BLOCK-TRANSITIVE DESIGNS IN AFFINE SPACES

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Abstract. This paper deals with block-transitive t-(v, k, λ) designs in affine spaces for large t, with a focus on the important index $\lambda = 1$ case. We prove that there are no non-trivial 5-$(v, k, 1)$ designs admitting a block-transitive group of automorphisms that is of affine type. Moreover, we show that the corresponding non-existence result holds for 4-$(v, k, 1)$ designs, except possibly when the group is one-dimensional affine. Our approach involves a consideration of the finite 2-homogeneous affine permutation groups.

1. Introduction

The construction and characterization of block-transitive t-(v, k, λ) designs in affine spaces is an interesting and beautiful topic of research. The situation when $t = 2$, in particular for the index $\lambda = 1$ case, has been studied in greater detail over the last decades. However, less is known when $t \geq 3$. Obvious natural examples exist for $t = 3$ and arbitrary λ, by using point 3-transitive affine groups over the field $GF(2)$ as groups of automorphisms. For general t-designs, it has been shown that block-transitivity implies point 2-homogeneity (and hence point-primitivity) when $t \geq 4$, while for $t < 4$ an infinite number of counter-examples demonstrate that block-transitivity does not necessarily imply point-primitivity (see Proposition 8; and [11] for the case $t < 4$).

Alltop [1] constructed in 1971 the first explicit example of a block-transitive 5-design in affine space, having $v = 256$ points and block-size $k = 24$. He showed that an orbit of a 3-transitive affine group over $GF(2)$ on the k-subsets of the underlying vector space is a 5-design whenever it is a 4-design, and derived a necessary and sufficient condition for this to take place. Alltop's construction has been extended by Cameron & Praeger [8],

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To Spyros Magliveras on the occasion of his 70th birthday.

yielding a flag-transitive 5-$(2^8, 24, \lambda)$ design with $\lambda = 2^{21}.3^2.5^2.7.31$. Moreover, Cameron and Praeger proved the non-existence of block-transitive t-designs for $t > 7$.

In this paper, we focus on block-transitive 4- and 5-designs in affine spaces for the important index $\lambda = 1$ case. We will generalize several arguments developed in our earlier work on flag-transitive Steiner designs ([16, 17, 18, 19], and [20] for a monograph) to the weaker condition of block-transitivity. Our approach involves a consideration of the finite 2-homogeneous affine permutation groups. We remark that in [21, 22], we already showed in particular that no block-transitive Steiner 6-design or 7-design admitting a 3-transitive affine group over $GF(2)$ as a group of automorphisms can exist.

We prove the following main result:

Main Theorem. *There is no non-trivial Steiner* 5*-design* $\mathcal{D}$ *admitting a block-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *of automorphisms that is of affine type. Moreover, there is no non-trivial Steiner* 4*-design* $\mathcal{D}$ *admitting a block-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *of automorphisms that is of affine type, except possibly when* $G \leq A\Gamma L(1, q)$.

The paper is organized as follows. In Section 2, we introduce the notation and preliminary results that are important for the remainder of the paper. A discussion on examples of block-transitive t-designs in affine spaces for $t \geq 3$ is followed by a proof of the non-existence of block-transitive Steiner 4- and 5-designs admitting a 3-transitive affine group over $GF(2)$ as a group of automorphisms. In Section 4, we investigate Steiner 4- and 5-designs with a group of affine type as a possibly block-transitive group of automorphisms. We may restrict here to finite 2-homogeneous affine permutation groups. This investigation completes the proof of the Main Theorem.

2. Notation and Preliminaries

Definition 1. For positive integers $t \leq k \leq v$ and λ, a t-(v, k, λ) *design* $\mathcal{D}$ is a pair $(X, \mathcal{B})$, satisfying the following properties:

(i) X is a set of v elements, called *points*,

(ii) $\mathcal{B}$ is a family of k-subsets of X, called *blocks*,

(iii) every t-subset of X is contained in exactly λ blocks.

We denote points by lower-case and blocks by upper-case Latin letters. Via convention, let $b := |\mathcal{B}|$ denote the number of blocks. A *flag* of $\mathcal{D}$ is an incident point-block pair (x, B), where $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. A t-design is called *simple*, if the same k-subset of points may not occur twice as a block. If not stated otherwise, we will restrict our attention to simple designs in this paper. If $t < k < v$, then we speak of a *non-trivial* t-design. For historical reasons, a t-(v, k, λ) design with index $\lambda = 1$ is called a *Steiner t-design* (sometimes also a *Steiner system*). There are many infinite classes of Steiner t-designs for $t = 2$ and 3, however for $t = 4$ and 5 only a finite number are known. For a detailed treatment of combinatorial designs, we refer to [3, 9, 13, 23, 30]. In particular, [3, 9] provide encyclopedic accounts of key results and contain existence tables with known parameter sets.

In this paper, we are investigating t-designs which admit groups of automorphisms with homogeneity properties such as transitivity on blocks or flags. We consider automorphisms of a t-design $\mathcal{D}$ as permutations on X which preserve $\mathcal{B}$, and call a group $G \leq \mathrm{Aut}(\mathcal{D})$ of automorphisms of $\mathcal{D}$ *block-transitive* (respectively *flag-transitive*, *point t-transitive*, *point t-homogeneous*) if G acts transitively on the blocks (respectively transitively on the flags, t-transitively on the points, t-homogeneously on the points) of $\mathcal{D}$. For short, $\mathcal{D}$ is said to be, e.g., block-transitive if $\mathcal{D}$ admits a block-transitive group of automorphisms.

For $\mathcal{D} = (X, \mathcal{B})$ a Steiner t-design with $G \leq \mathrm{Aut}(\mathcal{D})$, let G_x denote the stabilizer of a point $x \in X$, and G_B the setwise stabilizer of a block $B \in \mathcal{B}$. For $x, y \in X$, we define $G_{xy} = G_x \cap G_y$.

For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest positive integer which is at most x.

We state some basic combinatorial facts (see, for instance, [3]):

Proposition 2. *Let $\mathcal{D} = (X, \mathcal{B})$ be a t-(v, k, λ) design, and for a positive integer $s \leq t$, let $S \subseteq X$ with $|S| = s$. Then the total number of blocks incident with each element of S is given by*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

In particular, for $t \geq 2$, a t-(v, k, λ) design is also an s-(v, k, λ_s) design.

It is customary to set $r := \lambda_1$ denoting the total number of blocks incident with a given point.

Corollary 3. *Let $\mathcal{D}$ be a t-(v, k, λ) design. Then the following holds:*

(a) $bk = vr$.

(b) $\binom{v}{t}\lambda = b\binom{k}{t}$.

(c) $r(k-1) = \lambda_2(v-1)$ *for* $t \geq 2$.

Corollary 4. *Let $\mathcal{D}$ be a t-(v, k, λ) design. Then*

$$\lambda \binom{v-s}{t-s} \equiv 0 \left(\mathrm{mod} \ \binom{k-s}{t-s} \right)$$

for each positive integer $s \leq t$.

Proposition 5. ([6, 31]). *If $\mathcal{D}$ is a non-trivial Steiner t-design, then the following holds:*

(a) $v \geq (t+1)(k-t+1)$.

(b) $v - t + 1 \geq (k - t + 2)(k - t + 1)$ *for* $t > 2$. *If equality holds, then* $(t, k, v) = (3, 4, 8), (3, 6, 22), (3, 12, 112), (4, 7, 23)$, *or* $(5, 8, 24)$.

As we are in particular interested in the cases when $t = 4$ or 5, we obtain from (b) the following upper bound for the positive integer k.

Corollary 6. *Let $\mathcal{D}$ be a non-trivial Steiner t-design with $t = 4 + i$, where $i = 0, 1$. Then*

$$k \leq \left\lfloor \sqrt{v - \left(\tfrac{11}{4} + i\right)} + \tfrac{5}{2} + i \right\rfloor.$$

Remark 7. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on any non-trivial Steiner t-design $\mathcal{D}$ with $t \geq 2$, then G acts point transitively on $\mathcal{D}$ by a result of Block [5, Thm. 2]. In view of Corollary 3 (b), this gives the equation

$$b = \frac{\binom{v}{t}}{\binom{k}{t}} = \frac{v\,|G_x|}{|G_B|},$$

where x is a point in X and B a block in $\mathcal{B}$.

We also state a generalization of Block's result, which is due to Cameron & Praeger [8, Thm. 2.1].

Proposition 8. (Cameron & Praeger, 1993). *Let $\mathcal{D}$ be a t-(v, k, λ) design with $t \geq 2$. Then, the following holds:*

(a) *If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then G also acts point $\lfloor t/2 \rfloor$-homogeneously on $\mathcal{D}$.*

(b) *If $G \leq \mathrm{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then G also acts point $\lfloor (t+1)/2 \rfloor$-homogeneously on $\mathcal{D}$.*

3. Block-Transitive Designs and Triply Transitive Affine Linear Groups

In this section we discuss block-transitive designs which admit a d-dimensional affine group $G = AGL(d, 2)$ in its triply transitive action on the 2^d points of the underlying vector space $V = V(d, 2)$. As G is 3-transitive, clearly for every cardinality k, every orbit of G on k-subsets of V yields a 3-design. Alltop [1] showed that such an orbit is a 5-design whenever it is a 4-design, and derived a necessary and sufficient condition for this to take place. He constructed this way the first block-transitive t-design in affine space with $t > 3$.

Example 1. (Alltop, 1971). *There exists a 5-$(2^8, 24, \lambda)$ design $\mathcal{D}$ admitting a block-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ with $G = AGL(8, 2)$ (where $\lambda = 2^{21}.3^2.5^2.7.31$).*

Alltop's construction has been extended by Cameron & Praeger [8], yielding a flag-transitive design.

Example 2. (Cameron & Praeger, 1993). *There exists a* 5-$(2^8, 24, \lambda)$ *design* $\mathcal{D}$ *admitting a flag-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *with* $G = AGL(8,2)$ *(where* $\lambda = 2^{21}.3^2.5^2.7.31$*).*

Remark 9. Bierbrauer [4] has constructed an infinite family of non-simple 7-designs which are invariant under $AGL(d,2)$, but not block-transitive.

Cameron & Praeger [8] proved the non-existence of block-transitive t-designs for $t > 7$.

Theorem 10. (Cameron & Praeger, 1993). *Let* $\mathcal{D}$ *be a non-trivial* t*-design. If* $G \leq \mathrm{Aut}(\mathcal{D})$ *acts block-transitively on* $\mathcal{D}$ *then* $t \leq 7$*, while if* $G \leq \mathrm{Aut}(\mathcal{D})$ *acts flag-transitively on* $\mathcal{D}$ *then* $t \leq 6$*.*

Considering the index $\lambda = 1$ case, we have shown in [21, 22] in particular that there exists no non-trivial Steiner 6-design or 7-design $\mathcal{D}$ admitting a block-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ with $G = AGL(d,2)$, $v = 2^d, d \geq 3$.

In this section, we prove:

Proposition 11. *There is no non-trivial Steiner* 4*-design or* 5*-design* $\mathcal{D}$ *admitting a block-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$*, where* $G = AGL(d,2)$*,* $v = 2^d, d \geq 3$*.*

Proof. As trivial designs are excluded, let $v = 2^d > k > t$ for $t = 4$ and 5, respectively. Furthermore, we may assume that always $d > 3$ in view of Corollary 6. First, we consider the case when $t = 4$. Any three distinct points being non-collinear in $AG(d,2)$, they generate an affine plane. Let $\mathcal{E}$ be the 2-dimensional vector subspace spanned by the first two basis vectors e_1 and e_2 of the vector space $V = V(d,2)$. Then the pointwise stabilizer of $\mathcal{E}$ in $SL(d,2)$ (and therefore also in G) acts point-transitively on $V \setminus \mathcal{E}$. If the unique block $B \in \mathcal{B}$ which is incident with the 4-subset $\{0, e_1, e_2, e_1 + e_2\}$ contains some point outside $\mathcal{E}$, then B contains all points of $V \setminus \mathcal{E}$, and so $k \geq 2^d = v$, which is impossible. Hence B can be identified with $\mathcal{E}$, and by the block-transitivity of G, each block must be an affine plane. This implies that always $k = 4$, a contradiction.

Now, let $t = 5$. Any five distinct points being non-coplanar in $AG(d,2)$, they generate an affine subspace of dimension at least 3. Let $\mathcal{E}$ be the 3-dimensional vector subspace spanned by the first three basis vectors e_1, e_2, e_3 of V. Then the pointwise stabilizer of $\mathcal{E}$ in $SL(d,2)$ (and therefore also in G) acts point-transitively on $V \setminus \mathcal{E}$. If the unique block $B \in \mathcal{B}$ which is incident with the 5-subset $\{0, e_1, e_2, e_3, e_1 + e_2\}$ contains some point outside $\mathcal{E}$, then B contains all points of $V \setminus \mathcal{E}$, and so $k \geq 2^d - 3$, a contradiction to Corollary 6. The block-transitivity of G now implies that each block must be contained in a 3-dimensional affine subspace. This leads to a contradiction as any five distinct points that generate a 4-dimensional affine subspace must also be incident with a unique block by Definition 1. $\square$

4. Block-Transitive Designs and Further Groups of Affine Type

We investigate in this section Steiner 4- and 5-designs with a group of affine type as a possibly block-transitive group of automorphisms. We note that due to Proposition 8, we may restrict ourselves to the finite 2-homogeneous affine permutation groups.

Let G be a group acting 2-homogeneously on a finite set X of $v \geq 3$ points. If G is not 2-transitive on X, then $G \leq A\Gamma L(1,q)$ with $v = q \equiv 3 \pmod 4$ by a result of Kantor [26]. On the other hand, relying on the classification of the finite simple groups, all 2-transitive groups on X are known (cf. [10, 12, 14, 15, 24, 27, 28, 29]). By a classical result of Burnside, they split into two types of groups. In the context of our consideration, we will deal with the groups of affine type: A finite 2-transitive permutation group on X is called of *affine type*, if G contains a regular normal subgroup T which is elementary Abelian of order $v = p^d$, where p is a prime. If a divides d, and if we identify G with a group of affine transformations

$$x \mapsto x^g + u$$

of $V = V(d,p)$, where $g \in G_0$ and $u \in V$, then particularly one of the following occurs:

(1) $G \leq A\Gamma L(1,p^d)$
(2) $G_0 \trianglerighteq SL(\frac{d}{a},p^a)$, $d \geq 2a$
(3) $G_0 \trianglerighteq Sp(\frac{2d}{a},p^a)$, $d \geq 2a$
(4) $G_0 \trianglerighteq G_2(2^a)'$, $d = 6a$
(5) $G_0 \cong A_6$ or A_7, $v = 2^4$
(6) $G_0 \trianglerighteq SL(2,3)$ or $SL(2,5)$, $v = p^2$, $p = 5, 7, 11, 19, 23, 29$ or 59, or $v = 3^4$
(7) G_0 contains a normal extraspecial subgroup E of order 2^5, and G_0/E is isomorphic to a subgroup of S_5, $v = 3^4$
(8) $G_0 \cong SL(2,13)$, $v = 3^6$,

Proposition 12. *There is no non-trivial Steiner 5-design $\mathcal{D}$ admitting a block-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ with $G \leq A\Gamma L(1,p^d)$, $v = p^d$.*

Proof. Clearly, $|G| \bigm| |A\Gamma L(1,v)| = v(v-1)d$. From Remark 7 follows

$$(v-2)(v-3)(v-4)\,|G_B| \bigm| k(k-1)(k-2)(k-3)(k-4)d.$$

By Proposition 5 (b), we have

$$v - 4 \geq (k-3)(k-4).$$

Hence

$$(v-2)(v-3)\,|G_B| \leq k(k-1)(k-2)d.$$

However, as $d \leq \log_2 v$, this is always impossible in view of Corollary 6. $\square$

Proposition 13. *There is no non-trivial Steiner 4-design or 5-design $\mathcal{D}$ admitting a block-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ of affine type, where $G_0 \trianglerighteq SL(\frac{d}{a}, p^a)$, $d \geq 2a$.*

Proof. First, let $t = 4$. The case $p^a = 2$ has already been treated in Proposition 11. For $p^a = 3$, we may argue similarly. Hence, let us assume that $p^a > 3$. For $d = 2a$, let $U = U(\langle e_1 \rangle) \leq G_0$ denote the subgroup of all transvections with axis $\langle e_1 \rangle$. Obviously, U fixes as points only the elements of $\langle e_1 \rangle$. Thus, G_0 has point-orbits of length at least p^a outside $\langle e_1 \rangle$. Let $S = \{0, e_1, x, y\}$ be a 4-subset of distinct points with $x, y \in \langle e_1 \rangle$. Clearly, U fixes the unique block $B \in \mathcal{B}$ which is incident with S. Therefore, if B contains at least one point outside $\langle e_1 \rangle$, then $k \geq p^a + 4$, a contradiction in view of Corollary 6. Hence, B is completely contained in $\langle e_1 \rangle$. As G is block-transitive, we may conclude that each block lies in an affine line. However, by Definition 1, any four distinct non-collinear points must also be incident with a unique block, a contradiction. Thus, let us assume that $d \geq 3a$. Then $SL(\frac{d}{a}, p^a)_{e_1}$ (and hence also G_{0,e_1}) acts point-transitively on $V \setminus \langle e_1 \rangle$. As above, let $S = \{0, e_1, x, y\}$ be a 4-subset of distinct points with $x, y \in \langle e_1 \rangle$. If the unique block $B \in \mathcal{B}$ which is incident with S contains some point outside $\langle e_1 \rangle$, then B contains all points outside, and thus $k \geq p^d - p^a + 4$, contradicting Corollary 6. We conclude that B lies completely in $\langle e_1 \rangle$, and we can proceed with the same argument as above. For $t = 5$, our methods may be applied mutatis mutandis. □

Proposition 14. *There is no non-trivial Steiner 4-design $\mathcal{D}$ admitting a block-transitive group $G \leq \mathrm{Aut}(\mathcal{D})$ of affine type, where $G_0 \trianglerighteq Sp(\frac{2d}{a}, p^a)$, $d \geq 2a$.*

Proof. We only consider in detail the case $t = 4$. Our arguments work, mutatis mutandis, also for the case $t = 5$. First, let $p^a \neq 2$. The permutation group $PSp(\frac{2d}{a}, p^a)$ on the points of the associated projective space is a rank 3 group, and the orbits of the one-point stabilizer are known (e.g. [25, Ch. II, Thm. 9.15 (b)]). Thus, $G_0 \trianglerighteq Sp(\frac{2d}{a}, p^a)$ has exactly two orbits on $V \setminus \langle x \rangle$ $(0 \neq x \in V)$ of length at least

$$\frac{p^a(p^{2d-2a} - 1)}{p^a - 1} = \sum_{i=1}^{\frac{2d}{a}-2} p^{ia} > p^d.$$

Let $S = \{0, x, y, z\}$ be a 4-subset with $y, z \in \langle x \rangle$. If the unique block which is incident with S contains at least one point of $V \setminus \langle x \rangle$, then $k > p^d + 4$, a contradiction to Corollary 6. Therefore, we may continue with our argumentation as in Proposition 13.

Considering the case $p^a = 2$, let $v = 2^{2d} > k > 4$. For $d = 2$ (here $Sp(4, 2) \cong S_6$ as well-known), Corollary 6 implies that $k = 5$ or 6, each of which is not possible in view of Corollary 3 (c). Therefore, let $d > 2$. It is easily seen that there are $2^{2d-1}(2^{2d} - 1)$ hyperbolic pairs in the non-degenerate symplectic space $V = V(2d, 2)$, and by Witt's theorem, $Sp(2d, 2)$ is transitive on these hyperbolic pairs. Let $\{x, y\}$ denote a hyperbolic pair, and $\mathcal{E} = \langle x, y\rangle$ the hyperbolic plane spanned by $\{x, y\}$. As $\mathcal{E}$ is non-degenerate, we have the orthogonal decomposition

$$V = \mathcal{E} \perp \mathcal{E}^{\perp}.$$

Clearly, $Sp(2d, 2)_{\{x,y\}}$ stabilizes $\mathcal{E}^{\perp}$ as a subspace, which implies that $Sp(2d, 2)_{\{x,y\}} \cong Sp(2d-2, 2)$. As $\mathrm{Out}(Sp(2d, 2)) = 1$, we have therefore

$$Sp(2d-2, 2) \cong Sp(2d, 2)_{\{x,y\}} \trianglelefteq Sp(2d, 2)_{\mathcal{E}} = G_{0,\mathcal{E}}.$$

As $Sp(2d-2, 2)$ acts transitively on the non-zero vectors of the $(2d-2)$-dimensional symplectic subspace, it is easy to see that the smallest orbit on $V \setminus \mathcal{E}$ under $G_{0,\mathcal{E}}$ has length at least $2^{2d-2} - 1$. If the unique block $B \in \mathcal{B}$ which is incident with the 4-subset $\{0, x, y, x+y\}$ contains some point in $V \setminus \mathcal{E}$, then $k \geq 2^{2d-2} + 3$, which is impossible in view of Corollary 6. Thus, B can be identified with $\mathcal{E}$, leading again to a contradiction. $\square$

Proposition 15. *There is no non-trivial Steiner* 4*-design or* 5*-design* $\mathcal{D}$ *admitting a block-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *of affine type, where* $G_0 \trianglerighteq G_2(2^a)'$, $d = 6a$.

Proof. Let $t = 4$. For $t = 5$, we may argue mutatis mutandis. First, let $a = 1$. Then $v = 2^6 = 64$, and so $k \leq 10$ by Corollary 6. We have $|G_2(2)'| = 2^5 \cdot 3^3 \cdot 7$ and $|\mathrm{Out}(G_2(2)')| = 2$. Using Corollary 4 and Remark 7, we can easily rule out the possibilities for k. Now, let $a > 1$. As here $G_2(2^a)$ is simple non-Abelian, it is sufficient to consider $G_0 \trianglerighteq G_2(2^a)$. The permutation group $G_2(2^a)$ is of rank 4, and for $0 \neq x \in V$, the one-point stabilizer $G_2(2^a)_x$ has exactly three orbits $\mathcal{O}_i$ $(i = 1, 2, 3)$ on $V \setminus \langle x\rangle$ of length $2^{3a} - 2^a, 2^{5a} - 2^{3a}, 2^{6a} - 2^{5a}$ (cf., e.g., [2] or [7, Thm. 3.1]). Thus, G_0 has exactly three orbits on $V \setminus \langle x\rangle$ of length at least $|\mathcal{O}_i|$. Let $S = \{0, x, y, z\}$ be a 4-subset with $y, z \in \langle x\rangle$. Again, we will show that the unique block $B \in \mathcal{B}$ which is incident with S lies completely in $\langle x\rangle$. If B contains at least one point of $V \setminus \langle x\rangle$ in $\mathcal{O}_2$ or $\mathcal{O}_3$, then we obtain again a contradiction to Corollary 6. Thus, we only have to consider the case when B contains points of $V \setminus \langle x\rangle$ which all lie in $\mathcal{O}_1$. By [2], the orbit $\mathcal{O}_1$ is exactly known, and we have

$$\mathcal{O}_1 = x\Delta \setminus \langle x\rangle,$$

where $x\Delta = \{y \in V \mid f(x, y, z) = 0 \text{ for all } z \in V\}$ with an alternating trilinear form f on V. Then B consists, apart from elements of $\langle x\rangle$, exactly of $\mathcal{O}_1$. Since $|\mathcal{O}_1| \neq 1$, we can choose $\langle \overline{x}\rangle \in x\Delta$ with $\langle \overline{x}\rangle \neq \langle x\rangle$. However, for symmetric reasons, the 4-subset $\{0, \overline{x}, \overline{y}, \overline{z}\}$ with $\overline{y}, \overline{z} \in \langle \overline{x}\rangle$ must also be incident with the unique block B, a contradiction to the fact that $\overline{x}\Delta \neq x\Delta$ for $\langle \overline{x}\rangle \neq \langle x\rangle$. Consequently, B is completely contained in $\langle x\rangle$, and we may argue as in the Propositions above. $\square$

Proposition 16. *There is no non-trivial Steiner* 4*-design or* 5*-design* $\mathcal{D}$ *admitting a block-transitive group* $G \leq \mathrm{Aut}(\mathcal{D})$ *of affine type, where* G_0 *is as in Cases* $(5)-(8)$.

Proof. We have in these cases only finitely many possibilities for k to check, which can easily be ruled out by hand, combining Corollaries 3, 4, 6, and Remark 7. $\square$

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