A design and a geometry for the group Fi_{22}

P. J. Cameron

School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK

and

A. Rudvalis

Mathematics and Statistics, University of Massachusetts Amherst, Amherst MA 01003, USA

Abstract

The Fischer group Fi_{22} acts as a rank 3 group of automorphisms of a symmetric 2-$(14080, 1444, 148)$ design. This design does not have a doubly transitive automorphism group, since there is a partial linear space with lines of size 4 defined combinatorially from the design and preserved by its automorphism group. We investigate this geometry and determine the structure of various subspaces of it.

In this paper we construct and investigate a partial linear space admitting the Fischer group Fi_{22} and defined combinatorially from a symmetric design also admitting this group. For details about the Fischer group we refer to the ATLAS of Finite Groups [3].

The Fischer group Fi_{22} has two conjugacy classes of subgroups of index 14080 isomorphic to $O_7(3)$. One of these subgroups has orbits of size 1, 3159, and 10920 on its own conjugacy class, and 364, 1080, and 12636 on the other. Construct an incidence structure in which the elements of the two classes are points and blocks respectively, and a point and block are incident if one lies in an orbit of size 364 or 1080 of the other. In this situation, we would expect that the number of blocks incident with two points takes one of two possible values, depending on the orbit of the pair of points. Remarkably, it occurs that the two values are the same, namely 148. Thus, we have a square (or symmetric) 2-$(14080, 1444, 148)$ design (see [2, Chapter 1] for definitions). (This can be seen from the collapsed adjacency matrices on the Web page [10]. The orbits of the point stabilizer in Fi_{22} are numbered in order of increasing size; so the relevant fact is that the sum of the

$(4,2)$ and $(4,3)$ entries in the second and third matrices is equal to the sum of the $(5,2)$ and $(5,3)$ entries in these matrices (and is 148).)

The design was discovered by the second author, in collaboration with David C. Hunt, in the early 1970s. A brief description of it is given in [11, p.112]. The design was rediscovered by Dempwolff [6]. Further information on the topics presented here can be found in [12].

There is a remarkable parallel to this situation, which we now review.

The Mathieu group M_{22} has two conjugacy classes of subgroups of index 176 isomorphic to A_7. One of these subgroups has orbits of size 1, 70, and 105 on its own conjugacy class, and 15, 35, and 126 on the other. Construct an incidence structure in which the elements of the two classes are points and blocks respectively, and a point and block are incident if one lies in an orbit of size 15 or 35 of the other. Again, the two possible numbers of blocks incident with two points coincide, and we have a square (or symmetric) 2-$(176,50,14)$ design. Again this can be verified directly from the collapsed adjacency matrices.

Moreover, in this case, G. Higman [9] showed that the full automorphism group of the design is larger than the Mathieu group: it is the Higman–Sims group HS, which acts doubly transitively on the sets of points and blocks.

It follows from the Classification of Finite Simple Groups that the design obtained from the Fischer group cannot have a doubly transitive automorphism group, since the list of 2-transitive groups includes only the symmetric and alternating groups of degree 14080. Nevertheless, it seemed worthwhile to find a more direct proof of this fact.

Generators for $G = Fi_{22}$ as a permutation group on 14080 points are given in the on-line Atlas of Finite Group Representations [14], which also gives the images of the generators under an outer automorphism interchanging the two conjugacy classes of $O_7(3)$ subgroups. Using this information, we use GAP [7] to construct the design. (A block stabilizer is the image of a point stabilizer under the outer automorphism; a block is the union of its orbits of sizes 364 and 1080; and the remaining blocks are the images of this one under the Fischer group.)

Next, we computed, for two pairs (α_1,α_2) and (β_1,β_2) of points in the two orbits, the list of pairs (μ_i,n_i), where there are n_i points γ of the design for which the number of blocks containing the chosen points and γ is μ_i. If $|G_{\alpha_1} : G_{\alpha_1\alpha_2}| = 3159$ and $|G_{\beta_1} : G_{\beta_1\beta_2}| = 10920$, then the lists are

$$\begin{array}{rl} \text{for } (\alpha_1,\alpha_2): & (7,288),(13,5600),(28,630),(148,2) \\ \text{for } (\beta_1,\beta_2): & (13,5184),(16,8748),(40,144),(148,4). \end{array}$$

The inequality of these two lists shows that no automorphism of the design can carry (α_1,α_2) to (β_1,β_2).

(A similar calculation for M_{22} gives the same list, namely

$$(3,72),(4,90),(8,12),(14,2),$$

for both orbits; this must be so as the design does have a 2-transitive automorphism group in this case.)

This computation shows an unexpected feature. We see that, given β_1 and β_2, there are two further points β_3 and β_4 such that the 148 blocks incident with two of these points actually contain all four of them. We call such a set of four points a *line*. Dually there are sets of four blocks such that every point incident with two of them is incident with all four. We call such a set of four blocks a *coline*. The lines are precisely those lines of the design (in the sense of Dembowski [5, p.65]) which have four points; the remaining design lines have two points. Similarly the colines are the four-point lines in the dual design.

The setwise stabilizer of a line is a solvable maximal subgroup H of G with structure $3_+^{1+6}{:}2^{3+4}{:}3^2:2$. It is the normalizer of the cyclic subgroup generated by an element g of order 3 in the conjugacy class 3B (in ATLAS [3] notation). This group induces S_4 on the line, and is also the setwise stabilizer of a coline L' (so that there is a G-invariant bijection between lines and colines). The element g has 148 fixed points (these are the points incident with all the blocks of the coline L'), and 148 fixed blocks (those incident with all the points of L).

The points and lines form a partial linear space. If a point α is not on the line L, then it is collinear with 1, 3 or 4 points of L. This geometry is rich in subspaces (where a subspace is a set S of points such that the line through two collinear points of S is contained in S). Indeed, any intersection of blocks is clearly a subspace. The following table gives the lengths of the orbits of the stabilizer of L on points outside L and, for each orbit, the number $a(\alpha,L)$ of points of L collinear with a point α in that orbit, the number of blocks containing L and α and the size of their intersection (a subspace of the geometry), its setwise stabilizer, and the group induced on the subspace by the setwise stabilizer:

Orbit size	1296	8748	144	3888
$a(\alpha,L)$	1	3	4	4
Number of blocks	13	16	40	13
Size of intersection	7	16	13	40
Stabilizer	order $2^5 3^6$	order $2^{12}3^3$	$3^{3+3}{:}L_3(3)$	$3^{3+3}{:}L_3(3)$
Induced group	$3^2{:}2^2$	$2^4{:}(3\times S_4)$	$L_3(3)$	$3^3{:}L_3(3)$

The subspace in the first case consists of two lines intersecting in a point; in the second case, an affine plane of order 4 with one parallel class of lines removed; in the third case, a projective plane of order 3. In the last case, the geometry on the subspace is a 2-$(40,4,1)$ design, which has the same parameters as but is not isomorphic to the projective 3-space over $\mathrm{GF}(3)$. This remarkable design was first investigated by D. G. Higman [8]. (The isomorphism test was performed using the GAP package DESIGN [13].) The setwise stabilizers of the subspaces in the last two cases are isomorphic but not conjugate in Fi_{22}; their conjugacy classes are exchanged by the outer automorphism. The first fixes a block and the second a point, so they are contained in non-conjugate subgroups $O_7(3)$ of Fi_{22}; indeed, they are stabilizers of totally isotropic 3-spaces in $O_7(3)$.

In each case, the value of $a(\alpha,L)$ is the same for all non-incident pairs (α,L) in the subspace (that is, each subspace is a partial geometry [1] [2, Chapter 7], though the geometry is "degenerate" in the first case.)

The group induced on a subspace of size 40 is the full isomorphism group of this design, with structure $3^3{:}L_3(3)$, and fixes a point in the subspace (permuting the other 39 points transitively). The subspace contains 13 projective planes of order 3; all of them contain the fixed point, and any further point lies in 4 of them.

Any further "plane", or intersection of all the blocks containing three non-collinear points, must consist of pairwise non-collinear points. There are just two orbits on such planes; the following table gives information about them. "Orbit size" is the size of the orbit of a 2-point stabilizer from which the third point is chosen.

Orbit size	288	630
Number of blocks	7	28
Size of intersection	3	4
Stabilizer	$S_3 \times S_7$	order $2^{11}3^3$
Induced group	S_3	S_4

It would not be too hard to classify completely the subspaces of the geometry which arise as intersections of blocks. We mention one further example. There is a subspace S of size 28 arising as the intersection of the blocks containing four given points. It is the union of two planes of size 16 (each isomorphic to an affine plane with a parallel class of lines removed) intersecting in one of the "removed" lines. The values of $a(\alpha,L)$ within this geometry are 1 and 3 (that is, it is a $(1,3)$-geometry in the sense of [4]). Its automorphism group has order $2^6 3^3$, and has two orbits on points and one on lines.

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