

A Generalization of Lee Codes

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Abstract Motivated by a problem in computer architecture we introduce a notion of the perfect distance-dominating set, PDDS, in a graph. PDDSs constitute a generalization of perfect Lee codes, diameter perfect codes, as well as other codes and dominating sets. In this paper we initiate a systematic study of PDDSs. PDDSs related to the application will be constructed and the non-existence of some PDDSs will be shown. In addition, an extension of the long-standing Golomb-Welch conjecture, in terms of PDDS, will be stated. We note that all constructed PDDSs are lattice-like which is a very important feature from the practical point of view as in this case decoding algorithms tend to be much simpler.

This paper is dedicated to the memory of Lucia Gionfriddo.

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1 Introduction

We introduce a generalization of perfect Lee codes and other dominating notions, motivated by the following problem in computer architecture, see e.g. [4]. Processing elements in a supercomputer communicate through a network that has the topology of the Cartesian product of cycles. It is desirable to place the Input/Output devices into the network in such a way that the communication of all elements in the network is optimized; each element of the network should be at distance at most t from at least one I/O device, ideally from exactly one I/O device. It is not difficult to see that perfect error correcting Lee codes, if any, provide the optimal placement.

Unfortunately, the perfect t -error correcting Lee codes of block length n over \mathbb{Z} , and over \mathbb{Z}_q , $q \geq 2n + 1$, shortly PLC(n, t) and PLC(n, t, q) codes, respectively, have been constructed only for $n = 1, 2$, and any t , and for $n \geq 3$ and $t = 1$. Moreover, as suggested by the well-known and long-standing conjecture of Golomb and Welch [16], PLC(n, t) codes and PLC(n, t, q), $q \geq 2n + 1$, codes do not exist in other cases. To remedy this obstacle, perfect Lee codes have been generalized in several ways, see e.g. [3], where the quasi-perfect Lee codes have been introduced. A weakness of the quasi-perfect Lee codes is that some words cannot be decoded in a unique way, and so far the quasi-perfect Lee codes have been found only for $n = 2$.

In order to offer a new approach to the placement problem we will introduce yet another generalization of Lee codes. Instead of defining it only for the Cartesian product of cycles and the Cartesian product of two-way infinite paths, denoted by A_n (= infinite graph whose vertex set is \mathbb{Z}^n with two vertices being adjacent if their Euclidean distance is 1), we introduce the new concept for an arbitrary graph. However, having in mind the application we will mainly focus on the Cartesian product of cycles and A_n . As usual, $[S]$ stands for the subgraph induced by S , and the distance $d(v, C)$ of a vertex $v \in V$ to C is given by $d(v, C) = \min\{d(v, w); w \in C\}$.

Definition 1 Let $t \geq 1$ and $\Gamma = (V, E)$ be a graph. A set $S \subset V$ will be said to be a t -perfect distance-dominating set in Γ , a t -PDDS in Γ , if, for each $v \in V$, there is a unique component C_v of $[S]$, so that for the distance $d(v, C_v)$ from v to C_v it is $d(v, C_v) \leq t$, and there is in C_v a unique vertex w with $d(v, w) = d(v, C_v)$.

The first condition guaranties that to each element v of the network there is at least one I/O device at the distance at most t from v , while the second condition, that in C_v there is a unique vertex w with $d(v, w) = d(v, C_v)$, guarantees that to each element v in the communication network, there is a uniquely determined I/O device with which v will communicate.

Now we describe how the new domination concept of PDDS relates to other coding theory and graph domination notions. First of all we note that

PLC(n, t, q) codes and PLC(n, t) codes are t -PDDSs in the Cartesian product of cycles and in A_n , respectively, with all components of t -PDDS being isolated vertices. A notion of a diameter perfect code has been introduced in [1]. For d odd, the diameter- d perfect Lee code in A_n coincides with the perfect $\frac{d-1}{2}$ -error correcting Lee code. It follows from [2, 14] that, for d even, diameter- d perfect Lee code in A_n exists if and only if there is a $\frac{d-2}{2}$ -PDDS in A_n whose each component consists of two adjacent vertices. In [6] Biggs extended the concept of the perfect code from a metric space to a graph. A perfect t -code in a graph $\Gamma = (V, E)$ is a set $C \subset V$ such that t -neighborhoods $N_t(c) = \{u \in V; d(c, u) \leq t\}$ with $c \in C$ form a partition of V . Clearly, a t -perfect code C in Γ is a t -PDDS in Γ with all vertices in C being isolated. Further, Weichsel [28] defined a notion of the perfect dominating set, or PDS. In our terminology a PDS is a 1-PDDS. PDSs were studied in the hypercube graphs [28, 13, 11], in the star graphs [12], in A_2 , and in toroidal grids [10, 9]. In addition, Klostermeyer and Goldwasser [20] defined the total perfect code in a graph to be a subset of its vertex set with the property that each vertex is adjacent to exactly one vertex in the subset. The NP-completeness of finding a 1-perfect code of Γ and that of finding a minimal perfect dominating set in a planar graph were established in [5, 21], and in [15], respectively.

Now we prove a statement related to the structure of PDDSs in A_n . It turns out that the choice of components of a t -PDDS in A_n is quite limited. To facilitate our discussion we introduce some notation. If no ambiguity is possible, n -tuples representing elements of \mathbb{Z}^n will be written without external parentheses or commas. O will stand for the element $00 \dots 0$ and $e_1 = 10 \dots 0$, $e_2 = 010 \dots 0$, \dots , $e_n = 00 \dots 1$.

Theorem 1 *If S is a t -PDDS in A_n then each component of S is the Cartesian product of (possibly infinite) paths.*

Proof Let S_0 be a component of S in A_n . Assume that S_0 is not a product of paths. Then wlog we may assume that $O, e_1 + e_2 \in S_0$, and $e_1 \notin V(S_0)$. Now, $d(e_1, S_0) = d(e_1, O) = d(e_1, e_1 + e_2) = 1$. That is, the vertex v is at the minimum distance 1 from two different vertices of S , a contradiction.

A similar result, in the case when PDS of the n -dimensional cube were considered, has been proved in [28].

With respect to the application mentioned above we will confine ourselves to the most interesting case of t -PDDSs in A_n whose components are all isomorphic to a fixed finite graph H , denoted for short by t -PDDS[H]. It would be very useful to characterize all finite graphs H for which there is a t -PDDS[H]. This would show the strength but also limitations of the new concept for practical purposes. So far we are able to do it only for A_2 .

Remark 1 We point out that if R is a t -PDDS[H], $H = (V, E)$, then R can be seen as a tiling of \mathbb{Z}^n by the graph $H^* = (V^*, E^*)$ where H^* is the induced subgraph of A_n on the set V^* , where $v \in V^*$ if and only if $d(v, V) \leq t$.

As usual P_k will stand for the path on k vertices. Hence, P_1 is an isolated vertex. Further, the cartesian product of graphs G and H is denoted by $G \square H$. At the moment we do not have enough evidence to conjecture when a t -PDDS[H] exists in a general case. However, if H is a product of at most two paths then we strongly believe that:

Conjecture 1 Let H be a finite path or a Cartesian product of two finite paths. Then a t -PDDS[H] in Λ_n exists if and only if either (i) $t = 1$, $n \geq 2$, and $H = P_k$, $k \geq 1$; or (ii) $t \geq 1$, $n = 2$, and $H = P_k$, $k \geq 1$; or (iii) $t \geq 1$, $n = 2$, and $H = P_2 \square P_k$, $k \geq 2$; or (iv) $t = 1$, $n = 3r + 2$, $r \geq 0$, and $H = P_2 \square P_2$; or (v) $t = 2$, $n = 3$, and $H = P_2$.

We note that (i) and (ii) extend Golomb-Welch conjecture as well as a conjecture raised in [14] by Etzion. For $k = 1$, the existence of a t -PDDS[P_k] in (i) and (ii) was shown by several authors in terms of PLC codes, see e.g. Golomb and Welch [16], and, for $k = 2$, by Etzion [14] in terms of diameter perfect Lee codes. The existence of a 2-PDDS[P_2] in Λ_3 follows from a Minkowski's tiling [22].

The next theorem constitutes one of the main results of the paper.

Theorem 2 *A t -PDDS[H] exists for all graphs H described in Conjecture 1.*

The following theorem provides additional supporting evidence for Conjecture 1.

Theorem 3 *If $3 \leq s \leq r$ then there is no t -PDDS[$P_s \square P_r$] in Λ_2 for $t \geq 1$.*

Corollary 1 *A t -PDDS[H] in Λ_2 exists if and only if either $t \geq 1$, and $H = P_k$, $k \geq 1$, or $t \geq 1$, and $H = P_2 \square P_k$, $k \geq 2$.*

To show that a t -PDDS[H] exists also in the case when H is the Cartesian product of at least three paths we offer the following theorem:

Theorem 4 *There is a 1-PDDS[Q_3] in Λ_3 , where $Q_3 = P_2 \square P_2 \square P_2$ is the 3-dimensional hypercube.*

Recently we learnt that Buzaglo and Etzion proved that a 1-PDDS[Q_n] exists if and only if $n = 2^k - 1$, or $n = 3^k - 1$, c.f. [7]. They proved the statement in terms of tilings by crosses; see Remark 1.

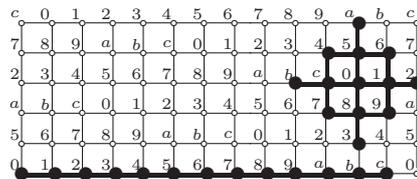
All t -PDDSs constructed in this paper are lattice-like, which is a very important feature from the practical point of view as in this case decoding algorithms tend to be much simpler. As the notion of lattice-like PDDS is a key one we provide a formal definition. Let $H = (V, E)$ be a subgraph of Λ_n , and let $z \in \mathbb{Z}^n$. Then $H + z$ denotes the graph $H' = (V', E')$, where $V' = V + z = \{w; \text{there exists } v \in V, w = v + z\}$, and $uv \in E$ if and only if $(u + z)(v + z) \in E'$. Let R be a t -PDDS[H] and $D \simeq H$ be a component of R . Then R will be called *lattice-like* if there exists a lattice L such that D' is a component of R

if and only if there is $z \in L$ so that $D' = D + z$. We recall, see Remark 1, that a t -PDDS[H] can be seen as a tiling. Thus a notion of a lattice-like tiling will be understood in the same way as a lattice-like PDDS.

All desired t -PDDS in Λ_n will be constructed by the same algebraic method. A PDDS constructed this way is lattice-like, which in turn implies that such a PDDS is periodic as well. That is, a suitable restriction of this PDDS constitutes a PDDS in the Cartesian product of cycles. This is the case of main interest because of the placement problem discussed above. We recall that a set $S \subset \mathbb{Z}^n$ is *periodic* if there are integers p_1, \dots, p_n such that $v \in S$ implies $v \pm p_i e_i \in S$ for all $i = 1, \dots, n$, where e_i is the unit vector in the direction of the i -axis. We recall that each lattice-like t -PDDS is periodic, but the converse is not true in general.

Now we describe a construction of a partition (tiling) of Λ_n . As far as we know Stein in [26] was the first one to use a group homomorphism to construct a lattice-like tiling; he did it in the case of a tiling by different types of crosses. Several variations of Stein’s construction can be found throughout the literature, see e.g. [26, 23, 27, 25, 17, 8, 24, 18]. For the reader’s convenience we provide a detailed description of this generalization. Let $(\mathbb{Z}^n, +)$ be the (component-wise) additive group on \mathbb{Z}^n . Consider a lattice L in $(\mathbb{Z}^n, +)$, i.e. a subgroup of $(\mathbb{Z}^n, +)$, generated by elements $u_1, \dots, u_n \in \mathbb{Z}^n$; hence $L = \{\alpha_1 u_1 + \dots + \alpha_n u_n; \alpha_i \in \mathbb{Z}, i = 1, \dots, n\}$. We denote by F the factor group $(\mathbb{Z}^n, +)/L$. Furthermore, let a set T of vertices in \mathbb{Z}^n contain one element from each coset of $(\mathbb{Z}^n, +)/L$. Then, $\mathcal{T} = \{T + u; u \in L\}$ constitutes a partition of \mathbb{Z}^n into parts of size $|F|$ and, for each $u \in L$, we have that $[T + u]$, the subgraph of Λ_n induced by $T + u$, is isomorphic to $[T]$. Clearly, for a given lattice L , we can partition the vertex set of Λ_n into parts such that the corresponding induced subgraphs have different shapes depending on the choice of T .

Example. Set $L = \{\alpha_1(13, 0) + \alpha_2(3, 2); \alpha_i \in \mathbb{Z}, i = 1, 2\}$. Then, $(\mathbb{Z}^2, +)/L = Z_{13}$. There are many options how to choose the graph $[T]$, e.g., $[T]$ might be a path of length 12, or a Lee sphere of radius 2, see the figure below where the both options are depicted in bold font. The numbers at the vertices of Λ_2 are elements of $Z_{13} = (\mathbb{Z}^2, +)/L$.



However, for our purpose, we will utilize an “inverse” process. Given an induced subgraph $D = (V, E)$ of Λ_n , find a partition (tiling) of Λ_n into copies

of D . Here we mean partitioning of the vertex set of Λ only, see Remark 1. Hence we need to find a suitable lattice L that would allow the required choice of the set T , i.e. $[T] = D$. It turns out that to do so one does not have to find the lattice L explicitly. We will show that the following construction leads to the desired tiling of Λ_n . We claim that if there exists an Abelian group $(G, +)$ of order $|V|$ and elements g_1, \dots, g_n of G such that the restriction of the homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$, $\Phi((a_1, \dots, a_n)) = a_1\Phi(e_1) + \dots + a_n\Phi(e_n) = a_1g_1 + \dots + a_ng_n$, to V is a bijection then there exists a partition of Λ_n into copies of D . In other words, we need to find an Abelian group G of order $|V|$ and assign elements g_1, \dots, g_n of G to the vertices e_1, \dots, e_n of Λ_n so that $\Phi((a_1, \dots, a_n)) = a_1\Phi(e_1) + \dots + a_n\Phi(e_n) = a_1g_1 + \dots + a_ng_n$, is a bijection on V . It is well known, that the ker of a homomorphism $\phi : A \rightarrow B$ is a subgroup of A . Thus, the elements w of \mathbb{Z}^n for which $\Phi(w) = 0$ form a lattice L in $(\mathbb{Z}^n, +)$. In addition, $(\mathbb{Z}^n, +)/L = G$ and the vertex set V comprises exactly one element from each coset of $(\mathbb{Z}^n, +)/L$; thus we can set $T = V$.

As the above method is the main tool in this paper, we summarize it as Corollary 2 (to Theorem 5 below)

Theorem 5 [19] Let $D = (V, E)$ be a subgraph of Λ_n . Then there is a lattice-like tiling of Λ_n by copies of D if and only if there is an Abelian group (G, \circ) and a homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$, so that the restriction of Φ to V is a bijection.

If the restriction of Φ to V is an injection, then Theorem 5 (in which D need not be connected) produces a packing of Λ_n by copies of D . This idea has been used in several papers, see e.g. [25, 17, 24]. The following corollary of Theorem 5 is tailored to our present needs:

Corollary 2 Let $t \geq 1$ and let H be a subgraph of Λ_n . Further, let H^* be an induced supergraph of H such that a vertex v belongs to H^* if and only if $d(v, H) \leq t$; let $D = (V, E)$ be a copy of H^* or a copy of a disjoint union of finitely many copies of H^* that contains vertices O, e_1, \dots, e_n . Then, there is a t -PDDS[H] if there exists an Abelian group G of order $|V|$ and a homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ such that the restriction of Φ to V is a bijection.

Remark 2 We will always choose D to contain vertices O, e_1, \dots, e_n . This is not a necessary condition but it will be added to simplify the exposition. A t -PDDS[H] constructed by means of Corollary 2 is lattice-like if D is isomorphic to H^* . If D consists of more copies of H^* , then we get a lattice tiling of \mathbb{Z}^n by D but this will not constitute a lattice-like t -PDDS[H].

The rest of the paper is organized as follows. Section 2 contains a proof of Theorem 2, while a proof of (i) of Theorem 3 will be given in Section 3. Theorem 4 will be proved in Section 4. To demonstrate the strength of the construction, in Section 5 we present a periodic 1-PDDS in Λ_2 that is not lattice-like.

2 Existence of t -PDDSs

In this section we prove Theorem 2, that is we prove the existence of t -PDDSs as described in Conjecture 1. For the sake of completeness we note that a Minkowski's tiling that proves part (v) can be obtained by Corollary 2 using the group $G = \mathbb{Z}_{38}$ and the homomorphism given by $\Phi(e_1) = 1$, $\Phi(e_2) = 11$ and $\Phi(e_3) = 7$.

2.1 Part (i)

Here we deal with the case when each component of a 1-PDDS is isomorphic to a path P_k of length $k - 1$, where $k \geq 2$. We start with the case when each component of a t -PDDS is an isolated vertex. Each 1-PDDS $[P_1]$ in A_n corresponds to a perfect 1-error correcting Lee code, PLC $(n, 1)$. The existence of such codes has been showed independently by several authors. Kártesz asked whether there exists a PLC $(3, 1)$. Feller, for $n = 3$, and then Korchmáros, and Golomb and Welch [16] showed that there is a PLC $(n, 1)$ for all $n \geq 2$. The following stronger theorem has been proved by Molnár [23].

Theorem 6 *The number of non-congruent lattice-like PLC $(n, 1)$ codes equals the number of Abelian groups of order $2n + 1$.*

To illustrate our method we prove the theorem. The following proof is shorter than the original one due to Molnár. Since in this case H is an isolated vertex, the graph H^* is of order $2n + 1$. We choose a copy of $D = (V, E)$ of H^* such that $V = \{\pm e_i; i = 1, \dots, n\} \cup \{O\}$. Let G be an Abelian group of order $2n + 1$. Choose a set $K = \{g_1, \dots, g_n\}$ formed by n distinct elements of G such that K contains exactly one element from each pair g, g^{-1} ; formally, $g \in K$ if and only if $g^{-1} \notin K$. Since no element of G is of order 2, the set K is well defined. Clearly, the restriction of the homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ given by $\Phi((a_1, \dots, a_n)) = \Phi(e_1)^{a_1} \circ \dots \circ \Phi(e_n)^{a_n}$ to V is a bijection. Thus, each Abelian group of order $2n + 1$ generates a PLC $(n, 1)$; this code is a periodic code where p_i s are orders of elements of G . It is not difficult to check that non-isomorphic groups generate non-congruent PLC $(n, 1)$ codes.

We note that Szabó [27] constructed, in the case when $2n + 1$ is not a prime, the first non-lattice-like PLC $(n, 1)$ code. This code is periodic though. In [18], for the same case, the first non-periodic PLC $(n, 1)$ code has been found. It has also been shown in [18] that there is a unique PLC $(n, 1)$ code for $n = 2, 3$.

The existence of 1-PDDS $[P_2]$ (called total perfect codes in [20]) has been proved in [14] in terms of diameter perfect codes.

Theorem 7 *A 1-PDDS $[P_k]$ in A_n exists for each $n \geq 2$ and each $k \geq 1$.*

Proof We will construct the desired PDDS by applying Corollary 2. Set $H = P_k$. We place the graph $D = (V, E)$ that is isomorphic to H^* in such a way that V comprises the vertices $O, e_1, 2e_1, \dots, (k-1)e_1$ of the path P_k and their $2nk - 2k + 2$ neighbors, namely $-e_1, ke_1$ and $\pm e_i, e_1 \pm e_i, \dots, (k-1)e_1 \pm e_i$ for $i = 2, \dots, n$. Thus, $|V| = 2nk - k + 2$ and D contains the vertices O and e_i , for $i = 1, \dots, n$, as required by Corollary 2. We choose $G = \mathbb{Z}_{2nk-k+2}$. The element g_i of G that is assigned to the vertex e_i , for $i = 1, \dots, n$, is $g_i = (i-1)k + 1$. To finish the proof, we need to show that the restriction of the mapping $\Phi((a_1, \dots, a_n)) = \Phi(e_1)^{a_1} \circ \dots \circ \Phi(e_n)^{a_n} = a_1g_1 + \dots + a_ng_n$ to the set V is a bijection. To see this, it suffices to note that $\Phi\{O, e_1, 2e_1, \dots, (k-1)e_1\} = \{0, 1, \dots, k-1\}$, $\Phi\{-e_1, ke_1\} = \{k, 2nk - k + 1\}$, and $\Phi\{\pm e_i, e_1 \pm e_i, \dots, (k-1)e_1 \pm e_i\} = \{\pm(i-1)k + 1, \pm(i-1)k + 2, \dots, \pm(i-1)k + k - 1, \pm ik\}$. In aggregate, $\Phi(V) = \{0, \dots, k\} \cup \bigcup_{i=2}^n \{(i-1)k + 1, \dots, ik\} \cup \bigcup_{i=2}^n \{(2n-i)k + 1, \dots, (2n-i-1)k + 2\} \cup \{2nk - k + 1\} = \{0, \dots, 2nk - k + 1\} = G$. For the reader convenience we illustrate the proof by means of three small examples for $k = 3$:

	$\langle e_1, e_2 \rangle$	$\langle e_1, e_3 \rangle$	$\langle e_1, e_4 \rangle$
$n=2$ \mathbb{Z}_{11}	$\begin{matrix} 7 & 8 & 9 \\ 10 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 4 & 5 & 6 \end{matrix}$		
$n=3$ \mathbb{Z}_{17}	$\begin{matrix} 13 & 14 & 15 \\ 16 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 4 & 5 & 6 \end{matrix}$	$\begin{matrix} 10 & 11 & 12 \\ 16 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 7 & 8 & 9 \end{matrix}$	
$n=4$ \mathbb{Z}_{23}	$\begin{matrix} 19 & 20 & 21 \\ 22 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 4 & 5 & 6 \end{matrix}$	$\begin{matrix} 16 & 17 & 18 \\ 22 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 7 & 8 & 9 \end{matrix}$	$\begin{matrix} 13 & 14 & 15 \\ 22 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 \\ 10 & 11 & 12 \end{matrix}$

2.2 Part (ii)

In this subsection we prove the existence of a t -PDDS in Λ_2 whose components are all isomorphic to a path P_k , where $t > 1$ and $k > 1$.

Theorem 8 *A t -PDDS $[P_k]$ in Λ_2 exists for each $t \geq 1$ and $k \geq 1$.*

Proof We provide a detailed proof as we use the same approach to prove this and the next theorem. Let H be a path P_k on vertices $\{O, e_2, 2e_2, \dots, (k-1)e_2\}$. Then H^* consists of vertices of H plus all vertices at distance at most t from H ; hence $|H^*| = 2t^2 + 2tk + k$. Clearly, $xe_1 + ye_2 \in H^*$ iff

$$\begin{aligned}
 & -t \leq x < 0 \text{ and } -x - t \leq y \leq x + t + k - 1 \\
 & \text{or} \\
 & 0 \leq x \leq t \text{ and } x - t \leq y \leq -x + t + k - 1
 \end{aligned}$$

We will construct the desired PDDS by applying Corollary 2 so that the graph $D = (V, E)$ consists of two disjoint copies of H^* ; a copy described above and

a translation of this copy by $(t, t + k)$. Thus, the other copy of H^* is given by

$$0 \leq x \leq t \text{ and } -x + t + k \leq x + t + 2k - 1$$

or

$$t + 1 \leq x \leq 2t \text{ and } x - t + k \leq y \leq -x + 3t + 2k - 1$$

In aggregate, $|V| = 4t^2 + 4tk + 2k$, and a vertex $xe_1 + ye_2 \in V$ iff

$$-t \leq x < 0 \text{ and } -x - t \leq y \leq x + t + k - 1 \quad (1)$$

$$\text{either} \quad (2)$$

$$0 \leq x \leq t \text{ and } x - t \leq y \leq x + t + 2k - 1 \quad (3)$$

$$\text{or} \quad (4)$$

$$t + 1 \leq x \leq 2t \text{ and } x - t + k \leq y \leq -x + 3t + 2k - 1 \quad (5)$$

To construct the desired lattice-like PDDS we choose the cyclic group $G = \mathbb{Z}_{4t^2+4tk+2k}$ and set $g_1 = 2t + 2k - 1$, and $g_2 = 1$. Hence $\Phi(xe_1 + ye_2) = ((2t + 2k - 1)x + y) \bmod (4t^2 + 4tk + 2k)$.

For fixed x , by (1), the set $I_x = \{y; xe_1 + ye_2 \in V\}$ is an interval. Therefore, as $g_2 = 1$, $\Phi(I_x)$ comprises $|I_x|$ consecutive elements of the group G , where we take that 0 follows the element $4t^2 + 4tk + 2k - 1$. To see that the mapping Φ is a bijection on V it is sufficient to show that the intervals $I_x, -t \leq x \leq 2t$ can be ordered in such a way that if I_z immediately precedes I_v in this order then $\Phi(\min I_v) = \Phi(\max I_z) + 1$. An order with this property is given implicitly below.

- (i) for each $-t \leq x \leq 0$, it is $\Phi(\min I_x) = \Phi(\max I_{x+2t}) + 1$;
- (ii) for each $1 \leq x \leq t$, it is $\Phi(\min I_x) = \Phi(\max I_{x-1}) + 1$;
- (iii) for each $t + 1 \leq x \leq 2t$, it is $\Phi(\min I_x) = \Phi(\max I_{-2t-1+x}) + 1$.

It is easy to prove (i)-(iii) by using (1) and simple calculations. For the readers convenience we work out details of (i). If $-t \leq x \leq 0$, then, from the first line of (1), $\Phi(\min I_x) = \Phi(xe_1 + (-x - t)e_2) = (x(2t + 2k - 1) + (-x - t)) \bmod (4t^2 + 4tk + 2k) = (2(t + k - 1)x - t) \bmod (4t^2 + 4tk + 2k)$.

For $-t + 1 \leq x \leq 0$, by the third line of (1), we get $\Phi(\max I_{x+2t}) = \Phi((x + 2t)e_1 + (-(x + 2t) + 3t + 2k - 1)e_2) = ((x + 2t)((2t + 2k - 1) + (-(x + 2t) + 3t + 2k - 1)) \bmod (4t^2 + 4tk + 2k) = ([2(t+k-1)x-t] + [4t^2+4tk+2k]-1) \bmod (4t^2+4tk+2k) = ([2(t+k-1)x-t]-1) \bmod (4t^2 + 4tk + 2k) = \Phi(\min I_x) - 1$.

Finally, for $x = -t$, by the second line of (1), $\Phi(\max I_{x+2t}) = \Phi((x + 2t)e_1 + (x + 2t + t + 2k - 1)e_2) = (t(2t + 2k - 1) + (2t + 2k - 1)) \bmod (4t^2 + 4tk + 2k) =$

1. $G = \mathbb{Z}_{2(t+1)(t+k)}$, $g_1 = t + 1$ and $g_2 = t + k$, if $m = 1$;
2. $G = \mathbb{Z}_m \times \mathbb{Z}_n$, where $n = \frac{2(t+1)(t+k)}{m}$, if $m \neq 1$; now take:
 - (a) $g_1 = (1, n)$ and $g_2 = (0, 1)$, if $m|t + k$;
 - (b) $g_1 = (1, \frac{n}{2(2t+1)})$ and $g_2 = (1, \frac{2t+1}{m})$, otherwise.

We leave the details of the proof of this approach of Theorem 9 to the reader and just give three small examples of it, for $(t, k) = (2, 2)$, $(2, 4)$, $(3, 3)$, where $G = \mathbb{Z}_{24}$, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_{24}$, respectively:

$$\begin{array}{cccc}
 & 6 & 10 & \\
 11 & 15 & 19 & 23 \\
 16 & 20 & \mathbf{0} & \mathbf{4} \quad 8 \quad 12 \\
 1 & 5 & \mathbf{9} & \mathbf{13} \quad 17 \quad 21 \\
 14 & 18 & 22 & 26 \\
 & 3 & 7 &
 \end{array}
 \left\| \begin{array}{cccc}
 & & 0,10 & 1,0 \\
 & 2,9 & 0,11 & 1,1 \quad 2,3 \\
 1,8 & 2,10 & \mathbf{0,0} & \mathbf{1,2} \quad 2,4 \quad 0,6 \\
 1,9 & 2,1 & \mathbf{0,1} & \mathbf{1,3} \quad 2,5 \quad 0,7 \\
 1,10 & 2,2 & \mathbf{0,2} & \mathbf{1,4} \quad 2,6 \quad 0,8 \\
 1,11 & 2,3 & \mathbf{0,3} & \mathbf{1,5} \quad 2,7 \quad 0,9 \\
 & 2,4 & \mathbf{0,4} & \mathbf{1,6} \quad 2,8 \\
 & & \mathbf{0,5} & \mathbf{1,7}
 \end{array} \right\|
 \begin{array}{cccc}
 & & & 1,18 \quad 0,21 \\
 & & 1,17 & 0,20 \quad 1,23 \quad 0,2 \\
 & 1,16 & 0,19 & 1,22 \quad 0,1 \quad 1,4 \quad 0,7 \\
 1,15 & 0,18 & 1,21 & \mathbf{0,0} \quad \mathbf{1,3} \quad 0,6 \quad 1,9 \quad 0,12 \\
 0,17 & 1,20 & 0,23 & \mathbf{1,2} \quad \mathbf{0,5} \quad 1,8 \quad 0,11 \quad 1,14 \\
 1,19 & 0,22 & 1,1 & \mathbf{0,4} \quad \mathbf{1,7} \quad 0,10 \quad 1,13 \quad 0,16 \\
 & & 1,0 & 0,3 \quad 1,6 \quad 0,9 \quad 1,12 \quad 0,15 \\
 & & & 1,5 \quad 0,8 \quad 1,11 \quad 0,14 \\
 & & & 1,10 \quad 0,13
 \end{array}$$

2.4 Part (iv)

In this subsection we discuss the existence of t -PDDSs in A_n whose components are isomorphic to $P_2 \square P_2$. Note that for $n = 2$ this case overlaps with the previous part.

Theorem 10 *Let $n = 3k + 2$, where $k \geq 0$. Then, there exists a lattice-like 1-PDDS in A_n whose components are isomorphic to $P_2 \square P_2$.*

Proof We will construct the desired PDDS by applying Corollary 2. Set $H = P_2 \square P_2$. We place the graph $D = (V, E)$ that is isomorphic to H^* in such a way that V comprises the vertices O , e_1 , e_2 and $e_1 + e_2$ and their $24k + 8$ neighbors; namely, $-e_1$, $2e_1$, $e_2 - e_1$, $e_2 + 2e_1$, $-e_2$, $2e_2$, $e_1 - e_2$, $e_1 + 2e_2$, and, if $k > 0$, then also vertices $\pm e_i$, $e_1 \pm e_i$, $e_2 \pm e_i$ and $e_1 + e_2 \pm e_i$ for $i = 3, \dots, 3k + 2$. Thus, $|V| = 24k + 12$, and D contains the vertices O and e_i , for $i = 1, \dots, n$, as required by Corollary 2. We set $G = \mathbb{Z}_{24k+12}$. The elements g_i of G that are assigned to the vertices e_i , for $i = 1, \dots, n$, are: $g_1 = 2 + 4k$, $g_2 = 3 + 6k$, and, if $k > 0$, then $g_{2+i} = 2 + 4k + i$, $g_{2+k+i} = 2 + 4k - i$ and $g_{2+2k+i} = 6 + 11k + i$, for $i = 1, \dots, k$. To finish the proof, we need to show that the restriction of the mapping $\Phi((a_1, \dots, a_n)) = \Phi(e_1)^{a_1} \circ \dots \circ \Phi(e_n)^{a_n} = a_1 g_1 + \dots + a_n g_n$ to the set V is a bijection. To see this, it suffices to check the table below (broken into two parts to be pasted together horizontally) that shows that each element of \mathbb{Z}_{24k+12} belongs to the set $\Phi(V)$. In the table the symbol $[a, b]$ stands for the set $\{a, a + 1, a + 2, \dots, b\}$. In all cells of the table, the index i runs through the interval $[1, 12 + 24k]$, where $12 + 24k \equiv 0$ in $G = \mathbb{Z}_{24k+12}$ and integers on the columns corresponding to G shown in increasing order from left to right, line

by line, and then from top to bottom:

V	$\Phi(V)$	G	
$e_1 - e_{2+k+i}$	i	$[1, k]$...
$e_2 - e_{2+k+i}$	$1+2k+i$	$[2+2k, 1+3k]$...
e_{1+k+i}	$2+4k+i$	$[3+4k, 2+5k]$...
$e_1 + e_2 - e_{2+k+i}$	$3+6k+i$	$[4+6k, 3+7k]$...
$e_1 + e_2 + i$	$4+8k+i$	$[5+8k, 4+9k]$...
$e_2 + e_{2+i}$	$5+10k+i$	$[6+10k, 5+11k]$...
$-e_2 + 2k+i$	$7+13k-i$	$[7+12k, 6+13k]$...
$e_1 + e_2 + e_{2+i}$	$7+14k+i$	$[8+14k, 7+15k]$...
$e_2 + e_{2+k+i}$	$9+17k-i$	$[9+16k, 8+17k]$...
$e_2 - e_{2+2k+i}$	$10+19k-i$	$[10+18k, 9+19k]$...
$-e_2 + k+i$	$10+20k+i$	$[11+20k, 10+21k]$...
$e_1 + e_2 - e_{2+2k+i}$	$12+23k-i$	$[12+22k, 11+23k]$...

V	$\Phi(V)$	G	V	$\Phi(V)$	G
...	$e_2 - e_{2+i}$	$[k+1, 2k]$	$e_2 - e_1$	$1+2k$	$1+2k$
...	e_{2+k+i}	$[2+3k, 1+4k]$	e_1	$2+4k$	$2+4k$
...	$e_1 + e_2 - e_{2+i}$	$[3+5k, 2+6k]$	e_2	$3+6k$	$3+6k$
...	$e_1 + e_2 + k+i$	$[4+7k, 3+8k]$	$2e_1$	$4+8k$	$4+8k$
...	$e_2 + e_{2+k+i}$	$[5+9k, 4+10k]$	$e_1 + e_2$	$5+10k$	$5+10k$
...	$e_2 + 2k+i$	$[6+11k, 5+12k]$	$2e_2$	$6+12k$	$6+12k$
...	$e_1 + e_2 + e_{2+k+i}$	$[7+13k, 6+14k]$	$2e_1 + e_2$	$7+14k$	$7+14k$
...	$e_1 + e_2 + 2k+i$	$[8+15k, 7+16k]$	$e_1 + 2e_2$	$8+16k$	$8+16k$
...	$e_2 + e_{2+2k+i}$	$[9+17k, 8+18k]$	$-e_2$	$9+18k$	$9+18k$
...	$-e_2 + i$	$[10+19k, 9+20k]$	$-e_1$	$10+20k$	$10+20k$
...	$e_1 + e_2 + e_{2+2k+i}$	$[11+21k, 10+22k]$	$e_1 - e_2$	$11+22k$	$11+22k$
...	$e_1 - e_{2+i}$	$[12+23k, 11+24k]$	O	$12+24k$	$12+24k$

As usual at the end of the proof we provide three small examples for $n = 2, 5,$ and $8,$ to illustrate it.

$\langle e_1, e_2 \rangle$			
	9	11	
10	0	2	4
1	3	5	7
	6	8	

$\langle e_1, e_2 \rangle$	$+ e_3$	$- e_3$	$+ e_4$	$- e_4$	$- e_5$	$+ e_5$
30	27	33	7	13	29	35
3	0	6	12	16	22	2
	9	15	21	2	8	14
	18	24		4	10	31
				1	17	23
				26	32	19
				28	34	25

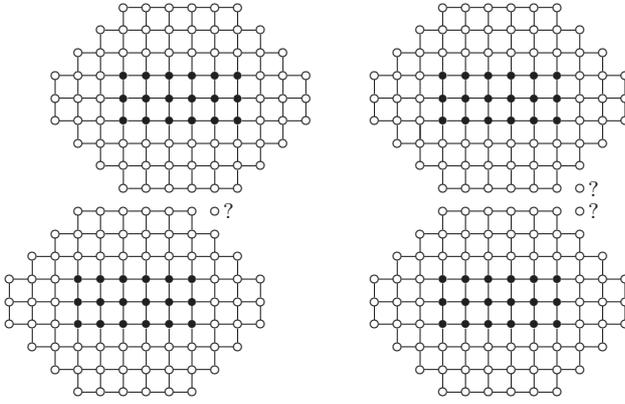
$\langle e_1, e_2 \rangle$	$+ e_3$	$- e_3$	$+ e_5$	$- e_5$	$- e_7$	$+ e_7$
50	45	55	11	21	49	59
5	0	10	20	26	36	4
	15	25	35	4	14	24
	30	40		24	34	6
				6	16	29
				44	54	39
				46	56	31
				41	51	1

	$+ e_4$	$- e_4$	$+ e_6$	$- e_6$	$- e_8$	$+ e_8$
	12	22	8	18	52	2
	27	37	3	13	23	33
			7	17	43	53
			28	38	32	42
			47	57	47	57

3 Proof of Theorem 3

In this section we prove Theorem 3.

Proof Suppose that there is a t -PDDS R in Λ_2 whose components are isomorphic to $P_k \square P_s$, where $k \geq s \geq 3$. Let H^* be an induced subgraph of Λ_n comprising the vertices of a copy H of $P_k \square P_s$ and all vertices at distance at most t from H . Clearly R generates a decomposition of \mathbb{Z}^2 into copies of H^* . Although R is not necessarily lattice-like, all components of R have to be either "parallel" to the x -axis, or to be "parallel" to the y -axis. Assume wlog that R contains a component $P_k \square P_s$ comprising vertices (x, y) , where $1 \leq x \leq k, t+1 \leq y \leq t+s$; see the figure below for examples of this situation for $k=6, s=3$ and $t=3$. Consider a set of vertices $A = \{(x, 0), 1 \leq x \leq k\}$. We will show that the vertices of A cannot be covered by vertex-disjoint copies of H^* . Assume that a copy of H^* covers only vertices $(x, 0), 1 \leq x \leq m, m < k$, see the left example below, where $m=4$. Then the vertex $(m+1, 0)$ cannot be covered in R . However, if all vertices in A are covered in R by the same copy of H^* (in this case the two copies of H^* have to be "parallel" as $k \geq s$), then the vertices $(k+1, 0)$ and $(k+1, 1)$ can be covered only if $s=2$, a contradiction as we consider the case $s \geq 3$. See the right example in the figure.



4 Proof of Theorem 4

Proof We will construct the desired PDDS by applying Corollary 2. Set $H = Q_3$. We place the graph $D = (V, E)$ that is isomorphic to H^* in such a way that V comprises the vertices $O, e_1, e_2, e_3, e_1+e_2, e_1+e_3, e_2+e_3$ and $e_1+e_2+e_3$ of Q_3 and their 24 neighbors. Thus, $|V| = 32$, and D contains the vertices O and e_i , for $i = 1, 2, 3$, as required by Corollary 2. We choose $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$. The elements g_i of G that are assigned to the vertices e_i are: $g_1 = 1, 3, 3, g_2 = 0, 1, 0$ and $g_3 = 0, 0, 1$. To finish the proof, we need to show that the restriction of the mapping $\Phi((a_1, a_2, a_3)) = \Phi(e_1)^{a_1} \circ \Phi(e_2)^{a_2} \circ \Phi(e_3)^{a_3} = a_1 g_1 + a_2 g_2 + a_3 g_3$ to the set V is a bijection. For the reader's convenience we provide all values of Φ on V in a table below. It suffices to note that all these values are distinct. The

Thus Corollary 2 provides a decomposition of \mathbb{Z}^2 into parts of order 32, each of them isomorphic to H^* . Further, as H^* can be decomposed into four copies of P_2 and its neighbors, we have constructed a 1-PDDS[P_2] R that is not lattice-like. However, it is straightforward that R is periodic. Therefore we have proved:

Theorem 11 *There exists a periodic non-lattice-like 1-PDDS[P_2] in Λ_2 .*

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