# A NOTE ON RATIONAL NORMAL CURVES TOTALLY TANGENT TO A HERMITIAN VARIETY

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ABSTRACT. Let q be a power of a prime integer p, and let X be a Hermitian variety of degree q + 1 in the *n*-dimensional projective space. We count the number of rational normal curves that are tangent to X at distinct q + 1 points with intersection multiplicity n. This generalizes a result of B. Segre on the permutable pairs of a Hermitian curve and a smooth conic.

#### 1. INTRODUCTION

Throughout this paper, we fix a power  $q := p^{\nu}$  of a prime integer p. Let k denote the algebraic closure of the finite field  $\mathbb{F}_{q^2}$ .

Let n be an integer  $\geq 2$ . We say that a hypersurface X of  $\mathbb{P}^n$  defined over  $\mathbb{F}_{q^2}$  is a *Hermitian variety* if X is projectively isomorphic over  $\mathbb{F}_{q^2}$ to the Fermat variety

$$X_I := \{x_0^{q+1} + \dots + x_n^{q+1} = 0\} \subset \mathbb{P}^n$$

of degree q + 1. (Strictly speaking, one should say that X is a Hermitian variety of rank n + 1. Since we treat only nonsingular Hermitian varieties in this paper, we omit the term "of rank n + 1".) We say that a hypersurface X of  $\mathbb{P}^n$  defined over k is a k-Hermitian variety if X is projectively isomorphic over k to  $X_I$ . By definition, the projective automorphism group  $\operatorname{Aut}(X) \subset \operatorname{PGL}_{n+1}(k)$  of a k-Hermitian variety X is conjugate to  $\operatorname{Aut}(X_I) = \operatorname{PGU}_{n+1}(\mathbb{F}_{q^2})$  in  $\operatorname{PGL}_{n+1}(k)$ .

Let X be a k-Hermitian variety in  $\mathbb{P}^n$ . A rational normal curve  $\Gamma$ in  $\mathbb{P}^n$  defined over k is said to be *totally tangent to* X if  $\Gamma$  is tangent

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to X at distinct q + 1 points and the intersection multiplicity at each intersection point is n.

A subset S of a rational normal curve  $\Gamma$  is called a *Baer subset* if there exists a coordinate  $t : \Gamma \cong \mathbb{P}^1$  on  $\Gamma$  such that S is the inverse image by t of the set  $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$  of  $\mathbb{F}_q$ -rational points of  $\mathbb{P}^1$ .

The purpose of this paper is to prove the following:

**Theorem 1.** Suppose that  $n \not\equiv 0 \pmod{p}$  and  $2n \leq q$ . Let X be a k-Hermitian variety in  $\mathbb{P}^n$ .

(1) The set  $R_X$  of rational normal curves totally tangent to X is nonempty, and  $\operatorname{Aut}(X)$  acts on  $R_X$  transitively with the stabilizer subgroup isomorphic to  $\operatorname{PGL}_2(\mathbb{F}_q)$ . In particular, we have

$$|R_X| = |\operatorname{PGU}_{n+1}(\mathbb{F}_{q^2})| / |\operatorname{PGL}_2(\mathbb{F}_q)|.$$

(2) For any  $\Gamma \in R_X$ , the points in  $\Gamma \cap X$  form a Baer subset of  $\Gamma$ .

(3) If X is a Hermitian variety, then every  $\Gamma \in R_X$  is defined over  $\mathbb{F}_{q^2}$  and every point of  $\Gamma \cap X$  is  $\mathbb{F}_{q^2}$ -rational.

The study of Hermitian varieties was initiated by B. Segre in [5]. Since then, Hermitian varieties have been intensively studied mainly from combinatorial point of view in finite geometry. (See, for example, Chapter 23 of [3]). B. Segre obtained Theorem 1 for the case n = 2 in the investigation of commutative pairs of polarities [5, n. 81]. We give a simple proof of the higher-dimensional analogue (Theorem 1) of his result using arguments of projective geometry over k.

**Notation.** (1) For simplicity, we put

 $\widetilde{G} := \operatorname{GL}_{n+1}(k)$  and  $G := \operatorname{PGL}_{n+1}(k)$ .

We let G act on  $\mathbb{P}^n$  from *right*. For  $T \in \widetilde{G}$ , we denote by  $[T] \in G$  the image of T by the natural homomorphism  $\widetilde{G} \to G$ . The entries  $a_{i,j}$  of a matrix  $A = (a_{i,j}) \in \widetilde{G}$  are indexed by

$$N := \{ (i, j) \in \mathbb{Z}^2 \mid 0 \le i \le n, \ 0 \le j \le n \}.$$

(2) Let M be a matrix with entries in k. We denote by  ${}^{t}M$  the transpose of M, and by  $\overline{M}$  the matrix obtained from M by applying  $a \mapsto a^{q}$  to the entries.

# 2. Proof of Theorem 1

The following well-known proposition is the main tool of the proof.

**Proposition 2.** The map  $\lambda : \widetilde{G} \to \widetilde{G}$  defined by  $\lambda(T) := T^{t}\overline{T}$  is surjective. The image of  $\operatorname{GL}_{n+1}(\mathbb{F}_{q^2}) \subset \widetilde{G}$  by  $\lambda$  is equal to the set

$$\mathcal{H} := \{ H \in \operatorname{GL}_{n+1}(\mathbb{F}_{q^2}) \mid H = {}^t\overline{H} \}$$

of Hermitian matrices over  $\mathbb{F}_{q^2}$ .

*Proof.* The first part is a variant of Lang's theorem (see [4] or [7]), and can be proved by means of differentials (see [1, 16.4]). The second part is due to B. Segre [5, n. 3]. See also [6, Section 1].

For a matrix  $A = (a_{i,j}) \in \widetilde{G}$ , we define a homogeneous polynomial  $f_A$  of degree q + 1 by

$$f_A := \sum_{(i,j) \in N} a_{i,j} x_i x_j^q$$

If  $g = [T] \in G$ , then the image  $X_I^g$  of the Fermat hypersurface  $X_I$  by g is defined by  $f_{\lambda(T^{-1})} = 0$ . Hence we obtain the following:

**Corollary 3.** A hypersurface X of  $\mathbb{P}^n$  is a k-Hermitian variety if and only if there exists a matrix  $A \in \widetilde{G}$  such that X is defined by  $f_A = 0$ . A hypersurface X is a Hermitian variety if and only if there exists a Hermitian matrix  $H \in \mathcal{H}$  over  $\mathbb{F}_{q^2}$  such that X is defined by  $f_H = 0$ .

Let  $\mathcal{V}$  denote the set of all k-Hermitian varieties in  $\mathbb{P}^n$ . For  $A \in \widetilde{G}$ , let  $X_A \in \mathcal{V}$  denote the hypersurface defined by  $f_A = 0$ . (The Fermat hypersurface  $X_I$  is defined by  $f_I = 0$ , where  $I \in \widetilde{G}$  is the identity matrix.) Remark that G acts on  $\mathcal{V}$  transitively.

Let  $\mathcal{R}$  denote the set of all rational normal curves in  $\mathbb{P}^n$ , and let  $\mathcal{P}$  be the set of pairs

$$[\Gamma, (Q_0, Q_1, Q_\infty)],$$

where  $\Gamma \in \mathcal{R}$ , and  $Q_0, Q_1, Q_\infty$  are ordered three distinct points of  $\Gamma$ . Let  $\Gamma_0 \in \mathcal{R}$  be the image of the morphism  $\phi_0 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$  given by

$$\phi_0(t) := [1:t:\cdots:t^n] \in \mathbb{P}^n.$$

We put

$$P_0 := \phi_0(0), \quad P_1 := \phi_0(1), \quad P_\infty := \phi_0(\infty).$$

Then we have  $[\Gamma_0, (P_0, P_1, P_\infty)] \in \mathcal{P}$ .

**Lemma 4.** The action of G on  $\mathcal{P}$  is simply transitive.

*Proof.* The action of G on  $\mathcal{R}$  is transitive by the definition of rational normal curves. Let  $\Sigma_0 \subset G$  denote the stabilizer subgroup of  $\Gamma_0 \in \mathcal{R}$ . Then we have a natural homomorphism

$$\psi: \Sigma_0 \to \operatorname{Aut}(\Gamma_0) \cong \operatorname{PGL}_2(k).$$

Note that  $\operatorname{Aut}(\Gamma_0)$  acts on the set of ordered three distinct points of  $\Gamma_0$  simple-transitively. Hence it is enough to show that  $\psi$  is an isomorphism. Since  $\Gamma_0$  contains n+2 points such that any n+1 of them are linearly independent in  $\mathbb{P}^n$ ,  $\psi$  is injective. Since  $\operatorname{PGL}_2(k)$  is generated by the linear transformations

 $t \mapsto at + b$  and  $t \mapsto 1/(t - c)$ , where  $a \in k^{\times}$  and  $b, c \in k$ ,

it is enough to find matrices  $M_{a,b} \in \widetilde{G}$  and  $N_c \in \widetilde{G}$  such that

$$[1, at + b, \dots, (at + b)^n] = [1, t, \dots, t^n] M_{a,b} \text{ and} [(t - c)^n, (t - c)^{n-1}, \dots, 1] = [1, t, \dots, t^n] N_c$$

hold for any  $a \in k^{\times}$  and  $b, c \in k$ . This is immediate.

We denote by  $\mathcal{I} \subset \mathcal{V} \times \mathcal{P}$  the set of all triples  $[X, \Gamma, (Q_0, Q_1, Q_\infty)]$ such that

- (1)  $X \in \mathcal{V}$  and  $[\Gamma, (Q_0, Q_1, Q_\infty)] \in \mathcal{P}$ ,
- (2)  $\Gamma$  is totally tangent to X, and
- (3)  $Q_0, Q_1, Q_\infty$  are contained in  $\Gamma \cap X$ .

We then consider the incidence diagram

$$egin{array}{ccc} \mathcal{I} & \stackrel{p_1}{\longrightarrow} & \mathcal{V} \\ & & \mathcal{P}. \end{array}$$

Note that G acts on  $\mathcal{I}$ , and that the projections  $p_1$  and  $p_{23}$  are G-equivariant.

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We consider the Hermitian matrix  $B = (b_{i,j}) \in \mathcal{H}$ , where

$$b_{i,j} := \begin{cases} \binom{n}{i} (-1)^i & \text{if } i+j=n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\phi_0^* f_B = \sum_{i=0}^n \binom{n}{i} (-1)^i t^i t^{q(n-i)} = (t^q - t)^n,$$

and hence  $[X_B, \Gamma_0, (P_0, P_1, P_\infty)]$  is a point of  $\mathcal{I}$ . Since  $\mathcal{I} \neq \emptyset$  and the action of G on  $\mathcal{V}$  is transitive, the G-equivariant map  $p_1$  is surjective. Thus  $R_X \neq \emptyset$  holds for any  $X \in \mathcal{V}$ .

The following proposition is proved in the next section.

**Proposition 5.** Suppose that  $n \neq 0 \pmod{p}$  and  $n \leq 2q$ . Then the fiber of  $p_{23}$  over  $[\Gamma_0, (P_0, P_1, P_\infty)] \in \mathcal{P}$  consists of a single point  $[X_B, \Gamma_0, (P_0, P_1, P_\infty)] \in \mathcal{I}$ . In particular,  $p_{23}$  is a bijection.

Theorem 1 follows from Proposition 5 as follows. First note that, for any  $X \in \mathcal{V}$ , the map  $[X, \Gamma, (Q_0, Q_1, Q_\infty)] \mapsto \Gamma$  gives a surjection

$$\rho_X : p_1^{-1}(X) \to R_X.$$

Proposition 5 implies that G acts on  $\mathcal{I}$  simple-transitively. If  $S \subset \Gamma$  is a Baer subset of  $\Gamma \in \mathcal{R}$ , then  $S^g \subset \Gamma^g$  is a Baer subset of  $\Gamma^g$  for any  $g \in G$ . Since  $\Gamma_0 \cap X_B$  is a Baer subset of  $\Gamma_0$ , we see that  $\Gamma \cap X$  is a Baer subset of  $\Gamma$  for any  $[X, \Gamma, (Q_0, Q_1, Q_\infty)] \in \mathcal{I}$ . Therefore the assertion (2) follows. Since  $p_1$  is G-equivariant, the stabilizer subgroup  $\operatorname{Aut}(X)$  of Xin G acts on the fiber  $p_1^{-1}(X)$  simple-transitively for any  $X \in \mathcal{V}$ . Note that  $\rho_X$  is  $\operatorname{Aut}(X)$ -equivariant. Hence the stabilizer subgroup  $\operatorname{Stab}(\Gamma)$ of  $\Gamma \in R_X$  in  $\operatorname{Aut}(X)$  acts on the fiber  $\rho_X^{-1}(\Gamma)$  simple-transitively. Moreover, since  $\Gamma$  contains n+2 points such that any n+1 of them are linearly independent,  $\operatorname{Stab}(\Gamma)$  is embedded into  $\operatorname{Aut}(\Gamma) \cong \operatorname{PGL}_2(k)$ . Since  $\rho_X^{-1}(\Gamma)$  is the set of ordered three distinct points of the Baer subset  $\Gamma \cap X$  of  $\Gamma$ , we see that  $\operatorname{Stab}(\Gamma)$  is conjugate to  $\operatorname{PGL}_2(\mathbb{F}_q)$  as a subgroup of  $\operatorname{Aut}(\Gamma) \cong \operatorname{PGL}_2(k)$ . Thus the assertion (1) follows. The assertion (3) is immediate from the facts that  $X_B$  is Hermitian, that  $\Gamma_0$ is defined over  $\mathbb{F}_{q^2}$ , and that every points of  $\Gamma_0 \cap X_B$  is  $\mathbb{F}_{q^2}$ -rational.  $\Box$ 

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#### 3. Proof of Proposition 5

Suppose that  $A = (a_{i,j}) \in \widetilde{G}$  satisfies  $[X_A, \Gamma_0, (P_0, P_1, P_\infty)] \in \mathcal{I}$ . We will show that A = c B for some  $c \in k^{\times}$ .

By the definition of  $\mathcal{I}$ , there exists a polynomial  $h \in k[t]$  such that the polynomial

$$\phi_0^* f_A = \sum_{(i,j) \in N} a_{i,j} t^{i+qj}$$

is equal to  $h^n$ , and that, regarded as a polynomial of degree q + 1, h has distinct q + 1 roots including 0, 1 and  $\infty$ . In particular, we have deg h = q and h(0) = 0. Thus we can set

$$h = \sum_{\nu=1}^{q} b_{\nu} t^{\nu}.$$

Since deg h = q and since t = 0 is a simple root of h = 0, we have

$$b_q \neq 0$$
 and  $b_1 \neq 0$ .

Let  $c_m$  denote the coefficient of  $t^m$  in  $\phi_0^* f_A$ . We have  $c_m = 0$  if no  $(i, j) \in N$  satisfy i + qj = m. By the assumption  $2n \leq q$ , we have

(3.1)  $c_m = 0$  if n < m < q or n + q(n-1) < m < qn.

We will show that

$$(3.2) b_{\mu} = 0 \quad \text{if} \quad n < \mu < q.$$

Let l be the largest integer such that l < q and  $b_l \neq 0$ . Since  $n \neq 0$ (mod p) and  $b_q \neq 0$ , the coefficient  $n b_q^{n-1} b_l$  of  $t^{l+q(n-1)}$  in  $h^n$  is nonzero. Therefore  $c_{l+q(n-1)} \neq 0$  follows from  $\phi_0^* f_A = h^n$ . By (3.1) and l < q, we have  $l \leq n$ . Hence (3.2) holds. In the same way, we will show that

(3.3) 
$$b_{\mu} = 0$$
 if  $1 < \mu < q - n + 1$ .

Let l be the smallest integer such that l > 1 and  $b_l \neq 0$ . Since  $n \neq 0$ (mod p) and  $b_1 \neq 0$ , the coefficient  $n b_1^{n-1} b_l$  of  $t^{n-1+l}$  in  $h^n$  is non-zero. Therefore  $c_{n-1+l} \neq 0$  follows from  $\phi_0^* f_A = h^n$ . By (3.1) and l > 1, we have  $n - 1 + l \geq q$ . Hence (3.3) holds. Combining (3.2), (3.3) with the assumption  $2n \leq q$ , we see that h is of the form  $b_q t^q + b_1 t$ . Since h(1) = 0, we have

$$h = b(t^q - t)$$
 for some  $b \in k^{\times}$ .

From  $\phi_0^* f_A = h^n$ , we see that  $A = b^n B$ .

*Remark* 6. In [2], another generalization of B. Segre's result [5, n. 81] is obtained.

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