# A NOTE ON RATIONAL NORMAL CURVES TOTALLY TANGENT TO A HERMITIAN VARIETY 

ICHIRO SHIMADA


#### Abstract

Let $q$ be a power of a prime integer $p$, and let $X$ be a Hermitian variety of degree $q+1$ in the $n$-dimensional projective space. We count the number of rational normal curves that are tangent to $X$ at distinct $q+1$ points with intersection multiplicity $n$. This generalizes a result of B. Segre on the permutable pairs of a Hermitian curve and a smooth conic.


## 1. Introduction

Throughout this paper, we fix a power $q:=p^{\nu}$ of a prime integer $p$. Let $k$ denote the algebraic closure of the finite field $\mathbb{F}_{q^{2}}$.

Let $n$ be an integer $\geq 2$. We say that a hypersurface $X$ of $\mathbb{P}^{n}$ defined over $\mathbb{F}_{q^{2}}$ is a Hermitian variety if $X$ is projectively isomorphic over $\mathbb{F}_{q^{2}}$ to the Fermat variety

$$
X_{I}:=\left\{x_{0}^{q+1}+\cdots+x_{n}^{q+1}=0\right\} \subset \mathbb{P}^{n}
$$

of degree $q+1$. (Strictly speaking, one should say that $X$ is a Hermitian variety of rank $n+1$. Since we treat only nonsingular Hermitian varieties in this paper, we omit the term "of rank $n+1$ ".) We say that a hypersurface $X$ of $\mathbb{P}^{n}$ defined over $k$ is a $k$-Hermitian variety if $X$ is projectively isomorphic over $k$ to $X_{I}$. By definition, the projective automorphism group $\operatorname{Aut}(X) \subset \mathrm{PGL}_{n+1}(k)$ of a $k$-Hermitian variety $X$ is conjugate to $\operatorname{Aut}\left(X_{I}\right)=\mathrm{PGU}_{n+1}\left(\mathbb{F}_{q^{2}}\right)$ in $\mathrm{PGL}_{n+1}(k)$.

Let $X$ be a $k$-Hermitian variety in $\mathbb{P}^{n}$. A rational normal curve $\Gamma$ in $\mathbb{P}^{n}$ defined over $k$ is said to be totally tangent to $X$ if $\Gamma$ is tangent

[^0]to $X$ at distinct $q+1$ points and the intersection multiplicity at each intersection point is $n$.

A subset $S$ of a rational normal curve $\Gamma$ is called a Baer subset if there exists a coordinate $t: \Gamma \leadsto \mathbb{P}^{1}$ on $\Gamma$ such that $S$ is the inverse image by $t$ of the set $\mathbb{P}^{1}\left(\mathbb{F}_{q}\right)=\mathbb{F}_{q} \cup\{\infty\}$ of $\mathbb{F}_{q}$-rational points of $\mathbb{P}^{1}$.

The purpose of this paper is to prove the following:
Theorem 1. Suppose that $n \not \equiv 0(\bmod p)$ and $2 n \leq q$. Let $X$ be $a$ $k$-Hermitian variety in $\mathbb{P}^{n}$.
(1) The set $R_{X}$ of rational normal curves totally tangent to $X$ is nonempty, and $\operatorname{Aut}(X)$ acts on $R_{X}$ transitively with the stabilizer subgroup isomorphic to $\mathrm{PGL}_{2}\left(\mathbb{F}_{q}\right)$. In particular, we have

$$
\left|R_{X}\right|=\left|\mathrm{PGU}_{n+1}\left(\mathbb{F}_{q^{2}}\right)\right| /\left|\mathrm{PGL}_{2}\left(\mathbb{F}_{q}\right)\right| .
$$

(2) For any $\Gamma \in R_{X}$, the points in $\Gamma \cap X$ form a Baer subset of $\Gamma$.
(3) If $X$ is a Hermitian variety, then every $\Gamma \in R_{X}$ is defined over $\mathbb{F}_{q^{2}}$ and every point of $\Gamma \cap X$ is $\mathbb{F}_{q^{2}}$-rational.

The study of Hermitian varieties was initiated by B. Segre in [5]. Since then, Hermitian varieties have been intensively studied mainly from combinatorial point of view in finite geometry. (See, for example, Chapter 23 of [3]). B. Segre obtained Theorem 1 for the case $n=2$ in the investigation of commutative pairs of polarities [5, n. 81]. We give a simple proof of the higher-dimensional analogue (Theorem 1) of his result using arguments of projective geometry over $k$.

Notation. (1) For simplicity, we put

$$
\widetilde{G}:=\mathrm{GL}_{n+1}(k) \quad \text { and } \quad G:=\mathrm{PGL}_{n+1}(k)
$$

We let $G$ act on $\mathbb{P}^{n}$ from right. For $T \in \widetilde{G}$, we denote by $[T] \in G$ the image of $T$ by the natural homomorphism $\widetilde{G} \rightarrow G$. The entries $a_{i, j}$ of a matrix $A=\left(a_{i, j}\right) \in \widetilde{G}$ are indexed by

$$
N:=\left\{(i, j) \in \mathbb{Z}^{2} \mid 0 \leq i \leq n, \quad 0 \leq j \leq n\right\} .
$$

(2) Let $M$ be a matrix with entries in $k$. We denote by ${ }^{t} M$ the transpose of $M$, and by $\bar{M}$ the matrix obtained from $M$ by applying $a \mapsto a^{q}$ to the entries.

## 2. Proof of Theorem 1

The following well-known proposition is the main tool of the proof.
Proposition 2. The map $\lambda: \widetilde{G} \rightarrow \widetilde{G}$ defined by $\lambda(T):=T^{t} \bar{T}$ is surjective. The image of $\mathrm{GL}_{n+1}\left(\mathbb{F}_{q^{2}}\right) \subset \widetilde{G}$ by $\lambda$ is equal to the set

$$
\mathcal{H}:=\left\{H \in \mathrm{GL}_{n+1}\left(\mathbb{F}_{q^{2}}\right) \mid H={ }^{t} \bar{H}\right\}
$$

of Hermitian matrices over $\mathbb{F}_{q^{2}}$.
Proof. The first part is a variant of Lang's theorem (see [4] or [7]), and can be proved by means of differentials (see [1, 16.4]). The second part is due to B. Segre [5, n. 3]. See also [6, Section 1].

For a matrix $A=\left(a_{i, j}\right) \in \widetilde{G}$, we define a homogeneous polynomial $f_{A}$ of degree $q+1$ by

$$
f_{A}:=\sum_{(i, j) \in N} a_{i, j} x_{i} x_{j}^{q} .
$$

If $g=[T] \in G$, then the image $X_{I}^{g}$ of the Fermat hypersurface $X_{I}$ by $g$ is defined by $f_{\lambda\left(T^{-1}\right)}=0$. Hence we obtain the following:

Corollary 3. A hypersurface $X$ of $\mathbb{P}^{n}$ is a $k$-Hermitian variety if and only if there exists a matrix $A \in \widetilde{G}$ such that $X$ is defined by $f_{A}=0$. A hypersurface $X$ is a Hermitian variety if and only if there exists a Hermitian matrix $H \in \mathcal{H}$ over $\mathbb{F}_{q^{2}}$ such that $X$ is defined by $f_{H}=0$.

Let $\mathcal{V}$ denote the set of all $k$-Hermitian varieties in $\mathbb{P}^{n}$. For $A \in \widetilde{G}$, let $X_{A} \in \mathcal{V}$ denote the hypersurface defined by $f_{A}=0$. (The Fermat hypersurface $X_{I}$ is defined by $f_{I}=0$, where $I \in \widetilde{G}$ is the identity matrix.) Remark that $G$ acts on $\mathcal{V}$ transitively.

Let $\mathcal{R}$ denote the set of all rational normal curves in $\mathbb{P}^{n}$, and let $\mathcal{P}$ be the set of pairs

$$
\left[\Gamma,\left(Q_{0}, Q_{1}, Q_{\infty}\right)\right]
$$

where $\Gamma \in \mathcal{R}$, and $Q_{0}, Q_{1}, Q_{\infty}$ are ordered three distinct points of $\Gamma$. Let $\Gamma_{0} \in \mathcal{R}$ be the image of the morphism $\phi_{0}: \mathbb{P}^{1} \hookrightarrow \mathbb{P}^{n}$ given by

$$
\phi_{0}(t):=\left[1: t: \cdots: t^{n}\right] \in \mathbb{P}^{n}
$$

We put

$$
P_{0}:=\phi_{0}(0), \quad P_{1}:=\phi_{0}(1), \quad P_{\infty}:=\phi_{0}(\infty) .
$$

Then we have $\left[\Gamma_{0},\left(P_{0}, P_{1}, P_{\infty}\right)\right] \in \mathcal{P}$.
Lemma 4. The action of $G$ on $\mathcal{P}$ is simply transitive.
Proof. The action of $G$ on $\mathcal{R}$ is transitive by the definition of rational normal curves. Let $\Sigma_{0} \subset G$ denote the stabilizer subgroup of $\Gamma_{0} \in \mathcal{R}$. Then we have a natural homomorphism

$$
\psi: \Sigma_{0} \rightarrow \operatorname{Aut}\left(\Gamma_{0}\right) \cong \mathrm{PGL}_{2}(k)
$$

Note that $\operatorname{Aut}\left(\Gamma_{0}\right)$ acts on the set of ordered three distinct points of $\Gamma_{0}$ simple-transitively. Hence it is enough to show that $\psi$ is an isomorphism. Since $\Gamma_{0}$ contains $n+2$ points such that any $n+1$ of them are linearly independent in $\mathbb{P}^{n}, \psi$ is injective. Since $\mathrm{PGL}_{2}(k)$ is generated by the linear transformations
$t \mapsto a t+b \quad$ and $\quad t \mapsto 1 /(t-c), \quad$ where $\quad a \in k^{\times} \quad$ and $b, c \in k$, it is enough to find matrices $M_{a, b} \in \widetilde{G}$ and $N_{c} \in \widetilde{G}$ such that

$$
\begin{aligned}
{\left[1, a t+b, \ldots,(a t+b)^{n}\right] } & =\left[1, t, \ldots, t^{n}\right] M_{a, b} \quad \text { and } \\
{\left[(t-c)^{n},(t-c)^{n-1}, \ldots, 1\right] } & =\left[1, t, \ldots, t^{n}\right] N_{c}
\end{aligned}
$$

hold for any $a \in k^{\times}$and $b, c \in k$. This is immediate.
We denote by $\mathcal{I} \subset \mathcal{V} \times \mathcal{P}$ the set of all triples $\left[X, \Gamma,\left(Q_{0}, Q_{1}, Q_{\infty}\right)\right]$ such that
(1) $X \in \mathcal{V}$ and $\left[\Gamma,\left(Q_{0}, Q_{1}, Q_{\infty}\right)\right] \in \mathcal{P}$,
(2) $\Gamma$ is totally tangent to $X$, and
(3) $Q_{0}, Q_{1}, Q_{\infty}$ are contained in $\Gamma \cap X$.

We then consider the incidence diagram


Note that $G$ acts on $\mathcal{I}$, and that the projections $p_{1}$ and $p_{23}$ are $G$ equivariant.

We consider the Hermitian matrix $B=\left(b_{i, j}\right) \in \mathcal{H}$, where

$$
b_{i, j}:= \begin{cases}\binom{n}{i}(-1)^{i} & \text { if } i+j=n \\ 0 & \text { otherwise }\end{cases}
$$

Then we have

$$
\phi_{0}^{*} f_{B}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i} t^{i} t^{q(n-i)}=\left(t^{q}-t\right)^{n},
$$

and hence $\left[X_{B}, \Gamma_{0},\left(P_{0}, P_{1}, P_{\infty}\right)\right]$ is a point of $\mathcal{I}$. Since $\mathcal{I} \neq \emptyset$ and the action of $G$ on $\mathcal{V}$ is transitive, the $G$-equivariant map $p_{1}$ is surjective. Thus $R_{X} \neq \emptyset$ holds for any $X \in \mathcal{V}$.

The following proposition is proved in the next section.
Proposition 5. Suppose that $n \not \equiv 0(\bmod p)$ and $n \leq 2 q$. Then the fiber of $p_{23}$ over $\left[\Gamma_{0},\left(P_{0}, P_{1}, P_{\infty}\right)\right] \in \mathcal{P}$ consists of a single point $\left[X_{B}, \Gamma_{0},\left(P_{0}, P_{1}, P_{\infty}\right)\right] \in \mathcal{I}$. In particular, $p_{23}$ is a bijection.

Theorem 1 follows from Proposition 5 as follows. First note that, for any $X \in \mathcal{V}$, the map $\left[X, \Gamma,\left(Q_{0}, Q_{1}, Q_{\infty}\right)\right] \mapsto \Gamma$ gives a surjection

$$
\rho_{X}: p_{1}^{-1}(X) \rightarrow R_{X} .
$$

Proposition 5 implies that $G$ acts on $\mathcal{I}$ simple-transitively. If $S \subset \Gamma$ is a Baer subset of $\Gamma \in \mathcal{R}$, then $S^{g} \subset \Gamma^{g}$ is a Baer subset of $\Gamma^{g}$ for any $g \in G$. Since $\Gamma_{0} \cap X_{B}$ is a Baer subset of $\Gamma_{0}$, we see that $\Gamma \cap X$ is a Baer subset of $\Gamma$ for any $\left[X, \Gamma,\left(Q_{0}, Q_{1}, Q_{\infty}\right)\right] \in \mathcal{I}$. Therefore the assertion (2) follows. Since $p_{1}$ is $G$-equivariant, the stabilizer subgroup $\operatorname{Aut}(X)$ of $X$ in $G$ acts on the fiber $p_{1}^{-1}(X)$ simple-transitively for any $X \in \mathcal{V}$. Note that $\rho_{X}$ is $\operatorname{Aut}(X)$-equivariant. Hence the stabilizer subgroup $\operatorname{Stab}(\Gamma)$ of $\Gamma \in R_{X}$ in $\operatorname{Aut}(X)$ acts on the fiber $\rho_{X}^{-1}(\Gamma)$ simple-transitively. Moreover, since $\Gamma$ contains $n+2$ points such that any $n+1$ of them are linearly independent, $\operatorname{Stab}(\Gamma)$ is embedded into $\operatorname{Aut}(\Gamma) \cong \mathrm{PGL}_{2}(k)$. Since $\rho_{X}^{-1}(\Gamma)$ is the set of ordered three distinct points of the Baer subset $\Gamma \cap X$ of $\Gamma$, we see that $\operatorname{Stab}(\Gamma)$ is conjugate to $\mathrm{PGL}_{2}\left(\mathbb{F}_{q}\right)$ as a subgroup of $\operatorname{Aut}(\Gamma) \cong \mathrm{PGL}_{2}(k)$. Thus the assertion (1) follows. The assertion (3) is immediate from the facts that $X_{B}$ is Hermitian, that $\Gamma_{0}$ is defined over $\mathbb{F}_{q^{2}}$, and that every points of $\Gamma_{0} \cap X_{B}$ is $\mathbb{F}_{q^{2}}$-rational.

## 3. Proof of Proposition 5

Suppose that $A=\left(a_{i, j}\right) \in \widetilde{G}$ satisfies $\left[X_{A}, \Gamma_{0},\left(P_{0}, P_{1}, P_{\infty}\right)\right] \in \mathcal{I}$. We will show that $A=c B$ for some $c \in k^{\times}$.

By the definition of $\mathcal{I}$, there exists a polynomial $h \in k[t]$ such that the polynomial

$$
\phi_{0}^{*} f_{A}=\sum_{(i, j) \in N} a_{i, j} t^{i+q j}
$$

is equal to $h^{n}$, and that, regarded as a polynomial of degree $q+1, h$ has distinct $q+1$ roots including 0,1 and $\infty$. In particular, we have $\operatorname{deg} h=q$ and $h(0)=0$. Thus we can set

$$
h=\sum_{\nu=1}^{q} b_{\nu} t^{\nu} .
$$

Since $\operatorname{deg} h=q$ and since $t=0$ is a simple root of $h=0$, we have

$$
b_{q} \neq 0 \quad \text { and } \quad b_{1} \neq 0
$$

Let $c_{m}$ denote the coefficient of $t^{m}$ in $\phi_{0}^{*} f_{A}$. We have $c_{m}=0$ if no $(i, j) \in N$ satisfy $i+q j=m$. By the assumption $2 n \leq q$, we have

$$
\begin{equation*}
c_{m}=0 \quad \text { if } \quad n<m<q \text { or } n+q(n-1)<m<q n . \tag{3.1}
\end{equation*}
$$

We will show that

$$
\begin{equation*}
b_{\mu}=0 \quad \text { if } \quad n<\mu<q \tag{3.2}
\end{equation*}
$$

Let $l$ be the largest integer such that $l<q$ and $b_{l} \neq 0$. Since $n \not \equiv 0$ $(\bmod p)$ and $b_{q} \neq 0$, the coefficient $n b_{q}^{n-1} b_{l}$ of $t^{l+q(n-1)}$ in $h^{n}$ is nonzero. Therefore $c_{l+q(n-1)} \neq 0$ follows from $\phi_{0}^{*} f_{A}=h^{n}$. By (3.1) and $l<q$, we have $l \leq n$. Hence (3.2) holds. In the same way, we will show that

$$
\begin{equation*}
b_{\mu}=0 \quad \text { if } \quad 1<\mu<q-n+1 \tag{3.3}
\end{equation*}
$$

Let $l$ be the smallest integer such that $l>1$ and $b_{l} \neq 0$. Since $n \not \equiv 0$ $(\bmod p)$ and $b_{1} \neq 0$, the coefficient $n b_{1}^{n-1} b_{l}$ of $t^{n-1+l}$ in $h^{n}$ is non-zero. Therefore $c_{n-1+l} \neq 0$ follows from $\phi_{0}^{*} f_{A}=h^{n}$. By (3.1) and $l>1$, we have $n-1+l \geq q$. Hence (3.3) holds.

Combining (3.2), (3.3) with the assumption $2 n \leq q$, we see that $h$ is of the form $b_{q} t^{q}+b_{1} t$. Since $h(1)=0$, we have

$$
h=b\left(t^{q}-t\right) \quad \text { for some } b \in k^{\times} .
$$

From $\phi_{0}^{*} f_{A}=h^{n}$, we see that $A=b^{n} B$.
Remark 6. In [2], another generalization of B. Segre's result [5, n. 81] is obtained.

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Department of Mathematics, Graduate School of Science, Hiroshima
University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN
E-mail address: shimada@math.sci.hiroshima-u.ac.jp


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