# Divisible designs from twisted dual numbers 

Andrea Blunck Hans Havlicek* Corrado Zanella

Dedicated to Helmut Mäurer on the occasion of his 70th birthday


#### Abstract

The generalized chain geometry over the local ring $K(\varepsilon ; \sigma)$ of twisted dual numbers, where $K$ is a finite field, is interpreted as a divisible design obtained from an imprimitive group action. Its combinatorial properties as well as a geometric model in 4-space are investigated. Mathematics Subject Classification (2000): 51E05, 51B15, 51E20, 51E25, 51A45. Key Words: divisible design, chain geometry, local ring, twisted dual numbers, geometric model.


## 1 Preliminaries

This paper deals with a special class of divisible designs, namely, those that are chain geometries over certain finite local rings, and their representation in projective space.
A finite geometry $\Sigma=(\mathcal{P}, \mathcal{B}, \|)$, consisting of a set $\mathcal{P}$ of points, a set $\mathcal{B}$ of blocks, and an equivalence relation $\|$ (parallel) on $\mathcal{P}$, is called a $t-\left(s, k, \lambda_{t}\right)$ divisible design ( $t$-DD for short), if there exist positive integers $t, s, k, \lambda_{t}$ such that the following axioms hold:

- Each block $B$ is a subset of $\mathcal{P}$ containing $k$ pairwise non-parallel points.
- Each parallel class consists of $s$ points.
- For each set $Y$ of $t$ pairwise non-parallel points there exist exactly $\lambda_{t}$ blocks containing $Y$.
- $t \leq k \leq v / s$, where $v:=|\mathcal{P}|$.

[^0]Note that sometimes DDs are called "group divisible designs".
A DD with trivial parallel relation, i.e. with $s=1$, is an ordinary design. A DD with $k=v / s$ is called transversal. In the subsequent sections we shall deal with transversal 3-DDs.
A method to construct DDs with a large group of automorphisms is due to A.G. Spera [11], using imprimitive group actions: Let $G$ be a group acting on a (finite) set $\mathcal{P}$ of "points" and leaving invariant an equivalence relation \| ("parallel"). Let $t$ be a positive integer such that there are at least $t$ parallel classes, and let $B_{0}$ be a set of $k \geq t$ pairwise non-parallel points (the "base block"). Assume that $G$ acts transitively on the set of $t$-tuples of pairwise non-parallel points. Let $\mathcal{B}$ be the orbit of $B_{0}$ under $G$, i.e. $\mathcal{B}=\left\{B_{0}^{g} \mid g \in G\right\}$. Then $\Sigma=(\mathcal{P}, \mathcal{B}, \|)$ is a $t$-DD with

$$
\begin{equation*}
\lambda_{t}=\frac{|G|}{\left|G_{B_{0}}\right|} \cdot \frac{\binom{k}{t}}{\binom{v / s}{t} s^{t}} \tag{1}
\end{equation*}
$$

where $G_{B_{0}}$ is the (setwise) stabilizer of $B_{0}$ in $G$ (see [11, Prop. 2.3]).
The projective line $\mathbb{P}(R)$ over a finite local ring $R$ is endowed with an equivalence relation (usually denoted by $\|$ ). It is invariant under the action of the general linear group $\mathrm{GL}_{2}(R)$ on $\mathbb{P}(R)$. Since $\mathrm{GL}_{2}(R)$ acts transitively on the set of triples of non-parallel points, any $k$-set $(k \geq 3)$ of mutually non-parallel points of $\mathbb{P}(R)$ can be chosen as base block $B_{0}$ in order to apply Spera's construction of a DD. This is, of course, a very general approach. Therefore, it is not surprising that not too much can be said about the corresponding 3 -DDs. It is straightforward to express their parameters $v$ and $s$, as well as the order of the group $\mathrm{GL}_{2}(R)$, in terms of $|R|$ and $|I|$, i.e. the cardinality of the unique maximal ideal $I$ of the given ring $R$. However, in order to calculate the parameter $\lambda_{3}$ by virtue of (1), one needs to know the order of the stabilizer of $B_{0}$ in $\mathrm{GL}_{2}(R)$. But it seems hopeless to calculate this order without further information about the base block $B_{0}$.
If $R$ is even a finite local algebra over a field $F$, say, then the projective line $\mathbb{P}(F)$ over $F$ can be considered as a subset of $\mathbb{P}(R)$, and it can be chosen as a base block. All 3-DDs obtained in this way satisfy $\lambda_{3}=1$; they are - up to notational differences - precisely the (classical) chain geometries $\Sigma(F, R)$; see [1], [6] or [9]. This was pointed out by Spera [11, Example 2.5]. In the cited paper also a series of interesting DDs are constructed from base blocks which are certain subsets of $\mathbb{P}(F)$. See also [7] for similar results. We mention in passing that higher-dimensional projective spaces over local algebras give rise to 2 -DDs [12].
The divisible designs which are constructed in the present paper arise also from chain geometries. However, we use this term in a more general form
which was introduced in [3] just a few years ago. The essential difference is as follows: We consider a finite local ring $R$ containing a subfield $K$ which is not necessarily in the centre of $R$. Thus $R$ need not be a $K$-algebra, but of course it is an algebra over some subfield of $K$. As before, we can define $\mathbb{P}(K) \subset \mathbb{P}(R)$ to be the base block. This gives a 3-DD which coincides with the (generalized) chain geometry $\Sigma(K, R)$. It is possible to express the parameter $\lambda_{3}$ of this DD in algebraic terms (see [3, Theorem 2.4]), but this is not very explicit in the general case. Therefore, we focus our attention on a particular class of local rings, namely twisted dual numbers. If the "twist" is non-trivial, then 3-DDs with parameter $\lambda_{3}=|K|$ are obtained.
In Section 4 we present an alternative description of our 3-DDs in a finite projective space over $K$.

## 2 Twisted dual numbers

Let $R$ be a (not necessarily commutative) local ring containing a (not necessarily central) subfield $K$. In view of our objective to construct DDs, we will later restrict ourselves to finite rings and fields, and hence we assume from the beginning that $K$ is commutative. As usual, we denote by $R^{*}$ the group of units (invertible elements) of $R$. We set $I:=R \backslash R^{*}$; since $R$ is local we have that $I$ is an ideal.
The ring $R$ is in a natural way a left vector space over $K$, sometimes written as ${ }_{K} R$. We assume that $\operatorname{dim}\left({ }_{K} R\right)=2$. Moreover, we assume that $R$ is not a field. We want to determine the structure of $R$. The ideal $I$ is a non-trivial subspace of the vector space ${ }_{K} R$. So $\operatorname{dim}\left({ }_{K} I\right)=1$, and $I=K \varepsilon$ for some $\varepsilon \in R \backslash K$. Then $1, \varepsilon$ is a basis of ${ }_{K} R$, and we may write $R=K+K \varepsilon$.
In order to describe the multiplication in $R$ we first observe that $\varepsilon^{2} \in I$, so $\varepsilon^{2}=b \varepsilon$ for some $b \in K$. This implies $(\varepsilon-b) \varepsilon=0$, whence also $\varepsilon-b \in I$ and so $b=0$. For each $x \in K$ we have $\varepsilon x \in I$, so there is a unique $x^{\prime} \in K$ such that $\varepsilon x=x^{\prime} \varepsilon$. One can easily check that $\sigma: x \mapsto x^{\prime}$ is an injective field endomorphism.
Conversely, given a field $K$ and an injective endomorphism $\sigma$ of $K$ we obtain a ring of twisted dual numbers $R=K(\varepsilon ; \sigma)=K+K \varepsilon$ with multiplication

$$
(a+b \varepsilon)(c+d \varepsilon)=a c+\left(a d+b c^{\sigma}\right) \varepsilon
$$

In the special case that $\sigma=\mathrm{id}$ this is the well known commutative ring $K(\varepsilon)$ of dual numbers over $K$.
The subfield $\operatorname{Fix}(\sigma)$ of $K$ fixed elementwise by $\sigma$ will be called $F$. So $F=K$ if, and only if, $\sigma=\mathrm{id}$.

The units of $R$ are exactly the elements of $R \backslash I=K^{*}+K \varepsilon$. One can easily check that the inverse of a unit $u=a+b \varepsilon$ (with $a, b \in K, a \neq 0$ ) is

$$
\begin{equation*}
u^{-1}=a^{-1}-a^{-1} b\left(a^{\sigma}\right)^{-1} \varepsilon . \tag{2}
\end{equation*}
$$

Later we shall need the following algebraic statements on $R=K(\varepsilon ; \sigma)$.
2.1 Lemma. The multiplicative group $R^{*}$ is the semi-direct product of $K^{*}$ and the normal subgroup

$$
\begin{equation*}
U=1+K \varepsilon=\{1+b \varepsilon \mid b \in K\} . \tag{3}
\end{equation*}
$$

Proof: Direct computation, using (2) for showing that $U$ is normal in $R^{*}$.
2.2 Lemma. Let $N$ be the normalizer of $K^{*}$ in $R^{*}$, i.e.,

$$
\begin{equation*}
N=\left\{n \in R^{*} \mid n^{-1} K^{*} n=K^{*}\right\} . \tag{4}
\end{equation*}
$$

Then $N=R^{*}$ if $\sigma=\mathrm{id}$ and $N=K^{*}$ otherwise.
Proof: For $\sigma=\mathrm{id}$ the assertion is clear. So let $\sigma \neq \mathrm{id}$ and $n=a+b \varepsilon \in N$. Take an element $x \in K \backslash F$. Using (2) we get $n^{-1} x n=x+a^{-1} b\left(x-x^{\sigma}\right) \varepsilon$, which must belong to $K$ since $n \in N$. Because of our choice of $x$ we have $x-x^{\sigma} \neq 0$, whence $b=0$, as desired.

## 3 The associated DD

In this section we construct a 3 -DD using the ring $R=K(\varepsilon ; \sigma)$. The construction is a special case of Spera's construction method described in Section 1 (see also [8, Section 2.3]). On the other hand, the resulting DD is nothing else than the (generalized) chain geometry over ( $K, R$ ) (compare [3], for details on ordinary chain geometries see [6], [9]).
From now on we assume that $R$, and hence also $K$ and $F$, are finite. Then $F=\mathrm{GF}(m)$ for some prime power $m$, and $K=\mathrm{GF}(q)$ with $q$ a power of $m$. Moreover, $\sigma$ now is an automorphism of $K$, namely, $\sigma: x \mapsto x^{m}$.
The construction is based on the action of the group $G=\mathrm{GL}_{2}(R)$ of invertible $2 \times 2$-matrices with entries in $R$ on the projective line over $R$, i.e., on the set

$$
\mathbb{P}(R)=\left\{R(a, b) \leq R^{2} \mid \exists c, d \in R:\left(\begin{array}{ll}
a & b  \tag{5}\\
c & d
\end{array}\right) \in G\right\} .
$$

Since $R$ is local, each pair $(a, b)$ as in (5) has the property that at least one of the two elements $a, b$ is invertible, because otherwise the existence of an
inverse matrix $\left(\begin{array}{ll}x & * \\ y & *\end{array}\right)$ would lead to the contradiction $1=a x+b y \in I$. So $\mathbb{P}(R)$ is the disjoint union

$$
\begin{equation*}
\mathbb{P}(R)=\{R(x, 1) \mid x \in R\} \cup\{R(1, z) \mid z \in I\} . \tag{6}
\end{equation*}
$$

On $\mathcal{P}=\mathbb{P}(R)$ we have an equivalence relation $\|$ given by

$$
R(a, b) \| R(c, d): \Longleftrightarrow\left(\begin{array}{ll}
a & b  \tag{7}\\
c & d
\end{array}\right) \notin G .
$$

More explicitly, this means for arbitrary $x, y \in R, z, w \in I$ :

$$
\begin{equation*}
R(1, z) \| R(1, w) ; R(x, 1) \nVdash R(1, z) ;(R(x, 1) \| R(y, 1) \Leftrightarrow x-y \in I) . \tag{8}
\end{equation*}
$$

Using the description in (8) one can see that $\|$ in fact is an equivalence relation.
Let us recall two facts (see [9, 1.2.2] and [9, Prop. 1.3.3], where non-parallel points are called "distant"): The group $G$ acts on $\mathcal{P}$ leaving $\|$ invariant. Moreover, $G$ acts transitively on the set of triples of pairwise non-parallel points of $\mathcal{P}$. By virtue of this action of $G$ and (8), any two parallel classes have the same cardinality $s=|I|$.
In order to apply Spera's method we now need a base block consisting of pairwise non-parallel points. As usual for chain geometries, we use the projective line over $K$.
Since $K$ is a subfield of $R$, the projective line $\mathbb{P}(K)$ can be seen as a subset $B_{0}$ of $\mathcal{P}=\mathbb{P}(R)$ as follows:

$$
\begin{equation*}
B_{0}=\mathbb{P}(K)=\{R(x, 1) \mid x \in K\} \cup\{R(1,0)\} \tag{9}
\end{equation*}
$$

Let $\mathcal{B}=B_{0}^{G}$. Then we get the following.
3.1 Theorem. The structure $\Sigma=(\mathcal{P}, \mathcal{B}, \|)$ is a transversal $3-D D$ with parameters $v=q^{2}+q, s=q, k=q+1(=v / s)$ and

$$
\lambda_{3}= \begin{cases}1 & \text { if } \quad \sigma=\mathrm{id} \\ q & \text { if } \sigma \neq \mathrm{id}\end{cases}
$$

Proof: From Spera's theorem we know that $\Sigma$ is a DD. The values of $v, s$, and $k$ are obtained from (6), (8), and (9), respectively. By [3, Theorem 2.4], we have $\lambda_{3}=\left|R^{*}\right| /|N|$, where $N$ is the normalizer defined in (4). By Lemma 2.2 we have two cases: If $\sigma=\mathrm{id}$, the normalizer $N$ coincides with $R^{*}$ and so $\lambda_{3}=1$. If $\sigma \neq \mathrm{id}$, the normalizer $N$ equals $K^{*}$, whence $\lambda_{3}=\left|R^{*}\right| /|N|=$ $(q-1) q /(q-1)=q$.

The equation $\sigma=$ id holds precisely when $K$ lies in the centre of $R$; in this case our DD is an ordinary chain geometry, namely, the Miquelian Laguerre plane over the algebra of dual numbers (see [1, I.2, II.4]).
We mention here that the parameter $\lambda_{3}$ could also be computed directly using the formula

$$
\begin{equation*}
\lambda_{3}=\frac{|G|}{\left|G_{B_{0}}\right|} \cdot \frac{1}{s^{3}} \tag{10}
\end{equation*}
$$

(see (1), note that $k=v / s$ ).
We add without proof that the 3 -DD $\Sigma$ can also be described as a lifted $D D$ in the sense of [5, Theorem 2.5], using the point set $\mathcal{P}$, the equivalence relation $\|$, the group

$$
H=\left\{\left.\left(\begin{array}{cc}
1+a \varepsilon & b \varepsilon \\
c \varepsilon & 1+d \varepsilon
\end{array}\right) \right\rvert\, a, b, c, d \in K\right\}
$$

which acts on $\mathcal{P}$, and the base block $B_{0}$ as (trivial) base DD. However, this alternative approach does not immediately show the large group of automorphisms given by the action of $G$ on $\mathcal{P}$.
We now have a closer look at the case $\sigma \neq \mathrm{id}$. We want to determine the $q$ blocks through three given pairwise non-parallel points more explicitly. Because of the transitivity properties of $G$ it suffices to consider the points $\infty=R(1,0), 0=R(0,1), 1=R(1,1)$. From [3, Theorem 2.4] we know the following: The blocks through $\infty, 0,1$ are exactly the images of $B_{0}$ under the group

$$
\widehat{R^{*}}=\left\{\left.\left(\begin{array}{ll}
u & 0  \tag{11}\\
0 & u
\end{array}\right) \right\rvert\, u \in R^{*}\right\}
$$

and two elements $\omega=\left(\begin{array}{ll}u & 0 \\ 0 & u\end{array}\right)$ and $\omega^{\prime}=\left(\begin{array}{cc}u^{\prime} & 0 \\ 0 & u^{\prime}\end{array}\right)$ of $\widehat{R^{*}}$ determine the same block if, and only if, $N u=N u^{\prime}$, with $N$ as in (4). So from Lemmas 2.2 and 2.1 we obtain:
3.2 Lemma. Let $\sigma \neq \mathrm{id}$. Then the blocks containing $\infty=R(1,0), 0=$ $R(0,1), 1=R(1,1)$ are exactly the $q$ sets

$$
B_{0}^{\omega}, \text { with } \omega=\left(\begin{array}{cc}
1+b \varepsilon & 0  \tag{12}\\
0 & 1+b \varepsilon
\end{array}\right), b \in K .
$$

We now give an explicit description of the action of the group

$$
\widehat{U}=\left\{\left.\left(\begin{array}{ll}
u & 0  \tag{13}\\
0 & u
\end{array}\right) \right\rvert\, u \in U\right\}=\left\{\left.\left(\begin{array}{cc}
1+b \varepsilon & 0 \\
0 & 1+b \varepsilon
\end{array}\right) \right\rvert\, b \in K\right\},
$$

associated to $U($ see (3)), on $\mathcal{P}=\mathbb{P}(R)$.

A direct calculation shows that each $\omega=\left(\begin{array}{ll}u & 0 \\ 0 & u\end{array}\right)$, with $u \in R^{*}$, acts on $\mathcal{P}$ via "conjugation" as follows:

$$
\begin{equation*}
\omega: R(x, 1) \mapsto R\left(u^{-1} x u, 1\right), \quad R(1, z) \mapsto R\left(1, u^{-1} z u\right) \tag{14}
\end{equation*}
$$

where, as before, $x \in R, z \in I$. For $u=1+b \varepsilon \in U$ this yields, using (2),

$$
\begin{equation*}
\omega: R(x, 1) \mapsto R\left(x+b\left(x_{1}-x_{1}^{\sigma}\right) \varepsilon, 1\right), \quad R(1, z) \mapsto R(1, z) \tag{15}
\end{equation*}
$$

where $x=x_{1}+x_{2} \varepsilon$. So the mapping $\omega \in \widehat{U}$ of (15) maps each point to a parallel one. Moreover, it fixes exactly those elements of the base block $B_{0}=\mathbb{P}(K)$ that belong to the subset $\mathbb{P}(F)$. This subset in turn is the intersection of all blocks through $\infty, 0,1$ (compare (12)); such intersections are also called traces (in German:"Fährten", see [1], [3]).
We consider a parallel class on which $\widehat{U}$ does not act trivially. By (15) this is the parallel class of some point $p=R\left(x_{1}, 1\right)$, where $x_{1} \in K \backslash F$ and consequently $p \in B_{0} \backslash \mathbb{P}(F)$. Then $\widehat{U}$ acts regularly on the parallel class under consideration. As a matter of fact, for each $p^{\prime}$ parallel to $p$, which has the form $p^{\prime}=R\left(x_{1}+x_{2} \varepsilon, 1\right)$, there is a unique $b \in K$ with $x_{2}=b\left(x_{1}-x_{1}^{\sigma}\right)$, so $p^{\omega}=p^{\prime}$, with $\omega$ as in (15). This means that for each $p^{\prime} \| p$ there is exactly one block through $\infty, 0,1$ that contains $p^{\prime}$ (and each block through $\infty, 0,1$ is obtained in this way, as each block meets all parallel classes).
All these results can be carried over to an arbitrary triple of pairwise nonparallel points, using the action of $G$. So we have the following.
3.3 Proposition. Let $\sigma \neq \mathrm{id}$. Let $p_{1}, p_{2}, p_{3} \in \mathcal{P}$ be pairwise non-parallel. Let $T$ be the intersection of all blocks through $p_{1}, p_{2}, p_{3}$, and let $C$ be a parallel class not meeting $T$. Then the following hold.
(a) There is a $g \in G$ such that $T=\mathbb{P}(F)^{g}$.
(b) Each block through $p_{1}, p_{2}, p_{3}$ meets $C$, and for each $x \in C$ there is a (unique) block through $p_{1}, p_{2}, p_{3}, x$.
3.4 Corollary. Let $p_{1}, p_{2}, p_{3}$ be pairwise non-parallel, let $T$ be the intersection of all blocks through $p_{1}, p_{2}, p_{3}$, and let $x \nVdash p_{1}, p_{2}, p_{3}$. Then the number of blocks through $p_{1}, p_{2}, p_{3}, x$ is

- $q$, if $x \in T$,
- 0 , if $x \notin T$, but $x \| x^{\prime}$ for some $x^{\prime} \in T$,
- 1, otherwise.

Finally, let us point out a particular case:
3.5 Corollary. Let $q$ be even and let $m=2$, i.e., $x^{\sigma}=x^{2}$ for all $x \in K$. Then $\Sigma=(\mathcal{P}, \mathcal{B}, \|)$ is a 4 -divisible design with parameter $\lambda_{4}=1$.

This result is immediate from Corollary 3.4, since $F=\operatorname{GF}(2)$ implies now $|T|=|\mathbb{P}(F)|=3$.

## 4 A geometric model

Now we are looking for a geometric point model of the DD $\Sigma$ defined above, i.e. a DD isomorphic to $\Sigma$ whose points are points of a suitable projective space. We find such a model on the Klein quadric $\mathcal{K}$ in $\operatorname{PG}(5, K)$ by using H. Hotje's representation [10].
4.1 Remark. One could also first find a line model of $\Sigma$ in $\operatorname{PG}(3, K)$ (where the points of $\Sigma$ are certain lines in 3 -space) and then apply the Klein correspondence. For details on such line models see [4], in particular Examples 5.2 and 5.4, and [2].

We embed the ring $R=K(\varepsilon ; \sigma)$ in the ring $M=\mathrm{M}(2, K)$ of $2 \times 2$-matrices with entries in $K$ via the ring monomorphism

$$
a+b \varepsilon \mapsto\left(\begin{array}{cc}
a & b  \tag{16}\\
0 & a^{\sigma}
\end{array}\right) .
$$

From now on we identify the ring $R$ with its image under this embedding. The projective line $\mathbb{P}(M)$ is defined, mutatis mutandis, according to (5). The points of $\mathbb{P}(M)$ are of the form $M(A, B)$, where $(A, B)$ are the first two rows of an invertible $4 \times 4$-matrix over $K$, because (up to notation) $\mathrm{GL}_{2}(M)$ equals $\mathrm{GL}_{4}(K)$. Then (16) allows to identify the point set $\mathbb{P}(R)$ of $\Sigma$ with a subset of $\mathbb{P}(M)$.
Now we establish the existence of a bijection $\Phi$ from $\mathbb{P}(M)$ onto the Klein quadric $\mathcal{K}$. For this we notice that $M$ is a $K$-algebra, with $K$ embedded in $M$ via $x \mapsto\left(\begin{array}{ll}x & 0 \\ 0 & x\end{array}\right)$, and that this algebra is kinematic, i.e., each element of $M$ satisfies a quadratic equation over $K$. Note that this embedding of $K$ in $M$ is different from the one obtained from (16), unless $\sigma=\mathrm{id}$. In [10] Hotje embeds the projective line over an arbitrary kinematic algebra in an appropriate quadric. For the matrix algebra $M$ this quadric is $\mathcal{K}$, and the embedding, which here is a bijection, is the following:

$$
\begin{equation*}
\Phi: \mathbb{P}(M) \rightarrow \mathcal{K}: M(A, B) \mapsto K(\widetilde{B} A, \operatorname{det} A, \operatorname{det} B), \tag{17}
\end{equation*}
$$

where $A, B$ are matrices in $M$, and for $B=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ we set $\widetilde{B}=\left(\begin{array}{cc}d & -b \\ -c & a\end{array}\right)$. The image of $\Phi$ is indeed the Klein quadric, because $M \times K \times K$ is a 6 dimensional vector space over $K$ endowed with the hyperbolic quadratic form $(C, x, y) \mapsto \operatorname{det} C-x y$.
We need the following additional statements:
4.2 Proposition. Consider the bijection $\Phi: \mathbb{P}(M) \rightarrow \mathcal{K}$ given in (17), and its restriction to $\mathbb{P}(R)$. Then
(a) The bijection $\Phi$ induces a homomorphism of group actions, mapping $\mathrm{GL}_{2}(M)$, acting on $\mathbb{P}(M)$, to a subgroup of the group of collineations of $\operatorname{PG}(5, K)$ leaving $\mathcal{K}$ invariant.
(b) This homomorphism maps the subgroup $\mathrm{GL}_{2}(R)$, acting on $\mathcal{P}=\mathbb{P}(R)$, to a subgroup of the group of collineations of $\operatorname{PG}(5, K)$ leaving $\mathcal{P}^{\Phi}$ invariant.
(c) Two points of $\mathbb{P}(R)$ are parallel if, and only if, their $\Phi$-images are joined by a line contained in $\mathcal{K}$.

Proof: For (a) see [10, (7.1/2/3)]; (b) follows from (a).
(c): This follows from $[10,(7.5)]$ and [4, Prop. 3.2].

Writing $K\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}\right)$ instead of $K\left(\left(\begin{array}{ll}x_{1} & x_{2} \\ x_{3} & x_{4}\end{array}\right), x_{5}, x_{6}\right)$, we obtain by a direct computation that the mapping $\Phi$ given in (17) acts on the points of $\mathcal{P}=\mathbb{P}(R) \subseteq \mathbb{P}(M)$ as follows:

$$
\begin{equation*}
R(a+b \varepsilon, 1) \mapsto K\left(a, b, 0, a^{\sigma}, a a^{\sigma}, 1\right) ; R(1, c \varepsilon) \mapsto K(0,-c, 0,0,1,0) \tag{18}
\end{equation*}
$$

We shall identify the elements of $\mathbb{P}(M)$ with their $\Phi$-images. Then, in particular, we have

$$
\begin{equation*}
B_{0}=\left\{K\left(a, 0,0, a^{\sigma}, a a^{\sigma}, 1\right) \mid a \in K\right\} \cup\{K(0,0,0,0,1,0)\} . \tag{19}
\end{equation*}
$$

In the next lemma we collect some observations, which can be seen directly using (18) and (19).
4.3 Lemma. Let $\mathcal{P}$ and $B_{0}$ be the point sets in $\operatorname{PG}(5, K)$ from above. Then the following hold:
(a) $\mathcal{P}=\mathcal{C} \backslash\{S\}$, where $\mathcal{C}$ is the cone with vertex $S=K(0,1,0,0,0,0)$ over $B_{0}$, i.e. the union of all lines joining $S$ with $B_{0}$.
(b) $\mathcal{P}$ is entirely contained in the hyperplane $\boldsymbol{H}$ with equation $x_{3}=0$, which is the tangent hyperplane to $\mathcal{K}$ at $S$.
(c) Two points of $\mathcal{P}$ are parallel if, and only if, they lie on a generator of $\mathcal{C}$, i.e. a line through $S$ contained in $\mathcal{C}$.

Now we describe the (image of) the base block $B_{0}$ more closely:
4.4 Lemma. Let $B_{0}$ be as in (19). Then the following hold:
(a) $B_{0}$ is a cap, i.e. a set of points no three of which are collinear.
(b) If $\sigma=\mathrm{id}$, then $B_{0}$ is a regular conic; in particular, $B_{0}$ is contained in a plane.
(c) If $\sigma \neq$ id, then $B_{0}$ spans the 3 -space $\boldsymbol{U}_{0}$, given by $x_{2}=0=x_{3}$, complementary to $S$ in $\boldsymbol{H}$.

Proof: (a): Assume that the line $L$ carries three points of $B_{0}$. Then $L \subseteq \mathcal{K}$. From Proposition 4.2(c) we see that the three points are pairwise parallel, a contradiction.
(b): Here $B_{0}=\left\{K\left(a, 0,0, a, a^{2}, 1\right) \mid a \in K\right\} \cup\{K(0,0,0,0,1,0)\}$, which obviously is a regular conic in the plane spanned by the points $K(1,0,0,1,0,0)$, $K(0,0,0,0,1,0), K(0,0,0,0,0,1)$ (namely, the intersection of this plane with the Klein quadric).
(c): In this case, the four vectors

$$
(0,0,0,0,1,0),(0,0,0,0,0,1),(1,0,0,1,1,1), \text { and }\left(a, 0,0, a^{\sigma}, a a^{\sigma}, 1\right)
$$

with $a \in K \backslash F$, are linearly independent, so the point set $B_{0}$ spans $\boldsymbol{U}_{0}$.
In case $\sigma=\mathrm{id}$, our geometric model is nothing else than the "cylinder model" of the Miquelian Laguerre plane $\Sigma$ : The points are the points of a cylinder in 3 -space (a quadratic cone minus its vertex), and the blocks are the regular conics on the cylinder (the intersections with planes complementary to the vertex). See, e.g., [1, I.2] for the real case.
We have a closer look at the special case that $\sigma^{2}=\mathrm{id}, \sigma \neq \mathrm{id}$. Then $q=m^{2}$ and $K$ is a quadratic extension of $F$. In this case there are Baer subspaces, i.e. spaces coordinatized by $F$, in each projective space over $K$.
4.5 Proposition. Let $\sigma^{2}=\mathrm{id}, \sigma \neq \mathrm{id}$. Then $B_{0}$ is an elliptic quadric in the Baer subspace $\mathbb{B} \cong \mathrm{PG}(3, m)$ of $\boldsymbol{U}_{0} \cong \mathrm{PG}(3, q)$ defined by the $F$-subspace

$$
\begin{equation*}
\left\{\left(x, 0,0, x^{\sigma}, f_{1}, f_{2}\right) \mid x \in K, f_{i} \in F\right\} . \tag{20}
\end{equation*}
$$

Proof: Obviously, the set in (20) is a 4-dimensional subspace of $K^{6}$, seen as a vector space over $F$, satisfying the equations $x_{2}=0=x_{3}$ and hence giving rise to a Baer subspace $\mathbb{B}$ of $\boldsymbol{U}_{0}$. The elements of $B_{0}$ all lie in $\mathbb{B}$. Moreover, by (19), $B_{0}$ equals the quadric in $\mathbb{B}$ determined by $N(x)=f_{1} f_{2}$, where $N(x)=x x^{\sigma}$ is the norm of $x \in K$ with respect to the field extension $K: F$ and, in particular, $N$ is a quadratic form on the vector space ${ }_{F} K$. Since $B_{0}$ is a cap by 4.4 (a), the quadric must be elliptic.

The quadratic form used in the above is just the restriction to $\mathbb{B}$ of the quadratic form describing the Klein quadric. The intersection of the Klein quadric and $\boldsymbol{U}_{0}$ is a hyperbolic quadric.
For the rest of this section we consider the case that $\sigma \neq \mathrm{id}$. We try to describe the geometric model of the DD $\Sigma$ more explicitly. From the above we know that our base block $B_{0}$ is a certain cap that spans a 3 -space $\boldsymbol{U}_{0}$ complementary to $S$ in the tangent hyperplane $\boldsymbol{H} \cong \mathrm{PG}(4, K)$ of $\mathcal{K}$ at $S$. In the next proposition we describe all blocks. Together with Lemma 4.3 this gives a description of $\Sigma$ in terms of $\operatorname{PG}(4, K)$.
4.6 Proposition. Let $\sigma \neq \mathrm{id}$. Then the blocks of $\Sigma$ are exactly the intersections of the cone $\mathcal{C}$ with the 3 -spaces complementary to $S$ in $\boldsymbol{H}$.

Proof: We know that $B_{0}=\mathcal{C} \cap \boldsymbol{U}_{0}$, with $\boldsymbol{U}_{0}$ complementary to $S$ in $\boldsymbol{H}$. Let $B$ be any block. Then $B=B_{0}^{g}$ for some $g \in G=\mathrm{GL}_{2}(R) \leq \mathrm{GL}_{2}(M)$. By Proposition 4.2(b), $g$ induces a collineation, say $\widetilde{g}$, of $\mathrm{PG}(5, K)$ leaving $\mathcal{K}$ and $\mathcal{P}$ invariant. This collineation fixes $S$ (which is the intersection of the lines corresponding to parallel classes) and its tangent hyperplane $\boldsymbol{H}$. So $B$, seen as a set of points in $\boldsymbol{H}$, is $B=B_{0}^{\tilde{g}}=\mathcal{C} \cap \boldsymbol{U}_{0}^{\widetilde{g}}$, where $\boldsymbol{U}_{0}^{\widetilde{g}}$ is a 3 -space complementary to $S$, as desired. The 3 -space $\boldsymbol{U}_{0}^{\widetilde{g}}$ is independent of the choice of $g$, as it is nothing else than the span of $B$.
So we have a mapping from the set of blocks to the set of complements of $S$ in $\boldsymbol{H}$, which is injective since each complement contains exactly one point of each generator of $\mathcal{C}$, i.e. of each parallel class of $\mathcal{P}$, and hence cannot belong to more than one block. A simple counting argument shows that the mapping is also surjective: The number of blocks is $b=|G| /\left|G_{B_{0}}\right|=q^{4}$ (this can be computed directly, or from (10) using $\lambda_{3}=q$ ), and the number of complements of $S$ in $\boldsymbol{H}$ also is $q^{4}$, because they form an affine 4 -space of order $|K|=q$.
4.7 Remark. The projective model of $\Sigma$ studied in this section is a special case of the lifted $t$-DDs described in [5, Cor. 3.3]. There, the following geometries are described as $t$-DDs obtained via the lifting process: Consider an
arbitrary finite projective space $\mathrm{PG}(n, q)$ and a set $B_{0}$ of $k$ points spanning a subspace $\boldsymbol{U}_{0}$ and having the property that any $t$ points of $B_{0}$ are independent. Let $\boldsymbol{S}$ be a complement of $B_{0}$. The point set of the $t$ - DD is the cone with basis $B_{0}$ and vertex $\boldsymbol{S}$, minus $\boldsymbol{S}$. The blocks are the intersections of the cone with subspaces complementary to $\boldsymbol{S}$, and two points are parallel if, and only if, together with $\boldsymbol{S}$ they span the same subspace.

The following is an obvious geometric analogue of Proposition 3.3 and Corollary 3.4.
4.8 Corollary. Let $p_{1}, p_{2}, p_{3}$ be pairwise non-parallel, let $T$ be the intersection of all blocks through $p_{1}, p_{2}, p_{3}$, and let $x \nVdash p_{1}, p_{2}, p_{3}$. Then
(a) $T$ is the intersection of the cone $\mathcal{C}$ with the plane $\boldsymbol{E}$ spanned by $p_{1}, p_{2}, p_{3}$.
(b) The blocks through $p_{1}, p_{2}, p_{3}, x$ are exactly the intersections of $\mathcal{C}$ with 3 -spaces through $\boldsymbol{E}$ complementary to $S$. The number of such 3 -spaces is

- $q$, if $x \in T$,
- 0 , if $x \notin T$, but $x \| x^{\prime}$ for some $x^{\prime} \in T$,
- 1, otherwise.


## References

[1] W. Benz. Vorlesungen über Geometrie der Algebren. Springer, Berlin, 1973.
[2] A. Blunck. The cross ratio for quadruples of subspaces. Mitt. Math. Ges. Hamburg, 22:81-97, 2003.
[3] A. Blunck and H. Havlicek. Extending the concept of chain geometry. Geom. Dedicata, 83:119-130, 2000.
[4] A. Blunck and H. Havlicek. Projective representations I. Projective lines over rings. Abh. Math. Sem. Univ. Hamburg, 70:287-299, 2000.
[5] A. Blunck, H. Havlicek, and C. Zanella. Lifting of divisible designs. Designs, Codes and Cryptography, 42:1-14, 2007.
[6] A. Blunck and A. Herzer. Kettengeometrien. Shaker, Aachen, 2005.
[7] S. Giese, H. Havlicek, and R.-H. Schulz. Some constructions of divisible designs from Laguerre geometries. Discrete Math., 301:74-82, 2005.
[8] H. Havlicek. Divisible Designs, Laguerre Geometry, and Beyond. Brescia, 2004. Summer School on Combinatorial Geometry and Optimisation "Giuseppe Tallini". Quaderni del Seminario Matematico di Brescia, 11, 2006 (electronic).
http://www.dmf.unicatt.it/cgi-bin/preprintserv/semmat/Quad2006n11
[9] A. Herzer. Chain geometries. In F. Buekenhout, editor, Handbook of Incidence Geometry, pages 781-842. Elsevier, Amsterdam, 1995.
[10] H. Hotje. Zur Einbettung von Kettengeometrien in projektive Räume. Math. Z., 151:5-17, 1976.
[11] A.G. Spera. t-divisible designs from imprimitive permutation groups. Europ. J. Combin., 13:409-417, 1992.
[12] A.G. Spera. On divisible designs and local algebras. J. Comb. Designs, 3:203-212, 1995.

Andrea Blunck, Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany, andrea.blunck@math.uni-hamburg.de

Hans Havlicek, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria, havlicek@geometrie.tuwien.ac.at

Corrado Zanella, Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università di Padova, Stradella S. Nicola, 3, I-36100 Vicenza, Italy, corrado.zanella@unipd.it


[^0]:    *Corresponding author.

