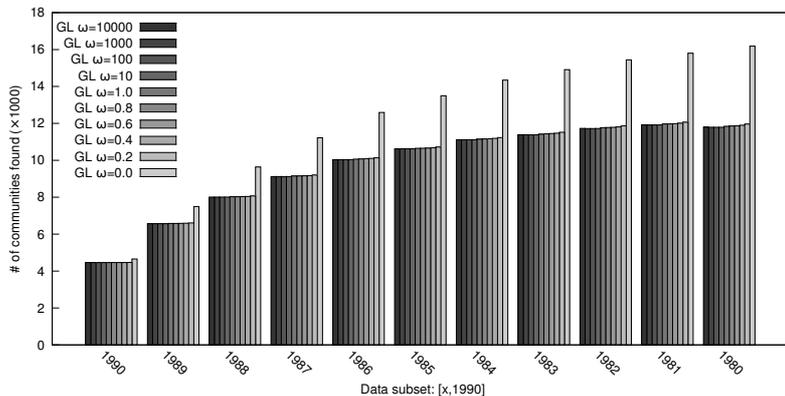


Fig. 11 Number of communities in the small nets returned by GL for different values of ω .

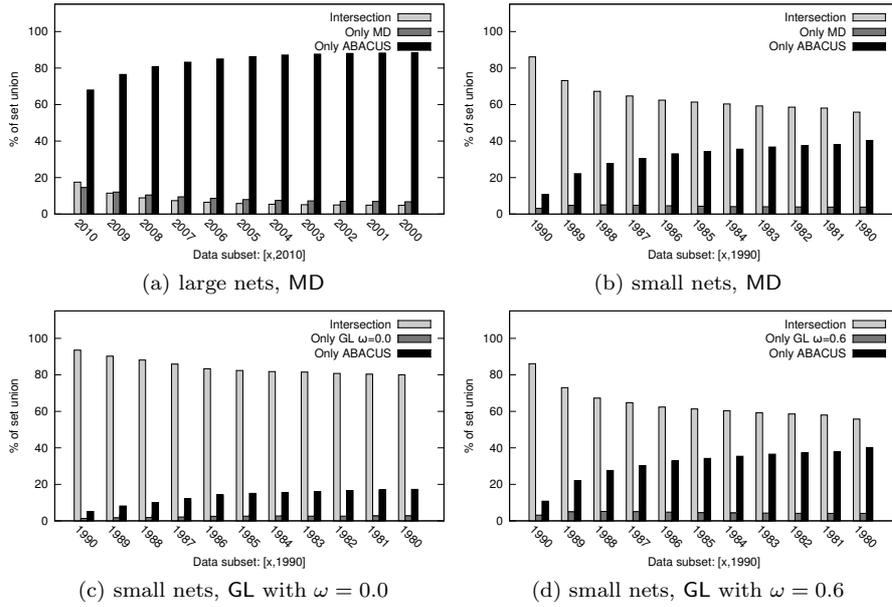


Fig. 12 Comparison between sets of communities found. Each plot compares ABACUS against a different baseline or net: large nets and MD in (a), small nets and MD in (b), small nets and GL with $\omega = 0.0$ in (c), small net and GL with $\omega = 0.6$ in (d). For each pair of sets of communities A (for ABACUS) and B (for baseline), we show, for each interval of years: $|A \cap B|/|A \cup B|$ -i.e. the portion of communities found by both ABACUS and the baseline with the light gray bar; $|A \setminus B|/|A \cup B|$ -i.e. the portion of communities found only by the ABACUS- with the black bar; $|B \setminus A|/|A \cup B|$ -i.e. the portion of communities found only by the baseline- with the dark gray bar. In these plots, MD or GL are always the leftmost bar within a stack, and ABACUS is always the rightmost one.

smaller. Moreover, the number of communities that is found only by ABACUS increases accordingly. This is clearly related to the type of communities that only ABACUS can find, i.e. the communities of unconnected nodes, or, more formally, communities formed by more than one connected component. In the following, we answer the above questions also from a qualitative perspective.

5.3.2 Qualitative evaluation

The two concepts of communities found by ABACUS and the baselines are different, without a clear winner (i.e., they just reflect different types of interactions among nodes). This situation can be also detected by the different classes of communities that only one of the two methods can find. Consider Figure 3: if that was the entire input, MD would collapse the network into a monodimensional one and possibly find only one community containing all the four nodes. GL would also collapse the multidimensional connectivity according to the parameter ω . This cannot happen in ABACUS, as the principle for which the nodes are found in the same multidimensional community is to

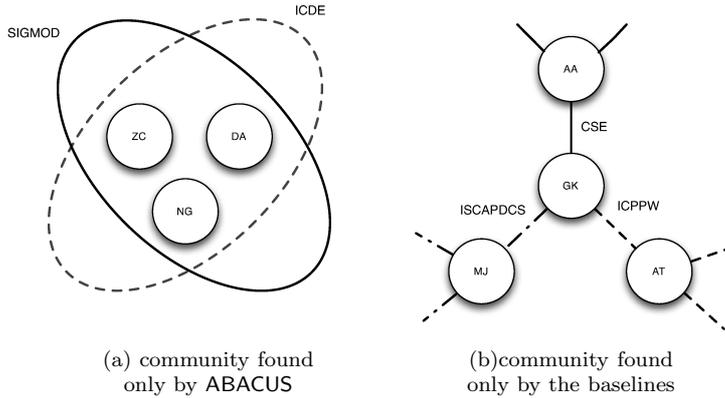


Fig. 13 Examples of communities found by (a) only ABACUS and (b) only the baselines. Nodes in (a): Deepak Agarwal, Zhiyuan Chen, Nitin Gupta. Nodes in (b): Anika Awwal, Matthew Jin, Gul N. Khan, Anita Tinoln. In (b) we also depicted the other outgoing edges from each nodes that were present in the input data. For the nature of the approaches, (a) could not be found by the baselines and (b) could not be found by ABACUS, as different nodes exist in different dimensions.

share memberships to monodimensional communities. That is, if Figure 3(a) was the entire input, ABACUS would find Jon Doe and John Smith in a multidimensional community, but not the entire set of nodes, as the remaining two do not share all the memberships to the other nodes (they do not exist in dimensions ICDM, CIKM and SIGMOD). Figure 13 shows two communities found during our comparison: (a) was found only by ABACUS, and (b) was found only by both the baselines. Note that we depict all the edges in the original input, if there were any, and we reported in (b) also the outgoing edges. While it is clear that (a) cannot be found by the baselines (as they rely on connectedness, but there are no edges among those nodes in the input), in order to confirm that (b) could not be found by ABACUS we had to investigate whether the four nodes were sharing memberships to the same communities in the depicted dimensions. That is, even if the image is showing a community that could not be detected by ABACUS if the depicted edges were the entire input data, there might be in the data other edges (and paths) connecting the nodes. After post-processing the data, we found that this was not the case for (b), as different nodes are connected in different dimensions (see also outgoing edges).

5.3.3 Scalability

The last part of our comparison regards scalability. Consider Figure 14, reporting the running time (in seconds) of ABACUS and MD on the large nets on the left, and ABACUS, MD and GL on the small nets on the right. As we see, even though by adding years we implicitly add also dimensions (not all the

conferences take place in all the years, see Table 1), this has a very low impact on the running time of ABACUS, and a very high impact for the baselines.

Note that here we report only the running times obtained with a minimum support of two. That is, we do not test the sensitivity to the minimum support parameter, as we already give the worst case. In reality, if looking for larger communities (depending on the application), the running times may be even lower.

To conclude, ABACUS is scalable, and able to process our data in 32 to 1200 seconds (20 minutes) on the large nets, while MD needed 380 (6 minutes) to 13500 seconds (225 minutes, i.e. almost 4 hours), and in less than a second to 2.5 seconds on the small nets, while MD needed up to 7.5 seconds and GL needed up to 86 seconds.

We also report that we were not able to run GL on networks larger than the small networks we used because of its memory occupation, that exceeded 4GB to process the large nets.

6 Conclusions and future work

In this paper, we have addressed the problem of multidimensional community discovery. We have given a definition of multidimensional community for which nodes sharing memberships to the same monodimensional communities in the different single dimensions are grouped together. This leads us to define a community extractor combining the use of

- A given monodimensional community discovery algorithm (that could also allow for overlapping communities)
- Frequent itemset pattern mining to allow merging discovered monodimensional communities into multidimensional ones

By browsing over the lattice generated by the frequent closed itemset mining algorithm, it is possible to extract multidimensional communities of different sizes (pattern lengths) and so navigate the complex multidimensional structure of a network, in a way that previous methods could not permit.

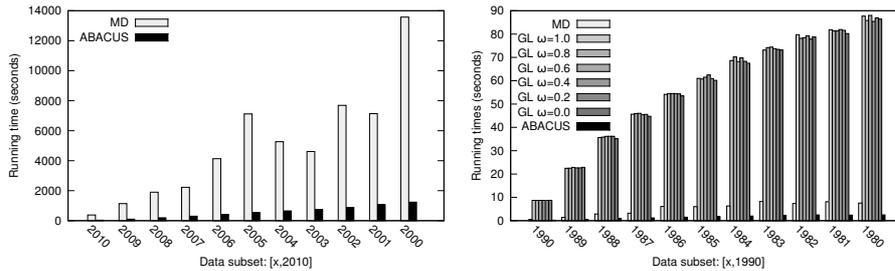


Fig. 14 Quantitative comparisons between ABACUS and the baselines. Running times (in seconds) of ABACUS and MD on the large net in (a), ABACUS and all the baselines on the small nets in (b).

The proposed method could lead to the development of analytical tools to characterize the redundancy in the dimensions, the impact of new dimensions on the network structure, and more in general to evaluate the interplay between dimensions. For these reasons, we see potential applications in real world problems including characterizing the interplay between mobility and communication dimensions in a place-to-place network [16], the similarity between users in a user-mobility profile network [45], or in the analysis spreading of infectious diseases [2]. We leave for future research the analysis of potential applications of ABACUS.

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