A pair of disjoint 3-GDDs of type $g^t u^1$ *[1]

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Abstract

Pairwise disjoint 3-GDDs can be used to construct some optimal constant-weight codes. We study the existence of a pair of disjoint 3-GDDs of type $g^t u^1$ and establish that its necessary conditions are also sufficient.

Keywords: group divisible design; disjoint; resolvable; modified group divisible design; idempotent Latin square; constant-weight code; constant-composition code

1 Introduction

Let X be a finite set of v elements and K a set of positive integers. A *group divisible design* K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of X into subsets (called *groups*); (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality from K, such that a group and a block contain at most one common point; (3) every pair of points from distinct groups occurs in exactly one block. If $\mathcal{G}$ contains u_i groups of size g_i for $1 \le i \le s$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ the *group type* (or *type*) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD. A k-GDD of type t^k is denoted by TD(k, t) and is called a *transversal design.* A K-GDD of type 1^v is commonly called a *pairwise balanced design*, denoted by $(v, K, 1)$-PBD. When $K = \{k\}$, a pairwise balanced design is just a Steiner system S$(2, k, v)$. It is well-known that an S$(2, 3, v)$ exists if and only if $v \equiv 1, 3 \pmod 6$.

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Colbourn et al. completely settle the necessary and sufficient conditions for the existence of 3-GDDs of type $g^t u^1$.

Lemma 1.1 ([10]) *Let g, t, and u be nonnegative integers. There exists a 3-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(1) *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*

(2) *$u \leq g(t-1)$ or $gt = 0$;*

(3) *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*

(4) *$gt \equiv 0 \pmod 2$ or $u = 0$;*

(5) *$\frac{1}{2}g^2 t(t-1) + gtu \equiv 0 \pmod 3$.*

Let $2 \notin K$. A *partial group divisible design* K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying conditions (1) and (2) of the definition of a K-GDD and (3') every pair of points from distinct groups occurs in at most one block. The *leave* of a partial K-GDD is a graph whose edges are all the pairs which belong to distinct groups and do not appear in any block. A K-GDD can be regarded as a partial K-GDD with an empty leave. Suppose that $(X, \mathcal{G}, \mathcal{B})$ and $(X, \mathcal{G}, \mathcal{B}')$ are two partial K-GDDs. If $\mathcal{B}$ and $\mathcal{B}'$ have no block in common, $(X, \mathcal{G}, \mathcal{B})$ and $(X, \mathcal{G}, \mathcal{B}')$ are said to be *disjoint*.

The purpose of this paper is to determine the existence spectrum of a pair of disjoint 3-GDDs of type $g^t u^1$. The problem is itself interesting in the theory of combinatorial designs. Also we have a motivation lying in a close relation between disjoint 3-GDDs and constant-weight codes. In Chee et al. [7], pairwise disjoint combinatorial designs of various types, including Steiner systems and group divisible designs, are utilized to construct optimal q-ary constant-weight codes with $q > 2$. In particular, a pair of disjoint 3-GDDs of type $1^{6t} 5^1$ is proved to exist for any positive integer t, which is used in constructing optimal 3-ary constant-weight codes of Hamming distance 4 and weight 3. In [8], the concept of group divisible design is generalized to a new code named group divisible code, which is shown useful in recursive constructions for constant-weight and constant-composition codes. One can also find applications of disjoint group divisible designs in the determination of more optimal constant-weight codes (see, for example, [20, 21]).

In order to study the existence of two disjoint 3-GDDs, we introduce some related notions and basic facts in this section. Let $(X, \mathcal{G}, \mathcal{A})$ be a K-GDD. A subset of the block set $\mathcal{A}$ is called a *parallel class* if it contains every element of X exactly once. If $\mathcal{A}$ can be partitioned into some parallel classes, the GDD is called *resolvable.* A resolvable $\mathrm{S}(2, 3, v)$ is the well-known Kirkman triple system of order v, denoted by $\mathrm{KTS}(v)$. A $\mathrm{KTS}(v)$ exists if and only if $v \equiv 3 \pmod 6$ (see [13]).

A *Latin square* of order t (briefly by $\mathrm{LS}(t)$) is a $t \times t$ array in which each cell contains a single element from a t-set, such that each element occurs exactly once in each row and exactly once in each column. Suppose that $L = (a_{ij})$ is an $\mathrm{LS}(t)$ defined on and indexed by a set T. If for each $i \in T$, $a_{ii} = i$, then the Latin square is called *idempotent.* If for any $i, j \in T$, $a_{ij} = a_{ji}$, then it is called *symmetric.* Suppose that $L = (a_{ij})$ and $L' = (b_{ij})$ are $\mathrm{LS}(t)$s on a set T. L and L' are *orthogonal* if every element of $T \times T$ occurs exactly once among the t^2 pairs (a_{ij}, b_{ij}), $1 \leq i, j \leq t$.

A $\mathrm{TD}(3,t)$ is often defined on $V \times I$ with groups $V \times \{i\}, i \in I$, where $|V| = t$, and $|I| = 3$. If the $\mathrm{TD}(3,t)$ has a parallel class $\{\{x\} \times I : x \in V\}$, then it is called *idempotent* and denoted by $\mathrm{ITD}(3,t)$. An $\mathrm{ITD}(3,t)$ is equivalent to an idempotent $\mathrm{LS}(t)$. So when $t \geq 4$, an $\mathrm{ITD}(3,t)$ exists. If the block set of an $\mathrm{ITD}(3,t)$ can be partitioned into t parallel classes, one of which is the idempotent one, we call it *resolvable* and denote by $\mathrm{RITD}(3,t)$. An $\mathrm{RITD}(3,t)$, which is equivalent to a pair of orthogonal $\mathrm{LS}(t)$s, exists if and only if $t \neq 2, 6$.

Let $(X, \mathcal{G}, \mathcal{B})$ and $(X, \mathcal{G}, \mathcal{B}')$ be two $\mathrm{ITD}(3,t)$s. They are called *disjoint* if $\mathcal{B}$ and $\mathcal{B}'$ have no block in common except the common idempotent parallel class. Similarly we have the definition of disjoint RITDs. Note that although a resolvable $\mathrm{TD}(3,t)$ can always be made idempotent, two disjoint $\mathrm{RTD}(3,t)$s do not always mean two disjoint $\mathrm{RITD}(3,t)$s. The existence result of a pair of disjoint $\mathrm{ITD}(3,t)$s and that of disjoint $\mathrm{RITD}(3,t)$s are given as follows.

Lemma 1.2 *For any integer $t \geq 4$, there exists a pair of disjoint ITD$(3,t)$s. For any integer $t \geq 4$ and $t \neq 6, 10$, there exists a pair of disjoint RITD$(3,t)$s.*

Proof By [11], for any integer $t \geq 4$, there exists a pair of disjoint idempotent Latin squares of order t. Equivalently, there is a pair of disjoint $\mathrm{ITD}(3,t)s$.

By [1], for any integer $t \geq 4$ and $t \neq 6, 10$, there exist three mutually orthogonal Latin squares defined on and indexed by I_t. By some permutations of rows and columns, we can form three new mutually orthogonal Latin squares, say L_1, L_2, L_3, in such a way that the main diagonal entries of L_3 are all 0's. Accordingly, the main diagonal of L_i ($i = 1, 2$) is a transversal. By renaming the symbols of L_1 and L_2, we obtain two idempotent Latin squares L_1' and L_2'. Further L_1', L_2' and L_3 are still mutually orthogonal. Let $L_1' = (a_{ij})$, $L_2' = (b_{ij})$, and $L_3 = (c_{ij})$. For each $0 \leq k \leq t-1$, let $T_k = \{(i,j) : c_{ij} = k\}$. Thus $T_0, T_1, \ldots, T_{t-1}$ form t disjoint transversals of L_1' and L_2', where T_0 consists of the main diagonal positions. Then we can construct a pair of disjoint $\mathrm{RITD}(3,t)s$ on $X = I_t \times I_3$ with group set $\mathcal{G} = \{I_t \times \{i\} : i \in I_3\}$. For $0 \leq k \leq t-1$, let $P_1^k = \{\{(i,0), (j,1), (a_{ij}, 2)\} : (i,j) \in T_k\}$, and $P_2^k = \{\{(i,0), (j,1), (b_{ij}, 2)\} : (i,j) \in T_k\}$. It is readily checked that each P_j^k ($0 \leq k \leq t-1$, $j = 1, 2$) is a parallel class of X and $P_1^0 = P_2^0$ is an idempotent parallel class. Let $\mathcal{B}_1 = \cup_{0 \leq k \leq t-1} P_1^k$ and $\mathcal{B}_2 = \cup_{0 \leq k \leq t-1} P_2^k$. Observing that $a_{ij} \neq b_{ij}$ if $i \neq j$, we obtain two disjoint $\mathrm{RITD}(3,t)s$ $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$. □

We next record some known results on disjoint 3-GDDs for later use.

Lemma 1.3 (1) ([6]) *Let $u = 0$, g, t, u satisfy all the conditions of Lemma 1.1, and $(g,t,u) \neq (1,3,0)$. Then there exists a pair of disjoint 3-GDDs of type g^t.*

(2) ([12]) *There exists a pair of disjoint 3-GDDs of type $1^t 3^1$, where $t \equiv 0, 4 \pmod 6$ and $t \geq 4$.*

It is trivial that there is a pair of disjoint 3-GDDs of type $g^t u^1$ if $gt = 0$. And Lemma 1.3 solves the case $u = g$ or $u = 0$. So we only need to consider the case g, u all positive, $u \neq g$, and $t \geq 3$. We call a triple (g,t,u) of positive integers with $u \neq g$ and $t \geq 3$ *admissible* provided that the five conditions in Lemma 1.1 all hold.

We shall utilize various methods to construct a pair of disjoint 3-GDDs of type $g^t u^1$ for any admissible triple (g, t, u). And we finally prove that the necessary conditions for the existence of a pair of 3-GDDs of type $g^t u^1$ are also sufficient. Our main result is:

Theorem 1.4 (Main Theorem) *Let g, t, and u be nonnegative integers. There exists a a pair of disjoint 3-GDDs of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(1) *if $g > 0$, then $t \geq 3$ and $(g, t, u) \neq (1, 3, 0)$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*

(2) *$u \leq g(t-1)$ or $gt = 0$;*

(3) *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*

(4) *$gt \equiv 0 \pmod 2$ or $u = 0$;*

(5) *$\frac{1}{2}g^2 t(t-1) + gtu \equiv 0 \pmod 3$.*

2 Recursive constructions

In this section we shall present several powerful recursive constructions for disjoint 3-GDDs.

The following construction is a variation of Wilson's Fundamental Construction in [19].

Construction 2.1 (Weighting Construction) *Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, and let $\omega : X \longmapsto Z^+ \cup \{0\}$ be a weight function. For every block $A \in \mathcal{A}$, suppose that there is a pair of disjoint 3-GDDs of type $\{\omega(x) : x \in A\}$. Then there exists a pair of disjoint 3-GDDs of type $\{\sum_{x \in G} \omega(x) : G \in \mathcal{G}\}$.*

Proof For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ "copies" of x. For any $Y \subseteq X$, let $S(Y) = \bigcup_{x \in Y} S(x)$. For every block $A \in \mathcal{A}$, construct a pair of disjoint 3-GDDs $(S(A), \{S(x) : x \in A\}, \mathcal{B}_A)$ and $(S(A), \{S(x) : x \in A\}, \mathcal{B}'_A\}$. Then it is readily checked that there exists a pair of disjoint 3-GDDs $(S(X),\ \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ and $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}'_A)$. $\square$

We also employ "Filling Construction" to break up the groups as follows:

Construction 2.2 (Filling Construction I) *Suppose that there is a pair of disjoint 3-GDDs of type $\{g_1, g_2, \ldots, g_t\}$. For each $1 \leq i \leq t-1$, if $g_i \equiv 0 \pmod s$ and there is a pair of disjoint 3-GDDs of type $s^{g_i/s} u^1$. Then there exists a pair of disjoint 3-GDDs of type $s^{\sum_{i=1}^{t-1} g_i/s} (g_t + u)^1$.*

Proof Let $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ be a pair of disjoint 3-GDDs of type $\{g_1,\ g_2, \ldots, g_t\}$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ with $|H_i| = g_i$ for $1 \leq i \leq t$, and Y be a set of cardinality u such that $X \cap Y = \emptyset$.

For each $1 \leq i \leq t-1$, we partition each H_i into g_i/s subsets H_{ij}, $1 \leq j \leq g_i/s$, such that $|H_{ij}| = s$. By assumption, there is a pair of 3-GDDs on $H_i \bigcup Y$ with $\{H_{ij} : 1 \leq j \leq g_i/s\} \cup \{Y\}$ as group set and $\mathcal{A}_i^1$ and $\mathcal{A}_i^2$ as the disjoint block sets. Let $\mathcal{G} = \{H_{ij} : 1 \leq i \leq t-1, 1 \leq j \leq g_i/s\} \cup \{H_t \cup Y\}$. It is readily checked that $(X \bigcup Y, \mathcal{G}, (\cup_{i=1}^{t-1} \mathcal{A}_i^1) \bigcup \mathcal{B}_1)$ and $(X \bigcup Y, \mathcal{G}, (\cup_{i=1}^{t-1} \mathcal{A}_i^2) \bigcup \mathcal{B}_2)$ are two disjoint 3-GDDs of type $s^{\sum_{i=1}^{t-1} g_i/s}(g_t+u)^1$. □

Corollary 2.3 *Let $t \geq 6$ be an even integer. If there exists a pair of disjoint 3-GDDs of type $(2g)^{t/2}u^1$, where $(g, t/2) \neq (1, 3)$, then so does a pair of disjoint 3-GDDs of type $g^t(u+g)^1$.*

Proof It follows from Filling Construction I since a pair of disjoint 3-GDDs of type g^3 exists by Lemma 1.3. □

Sometimes we only fill in one long group and use the following construction.

Construction 2.4 (Filling Construction II) *Suppose that there is a pair of disjoint 3-GDDs of type g^tu^1 and $u = sg + x$. If a pair of disjoint 3-GDDs of type g^sx^1 also exists, then there exists a pair of disjoint 3-GDDs of type $g^{s+t}x^1$.*

Proof Let $(X, \mathcal{H} \cup \{G\}, \mathcal{B}_1)$ and $(X, \mathcal{H} \cup \{G\}, \mathcal{B}_2)$ be a pair of disjoint 3-GDDs of type g^tu^1, where $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ and $G = (\cup_{i=1}^s G_i) \cup G_{s+1}$ with $|G_i| = g$ $(1 \leq i \leq s)$, $|G_{s+1}| = x$, and $|H_j| = g$ $(1 \leq j \leq t)$. Construct on G a pair of 3-GDDs of type g^sx^1 with same group set $\mathcal{G} = \{G_i : 1 \leq i \leq s+1\}$ and disjoint block sets $\mathcal{A}_1$ and $\mathcal{A}_2$. It is immediately checked that $(X, \mathcal{G} \cup \mathcal{H}, \mathcal{A}_1 \cup \mathcal{B}_1)$ and $(X, \mathcal{G} \cup \mathcal{H}, \mathcal{A}_2 \cup \mathcal{B}_2)$ are two disjoint 3-GDDs of type $g^{s+t}x^1$. □

What follows is a useful construction for generating 3-GDDs of type g^tu^1 with g relatively large.

Construction 2.5 *Suppose that there exists a 3-GDD of type $\{g_1, g_2, \ldots, g_s\}$. Let $t \geq 4$. If there is a pair of disjoint 3-GDDs of type ${g_i}^tu^1$ for each $1 \leq i \leq s$, then there exists a pair of disjoint 3-GDDs of type v^tu^1, where $v = \sum_{i=1}^s g_i$.*

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-GDD of type $\{g_1, g_2, \ldots, g_s\}$ and U be a set of cardinality u. We will construct the desired designs on $(X \times I_t) \cup U$ with group set $\mathcal{H} = \{X \times \{i\} : i \in I_t\} \cup \{U\}$.

For each block $B = \{x, y, z\} \in \mathcal{B}$, there is a pair of disjoint ITD$(3, t)$s by Lemma 1.2 on $B \times I_t$ with groups $\{a\} \times I_t$, $a \in B$. Delete the idempotent parallel class to form two disjoint block sets $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$.

For each group $G \in \mathcal{G}$, place on $(G \times I_t) \cup U$ a pair of disjoint 3-GDDs of type $|G|^tu^1$ with group set $\{G \times \{i\} : i \in I_t\} \cup \{U\}$ and block sets $\mathcal{C}_G^1$ and $\mathcal{C}_G^2$.

Then we produce on $(X \times I_t) \cup U$ a pair of disjoint 3-GDDs of type v^tu^1 with block sets $(\cup_{B \in \mathcal{B}} \mathcal{A}_B^1) \cup (\cup_{G \in \mathcal{G}} \mathcal{C}_G^1)$ and $(\cup_{B \in \mathcal{B}} \mathcal{A}_B^2) \cup (\cup_{G \in \mathcal{G}} \mathcal{C}_G^2)$. □

3 Direct constructions and preliminary results

In this section we shall involve some methods of direct construction. The "method of differences" will be used to construct some 3-GDDs of type $g^t u^1$, as is usually used in constructing cyclic designs. The cyclic partial Steiner triple systems also play a crucial role in constructing 3-GDDs.

The following result is simple but useful.

Lemma 3.1 *Suppose that there exists a pair of disjoint partial* 3*-GDDs of type* $g^t u^1$ *on* X*, where* $U \subseteq X$ *is the group of size* u*, and* L_1*,* L_2 *are their leaves respectively. If the pairs of the leave* L_j $(j = 1, 2)$ *can be partitioned into* s *disjoint* 1*-factors of* $X \setminus U$*, say,* $F_1^j, F_2^j, \dots, F_s^j$*, such that* $F_i^1 \cap F_i^2 = \emptyset$ *holds for each* $1 \le i \le s$*, then there exists a pair of disjoint* 3*-GDDs of type* $g^t(u + s)^1$*.*

Proof Let $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ be the assumed pair of disjoint partial 3-GDDs of type $g^t u^1$ with U as the group of size u. Define $V = \{\infty_1, \infty_2, \dots, \infty_s\}$, $\mathcal{C}_j = \cup_{i=1}^s \{\{\infty_i, x, y\} : \{x, y\} \in F_i^j\}$, and $\mathcal{H} = (\mathcal{G} \setminus \{U\}) \cup \{U \cup V\}$. Then $(X \cup V, \mathcal{H}, \mathcal{B}_1 \cup \mathcal{C}_1)$ and $(X \cup V, \mathcal{H}, \mathcal{B}_1 \cup \mathcal{C}_2)$ are two disjoint 3-GDDs of type $g^t(u + s)^1$. □

Each edge $\{a, b\}$ of a graph on vertices Z_v is assigned to an integer d between 1 and $[v/2]$, called its *difference*, if $|b - a| = d$ or $v - |b - a| = d$. A *difference triple* in Z_v is a set $\{a, b, c\}$ where $a + b \equiv c \pmod{v}$ or $a + b + c \equiv 0 \pmod{v}$. A difference d is called *good* in Z_v if $v/gcd(d, v)$ is even.

Lemma 3.2 ([17]) *Let* v *be even and* D *a subset of* $[1, v/2]$*. If* D *contains a good difference in* Z_v*, then the set of all unordered pairs of* Z_v *whose difference appears in* D *can be partitioned into 1-factors.*

Lemma 3.3 *Let* (g, t, u) *be an admissible triple with* $u \ge 2$ *and* $g(t-1) - u \equiv 0 \pmod{6}$*. Suppose that* $\{1, 2, \dots, gt/2\} \setminus \{t, 2t, \dots, [g/2]t\} = D_1 \cup D_2$*, where* D_1 *can be partitioned into* $(gt - g - u)/6$ *difference triples in* Z_{gt} *and* $gt/2 \in D_2$ *if* g *is odd, or* D_2 *contains a good difference in* Z_{gt} *if* g *is even, then there exists a pair of disjoint* 3*-GDDs of type* $g^t u^1$*.*

Proof Take $X = Z_{gt} \cup \{\infty_1, \infty_2, \dots, \infty_u\}$ as the point set and $\mathcal{G} = \{\{j, t + j, 2t + j, \dots, (g - 1)t + j\} : 0 \le j \le t - 1\} \cup \{\{\infty_1, \infty_2, \dots, \infty_u\}\}$ as the group set. Suppose that D_1 can be partitioned into difference triples $\{a_i, b_i, c_i\}$ in Z_{gt} such that $a_i + b_i \equiv c_i \pmod{v}$ or $a_i + b_i + c_i \equiv 0 \pmod{v}$, $1 \le i \le (gt - g - u)/6$. Let

$$\mathcal{A}_1 = \cup_{1 \le i \le (gt-g-u)/6} \{\{x, a_i + x, c_i + x\} : x \in Z_{gt}\},$$

and

$$\mathcal{A}_2 = \cup_{1 \le i \le (gt-g-u)/6} \{\{x, b_i + x, c_i + x\} : x \in Z_{gt}\}.$$

Then $(Z_{gt}, \mathcal{A}_1)$ and $(Z_{gt}, \mathcal{A}_2)$ form two disjoint partial 3-GDDs of type g^t. Their common leave $\mathcal{L}$ consists of all the pairs whose differences lie in D_2. By the assumption, D_2 contains a good difference in Z_{gt}. By Lemma 3.2, noting that g and u are both even or both odd, $\mathcal{L}$ can be partitioned into u 1-factors, say, $F_1, F_2, \dots, F_u$. Let $F_i' = F_{i+1}$ for $i = 1, 2, \dots, u$, where the subscripts are modulo u. Since $u \ge 2$, $F_i \cap F_i' = \emptyset$ $i = 1, 2, \dots, u$. Hence, there exists a pair of disjoint 3-GDDs of type $g^t u^1$ by Lemma 3.1. □

Corollary 3.4 *Let $u = g(t-1)$, where g and t are positive integers such that gt is even. Then there exists a pair of disjoint 3-GDDs of type $g^t u^1$.*

Proof The conclusion follows immediately by applying Lemma 3.3 with $D_1 = \emptyset$ and $D_2 = \{1, 2, \ldots, gt/2\} \setminus \{t, 2t, \ldots, [g/2]t\}$. $\square$

A partial $\mathrm{S}(2, 3, v)$ is called *cyclic* if it has an automorphism of order v. Usually, Z_v is taken as the point set of a cyclic design of order v and the corresponding automorphism is $i \to i+1 \pmod v$. So the blocks of a partial $\mathrm{S}(2, 3, v)$ can be partitioned into a number of orbits, each of which can be represented by a *starter block*. An orbit is called *full* if it consists of v different blocks and called *short* otherwise. In the proof of [10, Lemma 3.2], some cyclic partial Steiner triple systems are constructed.

Lemma 3.5 ([10]) *For $k \geq 1$ and $1 \leq s \leq 6$, let $r' = 7$ if $s = 2$ and $k \equiv 2, 3 \pmod 4$, or $r' = s - 1$ otherwise. Then there is a cyclic partial $S(2, 3, 6k+s)$ without short orbits whose leave is r-regular, where $r \equiv r' \pmod 6$, $r' \leq r \leq 6k + s - 1$. Further if $r < 6k + s - 1$, then the cyclic partial $S(2, 3, 6k+s)$ has a starter block containing a good difference.*

Lemma 3.6 *Suppose that (g, t, u) is an admissible triple with $u \geq 2$ and $g(t-1) - u \equiv 0 \pmod 6$. Further suppose $gt = 6k + s$, where $k \geq 1$ and $1 \leq s \leq 6$. Let $r = 7$ if $s = 2$ and $k \equiv 2, 3 \pmod 4$, or $r = s - 1$ otherwise. Whenever $u \geq 2g + r - 2$ if g is odd, or $u \geq 2g + r - 5$ if g is even, there exists a pair of disjoint 3-GDDs of type $g^t u^1$.*

Proof By Lemma 3.5, there is a cyclic partial $\mathrm{S}(2, 3, gt)$ without short orbit whose leave is r-regular. Moreover, it has a starter block containing a good difference. Let $\mathcal{F}$ be the set of difference triples associated with the starter blocks of this cyclic partial $\mathrm{S}(2, 3, gt)$. Let $\mathcal{F}_0$ be the set of difference triples of $\mathcal{F}$, each of which contains at least a multiple of t. Since $gt/2$ does not appear in a difference triple of the cyclic partial $\mathrm{S}(2, 3, gt)$, we have $|\mathcal{F}_0| \leq [(g-1)/2]$. Choose a subset $\mathcal{F}'$ such that $\mathcal{F}_0 \subset \mathcal{F}' \subset \mathcal{F}$ and $|\mathcal{F}'| = [(g-1)/2]$. Further for even g we can ensure that $\mathcal{F}'$ contains a difference triple which have a good difference not being a multiple of t. This can be done obviously if all the multiples of t appear in less than $(g-2)/2$ difference triples. Even if each difference triple of $\mathcal{F}'$ contains a multiple of t as a difference, it can be verified that the difference triple containing t also contains a good difference not being a multiple of t. Set $D_1 = \cup_{B \in \mathcal{F} \setminus \mathcal{F}'} B$ and let D_2 be the set of differences (between 1 and $gt/2$) neither appear in $\mathcal{F} \setminus \mathcal{F}'$ nor are multiples of t. Since the cyclic partial $\mathrm{S}(2, 3, gt)$ has no short orbit, we then have $D_1 \cup D_2 = \{1, 2, \ldots, gt/2\} \setminus \{t, 2t, \ldots, [g/2]t\}$. Furthermore, $|D_2| = g + (r-1)/2$ and $gt/2 \in D_2$ if g is odd, or $|D_2| = g - 2 + (r-1)/2$ and D_2 contains a good difference in Z_{gt} if g is even. By Lemma 3.3, there exists a pair of disjoint 3-GDDs of type $g^t u^1$, where $u = 2g + r - 2$ if g is odd and $u = 2g + r - 5$ if g is even. For other cases of larger u with $g(t-1) - u \equiv 0 \pmod 6$, diverting more differences produced by the difference triples in $\mathcal{F} \setminus \mathcal{F}'$ to D_2 works similarly. $\square$

Similar to Lemmas 3.1, 3.3, and 3.6, we can obtain the result of disjoint partial 3-GDDs of type $g^t u^1$, whose leaves are same, forming a 1-factor of the t groups of size g. We record this in a remark.

Remark 3.7 *Suppose that* (g,t,u) *is an admissible triple with* $u \neq 2$ *and* $g(t-1)-u \equiv 0 \pmod 6$. *Further suppose* $gt = 6k+s$, *where* $k \geq 1$ *and* $1 \leq s \leq 6$. *Let* $r = 7$ *if* $s = 2$ *and* $k \equiv 2,3 \pmod 4$, *or* $r = s-1$ *otherwise. Whenever* $u \geq 2g+r-2$ *if* g *is odd, or* $u \geq 2g+r-5$ *if* g *is even, there exists a pair of disjoint partial* 3*-GDDs of type* $g^t(u-1)^1$, *whose leaves are same, forming a* 1*-factor of the* t *groups of size* g.

Next we consider two small cases $g = 1$ and $g = 2$.

Lemma 3.8 ([9]) *There exists a pair of disjoint* 3*-GDDs of type* 1^tu^1 *whenever* $u \equiv 1,3 \pmod 6$, $u+t \equiv 1,3 \pmod 6$ *and* $7 \leq u \leq t-1$.

Lemma 3.9 *The Main Theorem holds for any admissible triple* $(1,t,u)$.

Proof Since $(1,t,u)$ is an admissible triple, u must be odd and $u \geq 3$. We distinguish the possibility of u to show the conclusion.

First if $u = 3$, then $t \equiv 0,4 \pmod 6$ and $t \geq 4$. A pair of disjoint 3-GDDs of type 1^t3^1 exists by Lemma 1.3.

Next if $u \equiv 1,3 \pmod 6$ and $u \geq 7$, then $u+t \equiv 1,3 \pmod 6$ and $u \leq t-1$. By Lemma 3.8, there exists a pair of disjoint 3-GDDs of type 1^tu^1.

Finally we treat $u \equiv 5 \pmod 6$. Then $t \equiv 0 \pmod 6$ and $u \leq t-1$. Corollary 3.4 solves the case $t = 6$ and $u = 5$. For $t \geq 12$, a pair of disjoint 3-GDDs of type 1^tu^1 is obtained by taking $g = 1$ and $r = 5$ in Lemma 3.6. □

Lemma 3.10 *The Main Theorem holds for any admissible triple* $(2,t,u)$ *with* $t \equiv 1,2 \pmod 3$.

Proof Since $(2,t,u)$ is an admissible triple, $t \equiv 1 \pmod 3$ requires $u \equiv 0 \pmod 6$ ($u \geq 6$), $t \equiv 2 \pmod 3$ demands $u \equiv 2 \pmod 6$ ($u \geq 8$), and $(1,2t,u+1)$ is also an admissible triple satisfying the equality $1\cdot(2t-1)-(u+1) \equiv 0 \pmod 6$. Let $2t = 6k+s$ and k,s,r be taken as in Remark 3.7. As $u+1 \geq 7 \geq r = 2\cdot 1+r-2$, there is a pair of partial 3-GDDs of type $1^{2t}u^1$ with U as the long group, whose leaves are same, forming a 1-factor of the $2t$ groups of size 1. Take this 1-factor together with U as new groups, we obtain a pair of disjoint 3-GDDs of type 2^tu^1. □

The complete solution for the case $g = 2$ is left to Section 5.

4 The case $t \equiv 3 \pmod 6$

A useful auxiliary design to construct 3-GDDs is resolvable $\{2,3\}$-GDD with 3 groups of even size, whose existence is investigated in [14]. We shall show in this section that two such GDDs with some restrictions also exist. Related results will be employed to solve the case $t \equiv 3 \pmod 6$ of the Main Theorem.

Lemma 4.1 *Let* g *and* u *be even,* $0 \leq u \leq 2g$, $(g,u) \neq (2,0)$ *or* $(6,0)$. *Then there is a pair of* $\{2,3\}$*-GDD of type* g^3 *with same groups and different block sets* $\mathcal{B}^1$ *and* $\mathcal{B}^2$ *satisfying all of the following conditions*:

(1) *Both* $\mathcal{B}^1$ *and* $\mathcal{B}^2$ *can be resolved into* u *parallel classes containing only blocks of size* 2 *and* $g-u/2$ *parallel classes containing only blocks of size* 3;

(2) $\mathcal{B}^1$ *and* $\mathcal{B}^2$ *have no block of size* 3 *in common*;

(3) *The* u *parallel classes containing only blocks of size* 2 *of* $\mathcal{B}^j$ $(j = 1,2)$ *can be arranged in sequence* $P_1^j, P_2^j, \ldots, P_u^j$, *in such a way that* $P_i^1 \cap P_i^2 = \emptyset$ *for each* $1 \le i \le u$.

Proof We follow the idea of Rees in [14]. Let $X = Z_g \times I_3$ be the point set and $\mathcal{G} = \{Z_g \times \{i\} \colon i \in I_3\}$ be the group set.

First we handle the case $u = 0$. Obviously when $g \ne 2, 6$, there exists a resolvable 3-GDD $(X, \mathcal{G}, \mathcal{C})$ of type g^3. Set $\mathcal{C}' = \{\{(x,0),(y,1),(z+1,2)\} : \{(x,0),(y,1),(z,2)\} \in \mathcal{C}\}$. Then $(X, \mathcal{G}, \mathcal{C}')$ is a resolvable 3-GDD disjoint with $(X, \mathcal{G}, \mathcal{C})$.

Next consider $u \ge 2$. Let $\mathcal{B}$ be the union of following $g+1$ parallel classes of X:

$$\begin{aligned}
S_i &= \{\{(x,0),(x+i,1),(x+2i,2)\} : x \in Z_g\},\ 0 \le i \le g/2-1,\\
S_i &= \{\{(x,0),(x+i,1),(x+2i+1,2)\} : x \in Z_g\},\ g/2 \le i \le g-2,\\
M_1 &= \{\{(x,0),(x-1,1)\},\{(x+g/2,0),(x+g/2-1,2)\},\\
&\qquad \{(x+g/2-1,1),(x-1,2)\} : 0 \le x \le g/2-1\},\\
M_2 &= \{\{(x,0),(x-1,1)\},\{(x+g/2,0),(x+g/2-1,2)\},\\
&\qquad \{(x+g/2-1,1),(x-1,2)\} : g/2 \le x \le g-1\}.
\end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{B})$ is a resolvable $\{2,3\}$-GDD with two parallel classes of blocks of size 2.

To generate more parallel classes, some transformations from parallel classes of triples to those of pairs are made.

(A) The pairs produced by $S_{g/2-1}$ and M_1 can be divided into three parallel classes P_{1l}, $1 \le l \le 3$, described below. Let

$$\begin{aligned}
M_{11} &= \{\{(x,0),(x-1,1)\} : 0 \le x \le g/2-1\},\\
M_{12} &= \{\{(x,0),(x-1,2)\} : g/2 \le x \le g-1 \text{ and } x \text{ is even}\}\\
&\qquad \cup \{\{(x+g/2-1,1),(x-1,2)\} : 0 \le x \le g/2-1 \text{ and } x \text{ is even}\},\\
M_{13} &= (M_1 \setminus M_{11}) \setminus M_{12}.
\end{aligned}$$

For each block B of $S_{g/2-1}$ and $1 \le l \le 3$, let $h_l^1(B)$ be the unique intersection of B and M_{1l} and let

$$P_{1l} = M_{1l} \cup (\cup\{B \setminus \{h_l^1(B)\} : B \in S_{g/2-1}\}).$$

Note: By replacing M_1 with M_2 and "x is even" with "x is odd" and interchanging the range $0 \le x \le g/2-1$ and $g/2 \le x \le g-1$ in M_{1l}, the pairs produced by $S_{g/2-1}$ and M_2 can also be divided into three parallel classes, which we denote by P_{2l}, $1 \le l \le 3$.

(B) For $0 \le i \le g/2-2$, all the pairs produced by the two classes S_i and $S_{g/2+i}$ can be divided into four parallel classes E_{ik}, $1 \le k \le 4$, as follows:

$$\begin{aligned}
E_{i1} &= \{\{(2x,0),(2x+i,1)\},\{(2x+1,0),(2x+2i+2,2)\},\\
&\qquad \{(2x+i+1,1),(2x+2i+1,2)\} : 0 \le x \le g/2-1\},\\
E_{i2} &= \{\{(2x+1,0),(2x+g/2+i+1,1)\},\{(2x,0),(2x+2i,2)\},\\
&\qquad \{(2x+g/2+i,1),(2x+2i+1,2)\} : 0 \le x \le g/2-1\}.
\end{aligned}$$

Setting $E_{i,k+2} = \{\{(x+1,s),(y+1,t)\} : \{(x,s),(y,t)\} \in E_{ik}\}$ for $k = 1,2$ yields another two parallel classes E_{i3} and E_{i4}.

Let ϕ be a bijection on $Z_g \times I_3$ such that $\phi((x,0)) = (x,0)$, $\phi((x,1)) = (x,1)$, and $\phi((x,2)) = (x+1,2)$. For a subset $\mathcal{A}$ of $\mathcal{B}$, define $\phi(\mathcal{A}) = \{\{\phi(a),\phi(b),\phi(c)\} : \{a,b,c\} \in \mathcal{A}\}$.

If $u/2$ is odd, then in $\mathcal{B}$ by replacing S_i and $S_{g/2+i}$ with E_{ik} (only if $u \geq 6$) for $0 \leq i \leq (u-6)/4$, $1 \leq k \leq 4$, we obtain a resolvable $\{2,3\}$-GDD $(X,\mathcal{G},\mathcal{B}^1)$ with exactly u parallel classes of pairs. $\mathcal{P}_1 = \{M_l : l = 1,2\} \cup \{E_{ik} : 0 \leq i \leq (u-6)/4, 1 \leq k \leq 4\}$ is the collection of the u parallel classes of pairs. And $\mathcal{P}_2 = \{S_i : (u-2)/4 \leq i \leq g/2-1$, or $(u-2)/4 + g/2 \leq i \leq g-2\}$ is the collection of the parallel classes of triples. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{B}^2 = \phi(\mathcal{B}^1)$. Apparently, $(X,\mathcal{G},\mathcal{B}^2)$ is a resolvable $\{2,3\}$-GDD with a collection of parallel classes $\{\phi(P) : P \in \mathcal{P}\}$. Besides, one can check that $\phi(M_1) \cap M_2 = \emptyset$, $\phi(M_2) \cap M_1 = \emptyset$, $\phi(E_{ik}) \cap E_{i,k+2} = \emptyset$ $(0 \leq i \leq (u-6)/4$, $k, k+2$ is modulo 4), and $\phi(Q) \cap R = \emptyset$ for any $Q, R \in \mathcal{P}_2$. So we prove the lemma for $u/2$ odd.

Otherwise, $u/2$ is even. Then in $\mathcal{B}$ by replacing S_i and $S_{g/2+i}$ with E_{ik} (only if $u \geq 8$) for $0 \leq i \leq (u-8)/4$, $1 \leq k \leq 4$, and replacing $S_{g/2-1}$ and M_1 with P_{1l}, $1 \leq l \leq 3$, we obtain a resolvable $\{2,3\}$-GDD $(X,\mathcal{G},\mathcal{B}^1)$ with exactly u parallel classes of pairs. $\mathcal{P}_1 = \{E_{ik} : 0 \leq i \leq (u-8)/4, 1 \leq k \leq 4\} \cup \{M_2\} \cup \{P_{1l} : l = 1,2,3\}$ contains the u parallel classes of pairs. And $\mathcal{P}_2 = \{S_i : (u-4)/4 \leq i \leq g/2-2$, or $(u-4)/4+g/2 \leq i \leq g-2\}$ contains all the parallel classes of triples. If we employ the same replacement except taking M_2 instead of M_1, then another resolvable $\{2,3\}$-GDD $(X,\mathcal{G},\mathcal{B}')$ is obtained. The collection of parallel classes are $\mathcal{P}' = ((\mathcal{P}_1 \cup \mathcal{P}_2) \setminus \{M_2, P_{11}, P_{12}, P_{13}\}) \cup \{M_1\} \cup \{P_{2l} : l = 1,2,3\}$. Let $\mathcal{B}^2 = \phi(\mathcal{B}')$. Then $(X,\mathcal{G},\mathcal{B}^2)$ is a resolvable $\{2,3\}$-GDD of type g^3 with a collection of parallel classes $\{\phi(P) : P \in \mathcal{P}'\}$. Further, $\mathcal{B}^1$ and $\mathcal{B}^2$ satisfy the three conditions required by the lemma, where $\phi(E_{ik}) \cap E_{i,k+2} = \emptyset$ $(0 \leq i \leq (u-8)/4$, $k, k+2$ is modulo 4), $\phi(M_1) \cap M_2 = \emptyset$, and $\phi(P_{2l}) \cap P_{1l} = \emptyset$ $(l = 1,2,3)$, $\phi(Q) \cap R = \emptyset$ for any $Q, R \in \mathcal{P}_2$. This completes the proof. □

Corollary 4.2 *The Main Theorem holds for any admissible triple (g,t,u) with $t \equiv 3 \pmod 6$.*

Proof (g,t,u) is admissible and $t \equiv 3 \pmod 6$, so $g \equiv 0 \pmod 2$, $u \equiv 0 \pmod 2$, and $2 \leq u \leq g(t-1)$.

We first treat $t = 3$. Suppose that $(X,\mathcal{G},\mathcal{A}_1 \cup \mathcal{B}_1)$ and $(X,\mathcal{G},\mathcal{A}_2 \cup \mathcal{B}_2)$ are two $\{2,3\}$-GDD of type g^3 satisfying all the three conditions in Lemma 4.1, where $\mathcal{A}_i$ $(i = 1,2)$ consists of u parallel classes of pairs, say, $F_1^i, F_2^i, \ldots, F_u^i$, and $\mathcal{B}_i$ $(i = 1,2)$ consists of parallel classes of triples. Further $F_j^1 \cap F_j^2 = \emptyset$ for $1 \leq j \leq u$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. By Lemma 3.1, there is a pair of disjoint 3-GDDs of type g^3u^1.

Next let $t = 6n+3$ where $n \geq 1$. There is a KTS(t) on a t-set Y having $3n+1$ parallel classes $P_1, P_2, \ldots, P_{3n+1}$. Since $u \equiv 0 \pmod 2$ and $u \leq g(t-1)$, we can take even integers u_j, $j = 1,2,\ldots,3n+1$, such that $0 \leq u_j \leq 2g$ and $u = \sum_{j=1}^{3n+1} u_j$. Let $U_j = \{\infty_1^j, \infty_2^j, \ldots, \infty_{u_j}^j\}$ and $U = \cup_{j=1}^{3n+1} U_j$. For every block $B = \{x,y,z\}$ of each parallel class P_j, $1 \leq j \leq 3n+1$, construct on $(B \times I_g) \cup U_j$ a pair of disjoint 3-GDDs of type $g^3{u_j}^1$ with group set $\{\{x\} \times I_g : x \in B\} \cup \{U_j\}$ and block sets $\mathcal{C}_B^1$ and $\mathcal{C}_B^2$. Set $Z = (Y \times I_g) \cup U$, $\mathcal{G} = \{\{x\} \times I_g : x \in Y\} \cup \{U\}$ and $\mathcal{C}^i = \bigcup_{B \in P_j, 1 \leq j \leq 3n+1} \mathcal{C}_B^i$ for $i = 1,2$. It is immediate that $(Z,\mathcal{G},\mathcal{C}^1)$ and $(Z,\mathcal{G},\mathcal{C}^2)$ are two disjoint 3-GDDs of type $g^t u^1$. □

Lemma 4.3 *Let g and u be even, $2 \le u \le 2g-2$. Then there is a pair of $\{2,3\}$-GDD of type g^3 with same groups and different block sets $\mathcal{B}^1$ and $\mathcal{B}^2$ satisfying all of the following conditions*:

(1) *Both $\mathcal{B}^1$ and $\mathcal{B}^2$ can be resolved into u parallel classes containing only blocks of size* 2 *and $g-u/2$ parallel classes containing only blocks of size* 3;

(2) *$\mathcal{B}^1$ and $\mathcal{B}^2$ have a common parallel class of size* 3 *but have no other triple in common*;

(3) *The u parallel classes containing only blocks of size* 2 *of $\mathcal{B}^j$ ($j = 1, 2$) can be arranged in sequence $P_1^j, P_2^j, \ldots, P_u^j$, in such a way that $P_i^1 \cap P_i^2 = \emptyset$ for each $1 \le i \le u$.*

Proof The proof is similar to that of Lemma 4.1. First we have a resolvable $\{2,3\}$-GDD $(X, \mathcal{G}, \mathcal{B})$ of type g^3 with M_1 and M_2 as the parallel classes of pairs, and S_i, $0 \le i \le g-2$, as the parallel classes of triples. The conclusion holds clearly for the case $(g, u) = (2, 2)$, so we assume that $g \ge 4$. We will use transformation of kind (B) and another three kinds to treat the parallel classes.

(C) The pairs produced by S_0 and M_1 can be divided into three parallel classes P_{0l}, $1 \le l \le 3$. Let

$$\begin{aligned} M_{01} &= \{\{(x+g/2-1,1),(x-1,2)\} : 0 \le x \le g/2-1\}, \\ M_{02} &= \{\{(x,0),(x-1,2)\} : g/2 \le x \le g-1 \text{ and } x \text{ is even}\} \\ &\quad \cup \{\{(x,0),(x-1,1)\} : 0 \le x \le g/2-1 \text{ and } x \text{ is even}\}, \\ M_{03} &= (M_1 \setminus M_{01}) \setminus M_{02}. \end{aligned}$$

For each block B of S_0 and $1 \le l \le 3$, let $h_l^0(B)$ be the unique intersection of B and M_{0l} and let

$$P_{0l} = M_{0l} \cup (\cup\{B \setminus \{h_l^0(B)\} : B \in S_0\}).$$

(D) The pairs produced by the two classes S_0 and S_{g-2} can be divided into four parallel classes F_k, $1 \le k \le 4$, as follows:

$$\begin{aligned} F_1 = \{&\{(2x+1,0),(2x-1,1)\}, \{(2x,0),(2x,2)\}, \\ &\{(2x,1),(2x-1,2)\} : 0 \le x \le g/2-1\}, \\ F_2 = \{&\{(2x,0),(2x,1)\}, \{(2x+1,0),(2x-2,2)\}, \\ &\{(2x+1,1),(2x+1,2)\} : 0 \le x \le g/2-1\}. \end{aligned}$$

Setting $F_{k+2} = \{\{(x+1,s),(y+1,t)\} : \{(x,s),(y,t)\} \in F_k\}$ for $k = 1, 2$ yields another two parallel classes F_3 and F_4.

(E) The pairs produced by the two classes $S_{g/2-2}$ and $S_{g/2-1}$ can be divided into four parallel classes H_k, $1 \le k \le 4$, as follows:

$$\begin{aligned} H_1 = \{&\{(2x+1,0),(2x+g/2-1,1)\}, \{(2x,0),(2x-4,2)\}, \\ &\{(2x+g/2,1),(2x-1,2)\} : 0 \le x \le g/2-1\}, \\ H_2 = \{&\{(2x,0),(2x+g/2-1,1)\}, \{(2x+1,0),(2x-1,2)\}, \\ &\{(2x+g/2-2,1),(2x-4,2)\} : 0 \le x \le g/2-1\}. \end{aligned}$$

Setting $H_{k+2} = \{\{(x+1,s),(y+1,t)\} : \{(x,s),(y,t)\} \in H_k\}$ for $k = 1, 2$ yields another two parallel classes H_3 and H_4.

Let ϕ be a bijection on $Z_g \times I_3$ such that $\phi((x,0)) = (x,0)$, $\phi((x,1)) = (x+1,1)$, and $\phi((x,2)) = (x+3,2)$. For a subset $\mathcal{A}$ of $\mathcal{B}$ define $\phi(\mathcal{A}) = \{\{\phi(a),\phi(b),\phi(c)\} : \{a,b,c\} \in \mathcal{A}\}$. Evidently, $\phi(S_{g/2-1}) = S_{g/2}$, which we will use as the common parallel class required by the lemma.

First let $u/2$ be odd. If more parallel classes of pairs are required, then replace step by step in $\mathcal{B}$ each pair S_0 and S_{g-2} with F_k, $S_{g/2-2}$ and $S_{g/2-1}$ with H_k, S_i and $S_{g/2+i}$ with E_{ik} ($1 \le i \le (u-10)/4$, $1 \le k \le 4$). Thus we obtain a resolvable $\{2,3\}$-GDD $(X, \mathcal{G}, \mathcal{B}^1)$ with a collection of parallel classes $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1 = \{M_i : i = 1,2\} \cup \{F_k : 1 \le k \le 4\} \cup \{H_k : 1 \le k \le 4\} \cup \{E_{ik} : 1 \le i \le (u-10)/4, 1 \le k \le 4\}$, $\mathcal{P}_2 = \{S_i : i = g/2$, or $(u-6)/4 \le i \le g/2-3$, or $(u-6)/4 + g/2 \le i \le g-3\}$ (observe that $S_{g/2} \in \mathcal{P}$). Similarly, replace in $\mathcal{B}$ each pair S_0 and $S_{g/2}$ with $E_{0,k}$, $S_{g/2-2}$ and S_{g-2} with $E_{g/2-2,k}$. And we still replace S_i and $S_{g/2+i}$ with E_{ik} ($1 \le i \le (u-10)/4$, $1 \le k \le 4$), then form another resolvable $\{2,3\}$-GDD $(X, \mathcal{G}, \mathcal{B}')$ with a collection of parallel classes $\mathcal{P}' = \mathcal{P}'_1 \cup \mathcal{P}'_2$, where $\mathcal{P}'_1 = \{M_i : i = 1,2\} \cup \{E_{ik} : 0 \le i \le (u-10)/4$, or $i = g/2-2, 1 \le k \le 4\}$, $\mathcal{P}'_2 = \{S_i : (u-6)/4 \le i \le g/2-3$, or $i = g/2-1$, or $(u-6)/4 + g/2 \le i \le g-3\}$. Let $\mathcal{B}^2 = \phi(\mathcal{B}')$. Obviously, $(X, \mathcal{G}, \mathcal{B}^2)$ is a resolvable $\{2,3\}$-GDD of type g^3 with a collection of parallel classes $\{\phi(P) : P \in \mathcal{P}'\}$ containing $\phi(S_{g/2-1})$. Besides, one can check that $\phi(P) \cap P = \emptyset$ for any $P \in \mathcal{P}'_1 \setminus \{E_{0k}, E_{g/2-2,k} : 1 \le k \le 4\}$, $\phi(E_{0k}) \cap F_k = \emptyset$, $\phi(E_{g/2-2,k}) \cap H_k = \emptyset$ (a slight difference when $g/2$ is odd: $\phi(E_{g/2-2,2}) \cap H_4 = \phi(E_{g/2-2,4}) \cap H_2 = \emptyset$), and $\phi(Q) \cap R = \emptyset$ for any $Q, R \in \mathcal{P}'_2$ except $\phi(S_{g/2-1}) = S_{g/2}$.

Finally let $u/2$ be even. For $1 \le i \le (u-4)/4$, $1 \le k \le 4$, replace in $\mathcal{B}$ each pair S_i and $S_{g/2+i}$ with E_{ik}, and replace S_0 and M_1 with P_{0l}, $1 \le l \le 3$. Thus we obtain a resolvable $\{2,3\}$-GDD $(X, \mathcal{G}, \mathcal{B}^1)$ with a collection of parallel classes $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1 = \{E_{ik} : 1 \le i \le (u-4)/4, 1 \le k \le 4\} \cup \{P_{0l} : l = 1,2,3\} \cup \{M_2\}$, $\mathcal{P}_2 = \{S_i : u/4 \le i \le g/2$, or $u/4 + g/2 \le i \le g-2\}$ (note that both $S_{g/2-1}$ and $S_{g/2}$ belong to $\mathcal{P}$). Similarly let $\mathcal{B}^2 = \phi(\mathcal{B}^1)$. Then $(X, \mathcal{G}, \mathcal{B}^2)$ is a resolvable $\{2,3\}$-GDD of type g^3 with a collection of parallel classes $\{\phi(P) : P \in \mathcal{P}\}$, which also satisfy all the conditions required by the lemma. □

Corollary 4.4 *Let g and u be even integers such that $0 \le u \le 2g-2$ and $(g,u) \ne (2,0)$. Then there exists a pair of 3-GDDs of type g^3u^1 with exactly g blocks in common and these g blocks form a parallel class of the union of the three groups of size g.*

Proof There is a pair of disjoint ITD$(3,g)$s for $g \ge 4$ by Lemma 1.2, so the conclusion holds if $u = 0$. If $2 \le u \le 2g-2$, there is a pair of $\{2,3\}$-GDDs meeting the conditions in Lemma 4.3. Analogous to the proof for $t = 3$ in Corollary 4.2, the conclusion follows. □

5 The case $g \equiv 0 \pmod 3$

In this section, we mainly examine the existence of a pair of disjoint 3-GDDs of type g^tu^1 for $g \equiv 0 \pmod 3$. We adopt a similar procedure as in Section 2 of [10], so we list some results on K-GDDs derived therein.

Lemma 5.1 ([4, 10, 13, 14, 15])

(1) *For odd integer* $t \geq 3$*, there is a* 4*-GDD of type* $3^t(\frac{3(t-1)}{2})^1$.

(2) *For even integer* $t \geq 6$*, there is a* $\{4,7\}$*-GDD of type* $3^t(\frac{3(t-2)}{2})^1$*, in which precisely one point of the long group belongs to blocks of size* 7*. Further this point does not belong to any block of size* 4 *if* $t \geq 8$.

(3) *There is a* 4*-GDD of type* 3^5.

(4) *For* $(t,m,k) = (4,6,3),(6,8,1)$*, there is a* $\{3,4\}$*-GDD of type* $3^t m^1$*, in which precisely* k *points of the long group belong to the blocks of size* 3.

The following three lemmas are all presented by utilizing the Weighting Construction. So we only point out the initial K-GDDs (all coming from Lemma 5.1), the weight function, and the input designs in the proof.

Lemma 5.2 *The Main Theorem holds for any admissible triple* (g,t,u) *with* $g \equiv 0 \pmod 6$ *and* $t \equiv 1 \pmod 2$.

Proof Let $g = 6x$ where $x \geq 1$. Start from a 4-GDD of type $3^t(\frac{3(t-1)}{2})^1$ with a long group $Y = \{y_1, y_2, \ldots, y_{3(t-1)/2}\}$ Then give even weight w_i between 0 and $4x$ to each point y_i of Y such that $u = \sum_{i=1}^{3(t-1)/2} w_i$. Next give weight $2x$ to any other point. By Lemma 1.3 and Corollary 4.2, for even $0 \leq w \leq 4x$ there is a pair of disjoint 3-GDDs of type $(2x)^3 w^1$. So the conclusion follows by the Weighting Construction. □

Lemma 5.3 *The Main Theorem holds for any admissible triple* (g,t,u) *with* $g \equiv 0 \pmod 6$*,* $t \equiv 0 \pmod 2$*, and* $t \geq 8$.

Proof Let $g = 6x$ where $x \geq 1$. Start from a $\{4,7\}$-GDD of type $3^t(\frac{3(t-2)}{2})^1$ with a long group $Y = \{y_1, y_2, \ldots, y_{3(t-2)/2}\}$, where only one point y_1 of Y belongs to the block of size 7, and y_1 does not belong to any block of size 4. We give y_1 weight $w_1 = 0$ or $10x$, give each $y_i \in Y$ with $i \geq 2$ even weight w_i, $0 \leq w_i \leq 4x$, such that $u = \sum_{i=1}^{3(t-2)/2} w_i$, and give each point not in Y weight $2x$. Since two disjoint 3-GDDs of type $(2x)^3 w^1$ (w even, $0 \leq w \leq 4x$), or $(2x)^6 v^1$ ($v = 0, 10x$) exist by Lemma 1.3, Corollaries 3.4 and 4.2, a pair of disjoint 3-GDDs of type $g^t u^1$ is obtained. □

Lemma 5.4 *The Main Theorem holds for any admissible triple* (g,t,u) *with* $g \equiv 0 \pmod 6$ *and* $t = 4, 6$.

Proof Let $g = 6x$ where $x \geq 1$. Set $(m,k) = (6,3)$ if $t = 4$ and $(m,k) = (8,1)$ if $t = 6$.

First we handle even u with $2kx \leq u \leq g(t-1)$. Start from a $\{3,4\}$-GDD of type $3^t m^1$ with a long group $Y = \{y_1, y_2, \ldots, y_m\}$, in which precisely k points $y_1, y_2, \ldots, y_k$ belong to the blocks of size 3. Give each y_i with $1 \leq i \leq k$ weight $2x$ and each y_i with $k+1 \leq i \leq m$ even weight $w_i, 0 \leq w_i \leq 4x$ such that $u = 2kx + \sum_{i=k+1}^{m} w_i$. Then weight $2x$ to every point not in Y. Since a pair of disjoint 3-GDDs of type $(2x)^3 w^1$ (w

even, $0 \le w \le 4x$) exists by Lemma 1.3 and Corollary 4.2, there is a pair of disjoint 3-GDDs of type $g^t u^1$ by the Weighting Construction.

Next we consider even u with $u < 2kx = 6x$ for $t = 4$. Start from a 4-GDD of type 3^5 with groups G_i, $1 \le i \le 5$, where $G_5 = \{y_1, y_2, y_3\}$. Weight $2x$ to each point of G_i with $1 \le i \le 4$ and weight even weight $w_j, 0 \le w_j \le 4x$, to each point y_j of G_5 such that $u = \sum_{j=1}^{3} w_j$. Utilize a pair of disjoint 3-GDDs of type $(2x)^4$ or $(2x)^3 w^1$ for even $0 \le w \le 4x$ and then obtain a pair of disjoint 3-GDDs of type $g^t u^1$ similarly.

Finally let u be even with $u < 2kx = 2x$ for $t = 6$. Start from a $\{4, 7\}$-GDD of type $3^6 6^1$ with a long group $Y = \{y_1, y_2, \ldots, y_6\}$, in which precisely one point y_1 in Y belongs to blocks of size 7. Assign y_i with $1 \le i \le 5$ weight 0, y_6 weight u, and each point of the group of size 3 weight $2x$. Utilize disjoint pairs of 3-GDDs of types $(2x)^s$ ($s = 3, 4, 6$) and $(2x)^3 u^1$ and then obtain a pair of disjoint 3-GDDs of type $(6x)^t u^1$. This completes the proof. □

We summarize the above results on $g \equiv 0 \pmod 6$ in a corollary.

Corollary 5.5 *The Main Theorem holds for any admissible triple* (g, t, u) *with* $g \equiv 0 \pmod 6$*.*

Then the solutions for $g = 2, 3, 4$ are ready-made.

Lemma 5.6 *The Main Theorem holds for any admissible triple* $(3, t, u)$*.*

Proof Since $(3, t, u)$ is admissible, t is even with $t \ge 4$, u is odd with $u \ne 3$, and $1 \le u \le 3(t-1)$. If $u \ge 5$ and $t \ge 6$, then by Corollary 5.5 there is a pair of disjoint 3-GDDs of type $6^{t/2}(u-3)^1$. Apply Corollary 2.3 to yield a pair of disjoint 3-GDDs of type $3^t u^1$.

If $t = 4$, then $u = 1, 5, 7, 9$. A pair of disjoint 3-GDDs of type $3^4 9^1$ exists by Corollary 3.4. The solutions for $u = 1, 5, 7$ are listed in the appendix.

For $u = 1$ and $t = 6, 8$, let $X = I_3 \times I_t$ and $\mathcal{G} = \{I_3 \times \{i\} : i \in I_t\} \cup \{\infty\}$. First construct on each $\{j\} \times I_t$ ($j \in I_3$) a pair of disjoint 3-GDDs of type 1^{t+1}. Then form a pair of disjoint ITD$(3, t)$s and delete their idempotent parallel class. Thus a pair of disjoint 3-GDDs of type $3^t 1^1$ is obtained.

For $u = 1$ and even t with $t \ge 10$, there are pairs of disjoint 3-GDDs of types $3^{t-4} 13^1$ and $3^4 1^1$ by the above arguments. Consequently a pair of disjoint 3-GDDs of types of $3^t 1^1$ is produced by Filling Construction II. □

Lemma 5.7 *The Main Theorem holds for any admissible triple* $(4, t, u)$*.*

Proof Note that $(4, t, u)$ is an admissible triple requires that $2 \le u \le 4(t-1)$, $u \ne 4$, $t \equiv 0 \pmod 3$ and $u \equiv 0 \pmod 2$, or $t \equiv 1 \pmod 3$ and $u \equiv 0 \pmod 6$, or $t \equiv 2 \pmod 3$ and $u \equiv 4 \pmod 6$.

Firstly, when $t \equiv 1 \pmod 3$ and $u \equiv 0 \pmod 6$, or $t \equiv 2 \pmod 3$ and $u \equiv 4 \pmod 6$, or $t \equiv 0 \pmod 3$ and $u \equiv 2 \pmod 6$, let $D = \{1, 2, \ldots, 2t-1\} \setminus \{t\}$. By Lemma 3.3, it suffices to show that D can be partitioned into a set D_1 of $(4t-4-u)/6$ difference triples and a set D_2 containing a good difference in Z_{4t}. This has been done in Section 4 of [16].

Secondly, let $t \equiv 0 \pmod 3$, $u \equiv 0, 4 \pmod 6$, $u \geq 6$, and $t \geq 9$. By Corollary 5.5 there is a pair of disjoint 3-GDDs of type $12^{t/3}(u-4)^1$. A pair of disjoint 3-GDDs of type 4^4 also exists by Lemma 1.3. Apply Filling Construction I to produce a pair of disjoint 3-GDDs of type $4^t u^1$.

Finally, we only need to handle $t = 3, 6$, $u \equiv 0, 4 \pmod 6$, and $u \geq 6$. The case $t = 3$ is solved by Corollary 4.2. There is a pair of disjoint 3-GDDs of type $8^3(u-4)^1$, so by Corollary 2.3, there exists a pair of disjoint 3-GDD of type $4^6 u^1$. □

Lemma 5.8 *The Main Theorem holds for any admissible triple* $(2, t, u)$.

Proof By Lemma 3.10, we only need to deal with the admissible triples $(2, t, u)$ with $t \equiv 0 \pmod 3$ and even u with $4 \leq u \leq 2(t-1)$. If $t \equiv 3 \pmod 6$, a pair of disjoint 3-GDDs of type $2^t u^1$ is obtained by Corollary 4.2. Otherwise, $t \equiv 0 \pmod 6$. There exists by Lemma 5.7 a pair of disjoint 3-GDDs of type $4^{t/2}(u-2)^1$. Then the conclusion follows by Corollary 2.3. □

To conclude this section we prove that the necessary conditions of the existence of two disjoint 3-GDDs of type $g^t u^1$ for $g \equiv 3 \pmod 6$ are also sufficient.

Lemma 5.9 *The Main Theorem holds for any admissible triple* (g, t, u) *with* $g \equiv 3 \pmod 6$.

Proof Since $g \equiv 3 \pmod 6$ and (g, t, u) is admissible, t must be even with $t \geq 4$, u be odd, and $u \leq g(t-1)$. Let $(X, \mathcal{A})$ be a KTS(g), where $\mathcal{A}$ can be resolved into $(g-1)/2$ parallel classes $P_1, P_2, \ldots, P_{(g-1)/2}$. Choose integers u_i, $1 \leq i \leq (g-1)/2$, such that u_1 is odd, $1 \leq u_1 \leq 3(t-1)$ and for each $2 \leq i \leq (g-1)/2$, u_i is even, $0 \leq u_i \leq 2(t-1)$. Let $U_1, U_2, \ldots, U_{(g-1)/2}$ be pairwise disjoint sets with $|U_i| = u_i$ and let $U = \cup_{i=1}^{(g-1)/2} U_i$. The desired two disjoint 3-GDDs will be constructed on the set $Y = (X \times I_t) \cup U$ with group set $\mathcal{G} = \{X \times \{i\} : i \in I_t\} \cup \{U\}$.

For each block $B = \{x, y, z\} \in P_1$, there is a pair of disjoint 3-GDDs $(X_B, \mathcal{G}_B, \mathcal{A}_B^1)$ and $(X_B, \mathcal{G}_B, \mathcal{A}_B^2)$ of type $3^t {u_1}^1$ by Lemmas 1.3 and 5.6, where $X_B = (B \times I_t) \cup U_1$ and $\mathcal{G}_B = \{B \times \{i\} : i \in I_t\} \cup \{U_1\}$.

For each block $B = \{x, y, z\} \in P_i$, $2 \leq i \leq (g-1)/2$, there is a pair of 3-GDDs of type $t^3 {u_i}^1$ with no block in common but a common parallel class $P = \{B \times \{i\} : i \in I_t\}$ of $B \times I_t$ by Corollary 4.4. Deleting the common parallel class P yields two disjoint block sets $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$.

For $i = 1, 2$, let $\mathcal{B}_i = \cup_{B \in P_j, 1 \leq j \leq (g-1)/2} \mathcal{A}_B^i$. It can be checked that $(Y, \mathcal{G}, \mathcal{B}_1)$ and $(Y, \mathcal{G}, \mathcal{B}_2)$ form a pair of disjoint 3-GDDs of type $g^t u^1$. □

6 Further constructions

In this section, we shall go a step further to employ cyclic partial S$(2, 3, v)$s to construct a pair of disjoint 3-GDDs.

Lemma 6.1 *Suppose that g is an even integer and there is a cyclic partial $S(2,3,g)$ which contains a starter block having a good difference and whose leave is r-regular. Let $t \geq 4$ and $t \neq 6, 10$, $0 \leq m \leq t-1$, and $0 \leq v \leq 2(t-1)$ such that a pair of disjoint 3-GDDs of type $2^t v^1$ exists. Then there is a pair of disjoint 3-GDDs of type $g^t((r-1)(t-1)+6m+v)^1$.*

Proof Let $G = \{\infty_1, \infty_2, \ldots, \infty_v\}$, $X = (Z_g \times I_t) \cup G$, and $\mathcal{G} = \{Z_g \times \{i\} : i \in I_t\} \cup \{G\}$. For $D \subseteq Z_g$, $x \in Z_g$, denote $D + x = \{d + x : d \in D\}$ and $dev(D) = \{D + x : x \in Z_g\}$. For $\Omega \subseteq Z_g \times I_t$, $x \in Z_g$, denote $\Omega + x = \{(d+x, i) : (d,i) \in \Omega\}$ and $dev(\Omega) = \{\Omega + x : x \in Z_g\}$.

Let $S_1, S_2, \ldots, S_n$ be the starter blocks of a cyclic partial S$(2,3,g)$ on Z_g, whose r-regular leave is L. Further suppose that S_1 contains a good difference. Clearly, $g/2$ appears as a difference in L but not in S_1. Let $L_1 = \bigcup_{\{a,b\} \subseteq S_1} dev(\{a,b\})$. By Lemma 3.2 and noting that S_1 contains a good difference, L has a 1-factorization with 1-factors $F_1, F_2, \ldots, F_r$ and L_1 has also a 1-factorization with $H_1, H_2, \ldots, H_6$, as 1-factors.

First for each pair $P \in F_1$, we can construct by the assumption on $(P \times I_t) \cup G$ a pair of disjoint 3-GDDs of type $2^t v^1$ with group set $\{P \times \{i\} : i \in I_t\} \cup \{G\}$ and two disjoint block sets $\mathcal{C}_P^0$ and $\mathcal{C}_P^1$. Set $\mathcal{C}^s = \bigcup_{P \in F_1} \mathcal{C}_P^s$ for $s = 0, 1$. (The other 1-factors are left for later use.)

Next we employ the starter block S_1. By Lemma 1.2, for $t \geq 4$ and $t \neq 6, 10$, there is a pair of disjoint RITD$(3,t)$s on $S_1 \times I_t$ with group set $\{\{x\} \times I_t : x \in S_1\}$. Let $P_0^s, P_1^s, \ldots, P_{t-1}^s$ $(s = 0,1)$ be their parallel classes, where P_0^s be the idempotent one. By deleting $m+1$ parallel classes, P_k^s, $0 \leq k \leq m$, we obtain two disjoint partial 3-GDDs with block sets $\mathcal{B}_1^0$ and $\mathcal{B}_1^1$.

Then we employ the starter block S_i $(i \neq 1)$. For each $2 \leq i \leq n$, construct on $S_i \times I_t$ two disjoint ITD$(3,t)$s with group set $\{\{x\} \times I_t : x \in S_i\}$. Delete the idempotent parallel class to form two disjoint block sets $\mathcal{B}_i^0$ and $\mathcal{B}_i^1$.

After that, for $s = 0, 1$, define $\mathcal{B}^s = \bigcup_{1 \leq i \leq n} dev(\mathcal{B}_i^s)$ and $\mathcal{A}^s = \mathcal{B}^s \cup \mathcal{C}^s$. One can check that $(X, \mathcal{G}, \mathcal{A}^0)$ and $(X, \mathcal{G}, \mathcal{A}^1)$ form two disjoint partial 3-GDDs of type $g^t v^1$ with leaves $\mathcal{L}^0$ and $\mathcal{L}^1$. If $(r-1)(t-1)+6m = 0$, then $\mathcal{L}^s$ is empty and we do have obtained a pair of disjoint 3-GDDs of type $g^t((r-1)(t-1)+6m+v)^1$. So we assume that $r \geq 2$ or $m \geq 1$. By the previous construction, for $s = 0, 1$, $\mathcal{L}^s$ consists of two parts $\mathcal{L}_1^s$ and $\mathcal{L}_2^s$, where $\mathcal{L}_1^0 = \mathcal{L}_1^1 = \{\{(a,i),(b,j)\} : \{a,b\} \in L \setminus F_1, i \neq j \in I_t\}$, and $\mathcal{L}_2^s$ contains all the pairs in $\bigcup_{k=1}^m dev(P_k^s)$.

Finally we partition each $\mathcal{L}^s$ into $(r-1)(t-1)+6m$ disjoint 1-factors of $Z_g \times I_t$ to complete the proof. For $\{a,b\} \in L \setminus F_1$ and $1 \leq i \leq t-1$, take $f_{ab}^i = \{\{(a,j),(b,j+i)\} : 0 \leq j \leq t-1\}$. Then we have $t-1$ disjoint 1-factors of $\{a,b\} \times I_t$. For $\{a,b\} \in L_1$ and $Q = dev(P_k^s)$ $(1 \leq k \leq m$ and $s = 0,1)$, take $f_{ab}^Q = \{\{(a,l),(b,u)\} : \{(a,l),(b,u),(c,w)\} \in Q\}$. Thus we have m disjoint 1-factors of $\{a,b\} \times I_t$ for each $s = 0, 1$, which for convenience we also denote in sequence by $f_{ab}^{s1}, f_{ab}^{s2}, \ldots, f_{ab}^{sm}$. Define

$$D_{ij} = \bigcup_{\{a,b\} \in F_j} \{\{\alpha,\beta\} : \{\alpha,\beta\} \in f_{ab}^i\}, \text{where } 1 \leq i \leq t-1 \text{ and } 2 \leq j \leq r,$$

$$E_{kl}^s = \bigcup_{\{a,b\} \in H_l} \{\{\alpha,\beta\} : \{\alpha,\beta\} \in f_{ab}^{sk}\}, \text{where } 1 \leq k \leq m \text{ and } 1 \leq l \leq 6.$$

It is readily checked that the union of these D_{ij}'s and E^s_{kl}'s equals $\mathcal{L}^s$, forming $(r-1)(t-1)+6m$ disjoint 1-factors of $Z_g \times I_t$. Obviously the number of these 1-factors is greater than 2 when $t \geq 4$ and $r \geq 2$ or $m \geq 1$, so we can arrange them such that Lemma 3.1 can be applied to form a pair of disjoint 3-GDDs of type $g^t((r-1)(t-1)+6m+v)^1$. $\square$

For any integer $g \geq 2$, there is a trivial cyclic $\mathrm{S}(2,3,g)$ (with no starter block) whose leave is $(g-1)$-regular. Then in a similar but simpler procedure than the proof of Lemma 6.1, we have an analogous result (the details of the proof are omitted).

Lemma 6.2 *Suppose that g is an even integer. Let $t \geq 4$, $t \neq 6, 10$, $0 \leq m \leq t-1$, and $0 \leq v \leq 2(t-1)$ such that a pair of disjoint 3-GDDs of type $2^t v^1$ exists. Then there is a pair of disjoint 3-GDDs of type $g^t((g-2)(t-1)+v)^1$.*

Lemma 6.3 ([18]) *Suppose that Γ is an abelian group of even order and $S \subseteq \Gamma \backslash \{0\}$. Let $G(\Gamma, S)$ be the graph with vertex set Γ and whose edge set is $\{\{x, x+s\} : x \in \Gamma, s \in S\}$. Then $G(\Gamma, S)$ has a 1-factorization whenever it is connected.*

Lemma 6.4 *Suppose that there is a cyclic partial $S(2,3,g)$ whose leave is r-regular with $r < g-1$. Let $t \geq 4$ be even, $0 \leq m \leq t-1$, and $1 \leq v \leq t-1$ such that a pair of disjoint 3-GDDs of type $1^t v^1$ exists. Then there is a pair of disjoint 3-GDDs of type $g^t(r(t-1)+6m+v)^1$.*

Proof Let $G = \{\infty_1, \infty_2, \ldots, \infty_v\}$, $X = (Z_g \times I_t) \cup G$, and $\mathcal{G} = \{Z_g \times \{i\} : i \in I_t\} \cup \{G\}$. We first construct two disjoint partial 3-GDDs of type $g^t v^1$ on X with group set $\mathcal{G}$. Then we partition their leaves into $r(t-1)+6m$ disjoint 1-factors. For $D \subseteq Z_g$, $\Omega \subseteq Z_g \times I_t$, and $x \in Z_g$, we use the notations $D + x$, $\Omega + x$, $dev(D)$, and $dev(\Omega)$ as in Lemma 6.1.

By the assumption, for each $i \in Z_g$, there is a pair of 3-GDDs of type $1^t v^1$ on $(\{i\} \times I_t) \cup G$ with G as the long group and disjoint block sets $\mathcal{D}^0_i$ and $\mathcal{D}^1_i$. For $s = 0, 1$, set $\mathcal{D}^s = \cup_{i \in Z_g} \mathcal{D}^s_i$.

Let $S_1, S_2, \ldots, S_n$ be the starter blocks of the cyclic partial $\mathrm{S}(2,3,g)$ on Z_g, whose r-regular leave is L. For each $2 \leq i \leq n$, construct on $S_i \times I_t$ two disjoint ITD$(3,t)$s with group set $\{\{x\} \times I_t : x \in S_i\}$ and delete the idempotent parallel class to form two disjoint block sets $\mathcal{C}^0_i$ and $\mathcal{C}^1_i$.

Next we handle S_1. Let $S_1 = \{a, b, c\}$. If $m = 0$, we deal with S_1 as S_i. So suppose $m \geq 1$. For $t \geq 6$ and $t \neq 12$, there is an RITD$(3, t/2)$ on $S_1 \times \{2k : 0 \leq k \leq t/2 - 1\}$ with group set $\{\{x\} \times \{2k : 0 \leq k \leq t/2-1\} : x \in S_1\}$ and $t/2$ parallel classes $P_1, P_2, \ldots, P_{t/2}$, where $P_1 = \{S_1 \times \{2k\} : 0 \leq k \leq t/2-1\}$. Define $M = (t-m+1)/2$ if m is odd, or $M = (t-m+2)/2$ if m is even. We proceed with M parallel classes as follows:

Take any block $B = \{(a, 2i), (b, 2j), (c, 2k)\} \in P_l$, $l = 1$ if m is odd, or $l = 1, 2$ if m is even. For $s = 0, 1$, form a partial 3-GDD of type 2^3 with group set $\{\{a\} \times \{2i + 2s, 2i+2s+1\}, \{b\} \times \{2j, 2j+1\}, \{c\} \times \{2k, 2k+1\}\}$ and block set $\mathcal{A}^s_B$, where

$$\mathcal{A}^s_B = \{\{(a, 2i+2s), (b, 2j), (c, 2k)\}, \{(a, 2i+2s+1), (b, 2j+1), (c, 2k+1)\}\}, \qquad (1)$$

and the second components are modulo t.

For any block $B = \{(a, 2i), (b, 2j), (c, 2k) \in P_l$, $2 \le l \le M$ if m is odd, or $3 \le l \le M$ if m is even, take a 3-GDD with group set $\{\{a\} \times \{2i + 2s, 2i + 2s + 1\}, \{b\} \times \{2j, 2j + 1\}, \{c\} \times \{2k, 2k + 1\}\}$ and block set $\mathcal{A}_B^s$, where $s = 0, 1$.

For $s = 0, 1$, define $\mathcal{C}_1^s = \bigcup_{B \in P_l, 1 \le l \le M} \{dev(A) : A \in \mathcal{A}_B^s\}$. Then by defining $\mathcal{C}^s = \bigcup_{i=1}^n \mathcal{C}_i^s$ and $\mathcal{B}^s = \mathcal{D}^s \bigcup \mathcal{C}^s$, we produce two disjoint partial 3-GDDs of type $g^t v^1$ $(X, \mathcal{G}, \mathcal{B}^0)$ and $(X, \mathcal{G}, \mathcal{B}^1)$. Denote their leaves by $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. By the construction, $\mathcal{L}_s$ $(s = 0, 1)$ consists of at most three parts. We partition the pairs in the leave into $r(t-1) + 6m$ disjoint 1-factors of $Z_g \times I_t$ to complete the proof for $t \ge 6$ and $t \ne 12$.

Part I: For $s = 0, 1$, $l = 1$ if m is odd, or $l = 1, 2$ if m is even, observe that we take a partial 3-GDD as in the expression (1) for each block $B = \{(a, 2i), (b, 2j), (c, 2k)\}$ of P_l, leading to the leave $\mathcal{L}_l^s = \mathcal{L}_{l0}^s \cup \mathcal{L}_{l1}^s \cup \mathcal{L}_{l2}^s$ with

$$\begin{aligned}
\mathcal{L}_{l0}^s &= \bigcup_{B \in P_l} (dev(\{(a, 2i + 2s), (b, 2j + 1)\}) \cup dev(\{(a, 2i + 2s + 1), (b, 2j)\})),\\
\mathcal{L}_{l1}^s &= \bigcup_{B \in P_l} (dev(\{(a, 2i + 2s), (c, 2k + 1)\}) \cup dev(\{(a, 2i + 2s + 1), (c, 2k)\})),\\
\mathcal{L}_{l2}^s &= \bigcup_{B \in P_l} (dev(\{(b, 2j), (c, 2k + 1)\}) \cup dev(\{(b, 2j + 1), (c, 2k)\})).
\end{aligned}$$

Observe that the second components of each pair in $\mathcal{L}_{li}^s$ $(i = 0, 1, 2)$ are not equivalent modulo 2. So the graph $\mathcal{L}_{li}^s$ consists of some cycles of even length. Thus each cycle has a 1-factorization with two 1-factors. By collecting the 1-factors corresponding to all the connected cycles of $\mathcal{L}_{li}^s$, we obtain two 1-factors of $Z_g \times I_t$, say $F_{l,2i}^s$ and $F_{l,2i+1}^s$. Furthermore, $F_{l,p}^0 \cap F_{l,p+2}^1 = \emptyset$, where $p \in I_6$ and $p + 2$ is reduced to I_6. Now for fixed s we have six 1-factors of $Z_g \times I_t$ for odd m or twelve 1-factors for even m.

Part II: This part of leave exists only if $m \ge 3$. For $s = 0, 1$, and $M + 1 \le l \le t/2$, observe that we do not use any block in P_l, which leads to leave $\mathcal{L}_l^s$ described below. For each $B = \{(a, 2i), (b, 2j), (c, 2k)\} \in P_l$, $\mathcal{L}_l^s$ contains the pairs in the 2-GDD with group set $\{\{a\} \times \{2i + 2s, 2i + 2s + 1\}, \{b\} \times \{2j, 2j + 1\}, \{c\} \times \{2k, 2k + 1\}\}$. By similar arguments, $\mathcal{L}_l^s$ can be partitioned into twelve disjoint 1-factors of $Z_g \times I_t$ and we obtain K 1-factors altogether, say, $G_0^s, G_1^s, \ldots, G_{K-1}^s$, where $K = 6(m-1)$ for odd m or $K = 6(m-2)$ for even m. Furthermore, we can arrange them such that $G_i^0 \cap G_i^1 = \emptyset$ holds for all $0 \le i \le K - 1$.

Part III: This part of leave exists only if $r \ne 0$. We consider the leave L of the cyclic partial $\mathrm{S}(2, 3, g)$. Observe that $dev(P)$ is a 2-regular graph consisting of some cycles for any pair $P \in L$. For each connected component C, the set $\{\{(u, i), (w, j)\} : \{u, w\} \in C, i \ne j \in I_t\}$ can be 1-factorized by Lemma 6.3 (taking $\Gamma = \{(i \bmod |C|, i \bmod t) : 0 \le i \le \mathrm{lcm}(|C|, t)\}$ and $S = \{0\} \times (Z_t \setminus \{0\})$). Thus $r(t-1)$ 1-factors of $Z_g \times I_t$ are obtained when taking P all over the r-regular leave L. These 1-factors, $H_0, H_1, \ldots, H_{r(t-1)-1}$, are all contained in both $\mathcal{L}_0$ and $\mathcal{L}_1$ and certainly $H_i \cap H_{i+1} = \emptyset$.

So we obtain $r(t-1) + 6m$ disjoint 1-factors altogether. By Lemma 3.1, there is a pair of disjoint 3-GDDs of type $g^t(r(t-1) + 6m + v)^1$ for $t \ge 6$ and $t \ne 12$.

If $t = 4$, we can utilize on $S_1 \times I_4$ an RITD$(3, 4)$ with the idempotent parallel class omitted and further empty some parallel classes. If $t = 12$, we use on $S_1 \times \{3k : 0 \le k \le 3\}$ an RITD$(3, 4)$ with the idempotent parallel class omitted. And then deal with its four parallel classes by two ways. Choose appropriate number of parallel classes to construct for each $s = 0, 1$ an RTD(3,3) with groups $\{a\} \times \{3i+3s, 3i+3s+1, 3i+3s+2\}$, $\{b\} \times \{3j+3s, 3j+3s+1, 3j+3s+2\}$, and $\{c\} \times \{3k+3s, 3k+3s+1, 3k+3s+2\}$, where $\{(a, 3i), (b, 3j), (c, 3k)\}$ is any block of the chosen parallel classes. And for each block of the remaining parallel classes of the RITD(3,4), also take RTD(3,3) similarly but delete some parallel classes of this RTD. Then in a very similar way, a pair of disjoint 3-GDDs of type $g^t(r(t-1)+6m+v)^1$ is constructed. This completes the proof. □

Parallel to Lemma 6.2, the following result also holds.

Lemma 6.5 *Suppose that g is a positive integer. Let $t \geq 4$ be even, $0 \leq m \leq t-1$, and $1 \leq v \leq t-1$ such that a pair of disjoint 3-GDDs of type $1^t v^1$ exists. Then there is a pair of disjoint 3-GDDs of type $g^t((g-1)(t-1)+v)^1$.*

Lemma 6.6 *Let (g, t, u) be any admissible triple with $g > 5$ and $t \geq 4$. Then there exists a pair of disjoint 3-GDDs of type $g^t u^1$ whenever one of the following conditions meets:*

(1) $g \equiv 2, 8 \pmod{24}$ *if* $t \neq 6, 10$;

(2) $g \equiv 14, 20 \pmod{24}$ *and* $u \geq 6(t-1)$ *if* $t \neq 6, 10$;

(3) $g \equiv 4 \pmod 6$ *and* $u \geq 2(t-1)$ *if* $t \neq 6, 10$;

(4) $g \equiv 1 \pmod 6$;

(5) $g \equiv 5 \pmod 6$ *and* $u > 4(t-1)$;

(6) *If* $t = 6, 10$*, then* $u > t-1$ *for* $g \equiv 2, 8 \pmod{24}$*, or* $u > 7(t-1)$ *for* $g \equiv 14, 20 \pmod{24}$*, or* $u > 3(t-1)$ *for* $g \equiv 4 \pmod 6$.

Proof Suppose that $g = 6k + s$, where $k \geq 1$ and $1 \leq s \leq 6$. Let $r' = 7$ if $s = 2$ and $k \equiv 2, 3 \pmod 4$, or $r' = s - 1$ otherwise.

For any admissible (g, t, u) with $g \equiv 2, 4 \pmod 6$, $t \geq 4$, $t \neq 6, 10$, and $u \geq (r'-1)(t-1)$, first take $0 \leq x < 6$, $x \equiv u - (r'-1)(t-1) \pmod 6$ (x must be even) and next choose $r \equiv r' \pmod 6$ and $0 \leq u - (r-1)(t-1) - x = 6m \leq 6(t-1)$, then $u = (r-1)(t-1)+6m+x$ and $r \leq g-1$. By Lemma 3.5, there is a cyclic partial S$(2, 3, g)$ with an r-regular leave. Moreover, if $r < g-1$, there is a starter block containing a good difference. And we can check that $(2, t, x)$ is an admissible triple and then obtain a pair of disjoint 3-GDDs of type $2^t x^1$ by Lemma 5.8. Consequently there is a pair of disjoint 3-GDDs of type $g^t u^1$ by Lemma 6.1. If $r = g-1$, then $(2, t, 6m+x)$ is admissible and a pair of disjoint 3-GDDs of type $2^t(6m+x)^1$ also exists. So the conclusion follows by Lemma 6.2. This handles (1)-(3).

For any admissible (g, t, u) with $g \equiv 1, 5 \pmod 6$ (or $g \equiv 2, 4 \pmod 6$ and $t = 6, 10$) and $u > r'(t-1)$, first take $0 \leq x < 6$, $x \equiv u - r'(t-1) \pmod 6$ (x must be odd) and next choose $r \equiv r' \pmod 6$ and $0 \leq u - r(t-1) - x = 6m \leq 6(t-1)$, then $u = r(t-1) + 6m + x$ and $r \leq g-1$. By Lemma 3.5, there is a cyclic partial S$(2, 3, g)$ with an r-regular leave. It can be checked that $(1, t, x)$ (if $r < g-1$) or $(1, t, 6m+x)$ (if $r = g-1$) is an admissible triple, so there is a pair of disjoint 3-GDDs of type $1^t x^1$ or $1^t(6m+x)^1$ by Lemma 3.9. Consequently there is a pair of disjoint 3-GDDs of type $g^t u^1$ by Lemma 6.4 or 6.5. This proves (4)-(6). □

7 The case $g \equiv 2, 4 \pmod 6$

We handle the remaining cases when $g \equiv 2, 4 \pmod 6$ in this section.

Lemma 7.1 *The Main Theorem holds for any admissible triple* (g, t, u) *with* $g \equiv 4 \pmod 6$.

Proof By Lemma 6.6, we need only to consider admissible triples with $u < 2(t-1)$ if $t \neq 6, 10$ and $u \leq 3(t-1)$ if $t = 6, 10$. Let $g = 6n + 4$. The case $n = 0$ or $t = 3$ is solved by Lemma 5.7 and Corollary 4.2 respectively. So suppose that $n \geq 1$ and $t \geq 4$. Since (g, t, u) is admissible, either $u \equiv 0 \pmod 2$ if $t \equiv 0 \pmod 3$, or $u \equiv 0 \pmod 6$ if $t \equiv 1 \pmod 3$, or $u \equiv 4 \pmod 6$ if $t \equiv 2 \pmod 3$. We distinguish all the possible cases.

Case 1: $n \geq 3$ and $u \leq 3(t-1)$. There is a 3-GDD of type $6^n 4^1$ by Lemma 1.1. There are pairs of disjoint 3-GDDs of types $6^t u^1$ and $4^t u^1$ by Corollary 5.5 and Lemma 5.7. So a pair of disjoint 3-GDDs of type $(6n+4)^t u^1$ is obtained by Construction 2.5.

Case 2: $n = 2$ and $u \leq 3(t-1)$. There is a 3-GDD of type 4^4 by Lemma 1.1. There is a pair of disjoint 3-GDDs of type $4^t u^1$ by Lemma 5.7. So there exists a pair of disjoint 3-GDDs of type $16^t u^1$ by Construction 2.5.

Case 3: $n = 1$, $t \equiv 2 \pmod 3$, and $u < 2(t-1)$. Then $g = 10$ and $u \equiv 4 \pmod 6$. First Lemma 3.6 solves such cases with $u \geq 2g + 2 = 22$, leaving $u = 4$ if $t \leq 8$ or $u = 4, 16$ if $t \geq 11$ to be settled. Next utilize Lemma 3.3 to deal with $t = 5$ and $u = 4$ by taking on Z_{50} the difference triples $\{1, 23, 24\}$, $\{4, 18, 22\}$, $\{6, 7, 13\}$, $\{8, 11, 19\}$, $\{9, 12, 21\}$ and $\{2, 14, 16\}$. Finally for $t = 8$ and $u = 4$, or $t \geq 11$ and $u = 4, 16$, the Filling Construction II works by filling a pair of disjoint 3-GDDs of type $10^{t-3}(30+u)^1$ with such pair of type $10^3 u^1$.

Case 4: $n = 1$, $t \equiv 0, 1 \pmod 3$, and $u < 2(t-1)$. There is a 3-GDD of type $2^3 4^1$ and disjoint pairs of 3-GDDs of types $2^t u^1$ and $4^t u^1$ exist by Lemmas 5.7 and 5.8. So we produce a pair of disjoint 3-GDDs of type $10^t u^1$ by Construction 2.5.

Case 5: $n = 1$, $t = 6, 10$, and $2(t-1) \leq u \leq 3(t-1)$. Then $u \geq 10$ if $t = 6$. So there exists a pair of disjoint 3-GDDs of type $10^6 u^1$ by Corollary 2.3 since there is a pair of disjoint 3-GDDs of type $20^3(u-10)^1$ by Corollary 4.2. If $t = 10$, then $u = 18, 24$. Thus a pair of disjoint 3-GDDs of type $10^{10} u^1$ exists by Lemma 3.6. □

Lemma 7.2 *The Main Theorem holds for any admissible triple* (g, t, u) *with* $g = 14, 20$.

Proof For $g = 14, 20$, the case $t \equiv 3 \pmod 6$ has been solved by Corollary 4.2, so let $t \not\equiv 3 \pmod 6$. If $t \geq 6$ is even and $u > g$, a pair of disjoint 3-GDDs of type $g^t u^1$ can be obtained by Corollary 2.3 since a pair of disjoint 3-GDDs of type $(2g)^{t/2}(u-g)^1$ exists by Lemma 7.1. Thus by Lemma 6.6 we need only to consider $u < g$ if $t \geq 6$ is even and $u < 6(t-1)$ if $t = 4$ or $t \equiv 1, 5 \pmod 6$. Since (g, t, u) is admissible, either $u \equiv 0 \pmod 2$ if $t \equiv 0 \pmod 3$, or $u \equiv 0 \pmod 6$ if $t \equiv 1 \pmod 3$, or $u \equiv 2 \pmod 6$ if $t \equiv 2 \pmod 3$.

(1) $g = 14$.

Case 1: $t \geq 5$ and $u < 14$. Then $u \leq 2(t-1)$ (noting that (g, t, u) is admissible) and there exist a 3-GDD of type 2^7 and a pair of disjoint 3-GDDs of type $2^t u^1$ by Lemma

5.8, yielding a pair of disjoint 3-GDDs of type $14^t u^1$ by Construction 2.5.

Case 2: $t = 4, 6, 7$ and $u < 6(t-1)$, or $t = 5$ and $14 \le u < 6(t-1) = 24$. Employ the Weighting Construction. Start from a TD$(t+1, 7)$. Assign weight 2 to each point of the first t groups and then assign appropriate weight w to the point of the last group, where $w \equiv 0 \pmod 2$ if $t = 6$, or $w \equiv 0 \pmod 6$ if $t \in \{4, 7\}$, or $w \equiv 2 \pmod 6$ if $t = 5$.

Case 3: $t \equiv 1, 5 \pmod 6$, $t \ge 9$, and $u < 6(t-1)$. First Lemma 3.6 solves such cases with $u \ge 2g + 2 = 30$, leaving $u \le 28$ to be settled. Then fill a pair of disjoint 3-GDDs of type $14^3 u^1$ in that of type $14^{t-3}(42+u)^1$ to obtain a pair of disjoint 3-GDDs of type $14^t u^1$.

(2) $g = 20$.

Case 1: $t \equiv 1, 5 \pmod 6$, $t \ge 11$ and $u < 6(t-1)$. Similarly Lemma 3.6 solves such cases with $u \ge 2g + 2 = 42$. For $u \le 40$, fill in the long group of a pair of disjoint 3-GDDs of type $20^{t-3}(60+u)^1$ with that of type $20^3 u^1$ to produce the desired pair of type $20^t u^1$.

Case 2: even $t \ge 10$ and $u < 20$, or $t = 5$ and $u < 6(t-1) = 24$. If $t = 5$ and $u = 14$, employ Lemma 3.3 on Z_{100} by taking difference triples $\{1, 2, 3\}$, $\{4, 7, 11\}$, $\{6, 8, 14\}$, $\{9, 12, 21\}$, $\{13, 16, 29\}$, $\{17, 19, 36\}$, $\{18, 23, 41\}$, $\{22, 24, 46\}$, $\{26, 27, 47\}$, $\{28, 33, 39\}$, and $\{31, 32, 37\}$. If $t \ne 5$ or $u \ne 14$, then $u \le 2(t-1)$. So these cases can be solved similarly to the Case 1 of $g = 14$, using a 3-GDD of type 2^{10} instead of 2^7.

Case 3: $t = 4$ and $u < 6(t-1) = 18$, or $t = 6$ and $u < 20$. Then $u \le 4(t-1)$ and we can apply Construction 2.5 to a 3-GDD of type $4^3 8^1$. A pair of disjoint 3-GDDs of type $4^t u^1$ exist by Lemmas 5.7. If $t = 4$, or $t = 6$ and $u \ge 6$, a pair of disjoint 3-GDDs of type $8^t u^1$ exists by Lemma 6.6. And if $t = 6$ and $u = 2, 4$, a pair of disjoint 3-GDDs of types $8^t u^1$ also exists since a 3-GDD of type 2^4 and a pair of disjoint 3-GDDs of type $2^t u^1$ exist. Thus Construction 2.5 gives a pair of disjoint 3-GDDs of type $20^t u^1$.

Case 4: $t = 8$ and $u < 20$. Then $u = 2, 8, 14$. Similar to Case 1, fill in the long group of a pair of disjoint 3-GDDs of type $20^5(60+u)^1$ with that of type $20^3 u^1$ to produce the desired pair of type $20^8 u^1$.

Case 5: $t = 7$ and $u < 6(t-1) = 36$. Then $u = 6, 12, 18, 24, 30$. As in Case 3, we can handle $u \le 24$. The last case $u = 30$ is treated as follows.

Let $(X, \mathcal{G}, \mathcal{B})$ be a $\{2,3\}$-GDD of type 4^5, which is obtained by deleting a group of a 3-GDD of type 4^6. So the blocks of size 2 of $\mathcal{B}$ is partitioned into four parallel classes of X. Let $U = \{\infty_1, \infty_2, \ldots, \infty_6\}$, $Y = (X \times I_7) \cup U$, and $\mathcal{H} = \{X \times \{i\} : i \in I_7\} \cup \{U\}$. For each $B \in \mathcal{B}$ and $|B| = 3$, construct on $B \times I_7$ a pair of disjoint RITD(3,7)s (but deleting the idempotent parallel class) with group set $\{\{x\} \times I_7 : x \in B\}$ and block sets $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$. For each $G \in \mathcal{G}$, construct on $(G \times I_7) \cup U$ a pair of disjoint 3-GDDs of type $4^7 6^1$ with group set $\{\{x\} \times I_7 : x \in G\} \cup \{U\}$ and block sets $\mathcal{C}_G^1$ and $\mathcal{C}_G^2$. Set $\mathcal{C}^i = (\cup_{B \in \mathcal{B}, |B|=3} \mathcal{A}_B^i) \cup (\cup_{G \in \mathcal{G}} \mathcal{C}_G^i)$ where $i = 1, 2$. Then $(Y, \mathcal{H}, \mathcal{C}^1)$ and $(Y, \mathcal{H}, \mathcal{C}^2)$ form a pair of disjoint partial 3-GDDs of type $20^7 6^1$. Their common leave is $\{((x,i),(y,j)) : \{x,y\} \in \mathcal{B}, i, j \in I_7, i \ne j\}$. Noting that the pairs of $\mathcal{B}$ is partitioned into four parallel classes, we can partition the leave into $6 \times 4 = 24$ disjoint 1-factors of $X \times I_7$. Hence there is a pair of disjoint 3-GDDs of type $20^7 30^1$ by Lemma 3.1. $\square$

Lemma 7.3 *The Main Theorem holds for any admissible triple* (g, t, u) *with* $g \equiv 2$

(mod 6).

Proof By Lemmas 6.6 and 7.2, for $g \equiv 2, 8 \pmod{24}$, we need only to consider $t = 6, 10$ and $u \leq t-1$. For $g \equiv 14, 20 \pmod{24}$, we need only to consider $g \geq 38$ and $u < 6(t-1)$, further $u \leq 7(t-1)$ if $t = 6, 10$. The possible cases are listed as follows:

Case 1: $g \equiv 2, 8 \pmod{24}$, $t = 6, 10$, and $u \leq t-1$. Let $g = 6n+2$. The case $n = 0$ is solved by Lemma 5.8. So let $n \geq 1$. Since there are a 3-GDD of type 2^{3n+1} and a pair of disjoint 3-GDDs of type $2^t u^1$ by Lemmas 1.1 and 5.8, there is a pair of disjoint 3-GDDs of type $(6n+2)^t u^1$ by Construction 2.5.

Case 2: $g \equiv 14, 20 \pmod{24}$, $g \geq 38$, and $u < 6(t-1)$. Let $g = 6l+8$, where $l \geq 5$. There exists a pair of disjoint 3-GDDs of type $(6l+8)^t 8^1$ by Construction 2.5 since there are a 3-GDD of type $6^l 8^1$ and disjoint pairs of 3-GDDs of types $6^t u^1$ and $8^t u^1$ by Corollary 5.5 and Lemma 6.6 or Case 1 of the proof.

Case 3: $g \equiv 14 \pmod{24}$, $t = 6, 10$, and $6(t-1) \leq u \leq 7(t-1)$, where $m \geq 1$. Employ a 3-GDD of type $8^{3m} 14^1$ and disjoint pairs of 3-GDDs of types $8^t u^1$ and $14^t u^1$ (whose existence is assured by Case 1 and Lemma 7.2). Then we obtain a pair of disjoint 3-GDDs of type $(24m+14)^t u^1$.

Case 4: $g \equiv 20 \pmod{24}$, $t = 6, 10$, and $6(t-1) \leq u \leq 7(t-1)$. Let $g = 24k+20$, where $k \geq 1$. Employ a 3-GDD of type $8^{3k+1} 12^1$ and disjoint pairs of 3-GDDs of types $8^t u^1$ and $12^t u^1$ (Case 1 and Corollary 5.5). Then obtain a pair of disjoint 3-GDDs of type $(24k+20)^t u^1$. □

8 The case $g \equiv 5 \pmod 6$

We shall solve the existence problem of a pair of disjoint modified group divisible designs in this section. By doing so, the case $g \equiv 5 \pmod 6$ will be completed.

Let X be a finite set of gt points and K a set of positive integers. A *modified group divisible design* (introduced by Assaf in [3]) K-GDD is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of X into t g-subsets $G_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,g-1}\}$, $0 \leq i \leq t-1$. Each G_i is called a *group*. $\mathcal{H}$ is a partition of X into g t-subsets $H_j = \{x_{0,j}, x_{1,j}, \ldots, x_{t-1,j}\}$, $0 \leq j \leq g-1$. Each H_j is called a *hole*; (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality from K, such that a block contains no more than one point of any group and any hole; (3) every pair of points from distinct groups and distinct holes occurs in exactly one block. A modified group divisible design $\{3\}$-GDD with t groups and g holes is denoted by 3-MGDD(g, t). Notice that a 3-MGDD(g, t) can also be regarded as a 3-MGDD(t, g). The necessary conditions of the existence of a 3-MGDD(g, t) are $g, t \geq 3$, $(g-1)(t-1) \equiv 0 \pmod 2$, and $gt(g-1)(t-1) \equiv 0 \pmod 6$. Similarly, a pair of disjoint 3-MGDD(g, t)s means two 3-MGDD(g, t)s having same group set and hole set but disjoint block sets. A 3-MGDD$(3, t)$ is actually same as an ITD$(3, t)$. So there does not exist a pair of disjoint 3-MGDD(3,3)s. We shall show that it is the only exception.

Lemma 8.1 *Suppose that there exists a $(v, K, 1)$-PBD. If there exists a pair of disjoint 3-MGDD(g, k)s for any $k \in K$, then so does a pair of disjoint 3-MGDD(g, v)s.*

Proof Let $(X, \mathcal{B})$ be a $(v, K, 1)$-PBD, $\mathcal{G} = \{\{x\} \times I_g : x \in X\}$, and $\mathcal{H} = \{X \times \{i\} :$

$i \in I_g\}$. For any block $B \in \mathcal{B}$, construct a pair of disjoint 3-MGDD$(g, |B|)$s with group set $\mathcal{G}_B = \{\{x\} \times I_g : x \in B\}$, hole set $\mathcal{H}_B = \{B \times \{i\} : i \in I_g\}$, and disjoint block sets $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$. Define $\mathcal{A}^1 = \cup_{B\in\mathcal{B}}\mathcal{A}_B^1$ and $\mathcal{A}^2 = \cup_{B\in\mathcal{B}}\mathcal{A}_B^2$. Then it is immediate that $(X, \mathcal{G}, \mathcal{H}, \mathcal{A}^1)$ and $(X, \mathcal{G}, \mathcal{H}, \mathcal{A}^2)$ are two disjoint 3-MGDD(g, v)s. □

Lemma 8.2 ([2]) (1) *There exists a* $(v, \{3, 4, 6\}, 1)$*-PBD for any* $v \equiv 0, 1 \pmod 3$. (2) *There exists a* $(v, \{3, 5\}, 1)$*-PBD for any* $v \equiv 1 \pmod 2$.

Lemma 8.3 *For* $t = 4, 6$*, there exists a pair of disjoint* 3*-MGDD*$(5, t)$*s.*

Proof (1) Let $\mathcal{G} = \{\{i, i+1, i+2, i+3, i+4\} : i = 0, 5, 10, 15\}$ and $\mathcal{H} = \{\{j, j+5, j+10, j+15\} : j = 0, 1, 2, 3, 4\}$. We construct directly a pair of disjoint 3-MGDD$(5, 4)$s $(I_{20}, \mathcal{G}, \mathcal{H}, \mathcal{A}_1)$ and $(I_{20}, \mathcal{G}, \mathcal{H}, \mathcal{A}_2)$, where the blocks are listed below.

$\mathcal{A}_1$:	$\{0,6,12\}$	$\{0,7,11\}$	$\{0,8,16\}$	$\{0,9,17\}$	$\{0,13,19\}$	$\{0,14,18\}$
	$\{1,5,12\}$	$\{1,7,18\}$	$\{1,8,14\}$	$\{1,9,15\}$	$\{1,10,19\}$	$\{1,13,17\}$
	$\{2,5,13\}$	$\{2,6,19\}$	$\{2,8,15\}$	$\{2,9,10\}$	$\{2,11,18\}$	$\{2,14,16\}$
	$\{3,5,16\}$	$\{3,6,14\}$	$\{3,7,19\}$	$\{3,9,11\}$	$\{3,10,17\}$	$\{3,12,15\}$
	$\{4,5,18\}$	$\{4,6,17\}$	$\{4,7,13\}$	$\{4,8,10\}$	$\{4,11,15\}$	$\{4,12,16\}$
	$\{5,11,19\}$	$\{5,14,17\}$	$\{6,10,18\}$	$\{6,13,15\}$	$\{7,10,16\}$	$\{7,14,15\}$
	$\{8,11,17\}$	$\{8,12,19\}$	$\{9,12,18\}$	$\{9,13,16\}$		
$\mathcal{A}_2$:	$\{0,6,13\}$	$\{0,7,14\}$	$\{0,8,17\}$	$\{0,9,16\}$	$\{0,11,18\}$	$\{0,12,19\}$
	$\{1,5,19\}$	$\{1,7,15\}$	$\{1,8,12\}$	$\{1,9,13\}$	$\{1,10,17\}$	$\{1,14,18\}$
	$\{2,5,18\}$	$\{2,6,15\}$	$\{2,8,14\}$	$\{2,9,11\}$	$\{2,10,16\}$	$\{2,13,19\}$
	$\{3,5,11\}$	$\{3,6,19\}$	$\{3,7,10\}$	$\{3,9,17\}$	$\{3,12,16\}$	$\{3,14,15\}$
	$\{4,5,12\}$	$\{4,6,10\}$	$\{4,7,18\}$	$\{4,8,16\}$	$\{4,11,17\}$	$\{4,13,15\}$
	$\{5,13,17\}$	$\{5,14,16\}$	$\{6,12,18\}$	$\{6,14,17\}$	$\{7,11,19\}$	$\{7,13,16\}$
	$\{8,10,19\}$	$\{8,11,15\}$	$\{9,10,18\}$	$\{9,12,15\}$		

(2) Let $X = (Z_5 \times I_5) \cup \{\infty_i : i \in I_5\}$, $\mathcal{G} = \{\{x\} \times I_5 : x \in Z_5\} \cup \{\infty_i : i \in I_5\}$, and $\mathcal{H} = \{(Z_5 \times \{i\}) \cup \{\infty_i\} : i \in I_5\}$. A 3-MGDD$(5, 6)$ is constructed on X in [3] with group set $\mathcal{G}$, hole set $\mathcal{H}$ and block sets $\mathcal{B}_1$ developed under (mod 5, $-$) by the following blocks:

$\{(0,0),(1,1),(3,2)\}$	$\{(0,0),(1,2),(2,4)\}$	$\{(0,1),(3,2),(2,3)\}$
$\{(0,0),(3,1),(1,3)\}$	$\{(0,2),(1,3),(4,4)\}$	$\{(0,1),(1,2),(3,4)\}$
$\{(0,0),(2,3),(1,4)\}$	$\{(0,0),(2,2),(4,3)\}$	$\{(0,0),(2,1),(3,4)\}$
$\{(0,1),(1,3),(2,4)\}$	$\{(0,0),(4,1),\infty_0\}$	$\{(0,2),(3,3),\infty_0\}$
$\{(0,0),(4,2),\infty_1\}$	$\{(0,1),(4,4),\infty_1\}$	$\{(0,0),(3,3),\infty_2\}$
$\{(0,2),(3,4),\infty_2\}$	$\{(0,0),(4,4),\infty_3\}$	$\{(0,1),(4,3),\infty_3\}$
$\{(0,1),(4,2),\infty_4\}$	$\{(0,3),(2,4),\infty_4\}$	

Let $\mathcal{B}_2 = \{\{(x, a+1), (y, b+1), (z, c+1)\} : \{(x, a), (y, b), (z, c)\} \in \mathcal{B}_1\}$, where $\infty_i + 1 = \infty_i$ for $i \in I_5$. It is readily checked that $\mathcal{B}_1$ and $\mathcal{B}_2$ form block sets of two disjoint 3-MGDD$(5, 6)$s. □

Lemma 8.4 *There exists a pair of disjoint* 3*-MGDD*(g, t)*s for any one of the following parameters*:

(1) $g \geq 4$ *and* $t = 3$;

(2) $g \equiv 1, 3 \pmod 6$*,* $g \geq 4$ *and* $t = 4, 5, 6$;

(3) $g \equiv 0, 4 \pmod 6$*,* $g \geq 4$ *and* $t = 5$;

(4) $g \equiv 5 \pmod 6$*,* $g \geq 5$ *and* $t = 4, 6$.

Proof A pair of disjoint 3-MGDD(g,3)s with $g \geq 4$ exists by Lemma 1.2.

For $g \equiv 1, 3 \pmod 6$, $g \geq 4$ and $t = 4, 5, 6$, since there are an $\mathrm{S}(2, 3, g)$ and a pair of disjoint 3-MGDD(t,3)s, we obtain a pair of disjoint 3-MGDD(g,t)s by Lemma 8.1.

For $g \equiv 0, 4 \pmod 6$, there is a $(g, \{3, 4, 6\}, 1)$-PBD by Lemma 8.2. A pair of disjoint 3-MGDD(5,3)s exists by the above discussion. And a pair of disjoint 3-MGDD(5,4)s and a pair of disjoint 3-MGDD(5,6)s are given in Lemma 8.3. So we obtain a pair of disjoint 3-MGDD(g,5)s by Lemma 8.1.

For $g \equiv 5 \pmod 6$ and $t = 4, 6$, there is a $(g, \{3, 5\}, 1)$-PBD by Lemma 8.2. Utilize pairs of disjoint 3-MGDD(t,3)s and disjoint 3-MGDD(t,5)s. And then obtain a pair of disjoint 3-MGDD(g, t)s again by Lemma 8.1. □

Lemma 8.5 *Let* g *and* t *be positive integers satisfying* $g, t \geq 3$*,* $(g, t) \neq (3, 3)$*,* $(g-1)(t-1) \equiv 0 \pmod 2$ *and* $gt(g-1)(t-1) \equiv 0 \pmod 6$*. Then there exists a pair of disjoint* 3*-MGDD*(g, t)*s.*

Proof The conclusion follows by using Lemmas 8.1, 8.2 and 8.4. So we only point out the main ingredients. For $t \equiv 1, 3 \pmod 6$, $t \geq 3$ and $g \geq 4$, use an $\mathrm{S}(2, 3, t)$ and a pair disjoint 3-MGDD$(g, 3)$s. If $t \equiv 2 \pmod 6$, then $t \geq 8$, $g \geq 3$ and $g \equiv 1, 3 \pmod 6$. Use an $\mathrm{S}(2, 3, g)$ and a pair disjoint 3-MGDD$(t, 3)$s. If $t \equiv 5 \pmod 6$, then $g \equiv 0, 1 \pmod 3$ and $g \geq 4$. Use a $(t, \{3, 5\}, 1)$-PBD and a pair of disjoint 3-MGDD(g, s)s for $s = 3, 5$. If $t \equiv 0, 4 \pmod 6$, then $g \geq 3$ is odd. Use a $(t, \{3, 4, 6\}, 1)$-PBD and a pair of disjoint 3-MGDD(g, s)s for $s = 3, 4, 6$. □

The following lemmas deal with the admissible triples (g, t, u) with $g \equiv 5 \pmod 6$, so either $u \equiv 1 \pmod 2$ if $t \equiv 0 \pmod 6$, or $u \equiv 5 \pmod 6$ if $t \equiv 2 \pmod 6$, or $u \equiv 3 \pmod 6$ if $t \equiv 4 \pmod 6$.

Lemma 8.6 *Let* (g, t, u) *be any admissible triple with* $g \equiv 1, 5 \pmod 6$*,* $t \equiv 0, 4 \pmod 6$*,* $g \geq 5$*,* $t \geq 4$*, and* $u \leq t - 1$*. Then there exists a pair of disjoint* 3*-GDDs of type* $g^t u^1$*.*

Proof For $g \equiv 1, 5 \pmod 6$, $t \equiv 0, 4 \pmod 6$, $g \geq 5$, and $t \geq 4$, by Lemma 8.5 there is a pair of disjoint 3-MGDD(g, t)s on a gt-set X with group set $\mathcal{G}$, hole set $\mathcal{H}$ and disjoint block sets $\mathcal{A}_1$ and $\mathcal{A}_2$. Further $(1, t, u)$ is also an admissible triple. Let U be a u-set disjoint with X. For each $H \in \mathcal{H}$, construct on $H \cup U$ a pair of disjoint 3-GDDs of type $1^t u^1$ with U as the long group and $\mathcal{B}_H^1$ and $\mathcal{B}_H^2$ as the block sets. For $i = 1, 2$, let $\mathcal{C}_i = \mathcal{A}_i \cup (\cup_{H \in \mathcal{H}} \mathcal{B}_H^i)$. Thus $(X, \mathcal{G} \cup \{U\}, \mathcal{C}_1)$ and $(X, \mathcal{G} \cup \{U\}, \mathcal{C}_2)$ form a pair of disjoint 3-GDDs of type $g^t u^1$. □

Lemma 8.7 *There exists a pair of disjoint* 3*-GDDs of type* $g^t u^1$*, where* $(g,t,u) \in \{(5,4,3),(5,4,9),(11,4,3),(11,4,9),(11,4,15),(11,4,21),(11,4,27),(11,8,5),(11,6,7),(11,6,9)\}$.

Proof For $(g,t,u) = (5,4,3),(5,4,9),(11,4,3),(11,4,9),(11,4,15),(11,4,21),(11,4,27),(11,8,5)$, let $D = \{1,2,\ldots, gt/2\} \setminus \{t,2t,\ldots,[g/2]t\}$. Since a partition of D into D_1 and D_2 satisfying the conditions of Lemma 3.3 is given in Section 5 of [10], there exists a pair of disjoint 3-GDDs of type $g^t u^1$. For $g = 11, t = 6$ and $u = 7,9$, apply the Weighting Construction to a TD(7,7) as in [10, Lemma 5.4]. Take a block of the TD(7,7) and weight 5 to six points and weight 1 or 3 to the other point of the block. Then weight 1 to all the other points. Since there is a pair of disjoint 3-GDDs of type 1^7, $1^6 3^1$, $1^6 5^1$, or $5^6 3^1$ (Lemmas 3.9 and 8.6), a pair of disjoint 3-GDDs of type $g^t u^1$ also exists. □

Lemma 8.8 *Let* (g,t,u) *be any admissible triple with* $g = 5, 11$*,* $u < g$*,* $t \equiv 2 \pmod 6$*, and* $t \geq 14$*. Then there exists a pair of disjoint* 3*-GDDs of type* $g^t u^1$*.*

Proof For $g = 5, 11$, $u < g$, $t \equiv 2 \pmod 6$, and $t \geq 14$, there is a pair of disjoint 3-GDDs of type $(2g)^{(t-6)/2}(5g+u)^1$ by Lemma 7.1. There exists a pair of disjoint 3-GDDs of type $g^{t-6}(6g+u)^1$ by Corollary 2.3. There exists a pair of disjoint 3-GDDs of type $g^6 u^1$ by Lemmas 8.6 and 8.7. So a pair of disjoint 3-GDDs of type $g^t u^1$ exists by Filling Construction II. □

Lemma 8.9 *The Main Theorem holds for any admissible triple* (g,t,u) *with* $g \equiv 5 \pmod 6$ *and* $5 \leq g \leq 29$*.*

Proof The case of $u > 4(t-1)$ is solved by Lemma 6.6. Also noting that for $t \geq 6$ (must be even) and $u > g$, there exists a pair of disjoint 3-GDDs of type $g^t u^1$ by Corollary 2.3 since there is a pair of disjoint 3-GDDs of type $(2g)^{t/2}(u-g)^1$ by Lemma 7.1, we only need to consider the cases $u \leq 12$ if $t = 4$ and $u \leq 4(t-1)$ and $u < g$ if $t \geq 6$. All the possibilities are exhausted as follows (with (g,t,u) admissible):

Case 1: $g = 5, 11$, and $u \leq 4(t-1)$, further $u < g$ if $t \geq 6$. There are several subcases of t. (i) $t = 4$. There is a pair of disjoint 3-GDDs of type $g^t u^1$ by Lemma 8.7. (ii) $t \equiv 2 \pmod 6$. If $t = 8$, then we use Lemma 8.7 to deal with the only possible triple $(11,8,5)$. Otherwise $t \geq 14$ and Lemma 8.8 gives the solution. (iii) $t \equiv 0, 4 \pmod 6$. If $u \leq t-1$, then we use Lemma 8.6 to obtain the desired pair of 3-GDDs. Otherwise $t-1 < u < g$. Thus all the possible admissible triples are $(11,6,7)$ and $(11,6,9)$, the solutions of which are listed in Lemma 8.7.

Case 2: $g = 17$, and $u \leq 4(t-1)$, and further $u < g$ if $t \geq 6$. Since (g,t,u) is admissible, it is readily checked that $u \leq 3(t-1)$. Hence a pair of disjoint 3-GDDs of type $g^t u^1$ exists by Construction 2.5 since a 3-GDD of type $3^4 5^1$ and disjoint pairs of 3-GDDs of types $3^t u^1$ and $5^t u^1$ exist.

Case 3: $g = 29$, and $u \leq 4(t-1)$. Then a pair of disjoint 3-GDDs of type $g^t u^1$ exists by Construction 2.5 since a 3-GDD of type $5^4 9^1$ and disjoint pairs of 3-GDDs of types $5^t u^1$ and $9^t u^1$ exist.

Case 4: $g = 23$, $u \leq 4(t-1)$, and $u < g$. If $u \leq 3(t-1)$, a pair of disjoint 3-GDDs of type $g^t u^1$ exists by Construction 2.5 since a 3-GDD of type $3^6 5^1$ and disjoint pairs of

3-GDDs of types 3^tu^1 and 5^tu^1 exist. Thus it remains only to deal with the cases $t = 6$ and odd u with $15 < u \leq 20$.

Similar to [10, Lemma 4.3], start from a $\{2,3\}$-GDD of type $1^{18}5^1$ $(X, \mathcal{G}, \mathcal{B})$, where $G \in \mathcal{G}$, $|G| = 5$, and the blocks of size 2 form four parallel classes of $X \setminus G$, say $\mathcal{P}_i, i \in I_4$. Let $U = \{\infty_1, \infty_2, \ldots, \infty_u\}$, $Y = (X \times I_6) \cup U$, and $\mathcal{H} = \{X \times \{i\} : i \in I_6\} \cup \{U\}$. First for each $B \in \mathcal{B}$ and $|B| = 3$, construct on $B \times I_6$ a pair of disjoint ITD(3, 6)s omitting the idempotent parallel class, whose group set is $\{\{x\} \times I_6 : x \in B\}$ and two block sets are $\mathcal{A}_B^1$ and $\mathcal{A}_B^2$. Then we deal with G, the group of size 5 in $\mathcal{G}$. Construct on $(G \times I_6) \cup U$ a pair of disjoint 3-GDDs of type 5^6u^1 with group set $\{G \times \{i\} : i \in I_6\} \cup \{U\}$ and block sets $\mathcal{D}^1$ and $\mathcal{D}^2$. After that let $U_k = \{\infty_{5k+1}, \infty_{5k+2}, \ldots, \infty_{5k+5}\}$, where $k = 0, 1, 2$, and $U_3 = U \setminus (U_0 \cup U_1 \cup U_2)$. For each pair $P \in \mathcal{P}_3$ construct on $(P \times I_6) \cup U_3$ a pair of disjoint 3-GDDs of type $2^6(u-15)^1$, whose group set is $\{\{x\} \times I_6 : x \in B\} \cup \{U_3\}$ and two block sets are $\mathcal{E}_P^1$ and $\mathcal{E}_P^2$. Finally for each $\mathcal{P}_k$, $k = 0, 1, 2$, the set $\{\{(x, i), (y, j)\} : \{x, y\} \in \mathcal{P}_k, i \neq j \in I_6\}$ can be partitioned into 5 disjoint 1-factors of $X \setminus I_6$, denoted by $F_{k0}, F_{k1}, \ldots, F_{k4}$. Let $\mathcal{F}_k^1 = \cup_{0 \leq l \leq 4}\{\{\infty_{2k+1+l}, \alpha, \beta\} : \{\alpha, \beta\} \in F_{kl}\}$ and $\mathcal{F}_k^2 = \cup_{0 \leq l \leq 4}\{\{\infty_{2k+1+l}, \alpha, \beta\} : \{\alpha, \beta\} \in F_{k,l+1}\}$. For $i = 1, 2$, let $\mathcal{C}^i = \mathcal{D}^i \cup (\cup_{B \in \mathcal{B}, |B|=3}\mathcal{A}_B^i) \cup (\cup_{P \in \mathcal{P}_3}\mathcal{E}_P^i) \cup (\cup_{0 \leq k \leq 2}\mathcal{F}_k^i)$. It can be checked that $(Y, \mathcal{H}, \mathcal{C}^1)$ and $(Y, \mathcal{H}, \mathcal{C}^2)$ form two disjoint 3-GDDs of type 23^6u^1. □

Lemma 8.10 *The Main Theorem holds for any admissible triple (g, t, u) with $g \equiv 5$ (mod 6).*

Proof We can employ Lemma 6.6 to treat $u > 4(t-1)$, Corollary 4.2 to treat $t = 3$, and Lemma 8.9 to treat $g \leq 29$. So let $g = 6n + 5$, $n \geq 5$, $t \geq 4$ and $u \leq 4(t-1)$. Apply induction on n. Suppose that there is a pair of 3-GDDs of type h^sv^1 for any admissible triple (h, s, v) with $h = 6l + 5$, and $l < n$. If $n \equiv 3, 5 \pmod 6$, then a 3-GDD of type n^65^1 exists by Lemma 1.1. And disjoint pairs of 3-GDDs of types n^tu^1 and 5^tu^1 also exist by Lemma 8.9 or by the assumption. So a pair of disjoint 3-GDDs of type $(6n+5)^tu^1$ exists by Construction 2.5. If $n \equiv 0, 4 \pmod 6$, or $n \equiv 1 \pmod 6$, or $n \equiv 2 \pmod 6$, also utilize Construction 2.5 but taking instead a 3-GDD of type $(n-1)^611^1$, or $(n-2)^617^1$, or $(n-3)^623^1$, and so on. This completes the proof. □

9 Conclusion

Summing up the results of Lemmas 1.3, 5.9, 6.6, 7.1, 7.3, 8.10, and Corollary 5.5, we obtain the Main Theorem.

To end this paper we mention a byproduct on group divisible codes, which play an important role in the determination of some optimal constant-weight and constant-composition codes. Here we do not dwell on relevant notations on coding theory and the interested readers are referred to [8, 20]. If $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ are a pair of disjoint 3-GDDs of type g^tu^1, from which we can naturally obtain a pair of disjoint $(n, 4, 3)_2$ codes $\mathcal{C}_1$ and $\mathcal{C}_2$ where $n = gt + u$. As in [7], replace each occurrence of 1 with i in each codeword of $\mathcal{C}_i$ to yield a new code $\mathcal{C}_i'$ $(i = 1, 2)$. Thus $\mathcal{C}_1' \cup \mathcal{C}_2'$ forms a ternary group divisible codes of weight three, distance four and size $2b$, where $b = \frac{1}{6}(g^2t(t-1) + 2gtu)$, the number of blocks in a 3-GDD of type g^tu^1.

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Appendix

We list a pair of disjoint 3-GDDs of type $g^t u^1$, where $(g,t,u) \in \{(3,4,1),(3,4,5),(3,4,7)\}$. The point set is I_{gt+u}. The groups are $\{it+j : i \in I_g\}$, $j \in I_t$, and $\{gt, gt+1, \ldots, gt+u-1\}$. And the disjoint block sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are as follows.

(1) $(g,t,u) = (3,4,1)$.

$\mathcal{A}_1$:	$\{12,0,1\}$	$\{12,2,3\}$	$\{12,4,6\}$	$\{12,5,7\}$	$\{12,8,11\}$	$\{12,9,10\}$
	$\{0,2,5\}$	$\{0,3,6\}$	$\{0,7,9\}$	$\{0,10,11\}$	$\{1,2,8\}$	$\{1,3,10\}$
	$\{1,4,11\}$	$\{1,6,7\}$	$\{2,4,7\}$	$\{2,9,11\}$	$\{3,4,9\}$	$\{3,5,8\}$
	$\{4,5,10\}$	$\{5,6,11\}$	$\{6,8,9\}$	$\{7,8,10\}$		
$\mathcal{A}_2$:	$\{12,0,2\}$	$\{12,1,3\}$	$\{12,4,5\}$	$\{12,6,9\}$	$\{12,7,8\}$	$\{12,10,11\}$
	$\{0,1,6\}$	$\{0,3,5\}$	$\{0,7,10\}$	$\{0,9,11\}$	$\{1,2,7\}$	$\{1,4,10\}$
	$\{1,8,11\}$	$\{2,3,4\}$	$\{2,5,11\}$	$\{2,8,9\}$	$\{3,6,8\}$	$\{3,9,10\}$
	$\{4,6,11\}$	$\{4,7,9\}$	$\{5,6,7\}$	$\{5,8,10\}$		

(2) $(g,t,u) = (3,4,5)$.

$\mathcal{A}_1:$	$\{12,0,1\}$	$\{12,2,3\}$	$\{12,4,5\}$	$\{12,6,7\}$	$\{12,8,9\}$	$\{12,10,11\}$
	$\{13,0,2\}$	$\{13,1,3\}$	$\{13,4,6\}$	$\{13,5,7\}$	$\{13,8,10\}$	$\{13,9,11\}$
	$\{14,0,3\}$	$\{14,1,2\}$	$\{14,4,7\}$	$\{14,5,6\}$	$\{14,8,11\}$	$\{14,9,10\}$
	$\{15,0,5\}$	$\{15,1,6\}$	$\{15,2,8\}$	$\{15,3,9\}$	$\{15,4,11\}$	$\{15,7,10\}$
	$\{16,0,10\}$	$\{16,1,11\}$	$\{16,2,7\}$	$\{16,3,4\}$	$\{16,5,8\}$	$\{16,6,9\}$
	$\{0,6,11\}$	$\{0,7,9\}$	$\{1,4,10\}$	$\{1,7,8\}$	$\{2,4,9\}$	$\{2,5,11\}$
	$\{3,5,10\}$	$\{3,6,8\}$				
$\mathcal{A}_2:$	$\{12,0,2\}$	$\{12,1,3\}$	$\{12,4,6\}$	$\{12,5,7\}$	$\{12,8,10\}$	$\{12,9,11\}$
	$\{13,0,1\}$	$\{13,2,3\}$	$\{13,4,5\}$	$\{13,6,7\}$	$\{13,8,9\}$	$\{13,10,11\}$
	$\{14,0,5\}$	$\{14,1,4\}$	$\{14,2,8\}$	$\{14,3,10\}$	$\{14,6,11\}$	$\{14,7,9\}$
	$\{15,0,11\}$	$\{15,1,10\}$	$\{15,2,5\}$	$\{15,3,4\}$	$\{15,6,9\}$	$\{15,7,8\}$
	$\{16,0,6\}$	$\{16,1,7\}$	$\{16,2,9\}$	$\{16,3,8\}$	$\{16,4,11\}$	$\{16,5,10\}$
	$\{0,3,9\}$	$\{0,7,10\}$	$\{1,2,11\}$	$\{1,6,8\}$	$\{2,4,7\}$	$\{3,5,6\}$
	$\{4,9,10\}$	$\{5,8,11\}$				

(3) $(g,t,u)=(3,4,7)$.

$\mathcal{A}_1:$	$\{12,0,1\}$	$\{12,2,3\}$	$\{12,4,5\}$	$\{12,6,7\}$	$\{12,8,9\}$	$\{12,10,11\}$
	$\{13,0,2\}$	$\{13,1,3\}$	$\{13,4,6\}$	$\{13,5,7\}$	$\{13,8,10\}$	$\{13,9,11\}$
	$\{14,0,3\}$	$\{14,1,2\}$	$\{14,4,7\}$	$\{14,5,6\}$	$\{14,8,11\}$	$\{14,9,10\}$
	$\{15,0,5\}$	$\{15,1,4\}$	$\{15,2,8\}$	$\{15,3,9\}$	$\{15,6,11\}$	$\{15,7,10\}$
	$\{16,0,6\}$	$\{16,1,7\}$	$\{16,2,9\}$	$\{16,3,8\}$	$\{16,4,10\}$	$\{16,5,11\}$
	$\{17,0,10\}$	$\{17,1,11\}$	$\{17,2,5\}$	$\{17,3,4\}$	$\{17,6,9\}$	$\{17,7,8\}$
	$\{18,0,11\}$	$\{18,1,10\}$	$\{18,2,7\}$	$\{18,3,6\}$	$\{18,4,9\}$	$\{18,5,8\}$
	$\{0,7,9\}$	$\{1,6,8\}$	$\{2,4,11\}$	$\{3,5,10\}$		
$\mathcal{A}_2:$	$\{12,0,2\}$	$\{12,1,3\}$	$\{12,4,6\}$	$\{12,5,7\}$	$\{12,8,10\}$	$\{12,9,11\}$
	$\{13,0,1\}$	$\{13,2,3\}$	$\{13,4,5\}$	$\{13,6,7\}$	$\{13,8,9\}$	$\{13,10,11\}$
	$\{14,0,5\}$	$\{14,1,4\}$	$\{14,2,8\}$	$\{14,3,9\}$	$\{14,6,11\}$	$\{14,7,10\}$
	$\{15,0,3\}$	$\{15,1,2\}$	$\{15,4,7\}$	$\{15,5,6\}$	$\{15,8,11\}$	$\{15,9,10\}$
	$\{16,0,10\}$	$\{16,1,11\}$	$\{16,2,4\}$	$\{16,3,5\}$	$\{16,6,8\}$	$\{16,7,9\}$
	$\{17,0,11\}$	$\{17,1,10\}$	$\{17,2,7\}$	$\{17,3,6\}$	$\{17,4,9\}$	$\{17,5,8\}$
	$\{18,0,7\}$	$\{18,1,6\}$	$\{18,2,9\}$	$\{18,3,8\}$	$\{18,4,11\}$	$\{18,5,10\}$
	$\{0,6,9\}$	$\{1,7,8\}$	$\{2,5,11\}$	$\{3,4,10\}$		