# PROOFS OF TWO CONJECTURES ON THE DIMENSIONS OF BINARY CODES 

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#### Abstract

Let $\mathcal{L}$ and $\mathcal{L}_{0}$ be the binary codes generated by the column $\mathbb{F}_{2}$-null spaces of the incidence matrices of external points versus passant lines and internal points versus secant lines with respect to a conic in $\mathrm{PG}(2, q)$, respectively. We confirm the conjectures on the dimensions of $\mathcal{L}$ and $\mathcal{L}_{0}$ using methods from both finite geometry and modular representation theory.


## 1. Introduction

Let $\mathbb{F}_{q}$ be the finite field of order $q$, where $q=p^{e}, p$ is a prime and $e \geq 1$ is an integer. Let $\operatorname{PG}(2, q)$ denote the classical projective plane of order $q$ represented via homogeneous coordinates. Namely, a point $\mathbf{P}$ of $\mathrm{PG}(2, q)$ can be written as $\left(a_{0}, a_{1}, a_{2}\right)$, where $\left(a_{0}, a_{1}, a_{2}\right)$ is a non-zero vector of $V$, and a line $\ell$ as $\left[b_{0}, b_{1}, b_{2}\right]$, where $b_{0}, b_{1}, b_{2}$ are not all zeros. The point $\mathbf{P}=\left(a_{0}, a_{1}, a_{2}\right)$ lies on the line $\ell=\left[b_{0}, b_{1}, b_{2}\right]$ if and only if

$$
a_{0} b_{0}+a_{1} b_{1}+a_{2} b_{2}=0
$$

A non-degenerate conic in $\operatorname{PG}(2, q)$ is the set of points satisfying a non-degenerate quadratic form. It is well known that the set of points

$$
\begin{equation*}
\mathcal{O}=\left\{\left(1, t, t^{2}\right) \mid t \in \mathbb{F}_{q}\right\} \cup\{(0,0,1)\}, \tag{1.1}
\end{equation*}
$$

which is also the set of projective solutions of the non-degenerate quadratic form

$$
\begin{equation*}
Q\left(X_{0}, X_{1}, X_{2}\right)=X_{1}^{2}-X_{0} X_{2} \tag{1.2}
\end{equation*}
$$

over $\mathbb{F}_{q}$, gives rise to a standard example of a non-degenerate conic in $\operatorname{PG}(2, q)$. It can be shown that every non-degenerate conic must has $q+1$ points and no three of them are collinear, which forms an oval (see [8, P. 157]). In the case where $q$ is odd, Segre [16] proved that an oval in $\mathrm{PG}(2, q)$ must be a non-degenerate conic. In this paper, $q=p^{e}$ is always assumed to be an odd prime power. For convenience, we fix the conic in (1.1) as the "standard" conic. A line $\ell$ is passant, tangent, or secant accordingly as $|\ell \cap \mathcal{O}|=0$, 1 , or 2 , respectively. It is clear that every line of $\operatorname{PG}(2, q)$ must be in one of these classes. A point P is an internal, absolute, or external point depending on whether it lies on 0 , 1 , or 2 tangent lines to $\mathcal{O}$. The sets of secant, tangent, and passant lines are denoted by $S e, T$ and $P a$, respectively; the sets of external and internal points are denoted by $E$ and $I$, respectively. The sizes of these sets are $|S e|=|E|=\frac{q(q+1)}{2},|P a|=|I|=\frac{q(q-1)}{2}$, and $|T|=q+1$ (see (2.2)). Moreover, it can be shown that the quadratic form $Q$ in (1.1) induces a polarity $\sigma$, a correlation of order 2 , under which $E$ and $S e, O$ and $T$, and $I$ and $P a$ are in one-to-one correspondence with each other, respectively.

Let $\mathbf{C}$ be a $0-1$ matrix; that is, $\mathbf{C}$ is a matrix whose entries are either 0 or 1 . Note that $\mathbf{C}$ can be viewed as a matrix over any ring with 1 . The $p$-rank of $\mathbf{C}$, denoted by $\operatorname{rank}_{p}(\mathbf{C})$, is defined to be the dimension of the column space of $\mathbf{C}$ over a field $F$ of characteristic $p$.

[^0]The column null space of $\mathbf{C}$ over $F$ determines a linear code whose dimension is defined to be the dimension of the corresponding column null space of $\mathbf{C}$ over $F$.

Let A be the $\left(q^{2}+q+1\right) \times\left(q^{2}+q+1\right)$ point-line incidence matrix of $\operatorname{PG}(2, q)$; namely, $\mathbf{A}$ is a $0-1$ matrix and the rows and columns of $\mathbf{A}$ are labeled by the points and lines of $\mathrm{PG}(2, q)$, respectively, and the $(\mathbf{P}, \ell)$-entry of $\mathbf{A}$ is 1 if and only if $\mathbf{P} \in \ell$. It can be shown that the 2-rank of $\mathbf{A}$ is $q^{2}+q[9]$ and the $p$-rank of $\mathbf{A}$ is $\binom{p+1}{2}^{e}\left[1\right.$, where $q=p^{e}$. The binary linear code generated by the column $\mathbb{F}_{2}$-null space of $\mathbf{A}$ has dimension 1. Therefore, it is not useful for any practical purpose.

In [5], Droms, Mellinger and Meyer partitioned $\mathbf{A}$ into the following 9 submatrices:

$$
\left(\begin{array}{lll}
\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13}  \tag{1.3}\\
\mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\
\mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33}
\end{array}\right)
$$

where the block of rows for $\left(\mathbf{A}_{11}, \mathbf{A}_{21}, \mathbf{A}_{31}\right)$ are labeled by the absolute, internal, and external points, respectively, and the block of columns for $\left(\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{13}\right)$ are labeled by the tangent, passant, and secant lines, respectively. They used the column null spaces of the submatrices $\mathbf{A}_{i, j}$ for $2 \leq i, j \leq 3$ over $\mathbb{F}_{2}$ to construct four low-density parity-check (LDPC) codes. Based on computational evidence, they made a conjecture on the dimensions of these codes. For convenience, we denote $\mathbf{A}_{23}$ and $\mathbf{A}_{32}$ by $\mathbf{B}$ and $\mathbf{B}_{0}$, respectively. From (1.3), it follows that $\mathbf{B}$ and $\mathbf{B}_{0}$ are the incidence matrices of internal points versus secant lines and external points versus passant lines, respectively. Note that $\mathbf{B}$ is a $\frac{q(q+1)}{2} \times \frac{q(q-1)}{2}$ matrix and $\mathbf{B}_{0}$ is a $\frac{q(q-1)}{2} \times \frac{q(q+1)}{2}$ matrix. The purpose of this article is to confirm the following conjecture on the dimensions of the LDPC codes $\mathcal{L}$ and $\mathcal{L}_{0}$ arising from the column $\mathbb{F}_{2}$-null spaces of $\mathbf{B}$ and $\mathbf{B}_{0}$, respectively.

Conjecture 1.1. (Droms, Mellinger and Meyer [5) Let $\mathcal{L}$ and $\mathcal{L}_{0}$ be the $\mathbb{F}_{2}$-null spaces of $\mathbf{B}$ and $\mathbf{B}_{0}$, respectively. Then

$$
\operatorname{dim}_{\mathbb{F}_{2}}(\mathcal{L})= \begin{cases}\frac{q^{2}-1}{4}-q, & \text { if } q \equiv 1 \quad(\bmod 4), \\ \frac{2^{2}-1}{4}-q+1, & \text { if } q \equiv 3 \quad(\bmod 4)\end{cases}
$$

and

$$
\operatorname{dim}_{\mathbb{F}_{2}}\left(\mathcal{L}_{0}\right)= \begin{cases}\frac{q^{2}-1}{4}, & \text { if } q \equiv 1 \quad(\bmod 4) \\ \frac{q^{2}-1}{4}+1, & \text { if } q \equiv 3 \quad(\bmod 4)\end{cases}
$$

Suppose that $\mathbf{P}_{1}, \ldots, \mathbf{P}_{q(q+1) / 2}$ and $\ell_{1}, \ldots, \ell_{q(q-1) / 2}$ are indexing the rows and columns of $\mathbf{B}$, respectively. Then we permute the rows and columns of $\mathbf{B}_{0}$ to obtain a new matrix $\mathbf{C}$ such that the rows and columns of $\mathbf{C}$ are indexed by $\ell_{1}, \ldots, \ell_{q(q-1) / 2}$ and $\mathbf{P}_{1}, \ldots, \mathbf{P}_{q(q+1) / 2}$, respectively. The matrix $\mathbf{C}$ is indeed equal to $\mathbf{B}^{\top}$, where $\mathbf{B}^{\top}$ is the transpose of $\mathbf{B}$. This implies that $\mathbf{B}$ and $\mathbf{B}_{0}$ have the same 2-rank. Therefore, in order to find the dimensions of the $\mathbb{F}_{2}$-null spaces of $\mathbf{B}$ and $\mathbf{B}_{0}$, it suffices to calculate the 2-rank of either $\mathbf{B}$ or $\mathbf{B}_{0}$. Recall that the subgroup $G$ of $\operatorname{PGL}(3, q)$ fixing $\mathcal{O}$ is isomorphic to $\operatorname{PGL}(2, q)$ [8, p. 158]. Further, $G$ has an index 2 normal subgroup $H$ isomorphic to $\operatorname{PSL}(2, q)$. It is known [7] that $H$ acts transitively on $E$ and $I$ as well as on $S e, T$ and $S k$.

In [17], Sin, Wu and Xiang calculate the 2-rank of $\mathbf{A}_{33}$ (i.e. the incidence matrix of external points and secant lines) using a combination of techniques from finite geometry and modular representation of $H$. In this article, we compute the 2-rank of $\mathbf{B}$ using similar representation theoretic results obtained in [17] and different geometric results. Therefore, between the current article and [17], the reader will expect to see some overlaps in the results and statements on modular representation of $H$ as well as the basic geometric facts about conics.

Let $F$ be an algebraic closure of $\mathbb{F}_{2}$. Let $F^{I}$ and $F^{E}$ be the free $F$-modules whose standard bases consist of the characteristic column vectors of $I$ and those of $E$, respectively. The actions of $H$ on $I$ and $E$ make the free $F$-modules $F^{I}$ and $F^{E}$ into $F H$-permutation modules. We define a map

$$
\begin{equation*}
\phi_{\mathbf{B}}: F^{I} \rightarrow F^{E} \tag{1.4}
\end{equation*}
$$

as follows: specify the images of the basis elements of $F^{I}$ under $\phi_{\mathbf{B}}$ first, i.e.

$$
\phi_{\mathbf{B}}\left(\mathcal{G}_{\mathbf{P}}\right)=\sum_{\mathbf{Q} \in \mathbf{P}^{\perp} \wedge_{\cap E}} \chi_{\mathbf{Q}}
$$

for each $\mathbf{P} \in I$, and then extend $\phi_{\mathbf{B}}$ linearly to $F^{I}$, where $\perp$ is the polarity induced by the quadratic form $Q, \mathcal{G}_{\mathbf{P}}$ and $\chi_{\mathbf{Q}}$ are the characteristic column vectors of the internal point $\mathbf{P}$ with respect to $I$ and the external point $\mathbf{Q}$ with respect to $E$, respectively. The matrix of $\phi_{\mathbf{B}}$ is a $0-1$ matrix of size $|E| \times|I|$. Up to permutations of the rows and columns, $\mathbf{B}$ regarded as a matrix over $F$, is the matrix of $\phi_{\mathbf{B}}$ with respect to the standard bases of $F^{I}$ and $F^{E}$. Moreover, $\phi_{\mathbf{B}}(\mathbf{x})=\mathbf{B x}$ for $\mathbf{x} \in F^{I}$. It can be shown that $\phi_{\mathbf{B}}$ is an $F H$-homomorphism. Hence, the column space of $\mathbf{B}$ over $F$ is equal to $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$, which is also an $F H$-submodule of $F^{E}$. This point of view enables us to use results from modular representation of $H$ to determine the dimension of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ and thus the 2-rank of $\mathbf{B}$. We remark that in the calculation of the 2-rank of $\mathbf{A}_{33}$ the authors of [17] view $\mathbf{A}_{33}$ as the matrix of an $F H$-homomorphism $\phi$ from $F^{E}$ to $F^{E}$ under which the characteristic vector of an external point $\mathbf{P}$ is mapped to the sum of the characteristic vectors of the external points on $\mathbf{P}^{\perp}$.

Our idea of calculating $\operatorname{dim}_{F}\left(\operatorname{Im}\left(\phi_{\mathbf{B}}\right)\right)$ is to find a decomposition of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ into a direct sum of its submodules whose dimensions can be computed easily. To this end, we apply Brauer's theory and compute the decomposition of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ into blocks. The silmilar idea was used in [17] to compute the decomposition of $\operatorname{Ker}(\phi)$ into blocks as well as $\operatorname{dim}_{F}(\operatorname{Ker}(\phi))$. Nevertheless, there are two major differences between the current article and [17]: (1) the geometric results used to compute the decomposition of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ into blocks are essentially different from these used to compute the decomposition of $\operatorname{Ker}(\phi)$; (2) the summands of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ in its block decomposition are more complicated than these of $\operatorname{Ker}(\phi)$, which indicates that more efforts are required to find $\operatorname{dim}_{F}\left(\operatorname{Im}\left(\phi_{\mathbf{B}}\right)\right)$.

In the following we will give a brief overview of this article. In Section 2, we first review the basic facts about $\mathcal{O}$ and then prove several crucial geometric results. From them, in Section 5, we show that the 2-rank of the incidence matrix $\mathbf{D}$ of external points and $N_{P a, E}(\mathbf{P})$ for $\mathbf{P} \in I$ (the set of external points on the passant lines through $\mathbf{P}$ ) is either $q$ or $q-1$, depending on $q$. The character of the complex permutation module $\mathbb{C}^{I}$ and its decomposition into a sum of the irreducible ordinary characters of $H$ were calculated in [19]; the decomposition of the characters of $H$ into 2-blocks was given by Burkhardt [3] and Landrock [13]. From them we see that $\mathbb{C}^{I}$ is a direct sum of $\mathbb{C} H$-modules consisting of one simple module from each block of defect zero, and some summands from blocks of positive defect. According to Brauer's theory, $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ is the direct sum

$$
\begin{equation*}
\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\bigoplus_{B} \operatorname{Im}\left(\phi_{\mathbf{B}}\right) e_{B} \tag{1.5}
\end{equation*}
$$

where $e_{B}$ is a primitive idempotent in the center of $F H$. The block idempotents $e_{B}$ are elements of $F H$ and were computed in [17]. In order to compute $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) e_{B}$ for each 2block $B$, we need detailed information concerning the action of group elements in various conjugacy classes on various geometric objects and on the intersections of certain special subsets of $H$ with various conjugacy classes of $H$. These computations are made in Sections 3 and 4 . These information also tell us that (i) $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) e_{B_{0}}$ is equal to the column space of

D over $F$, or this space plus an additional trivial module, depending on $q$, where $B_{0}$ is the principal 2-block of $H$, and (ii) block idempotents associated with non-principal 2-blocks of positive defect annihilate $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ (Lemma 6.2). Since the $B$-component of $F^{I}$ is the $\bmod 2$ reduction of the $B$-component of $\mathbb{C}^{I}$, using (i) and (ii), and the block decomposition of $\mathbb{C}^{I}$, we show that $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ is equal to the direct sum of the column space of $\mathbf{D}$ and the simple modules lying in the 2 -blocks of defect 0 , or this sum plus an additional trivial module, depending on $q$. Then the dimension formula of $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ follows instantly as a corollary.

## 2. Geometric Results

Recall that a collineation of $\mathrm{PG}(2, q)$ is an automorphism of $\mathrm{PG}(2, q)$, which is a bijection from the set of all points and all lines of $\mathrm{PG}(2, q)$ to itself that maps a point to a line and a line to a point, and preserves incidence. It is well known that each element of GL $(3, q)$, the group of all $3 \times 3$ non-singular matrices over $\mathbb{F}_{q}$, induces a collineation of $\operatorname{PG}(2, q)$. The proof of the following lemma is straightforward.
Lemma 2.1. Let $\mathbf{P}=\left(a_{0}, a_{1}, a_{2}\right)$ (respectively, $\ell=\left[b_{0}, b_{1}, b_{2}\right]$ ) be a point (respectively, a line) of $\mathrm{PG}(2, q)$. Suppose that $\theta$ is a collineation of $\mathrm{PG}(2, q)$ that is induced by $\mathbf{D} \in$ $G L(3, q)$. If we use $\mathbf{P}^{\theta}$ and $\ell^{\theta}$ to denote the images of $\mathbf{P}$ and $\ell$ under $\theta$, respectively, then

$$
\mathbf{P}^{\theta}=\left(a_{0}, a_{1}, a_{2}\right)^{\theta}=\left(a_{0}, a_{1}, a_{2}\right) \mathbf{D}
$$

and

$$
\ell^{\theta}=\left[b_{0}, b_{1}, b_{2}\right]^{\theta}=\left[c_{0}, c_{1}, c_{2}\right],
$$

where $c_{0}, c_{1}, c_{2}$ correspond to the first, the second, and the third coordinate of the vector $\mathbf{D}^{-1}\left(b_{0}, b_{1}, b_{2}\right)^{\top}$, respectively.

A correlation of $\mathrm{PG}(2, q)$ is a bijection from the set of points to the set of lines as well as the set of lines to the set of points that reverses inclusion. A polarity of $\mathrm{PG}(2, q)$ is a correlation of order 2. The image of a point $\mathbf{P}$ under a correlation $\sigma$ is denoted by $\mathbf{P}^{\sigma}$, and that of a line $\ell$ is denoted by $\ell^{\sigma}$. It can be shown [8, p. 181] that the non-degenerate quadratic form $Q\left(X_{0}, X_{1}, X_{2}\right)=X_{1}^{2}-X_{0} X_{2}$ induces a polarity $\sigma($ or $\perp)$ of $\mathrm{PG}(2, q)$, which can be represented by the matrix

$$
\mathbf{M}=\left(\begin{array}{ccc}
0 & 0 & -\frac{1}{2}  \tag{2.1}\\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & 0
\end{array}\right)
$$

Lemma 2.2. ([10, p. 47]) Let $\mathbf{P}=\left(a_{0}, a_{1}, a_{2}\right)$ (respectively, $\left.\ell=\left[b_{0}, b_{1}, b_{2}\right]\right)$ be a point (respectively, a line) of $\mathrm{PG}(2, q)$. If $\sigma$ is the polarity represented by the above non-singular symmetric matrix $\mathbf{M}$, then

$$
\mathbf{P}^{\sigma}=\left(a_{0}, a_{1}, a_{2}\right)^{\sigma}=\left[c_{0}, c_{1}, c_{2}\right]
$$

and

$$
\ell^{\sigma}=\left[b_{0}, b_{1}, b_{2}\right]^{\sigma}=\left(b_{0}, b_{1}, b_{2}\right) \mathbf{M}^{-1}
$$

where $c_{0}, c_{1}, c_{2}$ correspond to the first, the second, the third coordinate of the column vector $\mathbf{M}\left(a_{0}, a_{1}, a_{2}\right)^{\top}$, respectively.

For example, if $\mathbf{P}=(x, y, z)$ is a point of $\mathrm{PG}(2, q)$, then its image under $\sigma$ is $\mathbf{P}^{\sigma}=$ $[z,-2 y, x]$.

For convenience, we will denote the set of all non-zero squares of $\mathbb{F}_{q}$ by $\square_{q}$, and the set of non-squares by $\nabla_{q}$. Also, $\mathbb{F}_{q}^{*}$ is the set of non-zero elements of $\mathbb{F}_{q}$.
Lemma 2.3. ([8, p. 181-182]) Assume that $q$ is odd.
(i) The polarity $\sigma$ above defines three bijections; that is, $\sigma: I \rightarrow P a, \sigma: E \rightarrow S e$, and $\sigma: \mathcal{O} \rightarrow T$ are all bijections.
(ii) A line $\left[b_{0}, b_{1}, b_{2}\right]$ of $\mathrm{PG}(2, q)$ is a passant, a tangent, or a secant to $\mathcal{O}$ if and only if $b_{1}^{2}-4 b_{0} b_{2} \in \square_{q}, b_{1}^{2}-4 b_{0} b_{2}=0$, or $b_{1}^{2}-4 b_{0} b_{2} \in \square_{q}$, respectively.
(iii) A point $\left(a_{0}, a_{1}, a_{2}\right)$ of $\mathrm{PG}(2, q)$ is internal, absolute, or external if and only if $a_{1}^{2}-a_{0} a_{2} \in \square_{q}, a_{1}^{2}-a_{0} a_{2}=0$, or $a_{1}^{2}-a_{0} a_{2} \in \square_{q}$, respectively.

The results in the following lemma can be obtained by simple counting; see 8 for more details and related results.

Lemma 2.4. ([8, p. 170]) Using the above notation, we have

$$
\begin{equation*}
|T|=|\mathcal{O}|=q+1,|P a|=|I|=\frac{q(q-1)}{2}, \text { and }|S e|=|E|=\frac{q(q+1)}{2} \text {. } \tag{2.2}
\end{equation*}
$$

Also, we have the following tables:
Table 1. Number of points on lines of various types

| Name | Absolute points | External points | Internal points |
| :---: | :---: | :---: | :---: |
| Tangent lines | 1 | $q$ | 0 |
| Secant lines | 2 | $\frac{q-1}{2}$ | $\frac{q-1}{2}$ |
| Passant lines | 0 | $\frac{q+1}{2}$ | $\frac{q+1}{2}$ |

Table 2. Number of lines through points of various types

| Name | Tangent lines | Secant lines | Skew lines |
| :---: | :---: | :---: | :---: |
| Absolute points | 1 | $q$ | 0 |
| External points | 2 | $\frac{q-1}{2}$ | $\frac{q-1}{2}$ |
| Internal points | 0 | $\frac{q+1}{2}$ | $\frac{q+1}{2}$ |

2.1. More geometric results. Let $G$ be the automorphism group of $\mathcal{O}$ in $\operatorname{PGL}(3, q)$ (i.e. the subgroup of $\operatorname{PGL}(3, q)$ fixing $\mathcal{O}$ setwise). Then $G$ is the image in $\operatorname{PGL}(3, q)$ of $\mathrm{O}(3, q)=\mathrm{SO}(3, q) \times\langle-1\rangle$, hence also the image of $\mathrm{SO}(3, q)$, to which it is isomorphic. For our computations, we will describe $G$ in a slightly different way. The map $\tau: \operatorname{GL}(2, q) \rightarrow$ $\mathrm{GL}(3, q)$ sending the matrix $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ to

$$
\left(\begin{array}{ccc}
a^{2} & a b & b^{2}  \tag{2.3}\\
2 a c & a d+b c & 2 b d \\
c^{2} & c d & d^{2}
\end{array}\right)
$$

is a group homomorphism. The image of $\tau(\mathrm{GL}(2, q))$ in $\mathrm{PGL}(3, q)$ lies in $G$. Now, whether or not the group $\tau(\mathrm{GL}(2, q))$ contains $\mathrm{SO}(3, q)$ depends on $q$. Nevertheless, $\tau(\mathrm{GL}(2, q))$ always contains a subgroup of index 2 in $\mathrm{O}(3, q)$ whose image in $\operatorname{PGL}(3, q)$ is $G$. Thus, the induced homomorphism $\bar{\tau}: \operatorname{PGL}(2, q) \rightarrow \operatorname{PGL}(3, q)$ maps $\operatorname{PGL}(2, q)$ isomorphically onto $G$.

Let $H=\tau(\mathrm{SL}(2, q))$, the group of matrices of the form (2.3) such that $a d-b c=1$. Since the kernel of $\tau$ is $\left\langle-I_{2}\right\rangle$, it follows that $H \cong \operatorname{PSL}(2, q)$ and that $H$ is isomorphic to its image $\bar{H}$ in $\operatorname{PGL}(3, q)$. In fact, we have $H=\Omega(3, q)$.

Since

$$
\operatorname{PGL}(2, q)=\operatorname{PSL}(2, q) \cup\left(\begin{array}{cc}
1 & 0 \\
0 & \xi^{-1}
\end{array}\right) \cdot \operatorname{PSL}(2, q),
$$

our discussion shows that

$$
\begin{equation*}
H \cup \mathbf{d}\left(1, \xi^{-1}, \xi^{-2}\right) \cdot H \tag{2.4}
\end{equation*}
$$

is a full set of representative matrices for the elements of $G$. In our computations, it will often be convenient to refer to elements of $G$ by means of their representatives in the set (2.4). Additionally, a group element in (2.3) has the inverse equal to

$$
\left(\begin{array}{ccc}
d^{2} & -b d & b^{2}  \tag{2.5}\\
-2 c d & a d+b c & -2 a b \\
c^{2} & -a c & c^{2}
\end{array}\right) .
$$

Moreover, the following holds.
Lemma 2.5. 7] The group $G$ acts transitively on I and Pa as well as on E and Se.
We will refer to this lemma frequently in the rest of this section.
Lemma 2.6. [17, Lemma 2.9] Let $\mathbf{P}$ be a point not on $\mathcal{O}$, $\ell$ a non-tangent line, and $\mathbf{P} \in \ell$. Using the above notation, we have the following.
(i) If $\mathbf{P} \in I$ and $\ell \in P a$, then $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 1(\bmod 4)$, and $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 3$ $(\bmod 4)$.
(ii) If $\mathbf{P} \in I$ and $\ell \in S e$, then $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 1(\bmod 4)$, and $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 3$ $(\bmod 4)$.
(iii) If $\mathbf{P} \in E$ and $\ell \in P a$, then $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 1(\bmod 4)$, and $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 3$ $(\bmod 4)$.
(iv) If $\mathbf{P} \in E$ and $\ell \in S e$, then $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 1(\bmod 4)$, and $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 3$ $(\bmod 4)$.
Next we define $\square_{q}-1:=\left\{s-1 \mid s \in \square_{q}\right\}$ and $\square_{q}-1:=\left\{s-1 \mid s \in \square_{q}\right\}$.
Lemma 2.7. [18] Using the above notation,
(i) if $q \equiv 1(\bmod 4)$, then $\left|\left(\square_{q}-1\right) \cap \square_{q}\right|=\frac{q-5}{4}$ and $\left|\left(\square_{q}-1\right) \cap \not \square_{q}\right|=\left|\left(\not \square_{q}-1\right) \cap \square_{q}\right|$ $=\left|\left(\not \square_{q}-1\right) \cap \not \square_{q}\right|=\frac{q-1}{4}$;
(ii) if $q \equiv 3(\bmod 4)$, then $\left|\left(\not \square_{q}-1\right) \cap \square_{q}\right|=\frac{q+1}{4}$ and $\left|\left(\square_{q}-1\right) \cap \square_{q}\right|=\left|\left(\square_{q}-1\right) \cap \not \square_{q}\right|$ $=\left|\left(\square_{q}-1\right) \cap \not \square_{q}\right|=\frac{q-3}{4}$.
Definition 2.8. Let $\mathbf{P}$ be a point not on $\mathcal{O}$ and $\ell$ a line. We define $E_{\ell}$ (respectively, $I_{\ell}$ ) to be the set of external (respectively, internal) points on $\ell, P a_{\mathbf{P}}$ (respectively, $S e_{\mathbf{P}}$ ) the set of passant (respectively, secant) lines through $\mathbf{P}$, and $T_{\mathbf{P}}$ the set of tangent lines through P. Also, $N_{P a, E}(\mathbf{P})$ (respectively, $\left.N_{S e, E}(\mathbf{P})\right)$ is defined to be the set of external points on the passant (respectively, secant) lines through $\mathbf{P}$.

In the following lemma, we list the sizes of the above defined sets as well as the action of $G$ on these sets. Also, we adopt standard notation from permutation group theory. For instance, if $W \subseteq I$, then $W^{g}:=\left\{w^{g} \mid w \in W\right\}, G_{\mathbf{P}}$ is the stabilizer of $\mathbf{P}$ in $G$, and for $M \subseteq G, M^{g}$ is the conjugate of $M$ under $g$.
Lemma 2.9. Using the above notation, if $\mathbf{P} \in I$, we have
(i) $\left|E_{\mathbf{P} \perp}\right|=\left|S e_{\mathbf{P}}\right|=\frac{q+1}{2}$,
(ii) $\left|I_{\mathbf{P}^{\perp}}\right|=\left|P a_{\mathbf{P}}\right|=\frac{q+1}{2}$,
(iii) $\left|N_{P a, E}(\mathbf{P})\right|=\left|N_{S e, E}(\mathbf{P})\right|=\frac{(q+1)^{2}}{4}$;
moreover, if $\mathbf{P}$ is not a point on $\mathcal{O}, \ell$ is a non-tangent line, and $g \in G$, we have
(iv) $I_{\ell}^{g}=I_{\ell g}$ and $P a_{\mathbf{P}}^{g}=P a_{\mathbf{P}^{g}}$,
(v) $E_{\ell}^{g}=E_{\ell^{g}}$ and $S e_{\mathbf{P}}^{g}=S e_{\mathbf{P}^{g}}$,
(vi) $H_{\mathbf{P}}^{g}=H_{\mathbf{P}^{g}}$,
(vii) $N_{P a, E}^{g}(\mathbf{P})=N_{P a, E}\left(\mathbf{P}^{g}\right)$ and $N_{S e, E}^{g}(\mathbf{P})=N_{S e, E}\left(\mathbf{P}^{g}\right)$,
(viii) $\left(\mathbf{P}^{\perp}\right)^{g}=\left(\mathbf{P}^{g}\right)^{\perp}$, where $\perp$ is the polarity of $\mathrm{PG}(2, q)$ defined as above.

Proof: The above (i) - (iii) follow from from Tables 1 and 2 and simple counting, and (iv) - (vii) follow from the fact that $G$ preserves incidence.

By the defintion of $G$, it is clear that the following two lemmas are true.
Lemma 2.10. Let $\mathbf{P}$ be a point of $\mathrm{PG}(2, q)$. Then the polarity $\perp$ defines a bijection between $I_{\mathbf{P} \perp}$ and $P a_{\mathbf{P}}$, and also a bijection between $E_{\mathbf{P} \perp}$ and $S e_{\mathbf{P}}$.
Lemma 2.11. Let $W$ be a subgroup of $G$. Suppose that $g \in G$ and $\mathbf{P}$ is a point of $\mathrm{PG}(2, q)$. Then

$$
\left(W^{g}\right)_{\mathbf{P}^{g}}=W_{\mathbf{P}}^{g}
$$

Proposition 2.12. Let $\mathbf{P}$ be a point not on $\mathcal{O}$ and set $K=G_{\mathbf{P}}$. Then $K$ is transitive on each of $I_{\mathbf{P}^{\perp}}, E_{\mathbf{P}^{\perp}}, P a_{\mathbf{P}}$, and $S e_{\mathbf{P}}$. Moreover, if $\mathbf{P} \in E$, then $K$ is also transitive on $T_{\mathbf{P}}$.
Proof: The case where $\mathbf{P} \in I$ is Proposition 2.11 in [19]; the case where $\mathbf{P} \in E$ or $\mathcal{O}$ is Lemma 2.11(iii) in [17].

Lemma 2.13. [17, Corollary 2.16] Let $\mathbf{P}$ be a point of $\mathrm{PG}(2, q)$ and let $\perp$ be the polarity of $\mathrm{PG}(2, q)$ defined above. Then for $g \in G_{\mathbf{P}}$ we have $\mathbf{P}^{\perp}=\left(\mathbf{P}^{\perp}\right)^{g}$. Consequently, $\mathbf{P}^{\perp}$ is fixed setwise by $G_{\mathbf{P}}$. Moreover, $G_{\mathbf{P} \perp}=G_{\mathbf{P}}$.
Lemma 2.14. Assume that $\mathbf{P} \in I$ and $\ell=\mathbf{P}^{\perp}$. Let $\mathbf{Q} \in E_{\ell}$ and $\ell^{*} \in T_{\mathbf{Q}}$. Suppose that $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are two distinct external points on $\ell^{*}$ and let $\ell_{1}$ and $\ell_{2}$ be the tangent lines different from $\ell^{*}$ through $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$, respectively. Then $\ell_{1}$ and $\ell_{2}$ meet in an external point on a secant line through $\mathbf{P}$ if and only if one of the following two cases occurs:
(i) $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are on two passant lines through $\mathbf{P}$;
(ii) $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are on two secant lines through $\mathbf{P}$.

Proof: Since $G$ is transitive on $I$ and preserves incidence, without loss of generality, we may assume that $\mathbf{P}=(1,0,-\xi)$, and thus $\ell=\left[1,0,-\xi^{-1}\right]$. Since $K:=G_{\mathbf{P}}$ is transitive on $E_{\ell}$ by Proposition 2.12, we can assume that $\mathbf{Q}=(0,1,0)$. Let $\ell^{*}=[1,0,0]$ be a tangent line through $\mathbf{Q}$. It is clear that

$$
E_{\ell^{*}}=\left\{(0,1, m) \mid m \in \mathbb{F}_{q}\right\}
$$

Let $\mathbf{P}_{1}=\left(0,1, m_{1}\right)$ and $\mathbf{P}_{2}=\left(0,1, m_{2}\right)$ be two distinct external points on $\ell^{*}$. Then the tangent lines through $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ different from $\ell^{*}$ are $\ell_{1}=\left[m_{1}^{2},-4 m_{1}, 4\right]$ and $\ell_{2}=$ $\left[m_{2}^{2},-4 m_{2}, 4\right]$, respectively. So we have that $\mathbf{P}_{3}:=\ell_{1} \cap \ell_{2}=\left(1, \frac{m_{1}+m_{2}}{4}, \frac{m_{1} m_{2}}{4}\right) \in E$. Thus the line through $\mathbf{P}$ and $\mathbf{P}_{3}$ is

$$
\ell_{\mathbf{P}, \mathbf{P}_{3}}=\left[m_{1}+m_{2},-4\left(\frac{m_{1} m_{2}}{4 \xi}+1\right), \frac{m_{1}+m_{2}}{\xi}\right],
$$

which is a secant line if and only if

$$
16\left(\frac{m_{1} m_{2}}{4 \xi}+1\right)^{2}-\frac{4\left(m_{1}+m_{2}\right)^{2}}{\xi}=\frac{\left(m_{1}^{2}-4 \xi\right)\left(m_{2}^{2}-4 \xi\right)}{\xi^{2}} \in \square_{q}
$$

if and only if either $m_{i}^{2}-4 \xi \in \square_{q}$ for $i=1,2$ or $m_{i}^{2}-4 \xi \in \square_{q}$ for $i=1,2$. Since the line through $\mathbf{P}$ and $\mathbf{P}_{i}(i=1$ or 2$)$ is $\ell_{\mathbf{P}, \mathbf{P}_{i}}=\left[1,-\frac{m_{i}}{\xi}, \frac{1}{\xi}\right]$, and its discrimnant is $\frac{m_{i}^{2}-4 \xi}{\xi^{2}}$, we
conclude that $\ell_{\mathbf{P}, \mathbf{P}_{3}}$ is a secant line if and only if either (i) $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are on two passant lines through $\mathbf{P}$ or (ii) $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are on two secant lines through $\mathbf{P}$.

Definition 2.15. Let $N \subseteq E$. We define $\chi_{N}$ to be the characteristic (column) vector of $N$ with respect to $E$; that is, $\chi_{N}$ is a column vector of length $|E|$ whose entries are indexed by the external points such that if $\mathbf{P} \in N$ then the entry of $\chi_{N}$ indexed by $\mathbf{P}$ is 1 , 0 otherwise. For a line $\ell$, if no confusion occurs, we shoule use $\chi_{\ell}$ to replace $\ell_{E_{\ell}}$. Also, if $N=\{\mathbf{P}\}$ is a singleton set, we will frequently use $\chi_{\mathbf{P}}$ to replace $\chi_{\{\mathbf{P}\}}$.
Remark 2.16. In the rest of this section, $\chi_{N}$ for $N \subseteq E$ will be always viewed as a column vector over $\mathbb{Z}$, the ring of integer.
Corollary 2.17. Let $\mathbf{P} \in I$. Using the above notation, we have

$$
\chi_{N_{P a, E}(\mathbf{P})} \equiv \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell} \quad(\bmod 2),
$$

where $\ell(\mathbf{P})$ is a tangent line through an external point on $\mathbf{P}^{\perp}, T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines distinct from $\ell(\mathbf{P})$ through the external points that are on both $\ell(\mathbf{P})$ and the passant lines through $\mathbf{P}$, and the congruence means entrywise congruence.
Proof: It is clear that $|T(\mathbf{P}, \ell(\mathbf{P}))|=\frac{q+1}{2}$ since there are $\frac{q+1}{2}$ passant lines through $\mathbf{P}$ and each of them meets $\ell(\mathbf{P})$ in an external point. Let $\ell \in T(\mathbf{P}, \ell(\mathbf{P}))$. Then by Lemma 2.14, any tangent line other than $\ell$ in $T(\mathbf{P}, \ell(\mathbf{P}))$ meets $\ell$ in an external point on a secant line through $\mathbf{P}$, and if we use $I E(\ell, \ell(\mathbf{P}))$ to denote their intersections with $\ell$ then the points in $E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))$ must be on the passant lines through $\mathbf{P}$. Since

$$
\left(E_{\ell_{1}} \backslash I E\left(\ell_{1}, \ell(\mathbf{P})\right)\right) \cap\left(E_{\ell_{2}} \backslash I E\left(\ell_{2}, \ell(\mathbf{P})\right)\right)=\emptyset
$$

for two distinct lines $\ell_{1}, \ell_{2} \in T(\mathbf{P}, \ell(\mathbf{P}))$ and

$$
\left|E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))\right|=q-\frac{q-1}{2}=\frac{q+1}{2},
$$

it follows that

$$
\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))}\left|E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))\right|=\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \frac{q+1}{2}=\frac{(q+1)^{2}}{4}
$$

which is the same as the size of $N_{P a, E}(\mathbf{P})$ by Lemma 2.9(iii). Consequently, we must have

$$
\bigcup_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} E_{\ell} \backslash I E(\mathbf{P}, \ell(\mathbf{P}))=\bigcup_{\ell \in P a_{\mathbf{P}}} E_{\ell}=N_{P a, E}(\mathbf{P}) .
$$

Moreover, since each point in $I E(\ell, \ell(\mathbf{P}))$ lies on exactly two lines in $T(\mathbf{P}, \ell(\mathbf{P}))$ and each point in $E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))$ doesn't lie on any line other than $\ell$ in $T(\mathbf{P}, \ell(\mathbf{P}))$, we obtain

$$
\begin{align*}
\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell}} & =\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))}+\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \sum_{\mathbf{Q} \in I E(\ell, \ell(\mathbf{P}))} \chi_{\mathbf{Q}} \\
& =\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))}+2 \sum_{\mathbf{Q} \in M} \chi_{\mathbf{Q}} \\
& \equiv \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \backslash I E(\ell, \ell(\mathbf{P}))}  \tag{2.6}\\
& =\sum_{\ell \in P a_{\mathbf{P}}} \chi_{\ell} \\
& =\chi_{N_{P a, E}(\mathbf{P})}(\bmod 2)
\end{align*}
$$

where $M=\left\{\ell_{1} \cap \ell_{2} \mid \ell_{1}, \ell_{2} \in T(\mathbf{P}, \ell(\mathbf{P})), \ell_{1} \neq \ell_{2}\right\}$.

Lemma 2.18. Assume that $q \equiv 1(\bmod 4)$. Let $\mathbf{P} \in \mathcal{O}$. Then there exits a set $\mathcal{M}(\mathbf{P})$ consisting of an odd number of internal points such that, for each external point $\mathbf{Q} \in \mathbf{P}^{\perp}$, the number of passant lines through $\mathbf{Q}$ and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplicity, is odd.

Remark 2.19. In this lemma, it is possible that $\mathbf{Q}, \mathbf{Q}_{1}, \ldots, \mathbf{Q}_{k}$ are on the same passant line $\ell$, where $\mathbf{Q} \in E_{\mathbf{P}^{\perp}}$ and $\mathbf{Q}_{i} \in \mathcal{M}(\mathbf{P})$ for $1 \leq i \leq k$. If this circumstance occurs, then the line $\ell$ should be counted $k$ times.

Proof: Without loss of generality, we may assume that $\mathbf{P}=(0,0,1)$, and so $\ell:=\mathbf{P}^{\perp}=$ $[1,0,0]$. Using Lemma 2.1 and (2.5), we have

$$
K:=H_{\ell}=\left\{\left.\left(\begin{array}{ccc}
d^{2} & -b d & b^{2}  \tag{2.7}\\
0 & 1 & -\frac{2 b}{d} \\
0 & 0 & \frac{1}{d^{2}}
\end{array}\right) \right\rvert\, d \in \mathbb{F}_{q}^{*}, b \in \mathbb{F}_{q}\right\} .
$$

Since

$$
\left(\begin{array}{ccc}
1 & -b & b^{2}  \tag{2.8}\\
0 & 1 & -2 b \\
0 & 0 & 1
\end{array}\right)^{k}=\left(\begin{array}{ccc}
1 & -k b & (k b)^{2} \\
0 & 1 & -2 k b \\
0 & 0 & 1
\end{array}\right)
$$

for any positive integer $k$, it is obvious that

$$
\left\{\left.\left(\begin{array}{ccc}
1 & -b & b^{2}  \tag{2.9}\\
0 & 1 & -2 b \\
0 & 0 & 1
\end{array}\right) \right\rvert\, b \in \mathbb{F}_{q}\right\}
$$

is a collineation subgroup of order $q$ in $K$, which we denote by $T$.
For $\left(0,1, u_{1}\right),\left(0,1, u_{2}\right)$ where $u_{1}, u_{2} \in \mathbb{F}_{q}$ and $u_{1} \neq u_{2}$, we have

$$
\left(0,1, u_{1}\right)\left(\begin{array}{ccc}
1 & -\frac{u_{1}-u_{2}}{2} & \left(\frac{u_{1}-u_{2}}{2}\right)^{2} \\
0 & 1 & -\left(u_{1}-u_{2}\right) \\
0 & 0 & 1
\end{array}\right)=\left(0,1, u_{2}\right) ;
$$

this implies that $T$ is transitive on $E_{\ell}=\left\{(0,1, u) \mid u \in \mathbb{F}_{q}\right\}$.
Now let $\mathbf{P}_{1}=(1,0,-\xi) \in I$, set $\mathcal{M}(\mathbf{P}):=\left\{\mathbf{P}_{1}^{g} \mid g \in T\right\}$ which is the $T$-orbit of $\mathbf{P}_{1}$, and let $\mathbf{Q}=(0,1, u) \in \ell$. Then

$$
\mathcal{M}(\mathbf{P})=\left\{\left(1,-b, b^{2}-\xi\right) \mid b \in \mathbb{F}_{q}\right\}
$$

and the lines through both $\mathbf{Q}$ and the points in $\mathcal{M}(\mathbf{P})$ form the multiset

$$
L(\mathbf{Q})=\left\{\left[b^{2}+u b-\xi, u,-1\right] \mid b \in \mathbb{F}_{q}\right\} .
$$

Note that a line $\left[b^{2}+u b-\xi, u,-1\right] \in L(\mathbf{Q})$ is passant if and only if $\frac{(u+2 b)^{2}}{4 \xi}-1 \in \square_{q}$. Since the number of $t \in \square_{q}$ satisfying $t-1 \in \square_{q}$ is $\left|\left(\square_{q}-1\right) \cap \square_{q}\right|=\frac{q-1}{4}$ by Lemma $2.7(\mathrm{i})$, it follows that the number of $b \in \mathbb{F}_{q} \backslash\left\{-\frac{u}{2}\right\}$ satisfying $\frac{(u+2 b)^{2}}{4 \xi}-1 \in \square_{q}$ is $2\left(\frac{q-1}{4}\right)=\frac{q-1}{2}$. Moreover, when $b=-\frac{u}{2}, \frac{(u+2 b)^{2}}{4 \xi}-1=-1 \in \square_{q}$ as $q \equiv 1(\bmod 4)$. Hence, the number of $b \in \mathbb{F}_{q}$ satisfying $\frac{(u+2 b)^{2}}{4 \xi}-1 \in \square_{q}$ is $\frac{q-1}{2}+1=\frac{q+1}{2}$. Thus, counted with multiplicity, there are $\frac{q+1}{2}$ passant lines in $L(\mathbf{Q})$. Therefore, there are an odd number of internal points (precisely $\frac{q+1}{2}$ ) in $\mathcal{M}(\mathbf{P})$ connecting $\mathbf{Q}$ by a passant line as $q \equiv 1(\bmod 4)$. Since $T$ is transitive on both $\mathcal{M}(\mathbf{P})$ and $E_{\ell}$ and preserves incidence, we conclude that the number of passant lines through an external point on $\mathbf{P}^{\perp}$ and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplictiy, must be odd.

Remark 2.20. Let $\mathbf{P} \in \mathcal{O}$. In the rest of this article, without being further mentioned, $\mathcal{M}(\mathbf{P})$ always denotes a set of internal points associated with $\mathbf{P}$ satisfying the conditions in Lemma 2.18.

Corollary 2.21. Assume that $q \equiv 1(\bmod 4)$. Let $\ell$ be a tangent line. Then

$$
\begin{aligned}
\chi_{\ell} & =\sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \sum_{\ell^{\prime} \in P a_{\mathbf{P}}} \chi_{\ell^{\prime}} \\
& =\sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \chi_{N_{P a, E}(\mathbf{P})}(\bmod 2)
\end{aligned}
$$

where the congruence is entrywise congruence.
Proof: Let $\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)$. Then from Corollary 2.17, it follows that

$$
\begin{align*}
\chi_{N_{P a, E}(\mathbf{P})} & =\sum_{\ell^{\prime} \in P a_{\mathbf{P}}} \chi_{\ell^{\prime}} \\
& \equiv \sum_{\ell^{\prime} \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell^{\prime}}(\bmod 2), \tag{2.10}
\end{align*}
$$

where $\ell(\mathbf{P})$ is a tangent line through an external point on $\mathbf{P}^{\perp}$ and $T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines different from $\ell(\mathbf{P})$ through the external points that are both $\ell(\mathbf{P})$ and the passant lines through $\mathbf{P}$.

Further, if we take $\ell(\mathbf{P})=\ell$ for each $\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)$ and set $W(\mathbf{P}):=\left\{\ell \cap \ell_{1} \mid \ell_{1} \in P a_{\mathbf{P}}\right\}$, then

$$
\begin{align*}
\sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \sum_{\ell^{\prime} \in P a_{\mathbf{P}}} \chi_{\ell^{\prime}} & \equiv \sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \sum_{\ell^{\prime} \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell^{\prime}} \\
& =\sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \sum_{\mathbf{Q} \in W(\mathbf{P})} \sum_{\ell^{\prime} \in T_{\mathbf{Q}} \backslash\{\ell\}} \chi_{\ell^{\prime}} \\
& =\sum_{\mathbf{Q} \in E_{\ell}} \sum_{\ell^{\prime} \in T_{\mathbf{Q}} \backslash\{\ell\}} a_{\ell^{\prime}} \chi_{\ell^{\prime}}  \tag{2.11}\\
& =\sum_{\ell^{\prime} \in T \backslash\{\ell\}} a_{\ell^{\prime}} \chi_{\ell^{\prime}} \\
& \equiv \sum_{\ell^{\prime} \in T \backslash\{\ell\}} \chi_{\ell^{\prime}}(\bmod 2),
\end{align*}
$$

where $a_{\ell^{\prime}}$ for $\ell^{\prime} \in T \backslash\{\ell\}$ are odd. In (2.11), the second equality follows from the definition of $T(\ell, \ell(\mathbf{P}))$ and the third equality holds since the multiset

$$
\begin{equation*}
\bigcup_{\mathbf{Q} \in E_{\ell}} \bigcup_{L(\mathbf{Q})} T_{\mathbf{Q}} \backslash\{\ell\} \tag{2.12}
\end{equation*}
$$

where $L(\mathbf{Q}):=\left\{\ell_{\mathbf{P}_{1}, \mathbf{Q}} \in P a \mid \mathbf{P}_{1} \in \mathcal{M}\left(\ell^{\perp}\right)\right\}$, is the same as the multiset

$$
\bigcup_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \bigcup_{\mathbf{Q} \in W(\mathbf{P})} T_{\mathbf{Q}} \backslash\{\ell\},
$$

and the tangent line $\ell^{\prime}$ other than $\ell$ through an external point $\mathbf{Q}$ on $\ell$ occurs an odd number of times in (2.12) by Lemma 2.18.

Since $\sum_{\ell^{\prime} \in T} \chi_{\ell^{\prime}} \equiv 0(\bmod 2)$, it follows that

$$
\begin{aligned}
\sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \sum_{\ell^{\prime} \in P a_{\mathbf{P}}} \chi_{\ell^{\prime}} & \equiv \sum_{\mathbf{P} \in \mathcal{M}\left(\ell^{\perp}\right)} \chi_{N_{P a, E}(\mathbf{P})} \\
& \equiv \sum_{\ell^{\prime} \in T \backslash\{\ell\}} \chi_{\ell^{\prime}} \\
& \equiv \chi_{\ell}+\sum_{\ell^{\prime} \in T} \chi_{\ell^{\prime}} \\
& \equiv \chi_{\ell} \quad(\bmod 2) .
\end{aligned}
$$

Lemma 2.22. Let $\mathbf{P} \in E$ and let $T_{1}$ and $T_{2}$ be the two tangent lines through $\mathbf{P}$. Assume that $Z \subseteq\left(E_{T_{1}} \cup E_{T_{2}}\right) \backslash\{\mathbf{P}\}$. Then there is a set $\mathcal{M}^{\prime}(\mathbf{P})$ consisting of an even number of internal points such that, for any point $\mathbf{Q} \in Z$, the number of passant lines through $\mathbf{Q}$ and the points in $\mathcal{M}^{\prime}(\mathbf{P})$, counted with multiplicity, is odd, and the number of passant lines through $\mathbf{P}$ and the points in $\mathcal{M}^{\prime}(\mathbf{P})$, counted with mutiplicity, is even.
Proof: Since $G$ is transitive on $E$, without loss of generality, we may assume that $\mathbf{P}=(0,1,0)$, and thus $T_{1}=[1,0,0]$ and $T_{2}=[0,0,1]$ are two tangent lines through $\mathbf{P}$. Let $K:=G_{\mathbf{P}}$ be the stabilizer of $\mathbf{P}$ in $G$. Using (2.4), we have

$$
\begin{align*}
K & =\left\{\left.\mathbf{d}\left(d^{2}, 1, \frac{1}{d^{2}}\right) \right\rvert\, d^{2} \in \square_{q}\right\} \cup\left\{\left.\boldsymbol{\operatorname { a d }}\left(\frac{1}{c^{2}},-1, c^{2}\right) \right\rvert\, c^{2} \in \square_{q}\right\} \\
& \cup\left\{\left.\mathbf{d}\left(d^{2}, \frac{1}{\xi}, \frac{1}{d^{2} \xi^{2}}\right) \right\rvert\, d^{2} \in \square_{q}\right\} \cup\left\{\left.\operatorname{ad}\left(\frac{1}{c^{2}},-\frac{1}{\xi}, \frac{c^{2}}{\xi^{2}}\right) \right\rvert\, c^{2} \in \square_{q}\right\} . \tag{2.13}
\end{align*}
$$

Let $\mathbf{P}_{1}=(1,1, x)$, where $x \in \square_{q}$ (respectively, $x \in \square_{q}$ ) and $1-x \in \rrbracket_{q}$, be an internal point for $q \equiv 3(\bmod 4)($ respectively, $q \equiv 1(\bmod 4))$. (Note that such an $x$ in the last coordinate of $\mathbf{P}_{1}$ exists in $\mathbb{F}_{q}$.) Then the $K$-orbit of $\mathbf{P}_{1}$ is

$$
\mathcal{O}_{\mathbf{P}_{1}}=\left\{\left.\left(1, \frac{1}{d^{2}}, \frac{x}{d^{4}}\right) \right\rvert\, d^{2} \in \square_{q}\right\} \cup\left\{\left.\left(1, \frac{1}{\xi d^{2}}, \frac{x}{\xi^{2} d^{4}}\right) \right\rvert\, d^{2} \in \square_{q}\right\} .
$$

To prove the first part of the lemma, we need only show that it holds for

$$
Z=\left(E_{T_{1}} \cup E_{T_{2}}\right) \backslash\{\mathbf{P}\} .
$$

Let $\mathbf{Q}=(0,1,1) \in Z$. Using (2.13), we have that $K_{\mathbf{Q}}$ only contains the identity collineation. So $K$ is transitive on $Z$ as $|Z|=|K|=2(q-1)$. The lines through $\mathbf{Q}$ and the points in $\mathcal{O}_{\mathbf{P}_{1}}$ form the multiset

$$
L(\mathbf{Q})=\left\{\left[x-d^{2}, d^{4},-d^{4}\right] \mid d^{2} \in \square_{q}\right\} \cup\left\{\left[x-d^{2} \xi, d^{4} \xi^{2},-d^{4} \xi^{2}\right] \mid d^{2} \in \square_{q}\right\} .
$$

A line in $L(\mathbf{Q})$ is passant if and only if

$$
\frac{\left(d^{2}-2\right)^{2}}{4(1-x)}-1 \in \square_{q}
$$

or

$$
\frac{\left(d^{2} \xi-2\right)^{2}}{4(1-x)}-1 \in \square_{q},
$$

where $d^{2} \in \square_{q}$. The number of $d^{2}$ satisfying either of the above two equations is equal to that of $t \in \mathbb{F}_{q}^{*}$ satisfying $\frac{(t-2)^{2}}{4(1-x)}-1 \in \square_{q}$ since $\mathbb{F}_{q}^{*}=\square_{q} \cup \square_{q} \xi$, where $\square_{q} \xi=$ $\left\{d^{2} \xi \mid d^{2} \in \square_{q}\right\}$. For the case where $q \equiv 3(\bmod 4)$, since the number of $t \in \mathbb{F}_{q}$ satisfying $\frac{(t-2)^{2}}{4(1-x)}-1 \in \square_{q}$ is equal to $2\left|\left(\square_{q}-1\right) \cap \square_{q}\right|=2\left(\frac{q+1}{4}\right)=\frac{q+1}{2}$ by Lemma 2.9)(ii) and $t=0$ is one of them, we see that the number of passant lines in $L(\mathbf{Q})$, counted with multiplicity, is $\frac{q-1}{2}$ which is odd since $q \equiv 3(\bmod 4)$. For the case where $q \equiv 1(\bmod 4)$, since the number
of $t \in \mathbb{F}_{q} \backslash\{2\}$ satisfying $\frac{(t-2)^{2}}{4(1-x)}-1 \in \square_{q}$ is equal to $2\left|\left(\not \square_{q}-1\right) \cap \square_{q}\right|=2\left(\frac{q-1}{4}\right)=\frac{q-1}{2}$ by Lemma 2.9(i), $t=0$ is not one of the solutions and $t=2$ also satisfies $\frac{(t-2)^{2}}{4(1-x)}-1 \in \square_{q}$, we see that the number of passant lines in $L(\mathbf{Q})$, counted with multiplicity, is $\frac{q+1}{2}$, which is odd as $q \equiv 1(\bmod 4)$. Now we set $\mathcal{M}^{\prime}(\mathbf{P}):=\mathcal{O}_{\mathbf{P}_{1}}$, and so $\left|\mathcal{M}^{\prime}(\mathbf{P})\right|=q-1$ is even. Since $K$ is transitive on both $Z=\left(E_{T_{1}} \cup E_{T_{2}}\right) \backslash\{\mathbf{P}\}$ and the points in $\mathcal{M}^{\prime}(\mathbf{P})$, the number of passant lines through a point in $Z$ and the points in $M^{\prime}(\mathbf{P})$, counted with multiplicity, must be odd.

The lines through $\mathbf{P}$ and the points in $\mathcal{M}^{\prime}(\mathbf{P})$ form the multiset

$$
\left\{\left.\left[1,0,-\frac{d^{4}}{x}\right] \right\rvert\, d^{2} \in \square_{q}\right\} \cup\left\{\left.\left[1,0,-\frac{d^{4} \xi^{2}}{x}\right] \right\rvert\, d^{2} \in \square_{q}\right\},
$$

each or none of which is a passant line accordingly as $q \equiv 3(\bmod 4)$ or $q \equiv 1(\bmod 4)$. Hence, we conclude that the number of passant lines through $\mathbf{P}$ and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplicty, is even.

Remark 2.23. Let $\mathbf{P} \in E$. In the following discussion, without being further mentioned, $\mathcal{M}^{\prime}(\mathbf{P})$ will always denote a set consisting of an even number of internal points that satisfy the conditions with $Z=E_{T_{1}} \backslash\{\mathbf{P}\}$ in the above lemma, where $T_{1}$ is one of the two tangent lines through $\mathbf{P}$.

Corollary 2.24. Let $\mathbf{P} \in E$ and let $T_{1}$ and $T_{2}$ be the two tangent lines through $\mathbf{P}$. Then

$$
\begin{aligned}
\chi_{T_{1}}+\chi_{T_{2}} & \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \sum_{\ell \in S e_{\mathbf{Q}}} \chi_{\ell} \\
& \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \chi_{N_{S e, E}(\mathbf{Q})}(\bmod 2),
\end{aligned}
$$

where the congruence means entrywise congruence.
Proof: Let $\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})$. Then Corollary 2.17 gives

$$
\begin{align*}
\chi_{N_{P a, E}(\mathbf{Q})} & \equiv \sum_{\ell^{\prime} \in P a_{\mathbf{Q}}} \chi_{\ell^{\prime}} \\
& \equiv \sum_{\ell^{\prime} \in T(\mathbf{Q}, \ell(\mathbf{Q}))} \chi_{\ell^{\prime}}(\bmod 2), \tag{2.14}
\end{align*}
$$

where $\ell(\mathbf{Q})$ is a tangent line through an external point on $\mathbf{Q}^{\perp}$ and $T(\mathbf{Q}, \ell(\mathbf{Q}))$ is the set tangent lines through the external points that are on both $\ell(\mathbf{Q})$ and the passant lines through $\mathbf{Q}$. Let $\mathbf{1}$ be the all-one column vector of length $|E|$. Since

$$
\begin{equation*}
\mathbf{1}+\chi_{N_{P a, E}(\mathbf{Q})} \equiv \chi_{N_{S e, E}(\mathbf{Q})} \quad(\bmod 2) \tag{2.15}
\end{equation*}
$$

and $\left|\mathcal{M}^{\prime}(\mathbf{P})\right|$ is even, we have

$$
\begin{align*}
\sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \sum_{\ell \in S_{\mathbf{Q}}} \chi_{\ell} & \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})}\left(\mathbf{1}+\chi_{N_{P a, E}(\mathbf{Q})}\right) \\
& \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \mathbf{1}+\sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \chi_{N_{P a, E}(\mathbf{Q})}  \tag{2.16}\\
& \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \chi_{N_{P a, E}(\mathbf{Q})} \\
& \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \sum_{\ell \in T(\mathbf{Q}, \ell(\mathbf{Q}))} \chi_{\ell}(\bmod 2) .
\end{align*}
$$

Further, if we set $\ell(\mathbf{Q}):=T_{1}$ for each $\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})$ and set $W^{\prime}(\mathbf{Q}):=\left\{T_{1} \cap \ell_{1} \mid \ell_{1} \in P a_{\mathbf{Q}}\right\}$, since the multiset

$$
\bigcup_{\mathbf{P}_{1} \in E_{T_{1}}} \bigcup_{L^{\prime}\left(\mathbf{P}_{1}\right)} T_{\mathbf{P}_{1}} \backslash\left\{T_{1}\right\},
$$

where $L^{\prime}\left(\mathbf{P}_{1}\right)=\left\{\ell_{\mathbf{P}_{1}, \mathbf{P}_{2}} \in P a \mid \mathbf{P}_{2} \in \mathcal{M}^{\prime}(\mathbf{P})\right\}$, is the same as the multiset

$$
\begin{equation*}
\bigcup_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \bigcup_{\mathbf{P}_{1} \in W^{\prime}(\mathbf{Q})} T_{\mathbf{P}_{1}} \backslash\left\{T_{1}\right\}, \tag{2.17}
\end{equation*}
$$

and the tangent line $\ell$ other than $T_{1}$ through an external point $\mathbf{P}_{1} \neq \mathbf{P}$ (respectively, $\mathbf{P}_{1}=\mathbf{P}$ ) on $T_{1}$ occurs an odd (respectively, even) number of times in (2.17) by Lemma 2.22, we obtain

$$
\begin{align*}
\sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \sum_{\ell \in T(\mathbf{Q}, \ell(\mathbf{Q}))} \chi_{\ell} & \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \sum_{\mathbf{P}_{1} \in W^{\prime}(\mathbf{Q})} \sum_{\ell \in T_{\mathbf{P}_{1} \backslash\left\{T_{1}\right\}}} \chi_{\ell} \\
& =\sum_{\mathbf{P}_{1} \in E_{T_{1}}} \sum_{\ell \in T_{\mathbf{P}_{1}} \backslash\left\{T_{1}\right\}} b_{\ell} \chi_{\ell}  \tag{2.18}\\
& =b_{T_{2}} \chi_{T_{2}}+\sum_{\ell \in T \backslash\left\{T_{1}, T_{2}\right\}} b_{\ell} \chi_{\ell} \\
& \equiv \sum_{\ell \in T \backslash\left\{T_{1}, T_{2}\right\}} \chi \ell(\bmod 2),
\end{align*}
$$

where $b_{\ell}$ for $\ell \in T \backslash\left\{T_{1}, T_{2}\right\}$ are all odd integers and $b_{T_{2}}$ is an even integer.
Using (2.16), (2.18), and the fact that $\sum_{\ell \in T} \chi_{\ell}=\mathbf{0}(\bmod 2)$, we have

$$
\begin{aligned}
\chi_{T_{1}}+\chi_{T_{2}} & \equiv \sum_{\chi \in T \backslash\left\{T_{1}, T_{2}\right\}} \chi_{\ell} \\
& \equiv \sum_{\mathbf{Q} \in M^{\prime}(\mathbf{P})} \sum_{\ell \in P a_{\mathbf{Q}}} \chi_{\ell} \\
& \equiv \sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})}^{\chi_{N_{P a, E}(\mathbf{Q})}}(\bmod 2) .
\end{aligned}
$$

## 3. The Conjugacy Classes and Intersection Parity

In this section, we review the conjugacy classes of $H$ and study their intersections with some special subsets of $H$.
3.1. Conjugacy classes. Recall that

$$
H=\left\{\left.\left(\begin{array}{ccc}
a^{2} & a b & b^{2} \\
2 a c & a d+b c & 2 b d \\
c^{2} & c d & d^{2}
\end{array}\right) \right\rvert\, a, b, c, d \in \mathbb{F}_{q}, a d-b c=1\right\}
$$

is the subgroup of $G$ that is isomorphic to $\operatorname{PSL}(2, q)$. If we define $T=\operatorname{tr}(g)+1$, where $g \in H$ and $\operatorname{tr}(g)$ is the trace of $g$, then the conjugacy classes of $H$ can be read as follows.
Lemma 3.1. [17, Lemma 3.2] The conjugacy classes of $H$ are given as follows.
(i) $D=\{\mathbf{d}(1,1,1)\}$;
(ii) $F^{+}$and $F^{-}$, where $F^{+} \cup F^{-}=\{g \in H \mid T(g)=4, g \neq \mathbf{d}(1,1,1)\}$;
(iii) $\left[\theta_{i}\right]=\left\{g \in H \mid T(g)=\theta_{i}\right\}, 1 \leq i \leq \frac{q-5}{4}$ if $q \equiv 1(\bmod 4)$, or $1 \leq i \leq \frac{q-3}{4}$ if $q \equiv 3$ $(\bmod 4)$, where $\theta_{i} \in \square_{q}, \theta_{i} \neq 4$, and $\theta_{i}-4 \in \square_{q}$;
(iv) $[0]=\{g \in H \mid T(g)=0\}$;
(v) $\left[\pi_{k}\right]=\left\{g \in H \mid T(g)=\pi_{k}\right\}, 1 \leq k \leq \frac{q-1}{4}$ if $q \equiv 1(\bmod 4)$, or $1 \leq k \leq \frac{q-3}{4}$ if $q \equiv 3(\bmod 4)$, where $\pi_{i} \in \square_{q}, \pi_{k} \neq 4$, and $\pi_{k}-4 \in \square_{q}$.
Remark 3.2. The set $F^{+} \cup F^{-}$forms one conjugacy class of $G$, and splits into two equalsized classes $F^{+}$and $F^{-}$of $H$. For our purpose, we denote $F^{+} \cup F^{-}$by [4]. Also, each of $D,\left[\theta_{i}\right]$, [0], and $\left[\pi_{k}\right]$ forms a single conjugacy class of $G$. The class [0] consists of all the elements of order 2 in $H$.

In the following, for convenience, we frequently use $C$ to denote any one of $D,[0],[4]$, $\left[\theta_{i}\right]$, or $\left[\pi_{k}\right]$. That is,

$$
\begin{equation*}
C=D,[0],[4],\left[\theta_{i}\right], \text { or }\left[\pi_{k}\right] \tag{3.1}
\end{equation*}
$$

### 3.2. Intersection properties.

Definition 3.3. Let $\mathbf{P} \in I, \mathbf{Q} \in E, \ell \in P a$. We define

$$
\mathcal{H}_{\mathbf{P}, \mathbf{Q}}=\left\{h \in H \mid\left(\mathbf{P}^{\perp}\right)^{h} \in P a_{\mathbf{Q}}\right\} \quad \text { and } \mathcal{S}_{\mathbf{P}, \ell}=\left\{h \in H \mid\left(\mathbf{P}^{\perp}\right)^{h}=\ell\right\}
$$

That is, $\mathcal{H}_{\mathbf{P}, \mathbf{Q}}$ consists of all the elements of $H$ that map the passant line $\mathbf{P}^{\perp}$ to a passant line through $\mathbf{Q}$ and $\mathcal{S}_{\mathbf{P}, \ell}$ is the set of elements in $H$ that map $\mathbf{P}^{\perp}$ to the passant line $\ell$.

Using the above notation, since $G$ preserves incidence, for $g \in G, \mathbf{P} \in I$, and $\ell \in P a$, we have

$$
\begin{equation*}
\mathcal{H}_{\mathbf{P}, \mathbf{Q}}^{g}=\mathcal{H}_{\mathbf{P}^{g}, \mathbf{Q}^{g}}, \mathcal{S}_{\mathbf{P}, \ell}^{g}=\mathcal{S}_{\mathbf{P}^{g}, \ell} \tag{3.2}
\end{equation*}
$$

The following corollary is apparent.
Corollary 3.4. Let $g \in G$ and $C$ be given in (3.1) and let $\mathbf{P}$ and $\mathbf{Q}$ be two external points. Then $\left(C \cap \mathcal{H}_{\mathbf{P}, \mathbf{Q}}\right)^{g}=C \cap \mathcal{H}_{\mathbf{P}^{g}, \mathbf{Q}^{g}}$.

Next the size of the intersection of each conjugacy class of $H$ with $K$ which stabilizes an element of $I$ in $H$ is calculated.
Corollary 3.5. Let $\mathbf{P} \in I$ and $K=H_{\mathbf{P}}$. Then we have
(i) $|K \cap D|=1$;
(ii) $|K \cap[4]|=0$;
(iii) $\left|K \cap\left[\pi_{k}\right]\right|=2$ for each $k$;
(iv) $\left|K \cap\left[\theta_{i}\right]\right|=0$ for each $i$;
(v) $|K \cap[0]|=\frac{q+1}{2}$ or $\frac{q-1}{2}$ accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$.

Proof: Let $\mathbf{Q}=(1,0,-\xi)$ and $K_{1}=H_{\mathbf{Q}}$. Since $H$ is transitive on $I$, it follows $\mathbf{Q}^{g}=\mathbf{P}$ for some $g \in H$. By Lemma 2.11, we have $K_{1}^{g}=K$. Consequently,

$$
|K \cap C|=\left|\left(K_{1} \cap C\right)^{g}\right|
$$

Therefore, to prove the corollary, it is enough to consider $\mathbf{P}=\mathbf{Q}$. It is clear that $|D \cap K|=$ 1. Let $g \in K \cap C$. Then the quadruples $(a, b, c, d)$ determining $g$ satisfy the following equations

$$
\begin{array}{ccc}
b d-a c \xi & = & 0 \\
b^{2}-a^{2} \xi & = & -\xi\left(d^{2}-c^{2} \xi\right)  \tag{3.3}\\
a d-b c & = & 1 \\
a+d & = & s
\end{array}
$$

where $s^{2}=0,4, \pi_{k}, \theta_{i}$. The equations in (3.3) give (1) $a=d=\frac{s}{2}, c^{2}=\frac{s^{2}-4}{4 \xi}, b^{2}=\frac{\left(s^{2}-4\right) \xi}{4}$ and (2) $a=-d, s=0, c^{2} \xi-1=a^{2}$. From Case (1), we see that $\left|K \cap\left[\pi_{k}\right]\right|=2$ for each $\left[\pi_{k}\right]$ and $|K \cap C|=0$ for $C=\left[\theta_{i}\right]$, [4]; moreover, if $q \equiv 3(\bmod 4)$, we obtain one group element $\operatorname{ad}\left(-\xi,-1, \xi^{-1}\right) \in K \cap[0]$ in Case (1). Since the number of $t \in \not \varnothing_{q}$ satisfying $t-1 \in \square_{q}$ is $\frac{q-1}{4}$ or $\frac{q-3}{4}$ accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$ by Lemma 2.9,
the number of $c \in \mathbb{F}_{q}^{*}$ satisfying $c^{2} \xi-1 \in \square_{q}$ is $2\left|\left(\not \square_{q}-1\right) \cap \square_{q}\right|$ which is $\frac{q-1}{2}$ or $\frac{q-3}{2}$ accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$. When $q \equiv 1(\bmod 4), c=0$ also satisfies $c^{2} \xi-1 \in \square_{q}$. Therefore, Case (1) and Case (2) give $\frac{q+1}{2}$ or $\frac{q-1}{2}$ different group elements in $K \cap[0]$ depending on $q$. Now the corollary is proved.

In the following lemma, we investigate the parity of $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ for each $C \neq[0]$ and $\mathbf{P} \in I, \mathbf{Q} \in E$. Recall that $\ell_{\mathbf{P}, \mathbf{Q}}$ is the line through $\mathbf{P}$ and $\mathbf{Q}$.

Lemma 3.6. Assume that $q \equiv 1(\bmod 4)$. Let $\mathbf{P} \in I$ and $\mathbf{Q} \in E$. Suppose that $C=D$, [4], $\left[\pi_{k}\right]\left(1 \leq k \leq \frac{q-1}{4}\right),\left[\theta_{i}\right]\left(1 \leq i \leq \frac{q-5}{4}\right)$.
(i) If $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$, then $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is even for each $C$.
(ii) If $\ell_{\mathbf{P}, \mathbf{Q}} \in P a_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is odd if and only if $C=D$.
(iii) If $\ell_{\mathbf{P}, \mathbf{Q}} \in P a_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, for each class $\left[\pi_{k}\right]$ with $1 \leq k \leq \frac{q-1}{4}$, there are two different points $\mathbf{Q}_{1}, \mathbf{Q}_{2} \in E_{\ell_{\mathbf{P}, \mathbf{Q}}}$ such that $\left|\left[\pi_{k}\right] \cap \mathcal{H}_{\mathbf{P}, \mathbf{Q}_{j}}\right|$ is odd for $j=1,2$; moreover, the two points associated with one class $\left[\pi_{k_{1}}\right]$ are different from those associated with the other class $\left[\pi_{k_{2}}\right]$, where $\left[\pi_{k_{1}}\right] \neq\left[\pi_{k_{2}}\right]$.
Proof: $\quad$ Since $G$ acts transitively on $I$ and preserves incidence, without loss of generality, we may assume that $\mathbf{P}=(1,0,-\xi)$. From (2.4), it follow that

$$
\begin{align*}
& K:=G_{\mathbf{P}}= \\
& \left\{\left.\left(\begin{array}{ccc}
d^{2} & c d \xi & c^{2} \xi^{2} \\
2 c d & d^{2}+c^{2} \xi & 2 c d \xi \\
c^{2} & c d & d^{2}
\end{array}\right) \right\rvert\, d, c \in \mathbb{F}_{q}, d^{2}-c^{2} \xi=1\right\} \\
& \begin{array}{l}
\cup\left\{\left.\left(\begin{array}{ccc}
d^{2} & -c d \xi & c^{2} \xi^{2} \\
2 c d & -d^{2}-c^{2} \xi & 2 c d \xi \\
c^{2} & -c d & d^{2}
\end{array}\right) \right\rvert\, \begin{array}{l}
d, c \in \mathbb{F}_{q},-d^{2}+c^{2} \xi=1
\end{array}\right\} \\
\cup\left\{\left.\left(\begin{array}{ccc}
d^{2} & c d & c^{2} \\
2 c d \xi^{-1} & d^{2}+c^{2} \xi^{-1} & 2 c d \\
c^{2} \xi^{-2} & c d \xi^{-1} & d^{2}
\end{array}\right) \right\rvert\, d, c \in \mathbb{F}_{q}, d^{2} \xi-c^{2}=1\right\} \\
\cup\left\{\left.\left(\begin{array}{ccc}
d^{2} & -c d & c^{2} \\
2 c d \xi^{-1} & -d^{2}-c^{2} \xi^{-1} & 2 c d \\
c^{2} \xi^{-2} & -c d \xi^{-1} & d^{2}
\end{array}\right) \right\rvert\, d, c \in \mathbb{F}_{q},-d^{2} \xi+c^{2}=1\right\} .
\end{array} \tag{3.4}
\end{align*}
$$

Since $K$ is transitive on both $P a_{\mathbf{P}}$ and $S e_{\mathbf{P}}$ by Proposition 2.12 and

$$
\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|=\left|\left(\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right)^{g}\right|=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}^{g}} \cap C\right|
$$

by Corollary 3.4, we may assume that $\mathbf{Q}$ is on either $\ell_{1}$ or $\ell_{2}$, where $\ell_{1}=\left[1,0, \xi^{-1}\right] \in P a_{\mathbf{P}}$ and $\ell_{2}=[0,1,0] \in S e_{\mathbf{P}}$.

Case I. $\mathbf{Q} \in \ell_{1}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$.
In this case, $\mathbf{Q}=(1, x,-\xi)$ for some $x \in \mathbb{F}_{q}^{*}$ and $x^{2}+\xi \in \square_{q}$, and

$$
P a_{\mathbf{Q}}=\left\{\left[1, s,(1+s x) \xi^{-1}\right] \mid s \in \mathbb{F}_{q}, s^{2}-4(1+s x) \xi^{-1} \in \not \square_{q}\right\} .
$$

Using (3.4), we obtain that

$$
K_{\mathbf{Q}}=\left\{\mathbf{d}(1,1,1), \mathbf{a d}\left(1,-\xi^{-1}, \xi^{-2}\right)\right\} .
$$

It is apparent that $\mathbf{d}(1,1,1)$ fixes each line in $P a_{\mathbf{Q}}$. From

$$
\mathbf{a d}\left(1,-\xi^{-1}, \xi^{-2}\right)^{-1}\left(1, s,(1+s x) \xi^{-1}\right)^{\top}=((1+s x) \xi,-s \xi, 1)^{\top},
$$

it follows that $\left[1, s,(1+s x) \xi^{-1}\right] \in P a_{\mathbf{Q}}$ is fixed by $K_{\mathbf{Q}}$ if and only if $s=0$ or $s=$ $-2 x^{-1}$. Therefore, $K_{\mathbf{Q}}$ has two orbits of length 1 on $P a_{\mathbf{Q}}$, i.e. $\left\{\ell_{1}=\left[1,0, \xi^{-1}\right]\right\}$ and
$\left\{\ell_{3}=\left[1,-2 x^{-1},-\xi^{-1}\right]\right\}$, and all other orbits, whose representatives are $\mathcal{R}_{1}$, have length 2. From

$$
\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|=\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|+\left|\mathcal{S}_{\mathbf{P}, \ell_{3}} \cap C\right|+2 \sum_{\ell \in \mathcal{R}_{1}}\left|\mathcal{S}_{\mathbf{P}, \ell} \cap C\right|,
$$

it follows that the parity of $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is determined by that of $\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|+\left|\mathcal{S}_{\mathbf{P}, \ell_{3}} \cap C\right|$. Here we used the fact that $\left|\mathcal{S}_{\mathbf{P}, \ell} \cap C\right|=\left|\mathcal{S}_{\mathbf{P}, \ell^{\prime}} \cap C\right|$ if $\left\{\ell, \ell^{\prime}\right\}$ is an orbit of $K_{\mathbf{P}}$ on $P a_{\mathbf{Q}}$. It is clear that $\left|\mathcal{S}_{\mathbf{P}, \ell} \cap D\right|=\left|\mathcal{S}_{\mathbf{P}, \ell_{3}} \cap D\right|=0$.

Note that the quadruples $(a, b, c, d)$ that determine group elements in $\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C$ satisfy the following equations

$$
\begin{array}{ccc}
-2 c d+2 a b \xi^{-1} & = & 0  \tag{3.5}\\
c^{2}-a^{2} \xi^{-1} & = & \left(d^{2}-b^{2} \xi^{-1}\right) \xi^{-1} \\
a+d & = & s \\
a d-b c & = & 1
\end{array}
$$

where $s^{2}=4, \pi_{k}, \theta_{i}$. The first two equations in (3.5) give $c= \pm \sqrt{-1} c \xi^{-1}$ and $a=$ $\pm \sqrt{-1} d$. Combining them with the last two equationsin (3.5), we obtain 0,4 or 8 quadruples $(a, b, c, d)$ satisfying the above equations, among which, both ( $a, b, c, d$ ) and $(-a,-b,-c,-d)$ do appear at the same time. Therefore, $\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|$ is 0,2 , or 4. Particularly, in [0], there might be only 2 elements satisfying the above conditions.

Similarly, the quadruples $(a, b, c, d)$ that determine a group element in $\mathcal{S}_{\mathbf{P}, \ell_{3}} \cap C$ satisfy the following equations

$$
\begin{array}{ccc}
-2 c d+2 a b \xi^{-1} & = & -2 x^{-1}\left(d^{2}-b^{2} \xi^{-1}\right) \\
c^{2}-a^{2} \xi^{-1} & = & -\xi^{-1}\left(d^{2}-b^{2} \xi^{-1}\right)  \tag{3.6}\\
a+d & = & s \\
a d-b c & = & 1,
\end{array}
$$

where $s^{2}=4, \pi_{k}, \theta_{i}$. The first two equations in (3.6) give

$$
\begin{equation*}
d^{2}-b^{2} \xi^{-1}= \pm A \tag{3.7}
\end{equation*}
$$

where

$$
\begin{equation*}
A=\sqrt{\frac{1}{\left(x^{2}+\xi^{-1}\right) \xi}} \tag{3.8}
\end{equation*}
$$

From (3.7), $c^{2}-d^{2}=\mp A \xi^{-1}$ and $a^{2}+d^{2}=s^{2}-2-2 b c$, it follows that

$$
\begin{equation*}
\left(b \xi^{-1}+c\right)^{2}=-\left( \pm 2 A+2-s^{2}\right) \xi^{-1} \tag{3.9}
\end{equation*}
$$

Hence, if (3.6) determines an odd number of group elements, then

$$
-\left( \pm 2 A+2-s^{2}\right) \xi^{-1} \notin \not \nabla_{q}
$$

If $-\left( \pm 2 A+2-s^{2}\right) \xi^{-1} \in \square_{q}$ and we set $B_{( \pm)}:=\sqrt{-\left( \pm 2 A+2-s^{2}\right) \xi^{-1}}$, by $c^{2}-d^{2} \xi^{-1}=$ $\pm A \xi^{-1}$ and $a^{2}-b^{2} \xi^{-1}=\mp A \xi^{-1}$, we have

$$
\begin{equation*}
d=\frac{1}{2 s}\left[s^{2}+\left(\xi B_{( \pm)}^{2}-2 B_{( \pm)} b\right)\right]\left(\text { or } d=\frac{1}{2 s}\left[s^{2}+\left(\xi B_{( \pm)}^{2}+2 B_{( \pm)} b\right)\right]\right) \tag{3.10}
\end{equation*}
$$

and thus

$$
\begin{equation*}
a=\frac{1}{2 s}\left[s^{2}-\left(\xi B_{( \pm)}^{2}-2 B_{( \pm)} b\right)\right]\left(\text { or } a=\frac{1}{2 s}\left[s^{2}-\left(\xi B_{( \pm)}^{2}+2 B_{( \pm)} b\right)\right]\right) . \tag{3.11}
\end{equation*}
$$

Combining $b=\left( \pm B_{( \pm)}-c\right) \xi^{-1}$ and $a d-b c=1$, we have

$$
\begin{equation*}
\left(\xi-\frac{B_{( \pm)}^{2} \xi^{2}}{s^{2}}\right) c^{2}+\left(\frac{\xi B_{( \pm)}^{3} \xi^{2}}{s^{2}}-B_{( \pm)} \xi\right) c+\left(\frac{s^{2}}{4}-\frac{B_{( \pm)}^{4} \xi^{2}}{4 s^{2}}-1\right)=0 \tag{3.12}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(\xi-\frac{B_{( \pm)}^{2} \xi^{2}}{s^{2}}\right) c^{2}-\left(\frac{\xi B_{( \pm}^{3} \xi^{2}}{s^{2}}-B_{( \pm)} \xi\right) c+\left(\frac{s^{2}}{4}-\frac{B_{( \pm)}^{4} \xi^{2}}{4 s^{2}}-1\right)=0 \tag{3.13}
\end{equation*}
$$

The discriminant of (3.12) or (3.13) is

$$
\begin{equation*}
\Delta=\xi^{2}\left(\frac{\xi B_{( \pm)}^{2} \xi}{s^{2}}-1\right)\left(\frac{s^{2}}{\xi}-\frac{4}{\xi}-B_{( \pm)}^{2}\right)=\frac{4 x^{2} \xi}{\left(x^{-2}+\xi^{-1}\right) s^{2}} \in \square_{q} \tag{3.14}
\end{equation*}
$$

From (3.10), (3.11), (3.12), (3.13), and (3.14), it follows that (3.6) produces 2 or 4 group elements; that is, $\left|\mathcal{S}_{\mathbf{P}, \ell_{3}} \cap C\right|=2$ or 4 .

If $-\left( \pm 2 A+2-s^{2}\right) \xi^{-1}=0$, then $s^{2}$ is one of $2 A+2$ and $-2 A+2$ since

$$
(2 A+2)(-2 A+2)=\frac{4 x^{2}}{x^{2}+\xi^{-1}} \in \not \nabla_{q}
$$

Therefore, in this case, we have $\left|\left[s^{2}\right] \cap \mathcal{S}_{\mathbf{P}, \ell_{3}}\right|=1$. It is also clear that, for the same $\left[s^{2}\right]$,

$$
\left|\left[s^{2}\right] \cap \mathcal{S}_{\mathbf{P}, \mathbf{P}_{1}^{\perp}}\right|
$$

is odd, where $\mathbf{P}_{1}=(1,-x,-\xi) \in E_{\ell_{3}}$. Moreover, when $x$ runs over

$$
L:=\left\{x \in \mathbb{F}_{q}^{*} \mid x^{2}+\xi \in \square_{q}\right\}
$$

once, each $\left[\pi_{k}\right]$ with $1 \leq k \leq \frac{q-1}{4}$ appears exactly twice in the multiset

$$
\left\{2 \sqrt{\frac{1}{\left(x^{-2}+\xi^{-1}\right) \xi}}+2\right\} \bigcup\left\{-2 \sqrt{\frac{1}{\left(x^{-2}+\xi^{-1}\right) \xi}}+2\right\}
$$

Note that

$$
\pm \frac{2}{\sqrt{\left(x_{1}^{-2}+\xi^{-1}\right) \xi}}+2= \pm \frac{2}{\sqrt{\left(x_{2}^{-2}+\xi^{-1}\right) \xi}}+2
$$

if and only if $x_{1}= \pm x_{2}$. Therefore, for each class $\left[\pi_{k}\right]$ with $1 \leq k \leq \frac{q-1}{4}$, there are two different points $\mathbf{Q}_{1}, \mathbf{Q}_{2} \in E_{\ell_{\mathbf{P}, \mathbf{Q}}}$ such that $\left|\left[\pi_{k}\right] \cap \mathcal{H}_{\mathbf{P}, \mathbf{Q}_{j}}\right|$ is odd for $j=1,2$; further, the two points associated with one class [ $\pi_{k_{1}}$ ] are different from those associated with the other class $\left[\pi_{k_{2}}\right]$, where $\left[\pi_{k_{1}}\right] \neq\left[\pi_{k_{2}}\right]$. The proof of (iii) is completed.

Case II. $\mathbf{Q}=\ell_{1} \cap \mathbf{P}^{\perp}$.
In this case, $\mathbf{Q}=(0,1,0)$. From (3.4), it follows that

$$
K_{\mathbf{Q}}=\left\{\mathbf{d}(1,1,1), \mathbf{a d}(-1,1,-1), \mathbf{d}\left(-1,-\xi^{-1},-\xi^{-2}\right), \mathbf{a d}\left(1,-\xi^{-1}, \xi^{-2}\right)\right\} .
$$

Since $P a_{\mathbf{Q}}=\left\{[1,0,-x] \mid x \in \varnothing_{q}\right\}$, it follows that the passant lines through $\mathbf{Q}$ that are fixed by $K_{\mathbf{Q}}$ are $\ell_{1}=\left[1,0, \xi^{-1}\right]$ and $\ell_{4}=\left[1,0,-\xi^{-1}\right]$. Thus, $K_{\mathbf{Q}}$ has two orbits of length 1 on $P a_{\mathbf{Q}}$ and all the other orbits, whose representatives are $\mathcal{R}_{2}$, have length 2. By

$$
\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|=\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|+\left|\mathcal{S}_{\mathbf{P}, \ell_{4}} \cap C\right|+2 \sum_{\ell \in \mathcal{R}_{2}}\left|\mathcal{S}_{\mathbf{P}, \ell} \cap C\right|,
$$

we obtain that the parity of $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is determined by that of $\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|+\left|\mathcal{S}_{\mathbf{P}, \ell_{4}} \cap C\right|$. From the discussions in Case I, we know that $\left|\mathcal{S}_{\mathbf{P}, \ell_{1}} \cap C\right|$ always even. Since $\ell_{4}=\mathbf{P}^{\perp}$ and $G_{\mathbf{P}}=G_{\mathbf{P} \perp}$ by Lemma 2.13, it follows from Corollary 3.5 that $\left|\mathcal{S}_{\mathbf{P}, \ell_{4}} \cap C\right|$ is odd if and only if $C=D$. The proof of (ii) is completed.

Case III. $\mathbf{Q} \in \ell_{2}$.
In this case, $\mathbf{Q}=(1,0,-y)$ for some $y \in \square_{q}$. Using (3.4), we see that

$$
K_{\mathbf{Q}}=\{\mathbf{d}(1,1,1), \mathbf{d}(-1,1,-1)\} .
$$

Moreover, all the orbits of $K_{\mathbf{Q}}$ on $P a_{\mathbf{Q}}=\left\{\left[1, s, y^{-1}\right] \mid s \in \mathbb{F}_{q}^{*}, s^{2}-4 y^{-1} \in \square_{q}\right\}$ have length 2 , then $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is even for each $C$. Part (i) is proved.

Lemma 3.7. Assume that $q \equiv 3(\bmod 4)$. Let $\mathbf{P} \in I$ and $\mathbf{Q} \in E$. Suppose that $C=D$, $[4]$, $\left[\pi_{k}\right]\left(1 \leq k \leq \frac{q-3}{4}\right),\left[\theta_{i}\right]\left(1 \leq i \leq \frac{q-3}{4}\right)$.
(i) If $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, for each class $\left[\theta_{i}\right]$ with $1 \leq i \leq \frac{q-3}{4}$, there are two different points $\mathbf{Q}_{1}, \mathbf{Q}_{2} \in E_{\ell_{\mathbf{P}, \mathbf{Q}}}$ such that $\left|\left[\theta_{i}\right] \cap \mathcal{H}_{\mathbf{P}, \mathbf{Q}_{j}}\right|$ is odd for $j=1,2$; moreover, the two points associated with one class $\left[\theta_{i_{1}}\right]$ are different from those associated with the other class $\left[\theta_{i_{2}}\right]$, where $\left[\theta_{i_{1}}\right] \neq\left[\theta_{i_{2}}\right]$.
(ii) If $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is odd if and only if $C=D$.
(iii) If $\ell_{\mathbf{P}, \mathbf{Q}} \in P a_{\mathbf{P}}$, then $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|$ is even for each $C$.

Proof: The proof is essentially the same as the one of Lemma 3.6. We omit the details.

## 4. Group Algebra $F H$

4.1. 2-Blocks of $\mathbf{H}$. In this section we recall several results on the 2-blocks of $H \cong$ $P S L(2, q)$. We refer the reader to [14] or [2] for a general introduction on this subject.

Let $\mathbf{R}$ be the ring of algebraic integers in the complex field $\mathbb{C}$. We choose a maximal ideal $\mathbf{M}$ of $\mathbf{R}$ containing $2 \mathbf{R}$. Let $F=\mathbf{R} / \mathbf{M}$ be the residue field of characteristic 2 , and let $*: \mathbf{R} \rightarrow F$ be the natural ring homomorphism. Define

$$
\begin{equation*}
\mathbf{S}=\left\{\left.\frac{r}{s} \right\rvert\, r \in \mathbf{R}, s \in \mathbf{R} \backslash \mathbf{M}\right\} \tag{4.1}
\end{equation*}
$$

Then it is clear that the map $*: \mathbf{S} \rightarrow F$ defined by $\left(\frac{r}{s}\right)^{*}=r^{*}\left(s^{*}\right)^{-1}$ is a ring homomorphism with kernel $\mathcal{P}=\left\{\left.\frac{r}{s} \right\rvert\, r \in \mathbf{M}, s \in \mathbf{R} \backslash \mathbf{M}\right\}$. In the rest of this article, $F$ will always be the field of characteristic 2 constructed as above. Note that $F$ is an algebraic closure of $\mathbb{F}_{2}$.

Let $\operatorname{Irr}(H)$ and $\operatorname{IBr}(H)$ be the set of irreducible ordinary characters and the set of irreducible Brauer characters of $H$, respectively. In the following, we deduce the 2-blocks of $H$ from the known results on the 2-blocks of PSL $(2, q)$. For baisc results on blocks of finite groups, we refer the reader to Chapter 3 of [14].

The character tables of $\operatorname{PSL}(2, q)$ were obtained by Jordan and Schur independently; see [11], [12], or [15] for the detailed discussions. The irreducible characters of $H$ can be read off from the character tables of $\operatorname{PSL}(2, q)$ as follows.
Lemma 4.1. ([11, [12], [15]) The irreducible ordinary characters of $H$ are:
(i) $1=\chi_{0}, \gamma, \chi_{1}, \ldots, \chi_{\frac{q-1}{4}}, \beta_{1}, \beta_{2}, \phi_{1}, \ldots, \phi_{\frac{q-5}{4}}$ if $q \equiv 1(\bmod 4)$, where $1=\chi_{0}$ is the trivial character, $\gamma$ is the character of degree $q$, $\chi_{s}$ for $1 \leq s \leq \frac{q-1}{4}$ are the characters of degree $q-1$, $\phi_{r}$ for $1 \leq r \leq \frac{q-5}{4}$ are the characters of degree $q+1$, and $\beta_{i}$ for $i=1,2$ are the characters of degree $\frac{q+1}{2}$;
(ii) $1=\chi_{0}, \chi_{1}, \ldots, \chi_{\frac{q-3}{4}}, \beta_{1}, \eta_{2}, \eta_{1}, \ldots, \phi_{\frac{q-3}{4}}$ if $q \equiv 3(\bmod 4)$, where $1=\chi_{0}$ is the trivial character, $\gamma$ is the character of degree $q$, $\chi_{s}$ for $1 \leq s \leq \frac{q-1}{3}$ are the characters of degree $q-1$, $\phi_{r}$ for $1 \leq r \leq \frac{q-3}{4}$ are the characters of degree $q+1$, and $\eta_{i}$ for $i=1,2$ are the characters of degree $\frac{q-1}{2}$;
The following lemma tells us how the irreducible ordinary characters of $H$ are partitioned into 2-blocks.
Lemma 4.2. [17, Lemma 4.1] First assume that $q \equiv 1(\bmod 4)$ and $q-1=m 2^{n}$, where $2 \nmid m$.
(i) The principal block $B_{0}$ of $H$ contains $2^{n-2}+3$ irreducible characters

$$
\chi_{0}=1, \gamma, \beta_{1}, \beta_{2}, \phi_{i_{1}}, \ldots, \phi_{i_{\left(2^{n-2}-1\right)}}
$$

where $\chi_{0}=1$ is the trivial character of $H, \gamma$ is the irreducible character of degree $q$ of $H, \beta_{1}$ and $\beta_{2}$ are the two irreducible characters of degree $\frac{q+1}{2}$, and $\phi_{i_{k}}$ for $1 \leq k \leq 2^{n-2}-1$ are distinct irreducible characters of degree $q+1$ of $H$.
(ii) $H$ has $\frac{q-1}{4}$ blocks $B_{s}$ of defect 0 for $1 \leq s \leq \frac{q-1}{4}$, each of which contains an irreducible ordinary character $\chi_{s}$ of degree $q-1$.
(iii) If $m \geq 3$, then $H$ has $\frac{m-1}{2}$ blocks $B_{t}^{\prime}$ of defect $n-1$ for $1 \leq t \leq \frac{m-1}{2}$, each of which contains $2^{n-1}$ irreducible ordinary characters $\phi_{t_{i}}$ for $1 \leq i \leq 2^{n-1}$.
Now assume that $q \equiv 3(\bmod 4)$ and $q+1=m 2^{n}$, where $2 \nmid m$.
(iv) The principal block $B_{0}$ of $H$ contains $2^{n-2}+3$ irreducible characters

$$
\chi_{0}=1, \gamma, \eta_{1}, \eta_{2}, \chi_{i_{1}}, \ldots, \chi_{i_{\left(2^{n-2}-1\right)}}
$$

where $\chi_{0}=1$ is the trivial character of $H, \gamma$ is the irreducible character of degree $q$ of $H, \eta_{1}$ and $\eta_{2}$ are the two irreducible characters of degree $\frac{q-1}{2}$, and $\chi_{i_{k}}$ for $1 \leq k \leq 2^{n-2}-1$ are distinct irreducible characters of degree $q-1$ of $H$.
(v) H has $\frac{q-3}{4}$ blocks $B_{r}$ of defect 0 for $1 \leq r \leq \frac{q-3}{4}$, each of which contains an irreducible ordinary character $\phi_{r}$ of degree $q+1$.
(vi) If $m \geq 3$, then $H$ has $\frac{m-1}{2}$ blocks $B_{t}^{\prime}$ of defect $n-1$ for $1 \leq t \leq \frac{m-1}{2}$, each of which contains $2^{n-1}$ irreducible ordinary characters $\chi_{t_{i}}$ for $1 \leq i \leq 2^{n-1}$.
Moreover, the above blocks form all the 2-blocks of $H$.
Remark 4.3. Parts ( $i$ ) and (iv) are from Theorem 1.3 in [13] and their proofs can be found in Chapter 7 of III in [2]. Parts (ii) and (v) are special cases of Theorem 3.18 in [14]. Parts (iii) and (vi) are proved in Sections II and VIII of [3].
4.2. Block Idempotents. Let $B l(H)$ be the set of 2-blocks of $H$. If $B \in B l(H)$, we write

$$
f_{B}=\sum_{\chi \in \operatorname{Irr}(B)} e_{\chi}
$$

where $e_{\chi}=\frac{\chi(1)}{|H|} \sum_{g \in H} \chi\left(g^{-1}\right) g$ is a central primitive idempotent of $\mathbf{Z}(\mathbb{C} H)$ and $\operatorname{Irr}(B)=$ $\operatorname{Irr}(H) \cap B$. For future use, we define $\operatorname{IBr}(B)=I B r(H) \cap B$. Since $f_{B}$ is an element of $\mathbf{Z}(\mathbb{C} H)$, we may write

$$
f_{B}=\sum_{C \in c l(H)} f_{B}(\widehat{C}) \widehat{C}
$$

where $c l(H)$ is the set of conjugacy classes of $H, \widehat{C}$ is the sum of elements in the class $C$, and

$$
\begin{equation*}
f_{B}(\widehat{C})=\frac{1}{|H|} \sum_{\chi \in \operatorname{Irr}(B)} \chi(1) \chi\left(x_{C}^{-1}\right) \tag{4.2}
\end{equation*}
$$

with a fixed element $x_{C} \in C$.
Theorem 4.4. Let $B \in B l(H)$. Then $f_{B} \in \mathbf{Z}(\mathbf{S} H)$. In other words, $f_{B}(\widehat{C}) \in \mathbf{S}$ for each block of $H$.
Proof: It follows from Corollary 3.8 in [14].
We extend the ring homomorphism $*: \mathbf{S} \rightarrow F$ to a ring homomorphism $*: \mathbf{S H} \rightarrow$ $F H$ by setting $\left(\sum_{g \in H} s_{g} g\right)^{*}=\sum_{g \in H} s_{g}^{*} g$. Note that $* \operatorname{maps} \mathbf{Z}(\mathbf{S} H)$ onto $\mathbf{Z}(F H)$ via $\left(\sum_{C \in c l(H)} s_{C} \widehat{C}\right)^{*}=\sum_{C \in c l(H)} s_{C}^{*} \widehat{C}$. Now we define

$$
e_{B}=\left(f_{B}\right)^{*} \in \mathbf{Z}(F H)
$$

which is the block idempotent of $B$. Note that $e_{B} e_{B^{\prime}}=\delta_{B B^{\prime}} e_{B}$ for $B, B^{\prime} \in B l(H)$, where $\delta_{B B^{\prime}}$ equals 1 if $B=B^{\prime}, 0$ otherwise. Also $1=\sum_{B \in B l(H)} e_{B}$.

All the block idempotents of the 2-blocks of $H$ are given in the following lemma; see [17] for the detailed calculations.
Lemma 4.5. [17, Lemma 4.4] First assume that $q \equiv 1(\bmod 4)$ and $q-1=m 2^{n}$ with $2 \nmid m$.

1. Let $B_{0}$ be the principal block of $H$. Then
(a) $e_{B_{0}}(\widehat{D})=1$.
(b) $e_{B_{0}}\left(\widehat{F^{+}}\right)=e_{B_{0}}\left(\widehat{F^{-}}\right) \in F$.
(c) $e_{B_{0}}\left(\widehat{\left[\theta_{i}\right]}\right) \in F, e_{B_{0}}(\widehat{[0]})=0$.
(d) $e_{B_{0}}\left(\widehat{\left[\pi_{k}\right]}\right)=1$.
2. Let $B_{s}$ be any block of defect 0 of $H$. Then
(a) $e_{B_{s}}(\widehat{D})=0$.
(b) $e_{B_{s}}\left(\widehat{F^{+}}\right)=e_{B_{s}}\left(\widehat{F^{-}}\right)=1$.
(c) $e_{B_{s}}(\widehat{[0]})=e_{B_{s}}\left(\widehat{\left.\theta_{i}\right]}\right)=0$.
(d) $e_{B_{s}}\left(\widehat{\left[\pi_{k}\right]}\right) \in F$.
3. Suppose $m \geq 3$ and let $B_{t}^{\prime}$ be any block of defect $n-1$ of $H$. Then
(a) $e_{B_{t}^{\prime}}(\widehat{D})=0$.
(b) $e_{B_{t}^{\prime}}\left(\widehat{F^{+}}\right)=e_{B_{t}^{\prime}}\left(\widehat{F^{-}}\right)=1$.
(c) $e_{B_{t}^{\prime}}\left(\widehat{\left[\widehat{\theta_{i}}\right]}\right) \in F, e_{B_{t}^{\prime}}(\widehat{([0]})=0$.
(d) $e_{B_{t}^{\prime}}\left(\widehat{\left[\pi_{k}\right]}\right)=0$.

Now assume that $q \equiv 3(\bmod 4)$. Suppose that $q+1=m 2^{n}$ with $2 \nmid m$.
4. Let $B_{0}$ be the principal block of $H$. Then
(a) $e_{B_{0}}(\widehat{D})=1$.
(b) $e_{B_{0}}\left(\widehat{F^{+}}\right)=e_{B_{0}}\left(\widehat{F^{-}}\right) \in F$.
(c) $\left.e_{B_{0}}\left(\widehat{\theta_{i}}\right]\right)=1$.
(d) $e_{B_{0}}(\widehat{[0]})=0, e_{B_{0}}\left(\widehat{\left[\pi_{k}\right]}\right) \in F$.
5. Let $B_{r}$ be any block of defect 0 of $H$. Then
(a) $e_{B_{r}}(\widehat{D})=0$.
(b) $e_{B_{r}}\left(\widehat{F^{+}}\right)=e_{B_{r}}\left(\widehat{F^{-}}\right)=1$.
(c) $e_{B_{r}}(\widehat{[0]})=e_{B_{r}}\left(\widehat{\left[\pi_{k}\right]}\right)=0$.
(d) $e_{B_{r}}\left(\widehat{\left[\theta_{i}\right]}\right) \in F$.
6. Suppose that $m \geq 3$ and let $B_{t}^{\prime}$ be any block of defect $n-1$ of $H$. Then
(a) $e_{B_{t}^{\prime}}(\widehat{D})=0$.
(b) $e_{B_{t}^{\prime}}^{\left(\widehat{F^{+}}\right)}=e_{B_{t}^{\prime}}\left(\widehat{F^{-}}\right)=1$.
(c) $e_{B_{t}^{\prime}}\left(\widehat{\left.\theta_{i}\right]}\right)=0$.
(d) $e_{B_{t}^{\prime}}(\widehat{[0]})=0, e_{B_{t}^{\prime}}\left(\widehat{\left[\pi_{k}\right]}\right) \in F$.

The following corollary will be used in the proof of Lemma 6.2.
Corollary 4.6. Let $B_{s}\left(1 \leq s \leq \frac{q-1}{4}\right)$ or $B_{r}\left(1 \leq r \leq \frac{q-3}{4}\right)$ be the blocks of defect 0 of $H$ depending on whether $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$. Using the above notation,
(i) if $q \equiv 1(\bmod 4)$, for each $B_{s}$, there is a class $\left[\pi_{k}\right]$ such that $e_{B_{s}}\left(\widehat{\left[\pi_{k}\right]}\right) \neq 0$;
(ii) if $q \equiv 3(\bmod 4)$, for each $B_{r}$, there is a class $\left[\theta_{i}\right]$ such that $e_{B_{r}}\left(\widehat{\left.\theta_{i}\right]}\right) \neq 0$.

Proof: First we assume that $q \equiv 1(\bmod 4)$. From Theorem 8.9 in [12], we have $\chi_{s}\left(g_{k}\right)=-\delta^{(2 k) s}-\delta^{-(2 k) s}$ for $1 \leq k \leq \frac{q-1}{4}$, where $\chi_{s}$ is the irreducible ordinary character lying in $B_{s}, g_{k} \in\left[\pi_{k}\right]$, and $\delta$ is a primitive $(q+1)$-th root of unit in $\mathbb{C}$. Note that

$$
\begin{aligned}
f_{B_{s}}\left(\widehat{\left[\pi_{k}\right]}\right) & =\frac{1}{|H|} \sum_{\chi_{s} \in B_{s}} \chi_{s}(1) \chi_{s}\left(g_{k}^{-1}\right) \\
& =-\frac{q_{-1}}{|H|}\left(\delta^{(2 k) s}+\delta^{-(2 k) s}\right) .
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k=1}^{(q-1) / 4} e_{B_{s}}\left(\widehat{\left[\pi_{k}\right]}\right) & =\left(-\frac{q-1}{1 \mid} \sum_{k=1}^{(q-1) / 4} \delta^{(2 k) s}+\delta^{-(2 k) s}\right)^{*} \\
& =\left(\frac{\delta^{2 s}-\delta^{\frac{q+3}{2} s}}{1-\delta^{2 s}}+\frac{\delta^{-2 s}-\delta^{-\frac{q+3}{2} s}}{1-\delta^{-2 s}}\right)^{*} \\
& =\left(\frac{\delta^{2 s} s \delta^{\frac{q+3}{2} s}}{1-\delta^{2 s}}+\frac{\delta^{-2 s} s-\delta^{-\frac{q-1}{2} s}}{1-\delta^{-2 s}}\right)^{*} \\
& =\left(\frac{\delta^{2 s}-\delta^{\frac{q+3}{2} s}}{1-\delta^{2 s}}+\frac{1-\delta^{\frac{q+3}{2} s}}{\delta^{2 s}-1}\right)^{*} \\
& =1,
\end{aligned}
$$

we conclude that $e_{B_{s}}\left(\widehat{\left[\pi_{k}\right]}\right) \neq 0$ for some $k$. Part (i) is proved.
Part (ii) can be proved in the same fashion using Theorem 8.11 in [12]; we omit the details.

Let $M$ be an $\mathbf{S} H$-module. We denote the reduction $M / \mathcal{P} M$, which is an $F H$-module, by $\bar{M}$. Then the following lemma is apparent.
Lemma 4.7. Let $M$ be an $\mathbf{S} H$-module and $B \in B l(H)$. Using the above notation, we have

$$
\overline{f_{B} M}=e_{B} \bar{M}
$$

i.e. reduction commutes with projection onto a block $B$.

## 5. Linear Maps and Their Matrices

Let $F$ be the algebraic closure of $\mathbb{F}_{2}$ defined in Section 4 . From now on, $\chi_{N}$ for $N \subseteq E$ will be always regarded as a vector over $F$. Recall that for $\mathbf{P} \in I, N_{P a, E}(\mathbf{P})$ (respectively, $N_{S e, E}(\mathbf{P})$ ) is the set of external points on the passant (respectively, secant) lines through $\mathbf{P}$. We define $\mathbf{D}$ (respectively, $\mathbf{D}^{\prime}$ ) to be the incidence matrix of $E$ and $N_{P a, E}(\mathbf{P})$ (respectively, $\left.N_{S e, E}(\mathbf{P})\right)$ for $\mathbf{P} \in I$. Namely, the columns of $\mathbf{D}$ and $\mathbf{D}^{\prime}$ can be viewed as the characteristic vectors of $N_{P a, E}(\mathbf{P})$ and $N_{S e, E}(\mathbf{P})$, respectively. In the following, we always regard both $\mathbf{D}$ and $\mathbf{D}^{\prime}$ as matrices over $F$.
Definition 5.1. For $\mathbf{P} \in I$, we define $\mathcal{G}_{\mathbf{P}}$ to be the column characteristic vector of $\mathbf{P}$ with respect to $I$, i.e. $\mathcal{G}_{\mathbf{P}}$ is a 0-1 column vector of length $|I|$ with entries indexed by the internal points; the entry of $\mathcal{G}_{\mathbf{P}}$ is 1 if and only if it is indexed by $\mathbf{P}$.

Let $k$ be the complex field $\mathbb{C}$, the algebraic closure $F$ of $\mathbb{F}_{2}$, or the ring $\mathbf{S}$ in (4.1). Let $k^{I}$ and $k^{E}$ be the free $k$-modules with the bases $\left\{\mathcal{G}_{\mathbf{P}} \mid \mathbf{P} \in I\right\}$ and $\left\{\chi_{\mathbf{P}} \mid \mathbf{P} \in E\right\}$, respectively. If we extend the actions of $H$ on the bases of $k^{I}$ and $k^{E}$, which are defined by $\chi_{\mathbf{P}} \cdot h=\chi_{\mathbf{P}^{h}}$ and $\mathcal{G}_{\mathbf{Q}} \cdot h=\mathcal{G}_{\mathbf{Q}^{h}}$ for $\mathbf{P} \in I, \mathbf{Q} \in E$, and $h \in H$, linearly to $k^{I}$ and $k^{E}$ respectively, then both $k^{I}$ and $k^{E}$ are $k H$-permutation modules. Since $H$ is transitive on $I$, we have

$$
k^{I}=\operatorname{Ind}_{K}^{H}\left(1_{k}\right),
$$

where $K$ is the stabilizer of an element of $I$ in $H$ and $\operatorname{Ind}_{K}^{H}\left(1_{k}\right)$ is the $k H$-module induced by $1_{k}$.

The decomposition of $1 \uparrow_{K}^{H}$, the character of $\operatorname{Ind}_{K}^{H}\left(1_{k}\right)$, into a sum of the irreducible ordinary characters of $H$ is given as follows.

Lemma 5.2. [19, Lemma 5.2] Let $K$ be the stabilizer of an internal point in $H$.
Assume that $q \equiv 1(\bmod 4)$. Let $\chi_{s}, 1 \leq s \leq \frac{q-1}{4}$, be the irreducible ordinary characters of degree $q-1$, $\phi_{r}, 1 \leq r \leq \frac{q-5}{4}$, irreducible ordinary characters of degree $q+1$, $\gamma$ the irreducible of degree $q$, and $\beta_{j}, 1 \leq j \leq 2$, irreducible ordinary characters of degree $\frac{q+1}{2}$.
(i) If $q \equiv 1(\bmod 8)$, then

$$
1 \uparrow_{K}^{H}=1+\sum_{s=1}^{(q-1) / 4} \chi_{s}+\gamma+\beta_{1}+\beta_{2}+\sum_{j=1}^{(q-9) / 4} \phi_{r_{j}},
$$

where $\phi_{r_{j}}, 1 \leq j \leq \frac{q-9}{4}$, may not be distinct.
(ii) If $q \equiv 5(\bmod 8)$, then

$$
1 \uparrow_{K}^{H}=1+\sum_{s=1}^{(q-1) / 4} \chi_{s}+\gamma+\sum_{j=1}^{(q-5) / 4} \phi_{r_{j}}
$$

where $\phi_{r_{j}}, 1 \leq j \leq \frac{q-5}{4}$, may not be distinct.
Next assume that $q \equiv 3(\bmod 4)$. Let $\chi_{s}, 1 \leq s \leq \frac{q-3}{4}$, be the irreducible ordinary characters of degree $q-1, \phi_{r}, 1 \leq r \leq \frac{q-3}{4}$, the irreducible ordinary characters of degree $q+1$, $\gamma$ the irreducible character of degree $q$, and $\eta_{j}, 1 \leq j \leq 2$, the irreducible ordinary characters of degree $\frac{q-1}{2}$.
(iii) If $q \equiv 3(\bmod 8)$, then

$$
1 \uparrow_{K}^{H}=1+\sum_{r=1}^{(q-3) / 4} \phi_{r}+\eta_{1}+\eta_{2}+\sum_{j=1}^{(q-3) / 4} \chi_{s_{j}}
$$

where $\chi_{s_{j}}, 1 \leq j \leq \frac{q-3}{4}$, may not be distinct.
(iv) If $q \equiv 7(\bmod 8)$, then

$$
1 \uparrow_{K}^{H}=1+\sum_{r=1}^{(q-3) / 4} \phi_{r}+\sum_{j=1}^{(q+1) / 4} \chi_{s_{j}}
$$

where $\chi_{s_{j}}, 1 \leq j \leq \frac{q+1}{4}$, may not be distinct.
Corollary 5.3. Using the above notation,
(i) if $q \equiv 1(\bmod 4)$, then the character of $\operatorname{In} d_{K}^{H}\left(1_{\mathbb{C}}\right) \cdot f_{B_{s}}$ is $\chi_{s}$ for each block $B_{s}$ of defect 0 ;
(ii) if $q \equiv 3(\bmod 4)$, then the character of $\operatorname{In} d_{K}^{H}\left(1_{\mathbb{C}}\right) \cdot f_{B_{r}}$ is $\phi_{r}$ for each block $B_{r}$ of defect 0 .

Proof: The corollary follows from Lemma 4.2 and Lemma 5.2 .
Since $H$ preserves incidence, the following corollary is obvious.
Corollary 5.4. Let $\mathbf{P} \in I$. Using the above notation, we have

$$
\chi_{N_{P a, E}(\mathbf{P})} \cdot h=\chi_{N_{P a, E}\left(\mathbf{P}^{h}\right)}, \chi_{N_{S e, E}(\mathbf{P})} \cdot h=\chi_{N_{S e, E}\left(\mathbf{P}^{h}\right)}
$$

for $h \in H$.

In the rest of the article, we always view $\mathcal{G}_{\mathbf{P}}$ as a vector over $F$. Consider the maps $\phi_{\mathbf{B}}$, $\phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}^{\prime}}$ from $F^{I}$ to $F^{E}$ defined by extending

$$
\mathcal{G}_{\mathbf{P}} \mapsto \chi_{\mathbf{P}^{\perp}}, \mathcal{G}_{\mathbf{P}} \mapsto \chi_{N_{P a, E}(\mathbf{P})}, \mathcal{G}_{\mathbf{P}} \mapsto \chi_{N_{S e, E}(\mathbf{P})}
$$

linearly to $F^{I}$, respectively. Then it is clear that as $F$-linear maps, the marices of $\phi_{\mathbf{B}}, \phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}^{\prime}}$ are $\mathbf{B}, \mathbf{D}$, and $\mathbf{D}^{\prime}$, respectively, and for $\mathbf{x} \in F^{I}, \phi_{\mathbf{B}}(\mathbf{x})=\mathbf{B} \mathbf{x}, \phi_{\mathbf{D}}(\mathbf{x})=\mathbf{D} \mathbf{x}$ and $\phi_{\mathbf{D}^{\prime}}(\mathbf{x})=\mathbf{D}^{\prime} \mathbf{x}$. Moreover, we have the following result.
Lemma 5.5. The maps $\phi_{\mathbf{B}}, \phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}^{\prime}}$ are all $F H$-module homomorphisms from $F^{I}$ to $F^{E}$.
Proof: Let $\mathcal{G}_{\mathbf{P}}$ be a basis element of $F^{I}$. Then $\phi\left(\mathcal{G}_{\mathbf{P}} \cdot h\right)=\phi\left(\mathcal{G}_{\mathbf{P}}\right) \cdot h$ since

$$
\phi_{\mathbf{B}}\left(\mathcal{G}_{\mathbf{P}} \cdot h\right)=\chi_{\left(\mathbf{P}^{h}\right)^{\perp}}=\chi_{\left(\mathbf{P}^{\perp}\right)^{h}}=\chi_{\mathbf{P}^{\perp}} \cdot h=\phi_{\mathbf{B}}\left(\mathcal{G}_{\mathbf{P}}\right) \cdot h .
$$

By linearity of $\phi_{\mathbf{B}}$, we have $\phi_{\mathbf{B}}(\mathbf{x}) \cdot h=\phi_{\mathbf{B}}(\mathbf{x} \cdot h)$ for each $\mathbf{x} \in F^{I}$. The proof of the map $\phi_{\mathbf{B}}$ being $F H$-homomorphism is completed.

The proofs of the other two maps being homomorphisms are similar since

$$
\chi_{N_{P a, E}(\mathbf{P})} \cdot h=\chi_{N_{P a, E}\left(\mathbf{P}^{h}\right)}, \chi_{N_{S e, E}(\mathbf{P})} \cdot h=\chi_{N_{S e, E}\left(\mathbf{P}^{h}\right)}
$$

for $h \in H$ and $\mathbf{P} \in I$ by Corollary 5.4. We omit the details.
For convenience, we use $\operatorname{col}_{F}(\mathbf{C})$ to denote the column space of the matrix $\mathbf{C}$ over $F$.
Corollary 5.6. Using the above notation, we have $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\operatorname{col}_{F}(\mathbf{B}), \operatorname{Im}\left(\phi_{\mathbf{D}}\right)=\operatorname{col}_{F}(\mathbf{D})$, and $\operatorname{Im}\left(\phi_{\mathbf{D}^{\prime}}\right)=\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)$.

Now we define $\mathcal{M}_{1}:=\left\langle\chi_{\ell} \mid \ell \in T\right\rangle_{F}$ and $\mathcal{M}_{2}:=\left\langle\chi_{\ell_{i}}+\chi_{\ell_{j}} \mid \ell_{i} \neq \ell_{j} \in T\right\rangle_{F}$ to be the spans of the corresponding characteristic vectors over $F$.

Lemma 5.7. The dimensions of $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ over $F$ are $\operatorname{dim}_{F}\left(\mathcal{M}_{1}\right)=q$ and $\operatorname{dim}_{F}\left(\mathcal{M}_{2}\right)=$ $q-1$, respectively. Moreover, the all-one column vector $\mathbf{1}$ of length $|E|$ is neither in $\mathcal{M}_{1}$ nor in $\mathcal{M}_{2}$.
Proof: $\quad$ Since $\sum_{\ell \in T} \chi_{\ell}=\mathbf{0}$, where $\mathbf{0}$ is the zero column vector of $|E|$, it follows that $\left\{\chi_{\ell} \mid\right.$ $\ell \in T\}$ is linearly dependent over $F$, i.e. $\operatorname{dim}_{F}\left(\mathcal{M}_{1}\right) \leq q$. Now let $T^{\prime} \subset T$ with $\left|T^{\prime}\right|=q$ and suppose that $\left\{\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ is linearly dependent over $F$. Then $\sum_{\ell \in T^{\prime}} a_{\ell} \chi_{\ell}=\mathbf{0}$, where $a_{\ell} \in F$ and $a_{\ell_{1}} \neq 0$ for some $\ell_{1} \in T^{\prime}$. Since there are $q$ external points on $\ell_{1}$ and there are only $q-1$ tangent lines other than $\ell_{1}$ in $T^{\prime}$, some external point on $\ell_{1}$ must be passed only by $\ell_{1}$ among the tangent lines in $T^{\prime}$, which forces $a_{\ell_{1}}=0$, a contradiction. This shows that $T^{\prime}$ must be linearly independent over $F$, and so $\operatorname{dim}_{F}\left(\mathcal{M}_{1}\right)=q$. Moreover, if $T^{\prime} \subset T$ and $\left|T^{\prime}\right|=q$, then $\left\{\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ must be a basis for $\mathcal{M}_{1}$.

Next if $\ell_{1}$ is a tangent line, then $\mathcal{M}_{2}=\left\langle\chi_{\ell_{1}}+\chi_{\ell} \mid \ell \in T \backslash\left\{\ell_{1}\right\}\right\rangle_{F}$ since $\chi_{\ell_{i}}+\chi_{\ell_{j}}=$ $\left(\chi_{\ell_{1}}+\chi_{\ell_{i}}\right)+\left(\chi_{\ell_{1}}+\chi_{\ell_{j}}\right) . \operatorname{As} \sum_{\ell \in T \backslash\left\{\ell_{1}\right\}}\left(\chi_{\ell_{1}}+\chi_{\ell}\right)=\mathbf{0}, \operatorname{dim}_{F}\left(\mathcal{M}_{2}\right) \leq q-1$. Let $T^{\prime} \subset T \backslash\left\{\ell_{1}\right\}$ with $\left|T^{\prime}\right|=q-1$ and suppose that $\left\{\chi_{\ell_{1}}+\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ is linearly dependent over $F$. Then $\sum_{\ell \in T^{\prime}} a_{\ell}\left(\chi_{\ell_{1}}+\chi_{\ell}\right)=\sum_{\ell \in T^{\prime}} a_{\ell} \chi_{\ell}=\mathbf{0}$ since $\left|T^{\prime}\right|$ is even, where $a_{\ell} \in F$ and $a_{\ell_{2}} \neq 0$ for some $\ell_{2} \in T^{\prime}$. By applying the same argument in the first paragraph of this proof, again, we obtain that $a_{\ell_{2}}=0$ which is a contradiction. Therefore, $\left\{\chi_{\ell_{1}}+\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ is linearly independent over $F$, and so $\operatorname{dim}_{F}\left(\mathcal{M}_{2}\right)=q-1$. Moreover, if $T^{\prime} \subset T \backslash\left\{\ell_{1}\right\}$ and $\left|T^{\prime}\right|=q-1$, then $\left\{\chi_{\ell_{1}}+\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ must be a basis for $\mathcal{M}_{2}$.

Now we assume that $\mathbf{1} \in \mathcal{M}_{1}$ and $\left\{\chi_{\ell} \mid \ell \in T^{\prime}\right\}$ with $T^{\prime} \subset T$ and $\left|T^{\prime}\right|=q$ is a basis for $\mathcal{M}_{1}$. Then $\sum_{\ell \in T^{\prime}} a_{\ell} \chi_{\ell}=1$, where $a_{\ell} \in F$ for $\ell \in T^{\prime}$ and $a_{\ell_{k}} \neq 0$ for some $\ell_{k} \in T^{\prime}$. Since $\left|T^{\prime} \backslash\left\{\ell_{k}\right\}\right|=q-1$, some external point on $\ell_{k}$ must be only passed by $\ell_{k}$ among all
the tangent lines in $T^{\prime}$; this forces $a_{\ell_{k}}=1$. For each $\ell \in T^{\prime} \backslash\left\{\ell_{k}\right\}$, we have $a_{\ell_{k}}+a_{\ell}=1$, that is, $a_{\ell}=0$ for each $\ell \in T^{\prime} \backslash\left\{\ell_{k}\right\}$. Thus $\chi_{\ell_{k}}=\mathbf{1}$, which is impossible. Consequently, $\mathbf{1} \notin \mathcal{M}_{1}$. Similarly, we can show that $\mathbf{1} \notin \mathcal{M}_{2}$. We omit the details.

Lemma 5.8. If $q \equiv 1(\bmod 4)$, then $\operatorname{col}_{F}(\mathbf{D})=\mathcal{M}_{1}$; if $q \equiv 3(\bmod 4)$, then $\operatorname{col}_{F}(\mathbf{D})=$ $\mathcal{M}_{2}$.

Proof: Assume that $q \equiv 1(\bmod 4)$. Let $\chi_{N_{P a, E}(\mathbf{P})}$ be the column of $\mathbf{D}$ indexed by $\mathbf{P}$. Then $\chi_{N_{P a, E}(\mathbf{P})}$ is an $F$-linear combination of the generating elements of $\mathcal{M}_{1}$ by Corollary 2.17, Now if $\chi_{\ell}$ is a generating element of $\mathcal{M}_{1}$, then it is an $F$-linear combination of the columns of $\mathbf{D}$ by Corollary 2.21. Therefore, $\operatorname{col}_{F}(\mathbf{D})=\mathcal{M}_{1}$.

Now we assume that $q \equiv 3(\bmod 4)$. Let $\chi_{N_{P a, E}(\mathbf{P})}$ be the column of $\mathbf{D}$ indexed by $\mathbf{P}$. Suppose that $\ell(\mathbf{P})$ is a tangent line through an external point on $\mathbf{P}^{\perp}$ and $T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines through the external points on $\ell(\mathbf{P})$ that are also on the passant lines through $\mathbf{P}$. Then by Corollary 2.17 and the fact that $|T(\mathbf{P}, \ell(\mathbf{P}))|=\frac{q+1}{2}$ is even, we have

$$
\begin{aligned}
\chi_{N_{P a, E}(\mathbf{P})} & =\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell} \\
& =\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))}\left(\chi_{\ell}+\chi_{\ell(\mathbf{P})}\right) ;
\end{aligned}
$$

that is, $\chi_{N_{P a, E}(\mathbf{P})} \in \mathcal{M}_{2}$. Now let $\chi_{\ell_{1}}+\chi_{\ell_{2}}$ be a generating element of $\mathcal{M}_{2}$. Then we have

$$
\chi_{\ell_{1}}+\chi_{\ell_{2}}=\sum_{\mathbf{Q} \in \mathcal{M}^{\prime}(\mathbf{P})} \chi_{N_{P a, E}(\mathbf{Q})}
$$

by Corollary [2.24, where $\mathbf{P}=\ell_{1} \cap \ell_{2}$. Hence, $\operatorname{col}_{F}(\mathbf{D})=\mathcal{M}_{2}$.
Corollary 5.9. If $q \equiv 1(\bmod 4), \operatorname{rank}_{2}(\mathbf{D})=q$; if $q \equiv 3(\bmod 4), \operatorname{rank}_{2}(\mathbf{D})=q-1$.
Proof: It follows from Lemmas 5.7 and 5.8.
Further, we have the following decomposition of $\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)$.
Lemma 5.10. If $q \equiv 3(\bmod 4)$, then $\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)=\langle\mathbf{1}\rangle \oplus \operatorname{col}_{F}(\mathbf{D})$ as $F H$-modules, where $\langle\mathbf{1}\rangle$ is the trivial $F H$-module generated by the all-one column vector $\mathbf{1}$.
Proof: Since each row of $\mathbf{D}^{\prime}$ has $\frac{(q-1)^{2}}{4} 1$ s, then

$$
\sum_{\mathbf{P} \in I} \chi_{N_{S e, E}(\mathbf{P})}=\mathbf{1} .
$$

For $h \in H$,

$$
\mathbf{1} \cdot h=\left(\sum_{\mathbf{P} \in I} \chi_{N_{S e, E}(\mathbf{P})}\right) \cdot h=\sum_{\mathbf{P} \in I} \chi_{N_{S e, E}\left(\mathbf{P}^{h}\right)}=\sum_{\mathbf{P} \in I} \chi_{N_{S e, E}(\mathbf{P})}=\mathbf{1} \in \operatorname{col}_{F}(\mathbf{D}) .
$$

Consequently, $\langle\mathbf{1}\rangle$ is indeed a trivial submodule of $\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)$.
It is clear that $\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)=\langle\mathbf{1}\rangle+\operatorname{col}_{F}(\mathbf{D})$ since $\chi_{N_{S e, E}(\mathbf{P})} \in \operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)$ if and only if $\chi_{N_{S e, E}(\mathbf{P})}=\mathbf{1}+\chi_{N_{P a, E}(\mathbf{P})} \in\langle\mathbf{1}\rangle+\operatorname{col}_{F}(\mathbf{D})$. Further, $\langle\mathbf{1}\rangle \cap \operatorname{col}_{F}(\mathbf{D})=\mathbf{0}$ since $\operatorname{col}{ }_{F}(\mathbf{D})=\mathcal{M}_{2}$ and $\mathbf{1} \notin \mathcal{M}_{2}$ by Lemmas 5.7 and 5.8. Therefore, $\operatorname{col}_{F}\left(\mathbf{D}^{\prime}\right)=\langle\mathbf{1}\rangle \oplus \operatorname{col}_{F}(\mathbf{D})$.

## 6. Statement and Proof of Main Theorem

The main theorem is given as follows.
Theorem 6.1. Let $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ and $\operatorname{Im}\left(\phi_{\mathbf{D}}\right)$ be defined as above. As FH-modules,
(i) if $q \equiv 1(\bmod 4)$, then

$$
\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{s=1}^{(q-1) / 4} M_{s}\right)
$$

where $M_{s}$ for $1 \leq s \leq \frac{q-1}{4}$ are pairwise non-isomorphic simple FH-modules of dimension $q-1$;
(ii) if $q \equiv 3(\bmod 4)$, then

$$
\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\langle\mathbf{1}\rangle \oplus \operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{r=1}^{(q-3) / 4} M_{r}\right),
$$

where $M_{r}$ for $1 \leq s \leq \frac{q-3}{4}$ are pairwise non-isomorphic simple $F H$-modules of dimension $q+1$ and $\langle\mathbf{1}\rangle$ is the trivial $F H$-module generated by the all-one column vector of length $|E|$.
To prove the main theorem, we need the following lemma.
Lemma 6.2. Let $q-1=2^{n} m$ or $q+1=2^{n} m$ with $2 \nmid m$ depending on whether $q \equiv 1$ $(\bmod 4)$ or $q \equiv 3(\bmod 4)$. Using the above notation,
(i) if $q \equiv 1(\bmod 4)$, then $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}}=\operatorname{Im}\left(\phi_{\mathbf{D}}\right), \operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{s}} \neq \mathbf{0}$ for $1 \leq s \leq \frac{q-1}{4}$, and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}=\mathbf{0}$ for $m \geq 3$ and $1 \leq t \leq \frac{m-1}{2}$;
(ii) if $q \equiv 3(\bmod 4)$, then $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}}=\operatorname{Im}\left(\phi_{\mathbf{D}^{\prime}}\right), \operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{r}} \neq \mathbf{0}$ for $1 \leq r \leq \frac{q-3}{4}$, and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}=\mathbf{0}$ for $m \geq 3$ and $1 \leq t \leq \frac{m-1}{2}$.

Proof: It is clear that $\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$ is generated by $\left\{\chi_{\mathbf{P} \perp} \mid \mathbf{P} \in I\right\}$ over $F$. Let $B \in B l(H)$. Since

$$
\begin{aligned}
\chi_{\mathbf{P}^{\perp}} \cdot e_{B} & =\sum_{C \in c l(H)} e_{B}(\widehat{C}) \sum_{h \in C} \chi_{\mathbf{P}^{\perp}} \cdot h \\
& =\sum_{C \in c l(H)} e_{B}(\widehat{C}) \sum_{h \in C} \chi_{\left(\mathbf{P}^{\perp}\right)^{h}} \\
& =\sum_{C \in c l(H)} e_{B}(\widehat{C}) \sum_{h \in C} \sum_{\mathbf{Q} \in\left(\mathbf{P}^{\perp}\right)^{h} \cap E} \chi_{\mathbf{Q}},
\end{aligned}
$$

we have

$$
\chi_{\mathbf{P}^{\perp}} \cdot e_{B}=\sum_{\mathbf{Q} \in I} \mathcal{S}(B, \mathbf{P}, \mathbf{Q}) \chi_{\mathbf{Q}}
$$

where

$$
S(B, \mathbf{P}, \mathbf{Q}):=\sum_{C \in c l(H)}\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right| e_{B}(\widehat{C})
$$

Assume first that $q \equiv 1(\bmod 4)$. If $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$, then $S(B, \mathbf{P}, \mathbf{Q})=0$ for each $B \in$ $B l(H)$ since $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|=0$ in $F$ for each $C \neq[0]$ by Lemma 3.6(i) and $e_{B_{0}}(\widehat{[0]})=$ $e_{B_{s}}(\widehat{[0]})=e_{B_{t}^{\prime}}(\widehat{[0]})=0$ by $1(\mathrm{c}), 2(\mathrm{c}), 3(\mathrm{c})$ of Lemma 4.5.

If $\ell_{\mathbf{P}, \mathbf{Q}} \in P a_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then by Lemma 3.6(ii) and 1(a), 1(c), 2(a), 2(c), 3(a), 3(c) of Lemma 4.5,

$$
\begin{aligned}
S\left(B_{0}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{0}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{0}}(\widehat{D})=0+1=1, \\
S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{s}}([0])+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{s}}(\widehat{D})=0+0=0, \\
S\left(B_{t}^{\prime}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{t}^{\prime}}([0])+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{t}^{\prime}}(\widehat{D})=0+0=0 .
\end{aligned}
$$

If $\mathbf{Q}$ is on a passant line $\ell$ through $\mathbf{P}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, then by Lemma 3.6(iii) and 1 (c), $1(\mathrm{~d}), 2(\mathrm{c}), 2(\mathrm{~d}), 3(\mathrm{c}), 3(\mathrm{~d})$ of Lemma 4.5.

$$
\begin{aligned}
& S\left(B_{0}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{0}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\pi_{k}\right]\right| e_{B_{0}}\left(\widehat{\left[\pi_{k}\right]}\right) \\
&=1, \\
& S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{s}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\pi_{k}\right]\right| e_{B_{s}}\left(\left[\pi_{k}\right]\right) \\
& S\left(B_{t}^{\prime}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{s}^{\prime}}\left(\widehat{\left[\left[\pi_{k}\right]\right.}\right) \\
&([0])+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\pi_{k}\right]\right| e_{B_{t}^{\prime}}\left(\left[\pi_{k}\right]\right)=0 .
\end{aligned}
$$

By Lemma 3.6(iii) and the fact that there are $\frac{q-1}{4}$ classes of the form $\left[\pi_{k}\right]$ and there are $\frac{q-1}{2}$ points on $\ell$ that are not on $\mathbf{P}^{\perp}$, we have that for each $\left[\pi_{k}\right]$ there exist two external points $\mathbf{Q}_{1}$ and $\mathbf{Q}_{2}$ on $\ell$ such that $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}_{j}} \cap\left[\pi_{k}\right]\right|(j=1$ or 2$)$ is odd and for each $\mathbf{Q} \in \ell$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$ there is a class $\left[\pi_{k}\right]$ such that $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\pi_{k}\right]\right|$ is odd. Combining the above analysis with Lemma 4.6, we obtain that for each $B_{s}$, there is a $\mathbf{Q}$ and a class $\left[\pi_{k}\right]$ such that $S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right)=e_{B_{s}}\left(\widehat{\left.\pi_{k}\right]}\right) \neq 0$.

Therefore, we have shown that $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}}=\operatorname{Im}\left(\phi_{\mathbf{D}}\right)$ by definition, $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{s}} \neq \mathbf{0}$ for each $s$, and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}=\mathbf{0}$. The proof of (i) is completed.

Now assume that $q \equiv 3(\bmod 4)$. If $\ell_{\mathbf{P}, \mathbf{Q}} \in P a_{\mathbf{P}}$, then $S(B, \mathbf{P}, \mathbf{Q})=0$ for each $B \in B l(H)$ since $\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C\right|=0$ by $4(\mathrm{~d}), 5(\mathrm{c}), 6(\mathrm{~d})$ of Lemma 4.5.

Let $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then by Lemma 3.7 (ii) and $4(\mathrm{a}), 4(\mathrm{~d}), 5(\mathrm{a}), 5(\mathrm{c}), 5(\mathrm{~d})$ of Lemma 4.5

$$
\begin{aligned}
S\left(B_{0}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{0}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{0}}(\widehat{D})=0+1=1, \\
S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{s}}([0])+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{s}}(\widehat{D})=0+0=0, \\
S\left(B_{t}^{\prime}, \mathbf{P}, \mathbf{Q}\right) & =\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{t}^{\prime}}([0])+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D\right| e_{B_{t}^{\prime}}(\widehat{D})=0+0=0 .
\end{aligned}
$$

If $\ell_{\mathbf{P}, \mathbf{Q}} \in S e_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, then by Lemma $3.7(\mathrm{i}), 4(\mathrm{c}), 4(\mathrm{~d}), 5(\mathrm{c}), 5(\mathrm{~d}), 6(\mathrm{c}), 6(\mathrm{~d})$ of Lemma 4.5,

$$
\begin{aligned}
& S\left(B_{0}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{0}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\theta_{i}\right]\right| e_{B_{0}}(\widehat{[\widehat{i}]}) \\
& S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{s}}(\widehat{[0]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\theta_{i}\right]\right| e_{B_{B^{\prime}}}\left(\left[\pi_{k}\right]\right) \\
&=e_{B_{s}}\left(\widehat{\left.\theta_{i}\right]}\right), \\
& S\left(B_{t}^{\prime}, \mathbf{P}, \mathbf{Q}\right)=\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap[0]\right| e_{B_{t}^{\prime}}\left([\widehat{00]})+\left|\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap\left[\theta_{i}\right]\right| e_{B_{t}}\left(\left[\widehat{\left.\theta_{i}\right]}\right)\right.\right.
\end{aligned}=0 .
$$

From Lemma 3.7(i) and Lemma 4.6, we have that for each $B_{s}$, there is a $\mathbf{Q}$ and a class $\left[\theta_{i}\right]$ such that $S\left(B_{s}, \mathbf{P}, \mathbf{Q}\right)=e_{B_{s}}\left(\widehat{\left[_{i}\right]}\right) \neq 0$.

Therefore, we have shown that $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}}=\operatorname{Im}\left(\phi_{\mathbf{D}^{\prime}}\right)$ by definition, $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{s}} \neq \mathbf{0}$ for each $s$, and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}=\mathbf{0}$. The proof of (ii) is completed.
Proof of Theorem 6.1; Let $B$ be a 2-block of defect 0 of $H$. Then by Lemma 4.7, we have

$$
F^{I} \cdot e_{B}=\overline{\mathbf{S}^{I} \cdot f_{B}}
$$

Therefore, by Corollary $5.3, F^{I} \cdot e_{b}=N$, where $N$ is the simple $F H$-module of dimension $q-1$ or $q+1$ lying in $B$ accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$. It is clear that $\phi_{\mathbf{B}}\left(F^{I}\right)=\operatorname{Im}\left(\phi_{\mathbf{B}}\right)$.

Assume that $q \equiv 1(\bmod 4)$ and $q-1=m 2^{n}$ with $2 \nmid m$. Since

$$
1=e_{B_{0}}+\sum_{s=1}^{(q-1) / 4} e_{B_{s}}+\sum_{t=1}^{(m-1) / 2} e_{B_{t}^{\prime}}
$$

we have

$$
\begin{align*}
\operatorname{Im}\left(\phi_{\mathbf{B}}\right) & =\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}} \oplus\left(\bigoplus_{s=1}^{(q-1) / 4} \operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{s}}\right) \oplus\left(\bigoplus_{t=1}^{(m-1) / 2} \operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}\right) \\
& =\operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{\substack{s=1 \\
(q-1) / 4}}^{(q-1) / 4} \phi_{\mathbf{B}}\left(F^{I}\right) \cdot e_{B_{s}}\right) \\
& =\operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{\substack{s=1}}^{(q-1) / 4} \phi_{\mathbf{B}}\left(F^{I} \cdot e_{B_{s}}\right)\right) \\
& =\operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{\substack{s=1 \\
(q-1) / 4}} \phi_{\mathbf{B}}\left(N_{s}\right)\right)  \tag{6.1}\\
& =\operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{s=1} M_{s}\right)
\end{align*}
$$

where $N_{s}$ is the simple module of dimension $q-1$ lying in $B_{s}$ for each $s$ by the discussion in the first paragraph and $M_{s}:=\phi_{\mathbf{B}}\left(N_{s}\right)$ for each $s$. In (6.1), the terms $e_{B_{t}^{\prime}}$ for $1 \leq t \leq \frac{m-1}{2}$ and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}$ for $1 \leq t \leq \frac{m-1}{2}$ appear only when $m \geq 3$; the second equality holds since $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{t}^{\prime}}=\mathbf{0}$ for each $t$ and $\operatorname{Im}\left(\phi_{\mathbf{B}}\right) \cdot e_{B_{0}}=\operatorname{Im}\left(\phi_{\mathbf{D}}\right)$ by Lemma $6.2(\mathrm{i})$; and the third equality holds since $\phi_{\mathbf{B}}$ is an $F H$-homomorphism by Lemma 5.5 and $e_{B_{s}} \in F H$. Consider the map

$$
\lambda_{S}: N_{S} \rightarrow \phi_{\mathbf{B}}\left(N_{s}\right)
$$

defined by $\lambda_{s}(n)=\phi_{\mathbf{B}}(n)$ for $n \in N_{s}$, where $1 \leq s \leq \frac{q-1}{4}$. It is clear that $\lambda_{s}$ is the same as the resctriction of $\phi_{\mathbf{B}}$ to $N_{s}$. Consequently, $\lambda_{s}$ is a surjective $F H$-homomorphism. Moreover, $\operatorname{Ker}\left(\lambda_{s}\right)$ is either $\mathbf{0}$ or $N_{s}$ since, otherwise, $\operatorname{Ker}\left(\lambda_{s}\right)$ would be a non-trivial submodule of $N_{s}$ which is impossible. If $\operatorname{Ker}\left(\lambda_{s}\right)=N_{s}$, then $\phi_{\mathbf{B}}\left(N_{s}\right)=\phi_{\mathbf{B}}\left(F^{I}\right) \cdot e_{B_{s}}=\mathbf{0}$, which is not the case by Lemma 6.2(i). Thus, we must have $\operatorname{Ker}\left(\lambda_{s}\right)=\mathbf{0}$; that is, $\lambda_{s}$ is an $F H$-isomorphism. So we have shown that $M_{s}:=\operatorname{Im}\left(N_{s}\right) \cong N_{s}$ and thus $M_{s}$ for $1 \leq s \leq \frac{q-1}{4}$ are pairwise non-isomorphic simple modules of dimension $q-1$. The proof of (i) is finished.

Now assume that $q \equiv 3(\bmod 4)$. Applying the same argument as above, we have

$$
\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\operatorname{Im}\left(\phi_{\mathbf{D}^{\prime}}\right) \oplus\left(\bigoplus_{r=1}^{(q-3) / 4} M_{r}\right),
$$

where $M_{r}$ for $1 \leq r \leq \frac{q-3}{4}$ are pairwise non-isomorphic simple $F H$-modules of dimension $q+1$. Since $\operatorname{Im}\left(\phi_{\mathbf{D}^{\prime}}\right)=\langle\mathbf{1}\rangle \oplus \operatorname{Im}\left(\phi_{\mathbf{D}}\right)$ by Lemma 5.10, it follows that

$$
\operatorname{Im}\left(\phi_{\mathbf{B}}\right)=\langle\mathbf{1}\rangle \oplus \operatorname{Im}\left(\phi_{\mathbf{D}}\right) \oplus\left(\bigoplus_{r=1}^{(q-3) / 4} M_{r}\right) .
$$

Now Conjecture 1.1 follows as a corollary.

Corollary 6.3. Let $\mathcal{L}$ and $\mathcal{L}_{0}$ be the $\mathbb{F}_{2}$-null spaces of $\mathbf{B}$ and $\mathbf{B}_{0}$, respectively. Then
and

$$
\operatorname{dim}_{\mathbb{F}_{2}}\left(\mathcal{L}_{0}\right)= \begin{cases}\frac{q^{2}-1}{4}, & \text { if } q \equiv 1 \quad(\bmod 4) \\ \frac{q^{2}-1}{4}+1, & \text { if } q \equiv 3 \quad(\bmod 4)\end{cases}
$$

Proof: From Theorem 6.1] and Corollary [5.9, it follows that the 2-rank of B is

$$
\operatorname{rank}_{2}(\mathbf{B})=q+\frac{(q-1)^{2}}{4}
$$

or

$$
\operatorname{rank}_{2}(\mathbf{B})=1+(q-1)+\frac{(q-1)^{2}}{4}
$$

accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$. Therefore, the dimension of the $\mathbb{F}_{2}$-null space of $\mathbf{B}$ is

$$
\operatorname{dim}_{\mathbb{F}_{2}}(\mathcal{L})=\frac{q(q-1)}{2}-\left(q+\frac{(q-1)^{2}}{4}\right)=\frac{q^{2}-1}{4}-q
$$

or

$$
\operatorname{dim}_{\mathbb{F}_{2}}(\mathcal{L})=\frac{q(q-1)}{2}-\left(1+(q-1)+\frac{(q-1)(q-3)}{4}\right)=\frac{q^{2}-1}{4}-q+1
$$

accordingly as $q \equiv 1(\bmod 4)$ or $q \equiv 3(\bmod 4)$.
Since $\operatorname{rank}_{2}(\mathbf{B})=\operatorname{rank}_{2}\left(\mathbf{B}_{0}\right)$, the dimension of $\mathcal{L}_{0}$ can be calculated in the same way. We omit the details.

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