# Minimum distance of Hermitian two-point codes 

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#### Abstract

We prove a formula for the minimum distance of two-point codes on a Hermitian curve.


## 1 Introduction

Homma and Kim [5], [6], [7, [8] gave a complete determination of the minimum distance of two-point codes on a Hermitian curve. Their method is based on the method of Kumar and Yang[12]. It leads to a theorem with many distinct cases and a long proof divided over four papers. The objective of this paper is to give a short and easy proof of the minimum distance of the Hemitian two-points codes using a method based on Kirfel and Pellikaan [9. First we use the shift bound to find the lower bound for the minimum distance. Secondly we use certain types of conics and lines to show that the bound is sharp. Kirfel and Pellikaan gave a general method of finding a lower bound for the minimum distance of one-point codes based on the decoding algorithms by Feng and Rao [4], and Duursma [3]. The method gives a short proof for the minimum distance of the Hermitian one-point codes. There have been approaches using Kirfel and Pellikaan's method to find the lower bound for the minimum distance of general algebraic geometric codes. In [2], near order functions are used to find the lower bound for the minimum distance of a two-point algebraic geometric code. Beelen [1] gave a method of finding the lower bound for the minimum distance of general algebraic geometric codes using the generalized order bound. In fact, the lower bound gives the exact minimum distance of the Hermitian two-point codes in a large range. We define the multiplicity of the Hermitian two-point code and use the multiplicity to find a path that will give the minimum distance of the Hermitian two-point code. Our formulas for the minimum distance of the Hermitian two-point code are given for all ranges and they meet the formulas by Homma and Kim with a shorter proof and fewer cases for the formulas. Moreover, our approach can be used in majority coset decoding [3] which decodes up to half the actual minimum distance.

In Section 2 we give the definition of multiplicity and a method of finding the lower bounds for the minimum distance of Hermitian two-point codes. In Section 3 we state the formulas for the multiplicity and minimum distance of the Hermitian two-point codes. The proofs of the formulas for the multiplicity and minimum distance of the Hermitian two-point codes are given in Section 4 and Section 5, respectively. In Appendix, we state

[^0]both formulas for the minimum distance of Hermitian two-point codes given by Homma and $\operatorname{Kim}[5],[6],[7],[8]$ and given in this paper for comparison.

## 2 Multiplicities and the shift bound

We give the definition of multiplicity and a method of finding the lower bound for the minimum distance of the Hermitian two-point code.

Let $X$ be a Hermitian curve defined by $y^{q}+y=x^{q+1}$ over $\mathbb{F}_{q^{2}}$. Then $X$ has $q^{3}+1$ rational points and the genus is $q(q-1) / 2$. Let $P_{\infty}$ be the point at infinity of $X$ and $P_{0}$ the origin of $X$. The canonical divisor $K$ of a Hermitian curve is $K=(q-2) H$, where $H \sim(q+1) P_{\infty} \sim(q+1) P_{0}$. Let $\mathbb{F}_{q^{2}}(X)$ be the function field of $X$ over $\mathbb{F}_{q^{2}}$. For $f \in \mathbb{F}_{q^{2}}(X) \backslash\{0\},(f)_{\infty}$ denotes the pole divisor of $f,(f)_{0}$ the zero divisor of $f$ and $(f)=(f)_{0}-(f)_{\infty}$ the divisor of $f$. Given a divisor $G$ on $X$ defined over $\mathbb{F}_{q^{2}}$, let $L(G)$ denote the vector space over $\mathbb{F}_{q^{2}}$ consisting of functions $f \in \mathbb{F}_{q^{2}}(X) \backslash\{0\}$ with $(f)+G \geq 0$ and the zero function. Let $G=a P_{\infty}+b P_{0}$ and $D=P_{1}+\cdots+P_{n}$ be a divisor of $X$, where $\operatorname{supp}(G) \cap \operatorname{supp}(D)=\emptyset$ and $P_{1}, \ldots, P_{n}$ are pairwise distinct. We define a code $C(D, G)$ as the image of the evaluation map ev : $\mathrm{L}(\mathrm{G}) \rightarrow \mathbb{F}_{\mathrm{q}^{2}}^{\mathrm{n}}$ given by ev $(\mathrm{f})=\left(\mathrm{f}\left(\mathrm{P}_{1}\right), \ldots, \mathrm{f}\left(\mathrm{P}_{\mathrm{n}}\right)\right)$ for all $f \in L(G)$. For a fixed $D$, we will use the notation $C(G)=C(a, b)$ for $C(D, G)$, where $G=a P_{\infty}+b P_{0}$.

Definition 2.1. Let $M_{P_{\infty}}(a, b)$ be the set of pairs $(f, g)$ of rational functions such that
(1) $f g \in L\left(a P_{\infty}+b P_{0}\right) \backslash L\left((a-1) P_{\infty}+b P_{0}\right)$
(2) $f \in L\left(a P_{\infty}+b P_{0}\right)$
(3) $g \in L\left((a+b) P_{\infty}\right)$

The multiplicity $m_{P_{\infty}}(a, b)$ is defined as

$$
m_{P_{\infty}}(a, b)=\#\left\{-b \leq i \leq a+1: \exists(f, g) \in M_{P_{\infty}}(a+1, b) \text { with } \operatorname{ord}_{P_{\infty}}(f)=-i\right\}
$$

We apply the shift bound argument from [11], see also [10], to obtain a lower bound for the weight of a vector that is orthogonal to $C(a, b)$ but not orthogonal to $C(a+1, b)$ in terms of the multiplicity.

Theorem 2.2. Let $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right) \in \mathbb{F}_{q^{2}}^{n}$ be a vector that is orthogonal to $C(a, b)$ but not orthogonal to $C(a+1, b)$. Then the weight of $\mathbf{c}$ is at least $m_{P_{\infty}}(a, b)$.

Proof. Let $m=m_{P_{\infty}}(a, b)$ and let $\left(f_{1}, g_{1}\right), \ldots,\left(f_{m}, g_{m}\right)$ be pairs in $M_{P_{\infty}}(a+1, b)$ such that $f_{1}, f_{2}, \ldots, f_{m}$ have distinct pole orders at $P_{\infty}$. Let $\operatorname{ord}_{P_{\infty}}\left(f_{m}\right)<\cdots<\operatorname{ord}_{P_{\infty}}\left(f_{1}\right)$.

Then

$$
\begin{aligned}
& g_{j} f_{i} \in L(G) \text { for } i=1, \ldots, j-1 \\
& g_{j} f_{i} \in L\left(G+P_{\infty}\right) \backslash L(G) \text { for } i=j \\
& g_{j} f_{i} \notin L\left(G+P_{\infty}\right) \quad \text { for } i=j+1, \ldots, m
\end{aligned}
$$

Let $A$ be the $m \times n$ matrix with entries $g_{i}\left(P_{j}\right)$ and let $B$ be the $m \times n$ matrix with entries $f_{i}\left(P_{j}\right)$. The $m \times m$ matrix $A \operatorname{diag}\left(c_{1}, \ldots, c_{n}\right) B^{T}$ is zero below the diagonal and nonzero on the diagonal. Hence it is of rank $m$ and the number of nonzero coordinates in $\mathbf{c}$ is at least $m$.

Definition 2.3. Let $M_{P_{0}}(a, b)$ be the set of pairs $(f, g)$ of rational functions such that
(1) $f g \in L\left(a P_{\infty}+b P_{0}\right) \backslash L\left(a P_{\infty}+(b-1) P_{0}\right)$
(2) $f \in L\left(a P_{\infty}+b P_{0}\right)$
(3) $g \in L\left((a+b) P_{0}\right)$

The multiplicity $m_{P_{0}}(a, b)$ is defined as

$$
m_{P_{0}}(a, b)=\#\left\{-a \leq j \leq b+1: \exists(f, g) \in M_{P_{0}}(a, b+1) \text { with } \operatorname{ord}_{P_{0}}(f)=-j\right\}
$$

Theorem 2.4. Let $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right) \in \mathbb{F}_{q^{2}}^{n}$ be a vector that is orthogonal to $C(a, b)$ but not orthogonal to $C(a, b+1)$. Then the weight of $\mathbf{c}$ is at least $m_{P_{0}}(a, b)$.

Proof. The proof is similar to Theorem 2.2.
We present a method to find a lower bound for the minimum distance of the Hermitian two-point codes. Let $0 \neq c \in C(a, b)^{\perp}$. We want to find a lower bound for the weight of a word $c \neq 0$ which is orthogonal to $C(a, b)$. Consider Figure 1 .

We first consider the vector spaces

$$
C(a, b) \subseteq C(a+1, b) \subseteq \cdots \subseteq \mathbb{F}_{q}^{n}
$$

The weight for the words $c$ that are orthogonal to $C(a, b)$ but not orthogonal to $C(a+1, b)$ is at least $m_{P_{\infty}}(a, b)$. The weight for the words $c$ that are orthogonal to $C(a+1, b)$ but not orthogonal to $C(a+2, b)$ is at least $m_{P_{\infty}}(a+1, b)$. Since $c$ is nonzero, there is a vector space not equal to the full space $\mathbb{F}_{q}^{n}$, say $C(a+i, b)$, such that the word $c$ is orthogonal to $C(a+i, b)$ but not orthogonal to $C(a+i+1, b)$. The weight of the word $c$ that is orthogonal to $C(a, b)$ but not orthogonal to $\mathbb{F}_{q}^{n}$ is at least the minimum of the multiplicities $m_{P_{\infty}}(a+j, b)$, where $j=0,1, \ldots, l_{1}$ and $C\left(a+l_{1}, b\right)=\mathbb{F}_{q}^{n}$. Now, we fix $a$ and increase $b$. The weight for the words $c$ that are orthogonal to $C(a, b)$ but not orthogonal to $C(a, b+1)$ is at least $m_{P_{0}}(a, b)$. The weight for the words $c$ that are orthogonal to $C(a, b+1)$ but not orthogonal to $C(a, b+2)$ is at least $m_{P_{0}}(a, b+1)$. By a similar argument as above, the weight of the word $c$ that is orthogonal to $C(a, b)$ but not orthogonal to $\mathbb{F}_{q}^{n}$ is at least the minimum of the multiplicities $m_{P_{0}}(a, b+j)$, where $j=0,1, \ldots, l_{2}$
$C\left(a+l_{1}, b\right)=\mathbb{F}_{q}^{n} \quad \mathbb{F}_{q}^{n} \quad \mathbb{F}_{q}^{n} \quad \mathbb{F}_{q}^{n} \quad \mathbb{F}_{q}^{n} \quad \mathbb{F}_{q}^{n}=C\left(a, b+l_{2}\right)$

$$
C(a+2, b) \quad C(a+1, b+1) \quad C(a, b+2)
$$



$$
C(a+1, b)
$$


$C(a, b)$

Figure 1: Lower bound for the minimum distance
and $C\left(a, b+l_{2}\right)=\mathbb{F}_{q}^{n}$. We increase the divisors by increasing the pole order of $P_{0}$ or $P_{\infty}$ by 1 , and compute the $m_{P_{0}}(a+1, b), m_{P_{\infty}}(a+1, b), m_{P_{0}}(a, b+1)$, and $m_{P_{\infty}}(a, b+1)$. We apply the same process until the Riemann-Roch space of the divisor becomes the full space $\mathbb{F}_{q}^{n}$. In each step, we can make a choice for the divisor by adding $P_{0}$ or $P_{\infty}$, that is, we can choose a path to the full space $\mathbb{F}_{q}^{n}$. For each path $P$, we take the minimum of the multiplicities along the path and denote it by $\min (P)$. Let $S$ be the set of $\min (P)$ for all the paths. Each element of the set $S$ gives a lower bound for the weight of $c$ with $0 \neq c \perp C(a, b)$. The best lower bound for the weight of $c$ is obtained by taking the maximum of the set $S$. This maximum is a lower bound for the minimum distance of $C(a, b)^{\perp}$.

## 3 Formulas for multiplicity and minimum distance

We state the formulas for the multiplicity and minimum distance of the Hermitian twopoint codes. The formulas give the minimum distance of the Hermitian two-point codes for all ranges of $G$. We divide the ranges into two parts as follows:

1. $\{G: \operatorname{deg} \mathrm{G}>\operatorname{deg} \mathrm{K}+\mathrm{q}\} \cup\left\{\mathrm{G}: \operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q} \wedge \mathrm{G} \nsim \mathrm{sP}_{\infty} \wedge \mathrm{G} \nsim \mathrm{tP}_{0}\right\}$ and
2. $\{G: \operatorname{deg} G<\operatorname{deg} K\} \cup\left\{G:(\operatorname{deg} K \leq \operatorname{deg} G \leq \operatorname{deg} K+q) \wedge\left(G \sim \mathrm{sP}_{\infty} \vee \mathrm{G} \sim \mathrm{tP}_{0}\right)\right\}$, where $s, t \in \mathbb{Z}$.

Theorem 3.3 and Theorem 3.5 give the formulas for the minimum distance for the first part and the second part, respectively.

Proposition 3.1. Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q, \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q .
\end{aligned}
$$

1. If $a_{1}<a_{0}+b_{0}$, then
$m_{P_{\infty}}(2 g-2+a, b)=\left(a_{0}+b_{0}-a_{1}\right)(q+1)-b_{1}+a_{1} q$
2. If $a_{0}+b_{0} \leq a_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{\infty}}(2 g-2+a, b)=a_{1}\left(q+a_{0}+b_{0}-a_{1}\right)-\min \left\{a_{1}, b_{1}\right\}$.
3. If $a_{0}+b_{0}+q-1<a_{1}$, then
$m_{P_{\infty}}(2 g-2+a, b)=0$.
$1^{\prime}$. If $b_{1}<a_{0}+b_{0}$, then
$m_{P_{0}}(2 g-2+a, b)=\left(a_{0}+b_{0}-b_{1}\right)(q+1)-a_{1}+b_{1} q$
$2^{\prime}$. If $a_{0}+b_{0} \leq b_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{0}}(2 g-2+a, b)=b_{1}\left(q+a_{0}+b_{0}-b_{1}\right)-\min \left\{a_{1}, b_{1}\right\}$.
$3^{\prime}$. If $a_{0}+b_{0}+q-1<b_{1}$, then
$m_{P_{0}}(2 g-2+a, b)=0$.
Proof. The proof is given in Section 4
Theorem 3.2. Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & & 0 \leq a_{1} \leq q, \\
b & =b_{0}(q+1)-b_{1}, & & 0 \leq b_{1} \leq q .
\end{aligned}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$.

1. If $a_{1}<a_{0}+b_{0}$, then
$m_{P_{\infty}}(2 g-2+a, b)=d^{*}$
2. If $a_{0}+b_{0} \leq a_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{\infty}}(2 g-2+a, b)=d^{*}+\left(a_{1}-a_{0}-b_{0}\right)\left(q+1-a_{1}\right)+\max \left\{0, b_{1}-a_{1}\right\}$.
3. If $a_{0}+b_{0}+q-1<a_{1}$, then
$m_{P_{\infty}}(2 g-2+a, b)=0$.
$1^{\prime}$. If $b_{1}<a_{0}+b_{0}$, then
$m_{P_{0}}(2 g-2+a, b)=d^{*}$
$2^{\prime}$. If $a_{0}+b_{0} \leq b_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{0}}(2 g-2+a, b)=d^{*}+\left(b_{1}-a_{0}-b\right)\left(q+1-b_{1}\right)+\max \left\{0, a_{1}-b_{1}\right\}$.
$3^{\prime}$. If $a_{0}+b_{0}+q-1<b_{1}$, then
$m_{P_{0}}(2 g-2+a, b)=0$.
Proof. Follows from Proposition 3.1.
Theorem 3.3. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G>\operatorname{deg} K+q$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ and $\mathrm{G} \nsim \mathrm{sP}_{\infty}$ and $\mathrm{G} \nsim \mathrm{tP}_{0}$ for all $s, t \in \mathbb{Z}$.

Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{array}{rlr}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q
\end{array}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$.

1. If $0 \leq a_{1}, b_{1} \leq a_{0}+b_{0}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}$.
2. If $0 \leq b_{1} \leq a_{0}+b_{0}<a_{1}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}-\left(a_{0}+b_{0}\right)$.
$2^{\prime}$. If $0 \leq a_{1} \leq a_{0}+b_{0}<b_{1}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+b_{1}-\left(a_{0}+b_{0}\right)$.
3. If $a_{0}+b_{0}<a_{1} \leq b_{1}<q$, then

$$
d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)
$$

$3^{\prime}$. If $a_{0}+b_{0}<b_{1} \leq a_{1}<q$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$.
4. If $a_{0}+b_{0}<a_{1}, b_{1}$ and $a_{1}=q, b_{1}=q$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+q-\left(a_{0}+b_{0}\right)$.

Proof. The proof is given in Section 5
Remark 3.4. We can rewrite Theorem 3.3 as

$$
d\left(C(D, G)^{\perp}\right)=d^{*}+\max \left\{0, a_{1}-\left(a_{0}+b_{0}\right), b_{1}-\left(a_{0}+b_{0}\right), a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)\right\}
$$

for all cases except case 4.
Theorem 3.5. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G<\operatorname{deg} K$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ with $\mathrm{G} \sim \mathrm{sP}_{\infty}$ or $\mathrm{G} \sim \mathrm{tP}_{0}$ for some $s, t \in \mathbb{Z}$.

If $G=a P_{\infty}+b P_{0}$ with

$$
\begin{aligned}
a & =a_{0}(q+1)+a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)+b_{1}, & 0 \leq b_{1} \leq q
\end{aligned}
$$

then

$$
d\left(C(D, G)^{\perp}\right)=a_{0}+b_{0}+2
$$

Proof. The proof is given in Section 5

## 4 Proof of Proposition 3.1

Proposition 3.1 Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, \\
b & =b_{0}(q+1)-b_{1} \leq q \\
b_{0}, & 0 \leq b_{1} \leq q
\end{aligned}
$$

1. If $a_{1}<a_{0}+b_{0}$, then
$m_{P_{\infty}}(2 g-2+a, b)=\left(a_{0}+b_{0}-a_{1}\right)(q+1)-b_{1}+a_{1} q$
2. If $a_{0}+b_{0} \leq a_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{\infty}}(2 g-2+a, b)=a_{1}\left(q+a_{0}+b_{0}-a_{1}\right)-\min \left\{a_{1}, b_{1}\right\}$.
3. If $a_{0}+b_{0}+q-1<a_{1}$, then
$m_{P_{\infty}}(2 g-2+a, b)=0$.
$1^{\prime}$. If $b_{1}<a_{0}+b_{0}$, then
$m_{P_{0}}(2 g-2+a, b)=\left(a_{0}+b_{0}-b_{1}\right)(q+1)-a_{1}+b_{1} q$
$2^{\prime}$. If $a_{0}+b_{0} \leq b_{1} \leq a_{0}+b_{0}+q-1$, then
$m_{P_{0}}(2 g-2+a, b)=b_{1}\left(q+a_{0}+b_{0}-b_{1}\right)-\min \left\{a_{1}, b_{1}\right\}$.
$3^{\prime}$. If $a_{0}+b_{0}+q-1<b_{1}$, then
$m_{P_{0}}(2 g-2+a, b)=0$.

Proof. By renaming $P_{0}$ with $P_{\infty}$ and $P_{\infty}$ with $P_{0}$, it is enough to prove the cases 1,2 and 3. We may assume that $K=(2 g-2) P_{\infty}$. Let $G=K+a P_{\infty}+b P_{0}$, where

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q
\end{aligned}
$$



Figure 2: Counting multiplicity
Then $L\left(K+a P_{\infty}+b P_{0}\right)=L\left(\left(q-2+a_{0}+b_{0}\right)(q+1) P_{\infty}-a_{1} P_{\infty}-b_{1} P_{0}\right)$ is spanned by the monomials $x^{i} y^{j}$ with
(1) $0 \leq i \leq q, \quad 0 \leq j, \quad i+j \leq q-2+a_{0}+b_{0}$,
(2) $i \geq a_{1}$ for $i+j=q-2+a_{0}+b_{0}$,
(3) $i \geq b_{1} \quad$ for $\quad j=0$.

We determine the pairs of integers $(i, j)$ such that there exist

$$
\begin{aligned}
& f \in L\left(i P_{\infty}+b P_{0}\right) \backslash L\left((i-1) P_{\infty}+b P_{0}\right) \\
& g \in L\left(j P_{\infty}\right) \backslash L\left((j-1) P_{\infty}\right),
\end{aligned}
$$

with $f g \in L\left(K+(a+1) P_{\infty}+b P_{0}\right) \backslash L\left(K+a P_{\infty}+b P_{0}\right)$. Let $x^{i_{1}} y^{j_{1}}$ and $x^{i_{2}} y^{j_{2}}$ be the leading monomials of $f$ and $g$, respectively, with $0 \leq i_{1}, i_{2} \leq q, 0 \leq j_{1}, j_{2}$. The product $f g \in L\left(K+(a+1) P_{\infty}+b P_{0}\right) \backslash L\left(K+a P_{\infty}+b P_{0}\right)$ if

$$
\begin{aligned}
& \text { (1) } x^{i_{1}} y^{j_{1}} x^{i_{2}} y^{j_{2}}=x^{a_{1}-1} y^{q-1+a_{0}+b_{0}-a_{1}}, \quad \text { or } \\
& \text { (2) } x^{i_{1}} y^{j_{1}} x^{i_{2}} y^{j_{2}}=x^{q+a_{1}} y^{-1+a_{0}+b_{0}-a_{1}},
\end{aligned}
$$

The solutions for $\left(i_{1}, j_{1}\right)$ are
(1) $0 \leq i_{1} \leq a_{1}-1, \quad 0 \leq j_{1} \leq q-1+a_{0}+b_{0}-a_{1}$, such that $i_{1} \geq b_{1}$ for $j_{1}=0$.
(2) $a_{1} \leq i_{1} \leq q, \quad 0 \leq j_{1} \leq-1+a_{0}+b_{0}-a_{1}$.
or
(1') $0 \leq j_{1} \leq-1+a_{0}+b_{0}-a_{1}, \quad 0 \leq i_{1} \leq q$, such that $i_{1} \geq b_{1}$ for $j_{1}=0$.
(2') $a_{0}+b_{0}-a_{1} \leq j_{1} \leq q-1+a_{0}+b_{0}-a_{1}, \quad 0 \leq i_{1} \leq a_{1}-1$.

If $a_{0}+b_{0}+q-1 \geq a_{1} \geq a_{0}+b_{0}$, then the total number of pairs $(i, j)$ is $\left(a_{0}+b_{0}-\right.$ $\left.a_{1}\right)(q+1)-b_{1}+q a_{1}$. If $a_{1}>a_{0}+b_{0}+q-1$, then (1) and (2) have no solutions. For $a_{0}+b_{0}-a_{1} \leq 0$, if $a_{1}=0$, then (1) and (2) have no solutions. If $a_{1} \neq 0$, then there are no solutions in (2). In (1), there are

$$
a_{1}\left(q+a_{0}+b_{0}-a_{1}\right)-\min \left\{a_{1}, b_{1}\right\}
$$

solutions. Thus we have the multiplicity $a_{1}\left(q+a_{0}+b_{0}-a_{1}\right)-\min \left\{a_{1}, b_{1}\right\}$ if $a_{0}+b_{0}-a_{1} \leq$ 0 .

## 5 Proof of Theorem 3.3, 3.5

For each path, the minimum of the multiplicities along the path is a lower bound for the minimum distance of $C(D, G)^{\perp}$. In Theorem 3.3, we find a path that gives a lower bound of the minimum distance which is sharp. The following two lemmas give the minimum of the multiplicities of a certain part of the path chosen in Theorem 3.3.

Lemma 5.1. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G>\operatorname{deg} K+q$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ and $\mathrm{G} \nsim \mathrm{sP}_{\infty}$ and $\mathrm{G} \nsim \mathrm{tP}_{0}$ for all $s, t \in \mathbb{Z}$.

Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q
\end{aligned}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$.
If $0 \leq b_{1} \leq a_{0}+b_{0} \leq a_{1}$ and $I_{2}=\{(2 g-2+a, b),(2 g-2+a+1, b), \ldots,(2 g-2+a+$ $\left.\left.a_{1}-\left(a_{0}+b_{0}\right), b\right)\right\}$, then

$$
\min _{i \in I_{2}}\left(m_{P_{\infty}}(i)\right)=d^{*}+a_{1}-\left(a_{0}+b_{0}\right)=a_{0} q+b_{0} q-a_{1} .
$$

If $0 \leq a_{1} \leq a_{0}+b_{0} \leq b_{1}$ and $I_{2^{\prime}}=\{(2 g-2+a, b),(2 g-2+a, b+1), \ldots,(2 g-2+a, b+$ $\left.\left.b_{1}-\left(a_{0}+b_{0}\right)\right)\right\}$, then

$$
\min _{i \in I_{2^{\prime}}}\left(m_{P_{0}}(i)\right)=d^{*}+b_{1}-\left(a_{0}+b_{0}\right)=a_{0} q+b_{0} q-b_{1}
$$

Proof. For the case $0 \leq b_{1} \leq a_{0}+b_{0} \leq a_{1}$, we need to show that $m_{P_{\infty}}(i) \geq d^{*}+a_{1}-\left(a_{0}+b_{0}\right)$ for $i \in I_{2}$, that is, we need to show that $m_{P_{\infty}}\left(2 g-2+a_{0}(q+1)-a_{1}^{\prime}, b\right) \geq d^{*}+a_{1}-\left(a_{0}+b_{0}\right)$ for $a_{1}^{\prime}=a_{1}, a_{1}-1, \ldots, a_{0}+b_{0}$. By Proposition 3.1, we have

$$
\begin{aligned}
& m_{P_{\infty}}\left(2 g-2+a_{0}(q+1)-a_{1}^{\prime}, b\right)-\left(d^{*}+a_{1}-\left(a_{0}+b_{0}\right)\right) \\
= & \left(a_{1}^{\prime}-\left(a_{0}+b_{0}\right)\right)\left(q-a_{1}^{\prime}\right) \geq 0 .
\end{aligned}
$$

The other case follows by symmetry.

Lemma 5.2. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G>\operatorname{deg} K+q$ or
(2) $\operatorname{degK} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ and $\mathrm{G} \nsim \mathrm{sP}_{\infty}$ and $\mathrm{G} \nsim \mathrm{tP}_{0}$ for all $s, t \in \mathbb{Z}$.

Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q .
\end{aligned}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$.
If $a_{0}+b_{0} \leq a_{1} \leq b_{1}<q$ and $I_{3}=\{(2 g-2+a, b),(2 g-2+a+1, b), \ldots,(2 g-2+a+$ $\left.\left.a_{1}-\left(a_{0}+b_{0}\right), b\right)\right\}$, then

$$
\min _{i \in I_{3}}\left(m_{P_{\infty}}(i)\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)=\left(a_{0}+b_{0}\right) q-\left(a_{0}+b_{0}\right) .
$$

If $a_{0}+b_{0} \leq b_{1} \leq a_{1}<q$ and $I_{3^{\prime}}=\{(2 g-2+a, b),(2 g-2+a, b+1), \ldots,(2 g-2+a, b+$ $\left.\left.b_{1}-\left(a_{0}+b_{0}\right)\right)\right\}$, then

$$
\min _{i \in I_{3^{\prime}}}\left(m_{P_{0}}(i)\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)=\left(a_{0}+b_{0}\right) q-\left(a_{0}+b_{0}\right) .
$$

Proof. For the case $a_{0}+b_{0} \leq a_{1} \leq b_{1}<q$, we need to show that
$m_{P_{\infty}}(i) \geq d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$ for $i \in I_{3}$, that is, we need to show that $m_{P_{\infty}}\left(2 g-2+a_{0}(q+1)-a_{1}^{\prime}, b\right) \geq d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$ for $a_{1}^{\prime}=a_{1}, a_{1}-1, \ldots, a_{0}+b_{0}$. By Proposition 3.1, we have

$$
\begin{aligned}
& m_{P_{\infty}}\left(2 g-2+a_{0}(q+1)-a_{1}^{\prime}, b\right)-\left(d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)\right) \\
& =a_{1}^{\prime}\left(q+a_{0}+b_{0}-a_{1}^{\prime}\right)-a_{1}^{\prime}-\left(a_{0}+b_{0}\right) q+\left(a_{0}+b_{0}\right) \\
& =\left(a_{1}^{\prime}-\left(a_{0}+b_{0}\right)\right)\left(q-a_{1}^{\prime}\right)-\left(a_{1}^{\prime}-\left(a_{0}+b_{0}\right)\right) \geq 0
\end{aligned}
$$

The other case follows by symmetry.
In order to prove that the lower bounds of Theorem 3.3 and Theorem 3.5 are sharp, we need to show that there exist words that have weight equal to the lower bounds. This can be shown by constructing functions with certain properties. The functions consist of multiplications of conics and lines. The following lemmas show that there are enough conics and lines to construct such functions.

Lemma 5.3. The curves $y^{q}+y=x^{q+1}$ and $x^{2}=\alpha y$ share the following automorphisms,

$$
\sigma(x, y)=\left(x y^{-1}, y^{-1}\right), \quad \rho_{a}(x, y)=\left(a x, a^{2} y\right) \text { for } a \in \mathbb{F}_{q}^{*}
$$

The group generated by the automorphisms is the dihedral group of size $2(q-1)$.
Proof. The first claim is easily verified. Finally, $\sigma$ is of order two and $\sigma \rho_{a} \sigma=\rho_{a}^{-1}$.

Lemma 5.4. For a rational point $P=(u, v)$ with $u \neq 0$ and $v \notin \mathbb{F}_{q}$, the function $x^{2}-\alpha y$, for $\alpha=u^{2} / v$, has zeros in $P_{0}$ (with multiplicity two) and in $2 q-2$ other rational points including $P$. The number of such functions is $\left(q^{2}-1\right) / 2$ when $q$ is odd and $\left(q^{2}+q\right) / 2$ when $q$ is even.

Proof. The function $x^{2}-\alpha y$ has poles only at $P_{\infty}$ of order $2 q$. Thus, there are $2 q$ zeros, two of which are $P_{0}$. We claim that the remaining zeros form a single orbit under the action of the dihedral group in the previous lemma. The orbit includes the points

$$
\left\{\left(a u, a^{2} v\right),\left(a u v^{-1}, a^{2} v^{-1}\right): a \in \mathbb{F}_{q}^{*}\right\}
$$

For $P=(u, v)$ with $u \neq 0$, the set $\left\{\left(a u, a^{2} v\right): a \in \mathbb{F}_{q}^{*}\right\}$ consists of $q-1$ distinct points. To show that the second group of $q-1$ points is disjoint from the first group it suffices to show that $\left(u v^{-1}, v^{-1}\right) \notin\left\{\left(a u, a^{2} v\right): a \in \mathbb{F}_{q}^{*}\right\}$. But $\left(u v^{-1}, v^{-1}\right)=\left(a u, a^{2} v\right)$ if and only if $v=a^{-1}$ which is excluded by the assumption $v \notin \mathbb{F}_{q}$. We compute the number of points $N$ such that $u \neq 0$ and $v \notin \mathbb{F}_{q}$.

|  | $P=(u, v)$ | $P: v \in \mathbb{F}_{q}$ | $P: u=0$ | $P: v \in \mathbb{F}_{q} \wedge u=0$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $q$ odd | $q^{3}$ | $q^{2}$ | $q$ | 1 | $q^{3}-q^{2}-q+1$ |
| $q$ even | $q^{3}$ | $q$ | $q$ | $q$ | $q^{3}-q$ |

To each point corresponds a unique function, and the number of functions is obtained as $N /(2 q-2)$.

We rewrite Lemma 5.4 in terms of divisors in Remark 5.5.
Remark 5.5. Let $x^{2}-\alpha y$ be a conic over the field $\mathbb{F}_{q^{2}}$ such that

$$
\begin{equation*}
\left(x^{2}-\alpha y\right)=2 P_{\infty}+2 P_{0}+P_{1}+\cdots+P_{2(q-1)}-2 H_{\infty}, \tag{1}
\end{equation*}
$$

where $P_{i}$ 's are distinct $\mathbb{F}_{q^{2}}$-rational points for $i=1,2, \ldots, 2(q-1)$ and $H_{\infty}=(q+1) P_{\infty}$. If $q$ is odd then there are $\left(q^{2}-1\right) / 2$ number of conics that satisfies (1). If $q$ is even then there are $\left(q^{2}+q\right) / 2$ number of conics that satisfies (1).

Remark 5.6. For the Hermitian curve $y^{q}+y=x^{q+1}$ over $\mathbb{F}_{q^{2}}$, the line passing through any two rational points intersect the curve in $\mathrm{q}+1$ distinct rational points. Hence we can choose a line with divisors as below :

$$
\begin{aligned}
& (y-\beta x)=P_{0}+q \text { distinct points }-H_{\infty} . \\
& (x-\gamma)=P_{\infty}+q \text { distinct points }-H_{\infty} . \\
& (y-\delta)=q+1 \text { distinct points }-H_{\infty} .
\end{aligned}
$$

for some $\beta, \quad \gamma$ and $\delta$ in $\mathbb{F}_{q^{2}}$.

Theorem 3.3 Suppose that $G$ satisfies either
(1) $\operatorname{deg} G>\operatorname{deg} K+q$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ and $\mathrm{G} \nsim \mathrm{sP}_{\infty}$ and $\mathrm{G} \nsim \mathrm{tP}_{0}$ for all $s, t \in \mathbb{Z}$.

Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
a & =a_{0}(q+1)-a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)-b_{1}, & 0 \leq b_{1} \leq q
\end{aligned}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$.

1. If $0 \leq a_{1}, b_{1} \leq a_{0}+b_{0}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}$.
2. If $0 \leq b_{1} \leq a_{0}+b_{0}<a_{1}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}-\left(a_{0}+b_{0}\right)$.
$2^{\prime}$. If $0 \leq a_{1} \leq a_{0}+b_{0}<b_{1}$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+b_{1}-\left(a_{0}+b_{0}\right)$.
3. If $a_{0}+b_{0}<a_{1} \leq b_{1}<q$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$.
$3^{\prime}$. If $a_{0}+b_{0}<b_{1} \leq a_{1}<q$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$.
4. If $a_{0}+b_{0}<a_{1}, b_{1}$ and $a_{1}=q, b_{1}=q$, then $d\left(C(D, G)^{\perp}\right)=d^{*}+q-\left(a_{0}+b_{0}\right)$.

Proof. To prove the theorem we find a path for each case which will give a lower bound of the minimum distance. Then we show that it is sharp by finding a word of weight equal to the lower bound.
Case 1.
We fix $b$ and increase $a$. By Theorem 3.2, $m_{P_{\infty}}(2 g-2+a, b)=d^{*}=a+b$ and $m_{P_{\infty}}(2 g-$ $\left.2+a^{\prime}, b^{\prime}\right) \geq a^{\prime}+b^{\prime}$ for arbitrary $a^{\prime}$ and $b^{\prime}$. Thus the lower bound is $d^{*}$.
Case 2.
We fix $b$ and increase $a$ to $a+a_{1}-\left(a_{0}+b_{0}\right)$. Then Case 2 is reduced to Case 1 . By Lemma 5.1, the lower bound is $d^{*}+a_{1}-\left(a_{0}+b_{0}\right)$.
Case 2'.
By the symmetry of Case 2, the lower bound is $d^{*}+b_{1}-\left(a_{0}+b_{0}\right)$.
Case 3.
We fix $b$ and increase $a$ to $a+a_{1}-\left(a_{0}+b_{0}\right)$. Then Case 3 is reduced to Case $2^{\prime}$. By Lemma 5.2, the lower bound is $d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$.
Case $3^{\prime}$.
By the symmetry of Case 3 , the lower bound is $d^{*}+a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)$.


Figure 3: Cases for Theorem 3.3

Case 4.
We fix $b$ and increase $a$ by 1. Then Case 4 is reduced to Case 3. By Proposition 3.1,

$$
\begin{aligned}
& m_{P_{\infty}}(2 g-2+a, b)=q\left(a_{0}+b_{0}\right)-q \text { and } \\
& m_{P_{\infty}}(2 g-2+a+1, b)=(q-1)\left(a_{0}+b_{0}+1\right)-(q-1) \\
& \quad=q\left(a_{0}+b_{0}\right)-\left(a_{0}+b_{0}\right) .
\end{aligned}
$$

Thus the lower bound is obtained at $m_{P_{\infty}}(2 g-2+a, b)$. In order to prove that the lower bound of the code using the points $P \in X\left(\mathbb{F}_{q^{2}}\right) \backslash\left\{P_{0}, P_{\infty}\right\}$ is sharp, we need to find a word of weight $d$, where $d$ is the lower bound. We use the fact that there exist a word with support $P_{1}, P_{2}, \ldots, P_{d}$ if and only if $\Omega\left(G-P_{1} \cdots-P_{d}\right) \neq \Omega(G)$. Equivalently,

$$
L\left(P_{1}+\cdots+P_{d}-a P_{\infty}-b P_{0}\right) \neq L\left(-a P_{\infty}-b P_{0}\right) .
$$

Note that $L\left(-a P_{\infty}-b P_{0}\right)=0$ because $a+b>1$. We need to find $P_{1}, P_{2}, \ldots, P_{d}$ such that

$$
P_{1}+\cdots+P_{d}+a_{1} P_{\infty}+b_{1} P_{0} \sim\left(a_{0}+b_{0}\right) H_{\infty}+E, \text { where } E \geq 0
$$

Case $2^{\prime}$ reduces to Case 1 by taking $E=\left(b_{1}-\left(a_{0}+b_{0}\right)\right) P_{0}$. Case 2 reduces to Case 1 by taking $E=\left(a_{1}-\left(a_{0}+b_{0}\right)\right) P_{\infty}$. Case 3 and $3^{\prime}$ reduces to Case 1 by taking $E=$ $\left(a_{1}-\left(a_{0}+b_{0}\right)\right) P_{\infty}+\left(b_{1}-\left(a_{0}+b_{0}\right)\right) P_{0}$. Thus all cases reduce to case 1 except case 4 which we prove separately.
Case 4. $a_{0}+b_{0}<a_{1}, b_{1}$ and $a_{1}=q, b_{1}=q$.
We need to find $P_{1}, P_{2}, \ldots, P_{d}$, where $d=\left(a_{0}+b_{0}-1\right) q$, such that

$$
\begin{equation*}
P_{1}+\cdots+P_{d}+q P_{\infty}+q P_{0} \sim\left(a_{0}+b_{0}\right) H_{\infty}+E, \quad \text { where } E \geq 0 . \tag{2}
\end{equation*}
$$

Let $(x)=P_{\infty}+P_{0}+P_{1}+P_{2}+\cdots+P_{q-1}-H_{\infty}$. We take

$$
f=\frac{x}{y} \prod_{i=1}^{a_{0}+b_{0}-1} \frac{1}{y-y_{i}}
$$

where $y_{i}$ is the $y$-coordinate of $P_{i}$.
We have

$$
\begin{aligned}
& \left(\prod_{i=1}^{a_{0}+b_{0}-1} \frac{1}{y-y_{i}}\right)=\left(a_{0}+b_{0}-1\right) H_{\infty}-P_{1}-P_{2}-\cdots-P_{a_{0}+b_{0}-1}-\cdots-P_{\left(a_{0}+b_{0}-1\right)(q+1)} \text { and } \\
& \left(\frac{x}{y}\right)=P_{\infty}+P_{0}+P_{1}+\ldots+P_{q-1}-(q+1) P_{0}
\end{aligned}
$$

Hence
$(f)=\left(a_{0}+b_{0}\right) H_{\infty}-q P_{\infty}-q P_{0}-\left(a_{0}+b_{0}-1\right) q$ distinct points $+P_{a_{0}+b_{0}}+\cdots+P_{q-1}$.
Therefore (2) is satisfied with $E=P_{a_{0}+b_{0}}+\cdots+P_{q-1}$.
Case 1. $0 \leq a_{1}, b_{1} \leq a_{0}+b_{0}$.
We need to show that there exist a function in

$$
\begin{aligned}
& L\left(P_{1}+\cdots+P_{d}+a_{1} P_{\infty}+b_{1} P_{0}-\left(a_{0}+b_{0}\right) H_{\infty}\right), \quad \text { or } \\
& L\left(\left(a_{0}+b_{0}\right) H_{\infty}-P_{1}-\cdots-P_{d}-a_{1} P_{\infty}-b_{1} P_{0}\right) .
\end{aligned}
$$

We construct $f$ in $L\left(\left(a_{0}+b_{0}\right) H_{\infty}-P_{1}-\cdots-P_{d}-a_{1} P_{\infty}-b_{1} P_{0}\right)$ by using the functions $x^{2}-\alpha_{i} y, \quad y-\beta_{j} x, \quad x-\gamma_{k}$, and $y-\delta_{l}$. Since $0+K+a P_{\infty}+b P_{0} \sim K+D, a+b=q^{3}-1$. Then $a_{0}+b_{0} \leq q^{2}-q-1$. Thus we need at most $\left(q^{2}-q-1\right) / 2$ conics that satisfy the condition (1) of Remark 5.5. By Lemma [5.4, there are enough conics that satisfy (1) which can be used to construct the function $f$.
Case : $a_{1} \leq b_{1} \leq a_{0}+b_{0}$.
If $a_{1}=2 m$ is even then, take

$$
f=\prod_{i=1}^{m}\left(x^{2}-\alpha_{i} y\right) \times \prod_{i=1}^{b_{1}-a_{1}}\left(y-\beta_{i} x\right) \times \prod_{i=1}^{a_{0}+b_{0}-b_{1}}\left(y-\delta_{i}\right)
$$

If $a_{1}=2 m+1$ is odd, then take

$$
f=x \prod_{i=1}^{m}\left(x^{2}-\alpha_{i} y\right) \times \prod_{i=1}^{b_{1}-a_{1}}\left(y-\beta_{i} x\right) \times \prod_{i=1}^{a_{0}+b_{0}-b_{1}}\left(y-\delta_{i}\right)
$$

Case: $b_{1}<a_{1}$.
If $b_{1}=2 m$ is even then, take

$$
f=\prod_{i=1}^{m}\left(x^{2}-\alpha_{i} y\right) \times \prod_{i=1}^{a_{1}-b_{1}}\left(x-\gamma_{i}\right) \times \prod_{i=1}^{a_{0}+b_{0}-a_{1}}\left(y-\delta_{i}\right) .
$$

If $b_{1}=2 m+1$ is odd, then take

$$
f=x \prod_{i=1}^{m}\left(x^{2}-\alpha_{i} y\right) \times \prod_{i=1}^{a_{1}-b_{1}}\left(x-\gamma_{i}\right) \times \prod_{i=1}^{a_{0}+b_{0}-a_{1}}\left(y-\delta_{i}\right)
$$

Theorem 3.5 Suppose that $G$ satisfies either
(1) $\operatorname{deg} G<\operatorname{deg} K$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ with $\mathrm{G} \sim \mathrm{sP}_{\infty}$ or $\mathrm{G} \sim \mathrm{tP}_{0}$ for some $s, t \in \mathbb{Z}$.

If $G=a P_{\infty}+b P_{0}$ with

$$
\begin{aligned}
a & =a_{0}(q+1)+a_{1}, & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)+b_{1}, & 0 \leq b_{1} \leq q
\end{aligned}
$$

then

$$
d\left(C(D, G)^{\perp}\right)=a_{0}+b_{0}+2
$$

Proof. We may assume that $K=(q-2)(q+1) P_{\infty}$. Let $H_{0}=(q+1) P_{0}$.
It suffices to prove for the three cases (i) $a_{0}+b_{0} \leq q-3$, (ii) $a_{0}+b_{0}=q-2, \quad a_{1}=0$, and (iii) $a_{0}+b_{0}=q-2, \quad b_{1}=0$. If we assume $b_{0}=0$ then (i) contains the case $\operatorname{deg} G<\operatorname{degK}$,
(ii) contains the case $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ with $\mathrm{G} \sim \mathrm{tP}_{0}$ and (iii) contains the case $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ with $\mathrm{G} \sim \mathrm{sP}_{\infty}$.
Case 1. $a_{0}+b_{0} \leq q-3$
Let $G_{1}=a_{0} H_{\infty}+b_{0} H_{0}$ and $G_{2}=a_{0} H_{\infty}+q P_{\infty}+b_{0} H_{0}+q P_{0}$. By Proposition 3.1, we have that $d\left(C\left(D, G_{1}\right)^{\perp}\right) \geq a_{0}+b_{0}+2$. Since $C\left(D, G_{2}\right)^{\perp} \subseteq C\left(D, G_{1}\right)^{\perp}, d\left(C\left(D, G_{2}\right)^{\perp}\right) \geq$ $a_{0}+b_{0}+2$. We need to show that there exists a word of weight $a_{0}+b_{0}+2$ in $C\left(D, G_{2}\right)^{\perp}$. The code $C\left(D, G_{2}\right)^{\perp}=C_{\Omega}\left(D, G_{2}\right)$ has a word of weight $d=a_{0}+b_{0}+2$ if there exists $P_{1}, \ldots, P_{d}$ with $\Omega\left(G_{2}-P_{1} \cdots-P_{d}\right) \neq \Omega\left(G_{2}\right)$. Equivalently, if

$$
\begin{aligned}
& L\left((q-2) H_{\infty}-\left(a_{0}+b_{0}+2\right) H_{\infty}+P_{0}+P_{1}+\cdots+P_{d}\right) \\
& \neq L\left((q-2) H_{\infty}-\left(a_{0}+b_{0}+2\right) H_{\infty}+P_{\infty}+P_{0}\right) .
\end{aligned}
$$

We choose $P_{1}, \ldots, P_{d}$ on a line that pass through $P_{0}$ and $P_{\infty}$. Since $P_{1}+\cdots+P_{d}+P_{d+1}+$ $\cdots+P_{q-1}+P_{0}+P_{\infty} \sim(q+1) P_{\infty}$, it is enough to show that

$$
\begin{aligned}
& L\left((q-1) H_{\infty}-\left(a_{0}+b_{0}+2\right) H_{\infty}-P_{d+1} \cdots-P_{q-1}\right) \\
& \neq L\left((q-1) H_{\infty}-\left(a_{0}+b_{0}+2\right) H_{\infty}-P_{1}-P_{2} \cdots-P_{q-1}\right) .
\end{aligned}
$$

We take $f=\prod_{i=d+1}^{q-1}\left(y-y_{i}\right)$, where $y_{i}$ is the $y$-coordinate of $P_{i}$.
In both cases $a_{0}+b_{0}=q-2, \quad a_{1}=0$ and $a_{0}+b_{0}=q-2, \quad b_{1}=0$, we have that the lower bound is $d=a_{0}+b_{0}+2$ by Proposition 3.1. Now we show that the lower bound
is sharp for both cases.
Case 2. $a_{0}+b_{0}=q-2, \quad a_{1}=0$
Since $G=K+b_{1} P_{0}$, we need to show that there exist $P_{1}, \ldots, P_{d}$ with

$$
\begin{aligned}
& L\left(P_{1}+\cdots+P_{d}-b_{1} P_{0}\right) \\
& \neq L\left(-b_{1} P_{0}\right) .
\end{aligned}
$$

Take $f=y /(y-x)$.
Case 3. $a_{0}+b_{0}=q-2, \quad b_{1}=0$
Since $G=K+a_{1} P_{\infty}$, we need to show that there exist $P_{1}, \ldots, P_{d}$ with

$$
\begin{aligned}
& L\left(P_{1}+\cdots+P_{d}-a_{1} P_{\infty}\right) \\
& \neq L\left(-a_{1} P_{\infty}\right) .
\end{aligned}
$$

Take $f=x-1$.

## 6 Appendix

In this section we give formulas for the minimum distance obtained by Homma and Kim. Also, we give formulas for the minimum distance obtained by our method for comparison.

## Homma and Kim method

Let $X$ be a Hermitian curve defined by $y^{q}+y=x^{q+1}$ over $\mathbb{F}_{q^{2}}$. Let $P_{\infty}$ be the point at infinity of $X$ and $P_{0}$ the origin of $X$. We consider the code $C(m, n)$ in $\left(\mathbb{F}_{q^{2}}\right)^{q^{3}-1}$ defined by the image of the evaluation map

$$
\begin{aligned}
L\left(m P_{\infty}+n P_{0}\right) & \longrightarrow\left(\mathbb{F}_{q^{2}}\right)^{q^{3}-1} \\
f & \longmapsto(f(P))_{P \in X\left(\mathbb{F}_{q^{2}}\right) \backslash\left\{P_{\infty}, P_{0}\right\}}
\end{aligned}
$$

where $X\left(\mathbb{F}_{q^{2}}\right)$ denotes the set of $\mathbb{F}_{q^{2}}$-rational points of $X$. Our problem is to determine the minimum distance of $C(m, n)$ for $0 \leq n \leq q$.
For $n=0$ the following theorem holds.
Theorem 6.1. [5, Theorem 5.2] Let $m=a q+b=\left(q^{2}-\rho\right) q+b \in A(\mathbb{Z}, q)$ with $b \leq a \leq$ $\min \left\{b+q^{2}-1, q^{2}+q-3\right\}$.
(i) If $b \leq a \leq b+q^{2}-q-1$, then $d(C(m, 0))=q^{3}-1-m$.
(ii) If $1 \leq \rho \leq q$ and $0 \leq b \leq q-\rho$, then $d(C(m, 0))=\rho q-1$.
(iii) If $q^{2}-1 \leq a \leq \min \left\{b+q^{2}-1, q^{2}+q-3\right\}$, then $d(C(m, 0))=q^{2}+q-a-2$.

For $n$ with $1 \leq n \leq q-1$, we have the following theorems.
Theorem 6.2. [5], [7], [8] Fix an integer $n$ with $1 \leq n \leq q-1$. Let $m=a q+b$ be $a$ nonnegative integer with $0 \leq b<q$.
[I] If $b \leq a \leq q-(n+1)$, then $d(C(m, n))=q^{3}-1-m$.
[II] If m satisfies either
(i) $0 \leq b \leq q-2$ and $q^{2}-1 \leq a \leq b+q^{2}-1$, or
(ii) $b=q-1$ and $q^{2}-1 \leq a \leq q^{2}+q-(n+3)$,
then $d(C(m, n))=q^{2}+q-a-2$.
[III] If $m$ satisfies either
(i) $b=0$ and $q-n \leq a \leq q^{2}-(n+1)$, or
(ii) $1 \leq b \leq q-2$ and $\max \{b, q-n\} \leq a \leq \min \left\{b+q^{2}-(q+1), q^{2}-(n+2)\right\}$, or
(iii) $b=q-1$ and $q-(n+1) \leq a \leq q^{2}-(n+2)$, then $d(C(m, n))=q^{3}-1-(m+n)$.

In order to describe $d(C(m, n))$ for remaining $m$, we put $m=\left(q^{2}-\rho\right) q+b$ for convenience.
Theorem 6.3. [6, Theorem 1.4] Fix an integer $n$ with $1 \leq n \leq q-1$. Let $m=\left(q^{2}-\rho\right) q+b$ be an integer with $0 \leq b<q$.
[IV] If $1 \leq b, n+1 \leq \rho$ and $\rho+b \leq q$, then $d(C(m, n))=\rho q-(n+1)$.
[V] If $\rho \leq n+1$ and $q<\rho+b$, then $d(C(m, n))=\rho(q-1)-(b-1)$.
[VI] Assume that $2 \leq \rho \leq n$ and $\rho+b \leq q$.
[VI-1] If either " $n \leq q-2$ " or " $n=q-1$ and $\rho+b<q$ ", then $d(C(m, n))=\rho(q-1)$.
[VI-2] If $n=q-1$ and $\rho+b=q$, then $d(C(m, n))=(\rho-1) q$.
We denote by $A(\mathbb{Z}, q)$ the array of integers with the infinite length of column

$$
\begin{array}{cccc}
\vdots & \vdots & & \vdots \\
-q & -q+1 & \ldots & -q+(q-1) \\
0 & 1 & \ldots & (q-1) \\
q & q+1 & \ldots & q+(q-1) \\
2 q & 2 q+1 & \ldots & 2 q+(q-1)
\end{array}
$$

Let

$$
\tilde{I}_{q}=\{a q+b \in A(\mathbb{Z}, q) \mid b \leq a\} \cup\{a q+(q-1) \mid 0 \leq a \leq q-2\} \cup\{-1\}
$$

and

$$
J_{q}=\left\{a q+b \in A(\mathbb{Z}, q) \mid b+q^{2} \leq a\right\} \cup\left\{a q+(q-1) \mid q^{2}-2 \leq a\right\}
$$

The following is the formula for the minimum distance of $C(m, q)$.
Theorem 6.4. [5, Theorem 6.1] Let $m=a q+b(0 \leq b<q)$ be an integer in $\tilde{I}_{q} \backslash J_{q}$.
(A) If $m$ satisfies either
(i) $0 \leq b \leq q-2$ and $b \leq a \leq b+q^{2}-q-1$, or
(ii) $b=q-1$ and $-1 \leq a \leq q^{2}-3$,
then $d(C(m, q))=q^{3}-q-m-1$.
(B) If $m$ satisfies the condition

$$
b+q^{2}-q \leq a \leq q^{2}-2
$$

then $d(C(m, q))=\left(q^{2}-a-1\right) q$.
(C) If $m$ satisfies the condition

$$
0 \leq b \leq q-2 \quad \text { and } \quad q^{2}-1 \leq a \leq b+q^{2}-1
$$

then $d(C(m, q))=q^{2}+q-a-2$.

## Formulas using our method

Theorem 6.5. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G>\operatorname{deg} K+q$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ and $\mathrm{G} \nsim \mathrm{sP}_{\infty}$ and $\mathrm{G} \nsim \mathrm{tP}_{0}$ for all $s, t \in \mathbb{Z}$.

Let $G=K+a P_{\infty}+b P_{0}$, where $K$ is a canonical divisor,

$$
\begin{aligned}
& a=a_{0}(q+1)-a_{1}, 0 \leq a_{1} \leq q \\
& b=b_{0}(q+1)-b_{1}, \\
& 0 \leq b_{1} \leq q
\end{aligned}
$$

Let $d^{*}=\operatorname{deg}(G)-(2 g-2)=a+b$. Then

$$
d\left(C(D, G)^{\perp}\right)=d^{*}+\max \left\{0, a_{1}-\left(a_{0}+b_{0}\right), b_{1}-\left(a_{0}+b_{0}\right), a_{1}+b_{1}-2\left(a_{0}+b_{0}\right)\right\}
$$

except for the case when $\left(a_{0}+b_{0}<a_{1}, b_{1}\right.$ and $\left.a_{1}=q, b_{1}=q\right)$, for which

$$
d\left(C(D, G)^{\perp}\right)=d^{*}+q-\left(a_{0}+b_{0}\right)
$$

Theorem 6.6. Suppose that $G$ satisfies either
(1) $\operatorname{deg} G<\operatorname{deg} K$ or
(2) $\operatorname{deg} \mathrm{K} \leq \operatorname{deg} \mathrm{G} \leq \operatorname{deg} \mathrm{K}+\mathrm{q}$ with $\mathrm{G} \sim \mathrm{sP}_{\infty}$ or $\mathrm{G} \sim \mathrm{tP}_{0}$ for some $s, t \in \mathbb{Z}$.

If $G=a P_{\infty}+b P_{0}$ with

$$
\begin{aligned}
a & =a_{0}(q+1)+a_{1}, & & 0 \leq a_{1} \leq q \\
b & =b_{0}(q+1)+b_{1}, & & 0 \leq b_{1} \leq q
\end{aligned}
$$

then

$$
d\left(C(D, G)^{\perp}\right)=a_{0}+b_{0}+2
$$

Example 6.7. Let $X$ be a Hermitian curve defined by $y^{8}+y=x^{9}$ over $\mathbb{F}_{64}$. Let $K$ be a canonical divisor and $G=m P_{\infty}+n P_{0}$. We consider the code $C(m, n)^{\perp}$. We give two tables with $\operatorname{deg} \mathrm{G}<\operatorname{deg} \mathrm{K}$ and $\operatorname{deg} \mathrm{G}>\operatorname{deg} \mathrm{K}+\mathrm{q}$. The rows represent $m$ and the columns represent $n$. The entries of the first matrix are the minimum distance of $C(m, n)^{\perp}$ and the entries of the second matrix state which formula from Homma and Kim were used to find the minimum distance. If $G=82 P_{\infty}+3 P_{0}$ then the minimum distance of $C(82,3)^{\perp}=35$ and since the $(82,3)$ entry of the second matrix is 361 it means Theorem 6.3 VI-1 was used to find the minimum distance. The entries with zero mean that it is not in the range of Homma and Kim's formula.

Table 1 is an example of cases when $\operatorname{deg} \mathrm{G}<\operatorname{deg} \mathrm{K}$.

| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |$\quad$| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 222 | 222 | 222 | 13 |
| 0 | 221 | 221 | 221 | 221 | 221 | 221 | 221 | 13 |
| 0 | 221 | 221 | 221 | 221 | 221 | 221 | 221 | 13 |
| 0 | 221 | 221 | 221 | 221 | 221 | 221 | 221 | 13 |
| 0 | 221 | 221 | 221 | 221 | 221 | 221 | 221 | 13 |

Table 1: Minimum distance for codes $C(m, n)^{\perp}$ when $m=18, \ldots, 26$ and $n=0, \ldots, 8$.
Table 2 is an example of cases when degG $>\operatorname{degK}+\mathrm{q}$. The matrices are given in 10 by 10 for easy comparison. Homma and Kim used $n=0, \ldots, q$ and $m=a q+b$ where $b=0, \ldots, q$ which is the upper left 9 by 9 matrix. We used $m=a_{0}(q+1)-a_{1}$ and $n=b_{0}(q+1)-b_{1}$, where $0 \leq a_{1}, b_{1} \leq q$ which is the lower right 9 by 9 matrix.

| 27 | 32 | 32 | 32 | 32 | 32 | 33 | 34 | 35 | 36 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 32 | 32 | 35 | 35 | 35 | 36 | 37 | 38 | 39 | 40 |
| 32 | 35 | 35 | 35 | 35 | 36 | 37 | 38 | 39 | 40 |
| 32 | 35 | 35 | 35 | 35 | 36 | 37 | 38 | 39 | 40 |
| 32 | 35 | 35 | 35 | 35 | 36 | 37 | 38 | 39 | 40 |
| 32 | 36 | 36 | 36 | 36 | 37 | 38 | 39 | 40 | 41 |
| 33 | 37 | 37 | 37 | 37 | 38 | 39 | 40 | 41 | 42 |
| 34 | 38 | 38 | 38 | 38 | 39 | 40 | 41 | 42 | 43 |
| 35 | 39 | 39 | 39 | 39 | 40 | 41 | 42 | 43 | 44 |
| 36 | 40 | 40 | 40 | 40 | 41 | 42 | 43 | 44 | 45 |


| 411 | 35 | 35 | 35 | 35 | 232 | 232 | 232 | 11 | 411 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 362 | 361 | 361 | 34 | 34 | 34 | 34 | 12 | 42 |
| 42 | 361 | 361 | 361 | 34 | 34 | 34 | 34 | 12 | 42 |
| 42 | 361 | 361 | 361 | 34 | 34 | 34 | 34 | 12 | 42 |
| 42 | 361 | 361 | 361 | 231 | 231 | 231 | 231 | 12 | 412 |
| 412 | 35 | 35 | 35 | 233 | 233 | 233 | 233 | 11 | 411 |
| 411 | 35 | 35 | 35 | 232 | 232 | 232 | 232 | 11 | 411 |
| 411 | 35 | 35 | 35 | 232 | 232 | 232 | 232 | 11 | 411 |
| 411 | 35 | 35 | 35 | 232 | 232 | 232 | 232 | 11 | 411 |
| 411 | 35 | 35 | 35 | 232 | 232 | 232 | 232 | 11 | 411 |

Table 2: Minimum distance for codes $C(m, n)^{\perp}$ when $m=81, \ldots, 90$ and $n=0, \ldots, 9$.

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