# On the minimum distance graph of an extended Preparata code 

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#### Abstract

The minimum distance graph of an extended Preparata code $P(m)$ has vertices corresponding to codewords and edges corresponding to pairs of codewords that are distance 6 apart. The clique structure of this graph is investigated and it is established that the minimum distance graphs of two extended Preparata codes are isomorphic if and only if the codes are equivalent.


## 1 Introduction

Let $\mathbb{Z}_{2}^{n}$ be the $n$-dimensional binary vector space. The (Hamming) distance between two vectors $x, y \in \mathbb{Z}_{2}^{n}$ is the number of coordinates in which they differ and it is denoted by $d(x, y)$. The weight of a vector $x \in \mathbb{Z}_{2}^{n}$ is the number of its non-zero entries. Let $\mathcal{C} \subseteq \mathbb{Z}_{2}^{n}$ be a (binary) code of length $n$ and $c \in \mathcal{C}$ a codeword. The minimum distance of $\mathcal{C}$ is the minimum distance of its codewords. The support of $c$, denoted as $\operatorname{supp}(c)$, is the set of non-zero coordinates of $c$. Two codes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ are isometric if there exists a one-to-one map $I: \mathcal{C}_{1} \rightarrow$ $\mathcal{C}_{2}$, such that $d(x, y)=d(I(x), I(y))$, for all $x, y \in \mathcal{C}_{1}$. Two codes are called equivalent if one can be obtained from the other by translation and permutation of coordinates.

A 1-perfect code $\mathcal{C}$ of length $n$ is a code such that for any vector $v \in \mathbb{Z}_{2}^{n}$ there is a unique codeword $c \in \mathcal{C}$ at a distance of at most 1 of $v$. An extended 1-perfect code is obtained from a 1-perfect code by adding a parity check coordinate. The minimum distance of a 1-perfect code is 3 whereas the minimum distance of an extended 1-perfect code is 4 .

Let $\mathcal{C}$ be a code of length $n$ and minimum distance $d$. The minimum distance graph $D G$ of $\mathcal{C}$ is the graph whose vertices are the codewords of $\mathcal{C}$, where two vertices are adjacent if and only if the corresponding codewords are at distance $d$ apart. We will write the vertices of $D G$ as the support set of the corresponding codewords. Conversely, for every vertex $v=\left\{v_{1}, \ldots, v_{s}\right\} \subseteq\{1, \ldots, n\}$, we will denote the corresponding codeword as $c\left(\left\{v_{1}, \ldots, v_{s}\right\}\right)$. A clique in a graph $G$ is a set of vertices of $G$ such that every pair of these vertices are adjacent in $G$.

[^0]For $t \subseteq\{1, \ldots, n\}$, we denote $C(t)$ the clique in $D G$ such that $t=\bigcap v$, for all $v$ in the clique.

A $t-(n, k, \lambda)$ design is a set of $n$ points $V$ and a collection, $B$, of $k$-tuples called blocks, such that any $t$-tuple of elements of $V$ is in exactly $\lambda$ blocks. A $2-(n, 3,1)$ design is a called a Steiner triple system and a $3-(n, 4,1)$ is called a Steiner quadruple system. If $V$ is the set $\{1, \ldots, n\}$, then we denote a Steiner triple system as $S T S(n)$ and a Steiner quadruple system as $S Q S(n)$. In a $S Q S(n)$, any triple is included in exactly one block of $B$. Given a $S Q S(n)$ and a triple $\{i, j, k\}$, we define $X(\{i, j, k\})$ as the element in $\{1, \ldots, n\}$ such that $\{i, j, k, X(\{i, j, k\})\} \in B$. If we consider a code $\mathcal{C}$, then the support of codewords of the same weight may have a design structure. In [1 we can find sufficient conditions for the supports of the codewords of the same weight to form a $t$-design. In particular, for any 1 -perfect code of length $2^{m}-1$, the codewords of weight 3 form a $\operatorname{STS}\left(2^{m}-1\right)$ and the codewords of weight 4 of any extended 1-perfect code of length $2^{m}$ form a $S Q S\left(2^{m}\right)$.

The Preparata code is a nonlinear distance invariant code of length $2^{m}-1$, $m$ even, $m \geq 4$, and minimum distance 5 3. We denote as $P(m)$, the extended Preparata code of length $n=2^{m}$, obtained from the Preparata code by adding a parity check coordinate. Note that $P(m)$ has minimum distance 6. In [8] it was shown that any extended Preparata code is a subcode of an extended 1-perfect code of length $n$. We denote $\mathcal{C}_{P(m)}$ the extended perfect code containing the extended Preparata code $P(m)$.

Lemma 1 Let $P(m)$ be an extended Preparata code and $\mathcal{C}_{P(m)}$ the extended 1perfect code containing it. Then, if $c$ is a codeword of $\mathcal{C}_{P(m)}$ of weight 4 , there is no codeword $c^{\prime}$ in $P(m)$ of weight 6 such that $\operatorname{supp}(c) \subset \operatorname{supp}\left(c^{\prime}\right)$.

Proof: $\quad$ Otherwise, $c, c^{\prime} \in \mathcal{C}_{P(m)}$ and $d\left(c, c^{\prime}\right)=2$.

Theorem 1 ([8], Theorem 1) Let $P(m)$ be an extended Preparata code and $\mathcal{C}_{P(m)}$ the extended 1-perfect code containing it. Then, $\mathcal{C}_{P(m)}=P(m) \cup Z(P(m))$, where $Z(P(m))=\left\{z \in \mathbb{Z}_{2}^{n} \mid d(c, z) \geq 4, \forall c \in P(m)\right\}$.

Corollary 1 Let $P(m)$ be an extended Preparata code and $\mathcal{C}_{P(m)}$ the extended 1-perfect code containing it. Then, every word in $\mathcal{C}_{P(m)}$ is either in $P(m)$, or at distance 4 from a word in $P(m)$.

Corollary 2 Let $P(m)$ be an extended Preparata code, $\mathcal{C}_{P(m)}$ the extended 1perfect code containing it and $S Q S(n)$ the Steiner quadruple system corresponding to the minimum weight codewords of $\mathcal{C}_{P(m)}$. Let b be a 4 -tuple in $\{1, \ldots, n\}$. Then, either $b$ is a block in $S Q S(n)$ and $b$ is not included in the support of any codeword of weight 6 in $P(m)$ or $b$ is included in the support of exactly one codeword of weight 6 in $P(m)$.

Proof: Let $b$ be a 4 -tuple in $\{1, \ldots, n\}$. If $b$ is a block in $S Q S(n)$, then $c(b)$ is a codeword in $\mathcal{C}_{P(m)}$ and, by Lemma $b$ is not included in the support of any codeword of weight 6 in $P(m)$. Assume $b$ is not a block in $S Q S(n)$. Then, $c(b)$ is not a codeword in $\mathcal{C}_{P(m)}$ and, by Theorem 11 there exists a codeword $c$ in $P(m)$ such that $d(c, c(b)) \leq 3$. As the weight of such $c$ is at least 6 , then, necessarily $c$ is a codeword of weight $6, d(c, c(b))=2$, and hence $b \subset \operatorname{supp}(c)$. Finally, if
there exist $c^{\prime} \in P(m)$ of weight 6 such that $b \in \operatorname{supp}\left(c^{\prime}\right)$, then $d\left(c, c^{\prime}\right) \leq 4$ that is not possible i $c \neq c^{\prime}$.

The question of whether the minimum distance graph of a code uniquely determines it up to equivalence first arose 7 in the context of enumerating and isomorphism testing for 1-perfect codes [6. Later it was solved in [2 and [4] and used in subsequent studies (e.g. 5]). In this article, we will establish that the minimum distance graph of extended Preparata codes $P(m)$ uniquely determines the code up to equivalence. Alternatively, if the distance graphs of two extended Preparata codes are isomorphic, the codes are necessarily isometric and equivalent. This result could prove useful in subsequent enumeration studies as well.

In Section 2, we will identify the maximum size cliques in $D G$ and show that they correspond to triples of coordinates. We will also identify sets of maximum size cliques that correspond to pairs of coordinates. The maximum cliques will be labeled in Section 3 allowing for the labeling of all vertices corresponding to codewords of minimum weight and, eventually, all the vertices in the graph. Finally, conclusions are given in Section 4

## 2 Identification of all the weight 6 codewords in $P(m)$

Let $D G$ be the minimum distance graph of an extended Preparata code $P(m)$. Identify one vertex $u_{0} \in D G$ as the all-zero codeword, $c\left(u_{0}\right)=\mathbf{0}$. Consider $N\left(u_{0}\right)$ the set of neighbors of $u_{0}$. It will correspond to all codewords of weight 6 in the code. Two vertices $u, v \in N\left(u_{0}\right)$ are adjacent if and only if $|u \cap v|=3$.

Lemma 2 Let $C$ be a clique in $N\left(u_{0}\right)$ and let $v=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\} \in C$.
(i) If there is a vertex $u$ in $C$ such that $\left|u \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right| \leq 1$, then there are at most 2 vertices in $C$ containing $\left\{v_{1}, v_{2}, v_{3}\right\}$.
(ii) If there is a vertex $u$ in $C$ such that $\left|u \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right|=2$, then there are at most 4 vertices in $C$ containing $\left\{v_{1}, v_{2}, v_{3}\right\}$.

Proof: If $\left|u \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right| \leq 1$, then any vertex, apart from $v$, containing $\left\{v_{1}, v_{2}, v_{3}\right\}$ has to have intersection at least two with the triple $u \backslash\{u \cap v\}$, and therefore there is only one such vertex.

In the case $\left|u \cap\left\{v_{1}, v_{2}, v_{3}\right\}\right|=2$, then any vertex different to $v$ containing $\left\{v_{1}, v_{2}, v_{3}\right\}$ also has intersection 1 with the triple $u \backslash\{u \cap v\}$ and, hence, there are at most three of such vertices.

Proposition 1 Let $C$ be a clique in $N\left(u_{0}\right)$ such that there is no triple $t$ intersecting all the vertices of $C$. Then, $|C| \leq 13$.

Proof: Let $v=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\} \in C$. Consider $T$ the set of triples $\{u \cap v \mid u \in C\}$. First, assume there are triples in $T$ that do not have any element in common; for example, $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $\left\{v_{4}, v_{5}, v_{6}\right\}$. By Lemma 2, there is, apart from $v$, at most one codeword containing each triple and, as the
distance between them is 6 , then both vertices have to be $\left\{v_{1}, v_{2}, v_{3}, v_{7}, v_{8}, v_{9}\right\}$ and $\left\{v_{4}, v_{5}, v_{6}, v_{7}, v_{8}, v_{9}\right\}$. Any other triple in $T$ intersects either $\left\{v_{1}, v_{2}, v_{3}\right\}$ or $\left\{v_{4}, v_{5}, v_{6}\right\}$ in two elements. If it intersects $\left\{v_{1}, v_{2}, v_{3}\right\}$ in two elements, then it intersects $\left\{v_{7}, v_{8}, v_{9}\right\}$ in one element and, hence it has to intersect $\left\{v_{4}, v_{5}, v_{6}\right\}$ in two elements that is a contradiction. In the same way, there is a contradiction if it intersects in two elements with $\left\{v_{4}, v_{5}, v_{6}\right\}$ and, hence, there is no other vertex in the clique and there are at most 3 vertices in the clique.

Assume now that every two triples in $T$ have intersection. Let $N$ be the number of triple in $T$ that appear in more than two vertices in $C$. We will study four different cases depending on the number $N$.
(i) $N=0$.

If $t \in T$ then $\bar{t}=v \backslash t$ can not be in $T$. If $\bar{T}=\{\bar{t} \mid t \in T\}$, then $|T \cup \bar{T}| \leq 20$, and $|T|=|\bar{T}|$, thus $|T| \leq 10$. The clique $C$ then has at most $1+10=11$ vertices.
(ii) $N=1$.

If the triple $t$ appears in more than two vertices, then, by Lemma2, every other triple in $T$ has intersection two with $t$. So, the number of triples in $T$ is at most $\binom{3}{2}\binom{3}{1}=9$ and the number of vertices in $C$ is at most $1+3+9=13$.
(iii) $N=2$.

Assume $t_{1}$ and $t_{2}$ are these two triples. By Lemma 2 the intersection of them is two, and all the other triples in $T$ have intersection two with the both triples. Therefore, the number of such triples is at most 4,2 containing $t_{1} \cap t_{2}$ and 2 containing only one element of $t_{1} \cap t_{2}$. Hence, the number of vertices in $C$ is $1+2 \cdot 3+4=11$.
(vi) $N \geq 3$.

Consider $t_{1}, t_{2}$ and $t_{3}$, three triples included in more than two vertices. Any pair of them have two elements in common (Lemma 24), and they also intersect in two elements with any other triple in $T$. If $\left|t_{1} \cap t_{2} \cap t_{3}\right|=2$, then there is only one triple having intersection with them and also contains $t_{1} \cap t_{2} \cap t_{3}$. If $\left|t_{1} \cap t_{2} \cap t_{3}\right|=1$, then the only triple intersecting the three of them is $\left(t_{1} \cup t_{2} \cup t_{3}\right) \backslash\left(t_{1} \cap t_{2} \cap t_{3}\right)$. In both cases, the number of triples in $T$ is 4 and the new triple intersect with all the other triples of $T$ in two elements. Therefore, if there are more than three triples in more than two vertices in $C$, then there are exactly 4 of such triples and the number of vertices in the clique is $1+4 \cdot 3=13$.

If we consider all the cases, then the maximum number of vertices in the clique is $\max \{3,11,13,11,13\}=13$.

Proposition 2 If $D G$ is the minimum distance graph of an extended Preparata code of length $n=2^{m}, m \geq 6, m$ even, then the cliques of maximum size in $N\left(u_{0}\right)$ correspond to cliques $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$, for $\left\{v_{1}, v_{2}, v_{3}\right\} \subseteq\{1, \ldots, n\}$.

Proof: Let $\left\{v_{1}, v_{2}, v_{3}\right\}$ be a triple in $\{1, \ldots, n\}$. By Corollary 2 the 4 -set $\left\{v_{1}, v_{2}, v_{3}, X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right\}$ is not included in the support of any codeword of $P(m)$ and, therefore, $X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$ is not included in any vertex of $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$. Moreover, since every 4 -tuple is included in exactly one vertex (Corollary 2), the clique $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$ contains $\frac{n-4}{3}$ vertices. If $m \geq 6$, then $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$ contains at least 20 vertices and, by Proposition 1, it is a maximum size clique.

Having established a one-to-one correspondence between triples and maximum size cliques in $D G$, we now proceed to identify all triples having a pair in common.

Lemma 3 Let $t_{1}$ and $t_{2}$ be triples in $\{1, \ldots, n\}$ and consider $v \in C\left(t_{1}\right)$. If $\left|v \cap t_{2}\right|=1$, then $v$ has at most two neighbors in $C\left(t_{2}\right)$.

Proof: $\quad$ Since $\left|v \cap t_{2}\right|=1, v^{\prime}=\left\{v \backslash\left\{v \cap t_{2}\right\}\right\}$ is a 5 -tuple in $\{1, \ldots, n\}$. Any neighbor of $v$ in $C\left(t_{2}\right)$ contains two elements of $v^{\prime}$ and, therefore, there are at most two of such neighbors.

Proposition 3 Let $t_{1}$ and $t_{2}$ be triples in $\{1, \ldots, n\}, n \geq 16$. If $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have no vertex in common and $\left|t_{1} \cap t_{2}\right| \leq 1$, then there is a vertex in one of the cliques that has less than two neighbors in the other clique.

Proof: If $\left|t_{1} \cap t_{2}\right|=1$ and $n \geq 16$, then there exist a triple $t$ such that $\left|t \cap t_{2}\right|=0$ and $t \cup t_{1} \in C\left(t_{1}\right)$. Then, by Lemma 3 $t \cup t_{1}$ has at most two neighbors in $C\left(t_{2}\right)$.

Assume $t_{1}=\left\{v_{1}, v_{2}, v_{3}\right\}, t_{2}=\left\{v_{4}, v_{5}, v_{6}\right\},\left|t_{1} \cap t_{2}\right|=0$. As $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have no vertex in common, $t_{1} \cup t_{2}$ is not a vertex in $D G$. If there is one vertex in $C\left(t_{1}\right)$ with exactly one element of $t_{2}$, then such vertex has two neighbors in $C\left(t_{2}\right)$ by Lemma3 The same argument applies to clique $C\left(t_{2}\right)$. Otherwise, there is a vertex $u_{1}=t_{1} \cup s_{1}$ in $C\left(t_{1}\right)$ with $\left|s_{1} \cap t_{2}\right|=2$, and a vertex $u_{2}=t_{2} \cup s_{2}$ in $C\left(t_{2}\right)$ with two elements of $\left|s_{2} \cap t_{1}\right|=2$. In that case, $\left|u_{1} \cap u_{2}\right|=4$ and therefore $d\left(c\left(u_{1}\right), c\left(u_{2}\right)\right)<6$ which is not possible.

Proposition 4 Let $t_{1}$ and $t_{2}$ be triples in $\{1, \ldots, n\}, n \geq 16$, such that $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have no vertex in common. If every vertex in $C\left(t_{1}\right)$ has 3 neighbors in $C\left(t_{2}\right)$ and vice versa, then $\left|t_{1} \cap t_{2}\right|=2$ and $t_{1} \cup t_{2}$ is a block in $S Q S(n)$.

Proof: Assume $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have no vertex in common. If $\left|t_{1} \cap t_{2}\right| \leq 1$ then there exists one vertex in one of the cliques with less that 3 neighbors in the other clique. Hence, if every vertex in each clique has 3 neighbors in the other, then $\left|t_{1} \cap t_{2}\right|=2$. If $t_{1} \cup t_{2}$ is not a block in $S Q S(v)$, then the vertex containing $t_{1} \cup t_{2}$ belongs to the intersection of both cliques that is not possible.

Corollary 3 There is a one-to-one correspondence between blocks in SQS(n) and 4-partite 3 -regular graphs in the minimum distance graph of an extended Preparata code.

Proof: Let $t_{1}, t_{2}$ be triples such that $t_{1} \cup t_{2}=b=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\} \in S Q S(n)$. Since $b$ is not contained in any vertex of $D G$ (Lemma 11), $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have no vertex in common. Consider the vertex $v=t_{1} \cup\left\{v_{5}, v_{6}, v_{7}\right\} \in C\left(t_{1}\right)$. Then, by Corollary 2, the 4 -sets $t_{2} \cup\left\{v_{5}\right\}, t_{2} \cup\left\{v_{6}\right\}$ and $t_{2} \cup\left\{v_{7}\right\}$ must belong to three different vertices in $C\left(t_{2}\right)$ and, therefore, $v \in C\left(t_{1}\right)$ has three neighbors in $C\left(t_{2}\right)$. The converse is given by Proposition (4)

Proposition 5 Let $t_{1}$ and $t_{2}$ be triples in $\{1, \ldots, n\}, n \geq 16$, such that $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have one vertex in common. There is a vertex in each clique with 2 neighbors in the other clique and every other vertex in each clique, apart from the intersection, has 3 neighbors in the other clique if and only if $\left|t_{1} \cap t_{2}\right|=2$.

Proof: $\quad$ Since $C\left(t_{1}\right)$ and $C\left(t_{2}\right)$ have one vertex in common, $X\left(t_{1}\right)$ and $X\left(t_{2}\right)$ appear in $C\left(t_{2}\right)$ and $C\left(t_{1}\right)$ respectively. Assume $t_{1}$ and $t_{2}$ has 2 elements in common. The vertex in $C\left(t_{1}\right)$ containing $t_{1} \cup X\left(t_{2}\right)$ has exactly 2 neighbors in $C\left(t_{2}\right)$ and the vertex in $C\left(t_{2}\right)$ containing $t_{2} \cup X\left(t_{1}\right)$ has 2 neighbors in $C\left(t_{1}\right)$. Any other vertex, apart from the vertex in the intersection of the cliques, has exactly 3 neighbors in the other clique.

If $\left|t_{1} \cap t_{2}\right|=1$, then every vertex $v$ in $C\left(t_{1}\right)$ with 3 neighbors in $C\left(t_{2}\right)$ has intersection with the pair $t_{2} \backslash\left(t_{1} \cap t_{2}\right)$, by Lemma 3 but it is not possible if there are more than 2 vertices with 3 neighbors, that is the case if $n \geq 16$.

Finally, if $t_{1}$ and $t_{2}$ do not have intersection, then every vertex in each clique have, apart from the vertex in the intersection, at most one neighbor in the other clique.

Define the set of all the cliques determined by triples with two elements in common:

$$
S\left(\left\{v_{1}, v_{2}\right\}\right)=\left\{C(t)| | t \mid=3,\left\{v_{1}, v_{2}\right\} \subset t\right\}
$$

Corollary 4 Given $N\left(u_{0}\right)$, we can identify all the sets $S(t)$, where $t$ is a pair in $\{1, \ldots, n\}$.

Proof: Propositions 4 and 5

## 3 Reconstruction of the extended Preparata code

From last section, we have fixed a vertex $u_{0}$ as the all-zero codeword and we have determined the set of vertices corresponding to codewords of weight 6 . Moreover, we can identify all cliques of maximum size with triples. In order to reconstruct the extended Preparata code from its minimum distance graph, we will label each vertex in the graph, that is, we will associate a subset $v$ of $\{1, \ldots, n\}$ such that the corresponding codeword in the code will be $c(v)$. Since cliques of maximum size correspond to cliques of type $C(t)$, where $t$ is a triple in $\{1, \ldots, n\}$, then by labeling a clique $C$ we will mean associate a triple $t$ in $\{1, \ldots, n\}$ such that $C=C(t)$. Similarly, to label a set $S(p)$ is to determine the pair $p$.

We consider one clique of maximum size and we label it as $C(\{1,2,3\})$. We define $X(\{1,2,3\})=n$ and choose disjoint triples $t$ of $\{4, \ldots, n-1\}$. We then
label all the vertices in $C(\{1,2,3\})$ as $\{1,2,3\} \cup t$. In Subsection 3.1 we will label first all the maximum size cliques having intersection with $C(\{1,2,3\})$ and after that any maximum size clique in $N\left(u_{0}\right)$ and, hence, all codewords in $N\left(u_{0}\right)$. Finally, in Subsection 3.2 we will label all cliques of maximum size in the distance graph and therefore all the codewords of the extended Preparata code.

### 3.1 Identification of the maximum size cliques in $N\left(u_{0}\right)$

Consider the clique $C(\{1,2,3\})$. By Corollary 4 , we can identify three sets $S(p)$ containing $C(\{1,2,3\})$, where $p$ is a pair, $p \subset\{1,2,3\}$. We label these sets as $S(\{1,2\}), S(\{2,3\})$ and $S(\{1,3\})$.

Let $v=\left\{1,2,3, v_{4}, v_{5}, v_{6}\right\}$ be a vertex in $C(\{1,2,3\})$. There are $\binom{6}{3}=$ 20 different triples $t \subset v$ such that $v \in C(t)$ and, therefore, 20 cliques $C(t)$ intersecting $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$ in the vertex $v$.

Consider the set $S(\{1,2\})$. There are three maximum size cliques, apart from $C(\{1,2,3\})$, that intersect $C(\{1,2,3\})$ in $v$ and are included in $S(\{1,2\})$. We label them as $C\left(\left\{1,2, v_{4}\right\}\right), C\left(\left\{1,2, v_{5}\right\}\right)$ and $C\left(\left\{1,2, v_{6}\right\}\right)$. The clique $C\left(\left\{1,2, v_{4}\right\}\right)$ belong to the 3 sets $S(\{1,2\})$, already labeled, $S\left(\left\{1, v_{4}\right\}\right)$ and $S\left(\left\{2, v_{4}\right\}\right)$. The set $S\left(\left\{1, v_{4}\right\}\right)$ is the one that intersects with the set $S(\{1,3\})$ and the clique in the intersection is $C\left(\left\{1,3, v_{4}\right\}\right)$. Similarly, we also label $S\left(\left\{2, v_{4}\right\}\right)$ that intersects with $S(\{2,3\})$ and the intersection is $C\left(\left\{2,3, v_{4}\right\}\right)$.

Doing the same process with the cliques $C\left(\left\{1,2, v_{5}\right\}\right)$ and $C\left(\left\{1,2, v_{6}\right\}\right)$ we label all the sets $S(\{i, k\})$ and the 9 cliques $C(\{i, j, k\})$, where $i, j \in\{1,2,3\}$ and $k \in\left\{v_{4}, v_{5}, v_{6}\right\}$.

In order to label the cliques $C(\{i, k, l\})$, where $i \in\{1,2,3\}$ and $k, l \in$ $\left\{v_{4}, v_{5}, v_{6}\right\}$, we consider the intersection on the sets $S(\{i, k\})$ and $S(\{i, l\})$ that are already labeled. So, we can also label the sets $S(\{k, l\})$, where $k, l \in$ $\left\{v_{4}, v_{5}, v_{6}\right\}$ and finally the clique $C\left(\left\{v_{4}, v_{5}, v_{6}\right\}\right)$.

That way the 20 cliques containing $v$ are labeled, and we can repeat the same process for any vertex in $C(\{1,2,3\})$.
Proposition 6 Let $x$ be a vertex in the minimum distance graph $D G$ of an extended Preparata code and $N(x)$ the set of neighbors of $x$. Let $C$ be a clique of maximum size in $N(x)$. If all the vertices in $C$ and all the maximum size cliques intersecting $C$ are labeled, then all the vertices in $N(x)$ are determined.

Proof: Assume all the vertices of $C$ and all the cliques of maximum size intersecting $C$ are labeled. That way, all the cliques $C(t)$ are labeled, where $t$ is a triple contained in some vertex of $C$. Moreover, we can identify and label all the sets $S(p)$, where $p$ is a pair included in some vertex of $C$. Consider the triple $\left\{v_{1}, v_{2}, v_{3}\right\}$ such that $C=C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$. Let $t^{\prime}=\left\{v_{i}, v_{4}, v_{5}\right\}$ be a triple where $v_{i} \in\left\{v_{1}, v_{2}, v_{3}\right\}$ and $v_{4}$ and $v_{5}$ belong to different vertices in $C$. Then, $S\left(\left\{v_{i}, v_{4}\right\}\right)$ and $S\left(\left\{v_{i}, v_{5}\right\}\right)$ are labeled and $C\left(\left\{v_{i}, v_{4}, v_{5}\right\}\right)=S\left(\left\{v_{i}, v_{4}\right\}\right) \cap$ $S\left(\left\{v_{i}, v_{5}\right\}\right)$. After labeling all the cliques of this type, we can identify the sets $S(p)$ where $p$ is any pair in $\{1, \ldots, n\} \backslash X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$. Hence, for any triple $\{i, j, k\} \in\{1, \ldots, n\} \backslash X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right), C(\{i, j, k\})=S(\{i, j\}) \cap S(\{j, k\})$. Finally, for any pair $p$ in $\{1, \ldots, n\} \backslash X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$, all the maximum size cliques in $S(p)$ are labeled except one, that is the clique $C\left(p \cup X\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right.$ and, therefore all the maximum size cliques are labeled and, hence, all vertices in $N(x)$.

### 3.2 Identification of all the vertices in the distance graph

In last subsection we have labeled all the vertices in $D G$ corresponding to weight 6 codewords in the extended Preparata code. For $u \in D G$, denote $N(u)$ be the set of neighbors of $u, N(u)^{+}=N(u) \cup u$ and $c(N(u))$ the subcode $\{c(v) \mid v \in$ $N(u)\}$ of $P(m)$. Then, we define $N(u)+u$ as $\{\operatorname{supp}(v) \mid v \in c(N(u))+u\}$.

Let $u=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ be one of vertex corresponding to weight 6 codewords in the extended Preparata code and $\bar{u}$ the set $\{1, \ldots, n\} \backslash u$. Note that all the codewords in $c(N(u))+u$ have weight 6 and their supports will be subsets of size 6 of $\{1, \ldots, n\}$. We will label $N(u)+u$ and, therefore, it will be labeled $N(u)$.

The vertex $u$ is included in $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right) \subset N\left(u_{0}\right)^{+} \cap N(u)^{+}$. That clique is labeled as $C\left(\left\{v_{4}, v_{5}, v_{6}\right\}\right)$ in $N(u)+u$ and all the vertices $\left\{v_{1}, v_{2}, v_{3}, v_{7}, v_{8}, v_{9}\right\} \in$ $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right) \subset N\left(u_{0}\right)^{+} \cap N(u)^{+}$are also labeled as $\left\{v_{4}, v_{5}, v_{6}, v_{7}, v_{8}, v_{9}\right\}$ in $C\left(\left\{v_{4}, v_{5}, v_{6}\right\}\right) \subset N(u)+u$. Moreover, any maximum size clique $C(t) \subset$ $N\left(u_{0}\right)^{+} \cap N(u)^{+}$, where $t$ is a triple, is labeled in $N(u)+u$ and also all its vertices.

If we identify $S(p)$, for any pair $p \subset\{1, \ldots, n\}$, then we will have labeled all vertices in $N(u)+u$. Since all the maximum size cliques $C(t)$ where $t \in u$ are labeled, we can identify the sets $S(p)$, where $p$ is a pair in $u$. In order to identify $S\left(\left\{v_{i}, v_{s}\right\}\right)$ where $v_{i} \in u, v_{s} \in \bar{u}$, consider $v_{j} \in u, v_{j} \neq v_{i}$. For all $v_{k} \in u \backslash\left\{v_{i}, v_{j}\right\}$, the vertex containing $\left\{v_{i}, v_{j}, v_{k}, v_{s}\right\}$ is included in $C\left(\left\{v_{i}, v_{j}, v_{k}\right\}\right)$ and, hence it is already labeled. That way, $\left\{v_{i}, v_{j}, v_{k}, v_{s}\right\}$, for $v_{k} \in u \backslash\left\{v_{i}, v_{j}\right\}$, are vertices included in a maximum size clique that correspond to $C\left(\left\{v_{i}, v_{j}, v_{s}\right\}\right)$. Hence, we can label all the maximum size cliques $C\left(\left\{v_{i}, v_{j}, v_{s}\right\}\right)$, for $v_{i}, v_{j} \in u, v_{s} \in \bar{u}$ and therefore, all the sets $S\left(\left\{v_{i}, v_{s}\right\}\right)$ where $v_{i} \in u, v_{s} \in \bar{u}$. Cliques $C\left(\left\{v_{i}, v_{j}, v_{s}\right\}\right)$, where $v_{i} \in u, v_{j}, v_{s} \in \bar{u}$ can be identified by $S\left(\left\{v_{i}, v_{s}\right\}\right) \cap S\left(\left\{v_{i}, v_{j}\right\}\right)$. Finally, with last cliques labeled, we identify $S\left(\left\{v_{i}, v_{j}\right\}\right), v_{i}, v_{j} \in \bar{u}$, and $C\left(\left\{v_{i}, v_{j}, v_{s}\right\}\right)=$ $S\left(\left\{v_{i}, v_{s}\right\}\right) \cap\left(\left\{v_{j}, v_{s}\right\}\right)$ where $v_{i}, v_{j}, v_{s} \in \bar{u}$.

In $N(u)+u$ we have labeled the clique $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$, all the vertices in the clique and all the maximum size cliques intersecting $C\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)$. Hence, by Proposition 6, we can identify all the vertices in $N(u)+u$ and, therefore, in $N(u)$. We can repeat this process with any vertex in $N\left(u_{0}\right)$ and any new vertex labeled in the graph and we obtain all the vertices labeled in the minimum distance graph $D G$.

## 4 Conclusions

As a consequence of the previous arguments, we have established the following results.

Theorem 2 The minimum distance graph of extended Preparata codes $P(m)$ uniquely determines the code up to equivalence. Alternatively, the distance graphs of two extended Preparata codes are isomorphic if and only if the codes are equivalent.

Corollary 5 Given the minimum distance graph for $P(m)$, one can reconstruct the corresponding extended perfect code $C_{P(m)}$ and its distance 4 graph.

One could also ask what relation, if any, there is between the minimum distance graph of $P(m)$ and that of $C_{P(m)}$.

## References

[1] E. F. Assmus and H.F. Mattson, Jr., New 5-designs, J. Combin. Theory, vol. 6, pp. 122-151, 1969.
[2] S. V. Avgustinonvich, Perfect binary (n,3) codes: the structure of graphs of minimum distances, Discrete Appl. Math. vol 114,pp. 9-11, 2001.
[3] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes, North-Holland Publishing Company, 1977.
[4] I. Y. Mogilnykh, P. R. J. Östergård, O. Pottonen and F. I. Soloveva, Reconstructing Extended Perfect Binary One-Error-Correcting Codes from Their Minimum Distance Graphs, Arxiv preprint arXiv:0810.5633, 2008.
[5] P. R. J. Östergård and O. Pottonen, The perfect binary one-error correcting codes of length 15- Part I-classification, submitted for publication.
[6] K. T. Phelps, An enumeration of 1-perfect binary codes, Australas. J. Combin., vol 21, pp. 287-298, 2000.
[7] K. T. Phelps and M. LeVan, Switching classes of perfect codes, Des. Codes Cryptogr., vol 16, pp. 179-184, 1999.
[8] G. V. Zaitsev, V. A. Zinoviev and N. V. Semakio, Interrelation of Preparata and Hamming codes and extension of Hamming codes to new double-errorcorrecting codes In 2nd International Symposium on Information Theory, Tsahkadzor, Armenia, USSR, 1971. Akademia Kiado, Budapest, pp. 257 263, 1973.


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