PROOFS OF TWO CONJECTURES ON THE DIMENSIONS OF BINARY CODES

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ABSTRACT. Let \mathcal{L} and \mathcal{L}_0 be the binary codes generated by the column \mathbb{F}_2 -null spaces of the incidence matrices of external points versus passant lines and internal points versus secant lines with respect to a conic in PG(2, q), respectively. We confirm the conjectures on the dimensions of \mathcal{L} and \mathcal{L}_0 using methods from both finite geometry and modular representation theory.

1. INTRODUCTION

Let \mathbb{F}_q be the finite field of order q, where $q = p^e$, p is a prime and $e \ge 1$ is an integer. Let $\mathrm{PG}(2,q)$ denote the classical projective plane of order q represented via homogeneous coordinates. Namely, a point \mathbf{P} of $\mathrm{PG}(2,q)$ can be written as (a_0, a_1, a_2) , where (a_0, a_1, a_2) is a non-zero vector of V, and a line ℓ as $[b_0, b_1, b_2]$, where b_0, b_1, b_2 are not all zeros. The point $\mathbf{P} = (a_0, a_1, a_2)$ lies on the line $\ell = [b_0, b_1, b_2]$ if and only if

$$a_0b_0 + a_1b_1 + a_2b_2 = 0$$

A non-degenerate conic in PG(2,q) is the set of points satisfying a non-degenerate quadratic form. It is well known that the set of points

$$\mathcal{O} = \{ (1, t, t^2) \mid t \in \mathbb{F}_q \} \cup \{ (0, 0, 1) \},$$
(1.1)

which is also the set of projective solutions of the non-degenerate quadratic form

$$Q(X_0, X_1, X_2) = X_1^2 - X_0 X_2$$
(1.2)

over \mathbb{F}_q , gives rise to a standard example of a non-degenerate conic in $\mathrm{PG}(2,q)$. It can be shown that every non-degenerate conic must has q + 1 points and no three of them are collinear, which forms an oval (see [8, P. 157]). In the case where q is odd, Segre [16] proved that an oval in $\mathrm{PG}(2,q)$ must be a non-degenerate conic. In this paper, $q = p^e$ is always assumed to be an odd prime power. For convenience, we fix the conic in (1.1) as the "standard" conic. A line ℓ is passant, tangent, or secant accordingly as $|\ell \cap \mathcal{O}| = 0$, 1, or 2, respectively. It is clear that every line of $\mathrm{PG}(2,q)$ must be in one of these classes. A point P is an internal, absolute, or external point depending on whether it lies on 0, 1, or 2 tangent lines to \mathcal{O} . The sets of secant, tangent, and passant lines are denoted by Se, T and Pa, respectively; the sets of external and internal points are denoted by E and I, respectively. The sizes of these sets are $|Se| = |E| = \frac{q(q+1)}{2}$, $|Pa| = |I| = \frac{q(q-1)}{2}$, and |T| = q + 1 (see (2.2)). Moreover, it can be shown that the quadratic form Q in (1.1) induces a polarity σ , a correlation of order 2, under which E and Se, O and T, and I and Pa are in one-to-one correspondence with each other, respectively.

Let **C** be a 0-1 matrix; that is, **C** is a matrix whose entries are either 0 or 1. Note that **C** can be viewed as a matrix over any ring with 1. The *p*-rank of **C**, denoted by $\operatorname{rank}_p(\mathbf{C})$, is defined to be the dimension of the column space of **C** over a field *F* of characteristic *p*.

Key words and phrases. Block idempotent, Brauer's theory, character, conic, general linear group, incidence matrix, low-density parity-check code, module, 2-rank.

The column null space of \mathbf{C} over F determines a linear code whose dimension is defined to be the dimension of the corresponding column null space of \mathbf{C} over F.

Let **A** be the $(q^2 + q + 1) \times (q^2 + q + 1)$ point-line incidence matrix of PG(2, q); namely, **A** is a 0-1 matrix and the rows and columns of **A** are labeled by the points and lines of PG(2, q), respectively, and the (**P**, ℓ)-entry of **A** is 1 if and only if **P** $\in \ell$. It can be shown that the 2-rank of **A** is $q^2 + q$ [9] and the p-rank of **A** is $\binom{p+1}{2}^e$ [1], where $q = p^e$. The binary linear code generated by the column \mathbb{F}_2 -null space of **A** has dimension 1. Therefore, it is not useful for any practical purpose.

In [5], Droms, Mellinger and Meyer partitioned **A** into the following 9 submatrices:

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}$$
(1.3)

where the block of rows for $(\mathbf{A}_{11}, \mathbf{A}_{21}, \mathbf{A}_{31})$ are labeled by the absolute, internal, and external points, respectively, and the block of columns for $(\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{13})$ are labeled by the tangent, passant, and secant lines, respectively. They used the column null spaces of the submatrices $\mathbf{A}_{i,j}$ for $2 \leq i, j \leq 3$ over \mathbb{F}_2 to construct four low-density parity-check (LDPC) codes. Based on computational evidence, they made a conjecture on the dimensions of these codes. For convenience, we denote \mathbf{A}_{23} and \mathbf{A}_{32} by \mathbf{B} and \mathbf{B}_0 , respectively. From (1.3), it follows that \mathbf{B} and \mathbf{B}_0 are the incidence matrices of internal points versus secant lines and external points versus passant lines, respectively. Note that \mathbf{B} is a $\frac{q(q+1)}{2} \times \frac{q(q-1)}{2}$ matrix and \mathbf{B}_0 is a $\frac{q(q-1)}{2} \times \frac{q(q+1)}{2}$ matrix. The purpose of this article is to confirm the following conjecture on the dimensions of the LDPC codes \mathcal{L} and \mathcal{L}_0 arising from the column \mathbb{F}_2 -null spaces of \mathbf{B} and \mathbf{B}_0 , respectively.

Conjecture 1.1. (Droms, Mellinger and Meyer [5]) Let \mathcal{L} and \mathcal{L}_0 be the \mathbb{F}_2 -null spaces of **B** and **B**₀, respectively. Then

$$\dim_{\mathbb{F}_2}(\mathcal{L}) = \begin{cases} \frac{q^2 - 1}{4} - q, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q^2 - 1}{4} - q + 1, & \text{if } q \equiv 3 \pmod{4}; \end{cases}$$

and

$$\dim_{\mathbb{F}_2}(\mathcal{L}_0) = \begin{cases} \frac{q^2 - 1}{4}, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q^2 - 1}{4} + 1, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Suppose that $\mathbf{P}_1,..., \mathbf{P}_{q(q+1)/2}$ and $\ell_1,..., \ell_{q(q-1)/2}$ are indexing the rows and columns of **B**, respectively. Then we permute the rows and columns of **B**₀ to obtain a new matrix **C** such that the rows and columns of **C** are indexed by $\ell_1,..., \ell_{q(q-1)/2}$ and $\mathbf{P}_1,..., \mathbf{P}_{q(q+1)/2}$, respectively. The matrix **C** is indeed equal to \mathbf{B}^{\top} , where \mathbf{B}^{\top} is the transpose of **B**. This implies that **B** and \mathbf{B}_0 have the same 2-rank. Therefore, in order to find the dimensions of the \mathbb{F}_2 -null spaces of **B** and \mathbf{B}_0 , it suffices to calculate the 2-rank of either **B** or \mathbf{B}_0 . Recall that the subgroup G of PGL(3, q) fixing \mathcal{O} is isomorphic to PGL(2, q) [8, p. 158]. Further, G has an index 2 normal subgroup H isomorphic to PSL(2, q). It is known [7] that H acts transitively on E and I as well as on Se, T and Sk.

In [17], Sin, Wu and Xiang calculate the 2-rank of \mathbf{A}_{33} (i.e. the incidence matrix of external points and secant lines) using a combination of techniques from finite geometry and modular representation of H. In this article, we compute the 2-rank of \mathbf{B} using similar representation theoretic results obtained in [17] and different geometric results. Therefore, between the current article and [17], the reader will expect to see some overlaps in the results and statements on modular representation of H as well as the basic geometric facts about conics.

Let F be an algebraic closure of \mathbb{F}_2 . Let F^I and F^E be the free F-modules whose standard bases consist of the characteristic column vectors of I and those of E, respectively. The actions of H on I and E make the free F-modules F^I and F^E into FH-permutation modules. We define a map

$$\phi_{\mathbf{B}}: F^I \to F^E \tag{1.4}$$

as follows: specify the images of the basis elements of F^{I} under $\phi_{\mathbf{B}}$ first, i.e.

$$\phi_{\mathbf{B}}(\mathcal{G}_{\mathbf{P}}) = \sum_{\mathbf{Q} \in \mathbf{P}^{\perp} \cap E} \chi_{\mathbf{Q}}$$

for each $\mathbf{P} \in I$, and then extend $\phi_{\mathbf{B}}$ linearly to F^{I} , where \bot is the polarity induced by the quadratic form Q, $\mathcal{G}_{\mathbf{P}}$ and $\chi_{\mathbf{Q}}$ are the characteristic column vectors of the internal point \mathbf{P} with respect to I and the external point \mathbf{Q} with respect to E, respectively. The matrix of $\phi_{\mathbf{B}}$ is a 0-1 matrix of size $|E| \times |I|$. Up to permutations of the rows and columns, \mathbf{B} regarded as a matrix over F, is the matrix of $\phi_{\mathbf{B}}$ with respect to the standard bases of F^{I} and F^{E} . Moreover, $\phi_{\mathbf{B}}(\mathbf{x}) = \mathbf{B}\mathbf{x}$ for $\mathbf{x} \in F^{I}$. It can be shown that $\phi_{\mathbf{B}}$ is an FH-homomorphism. Hence, the column space of \mathbf{B} over F is equal to $\mathrm{Im}(\phi_{\mathbf{B}})$, which is also an FH-submodule of F^{E} . This point of view enables us to use results from modular representation of H to determine the dimension of $\mathrm{Im}(\phi_{\mathbf{B}})$ and thus the 2-rank of \mathbf{B} . We remark that in the calculation of the 2-rank of \mathbf{A}_{33} the authors of [17] view \mathbf{A}_{33} as the matrix of an FH-homomorphism ϕ from F^{E} to F^{E} under which the characteristic vector of an external point \mathbf{P} is mapped to the sum of the characteristic vectors of the external points on \mathbf{P}^{\perp} .

Our idea of calculating $\dim_F(\operatorname{Im}(\phi_{\mathbf{B}}))$ is to find a decomposition of $\operatorname{Im}(\phi_{\mathbf{B}})$ into a direct sum of its submodules whose dimensions can be computed easily. To this end, we apply Brauer's theory and compute the decomposition of $\operatorname{Im}(\phi_{\mathbf{B}})$ into blocks. The silmilar idea was used in [17] to compute the decomposition of $\operatorname{Ker}(\phi)$ into blocks as well as $\dim_F(\operatorname{Ker}(\phi))$. Nevertheless, there are two major differences between the current article and [17]: (1) the geometric results used to compute the decomposition of $\operatorname{Im}(\phi_{\mathbf{B}})$ into blocks are essentially different from these used to compute the decomposition of $\operatorname{Ker}(\phi)$; (2) the summands of $\operatorname{Im}(\phi_{\mathbf{B}})$ in its block decomposition are more complicated than these of $\operatorname{Ker}(\phi)$, which indicates that more efforts are required to find $\dim_F(\operatorname{Im}(\phi_{\mathbf{B}}))$.

In the following we will give a brief overview of this article. In Section 2, we first review the basic facts about \mathcal{O} and then prove several crucial geometric results. From them, in Section 5, we show that the 2-rank of the incidence matrix **D** of external points and $N_{Pa,E}(\mathbf{P})$ for $\mathbf{P} \in I$ (the set of external points on the passant lines through **P**) is either q or q-1, depending on q. The character of the complex permutation module \mathbb{C}^I and its decomposition into a sum of the irreducible ordinary characters of H were calculated in [19]; the decomposition of the characters of H into 2-blocks was given by Burkhardt [3] and Landrock [13]. From them we see that \mathbb{C}^I is a direct sum of $\mathbb{C}H$ -modules consisting of one simple module from each block of defect zero, and some summands from blocks of positive defect. According to Brauer's theory, $\operatorname{Im}(\phi_{\mathbf{B}})$ is the direct sum

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \bigoplus_{B} \operatorname{Im}(\phi_{\mathbf{B}})e_{B} \tag{1.5}$$

where e_B is a primitive idempotent in the center of FH. The block idempotents e_B are elements of FH and were computed in [17]. In order to compute $\text{Im}(\phi_{\mathbf{B}})e_B$ for each 2block B, we need detailed information concerning the action of group elements in various conjugacy classes on various geometric objects and on the intersections of certain special subsets of H with various conjugacy classes of H. These computations are made in Sections 3 and 4. These information also tell us that (i) $\text{Im}(\phi_{\mathbf{B}})e_{B_0}$ is equal to the column space of **D** over F, or this space plus an additional trivial module, depending on q, where B_0 is the principal 2-block of H, and (ii) block idempotents associated with non-principal 2-blocks of positive defect annihilate $\operatorname{Im}(\phi_{\mathbf{B}})$ (Lemma 6.2). Since the *B*-component of F^I is the mod 2 reduction of the *B*-component of \mathbb{C}^I , using (i) and (ii), and the block decomposition of \mathbb{C}^I , we show that $\operatorname{Im}(\phi_{\mathbf{B}})$ is equal to the direct sum of the column space of \mathbf{D} and the simple modules lying in the 2-blocks of defect 0, or this sum plus an additional trivial module, depending on q. Then the dimension formula of $\operatorname{Im}(\phi_{\mathbf{B}})$ follows instantly as a corollary.

2. Geometric Results

Recall that a *collineation* of PG(2, q) is an automorphism of PG(2, q), which is a bijection from the set of all points and all lines of PG(2, q) to itself that maps a point to a line and a line to a point, and preserves incidence. It is well known that each element of GL(3, q), the group of all 3×3 non-singular matrices over \mathbb{F}_q , induces a collineation of PG(2, q). The proof of the following lemma is straightforward.

Lemma 2.1. Let $\mathbf{P} = (a_0, a_1, a_2)$ (respectively, $\ell = [b_0, b_1, b_2]$) be a point (respectively, a line) of $\mathrm{PG}(2,q)$. Suppose that θ is a collineation of $\mathrm{PG}(2,q)$ that is induced by $\mathbf{D} \in \mathrm{GL}(3,q)$. If we use \mathbf{P}^{θ} and ℓ^{θ} to denote the images of \mathbf{P} and ℓ under θ , respectively, then

$$\mathbf{P}^{\theta} = (a_0, a_1, a_2)^{\theta} = (a_0, a_1, a_2)\mathbf{D}$$

and

$$\ell^{\theta} = [b_0, b_1, b_2]^{\theta} = [c_0, c_1, c_2],$$

where c_0, c_1, c_2 correspond to the first, the second, and the third coordinate of the vector $\mathbf{D}^{-1}(b_0, b_1, b_2)^{\top}$, respectively.

A correlation of PG(2,q) is a bijection from the set of points to the set of lines as well as the set of lines to the set of points that reverses inclusion. A polarity of PG(2,q) is a correlation of order 2. The image of a point **P** under a correlation σ is denoted by \mathbf{P}^{σ} , and that of a line ℓ is denoted by ℓ^{σ} . It can be shown [8, p. 181] that the non-degenerate quadratic form $Q(X_0, X_1, X_2) = X_1^2 - X_0 X_2$ induces a polarity σ (or \perp) of PG(2,q), which can be represented by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$
 (2.1)

Lemma 2.2. ([10, p. 47]) Let $\mathbf{P} = (a_0, a_1, a_2)$ (respectively, $\ell = [b_0, b_1, b_2]$) be a point (respectively, a line) of PG(2, q). If σ is the polarity represented by the above non-singular symmetric matrix \mathbf{M} , then

$$\mathbf{P}^{\sigma} = (a_0, a_1, a_2)^{\sigma} = [c_0, c_1, c_2]$$

and

$$\ell^{\sigma} = [b_0, b_1, b_2]^{\sigma} = (b_0, b_1, b_2)\mathbf{M}^{-1},$$

where c_0, c_1, c_2 correspond to the first, the second, the third coordinate of the column vector $\mathbf{M}(a_0, a_1, a_2)^{\top}$, respectively.

For example, if $\mathbf{P} = (x, y, z)$ is a point of PG(2, q), then its image under σ is $\mathbf{P}^{\sigma} = [z, -2y, x]$.

For convenience, we will denote the set of all non-zero squares of \mathbb{F}_q by \Box_q , and the set of non-squares by \mathbb{P}_q . Also, \mathbb{F}_q^* is the set of non-zero elements of \mathbb{F}_q .

Lemma 2.3. ([8, p. 181–182]) Assume that q is odd.

- (i) The polarity σ above defines three bijections; that is, $\sigma: I \to Pa, \sigma: E \to Se$, and $\sigma: \mathcal{O} \to T$ are all bijections.
- (ii) A line $[b_0, b_1, b_2]$ of PG(2, q) is a passant, a tangent, or a secant to \mathcal{O} if and only
- (ii) If the left $[b_1, b_1] = [b_1, b_1] =$

The results in the following lemma can be obtained by simple counting; see [8] for more details and related results.

Lemma 2.4. ([8, p. 170]) Using the above notation, we have

$$|T| = |\mathcal{O}| = q+1, \ |Pa| = |I| = \frac{q(q-1)}{2}, \ and \ |Se| = |E| = \frac{q(q+1)}{2}.$$
 (2.2)

Also, we have the following tables:

TABLE 1. Number of points on lines of various types

Name	Absolute points	External points	Internal points
Tangent lines	1	q	0
Secant lines	2	$\frac{q-1}{2}$	$\frac{q-1}{2}$
Passant lines	0	$\frac{q\mp 1}{2}$	$\frac{q\mp 1}{2}$

TABLE 2. Number of lines through points of various types

Name	Tangent lines	Secant lines	Skew lines
Absolute points	1	q	0
External points	2	$\frac{q-1}{2}$	$\frac{q-1}{2}$
Internal points	0	$\frac{q \pm 1}{2}$	$\frac{q \pm 1}{2}$

2.1. More geometric results. Let G be the automorphism group of \mathcal{O} in PGL(3,q) (i.e. the subgroup of PGL(3,q) fixing \mathcal{O} setwise). Then G is the image in PGL(3,q) of $O(3,q) = SO(3,q) \times \langle -1 \rangle$, hence also the image of SO(3,q), to which it is isomorphic. For our computations, we will describe G in a slightly different way. The map $\tau : \mathrm{GL}(2,q) \to 0$ GL(3,q) sending the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$
(2.3)

is a group homomorphism. The image of $\tau(GL(2,q))$ in PGL(3,q) lies in G. Now, whether or not the group $\tau(\operatorname{GL}(2,q))$ contains $\operatorname{SO}(3,q)$ depends on q. Nevertheless, $\tau(\operatorname{GL}(2,q))$ always contains a subgroup of index 2 in O(3,q) whose image in PGL(3,q) is G. Thus, the induced homomorphism $\overline{\tau}$: PGL(2,q) \rightarrow PGL(3,q) maps PGL(2,q) isomorphically onto G.

Let $H = \tau(SL(2,q))$, the group of matrices of the form (2.3) such that ad - bc = 1. Since the kernel of τ is $\langle -I_2 \rangle$, it follows that $H \cong \text{PSL}(2,q)$ and that H is isomorphic to its image \overline{H} in PGL(3, q). In fact, we have $H = \Omega(3, q)$.

Since

$$\operatorname{PGL}(2,q) = \operatorname{PSL}(2,q) \cup \begin{pmatrix} 1 & 0 \\ 0 & \xi^{-1} \end{pmatrix} \cdot \operatorname{PSL}(2,q)$$

our discussion shows that

$$H \cup \mathbf{d}(1,\xi^{-1},\xi^{-2}) \cdot H$$
 (2.4)

is a full set of representative matrices for the elements of G. In our computations, it will often be convenient to refer to elements of G by means of their representatives in the set (2.4). Additionally, a group element in (2.3) has the inverse equal to

$$\begin{pmatrix} d^2 & -bd & b^2 \\ -2cd & ad + bc & -2ab \\ c^2 & -ac & c^2 \end{pmatrix}.$$
 (2.5)

Moreover, the following holds.

Lemma 2.5. [7] The group G acts transitively on I and Pa as well as on E and Se.

We will refer to this lemma frequently in the rest of this section.

Lemma 2.6. [17, Lemma 2.9] Let **P** be a point not on \mathcal{O} , ℓ a non-tangent line, and $\mathbf{P} \in \ell$. Using the above notation, we have the following.

- (i) If $\mathbf{P} \in I$ and $\ell \in Pa$, then $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 1 \pmod{4}$, and $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 3$ (mod 4).
- (ii) If $\mathbf{P} \in I$ and $\ell \in Se$, then $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 1 \pmod{4}$, and $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 3$ (mod 4).
- (iii) If $\mathbf{P} \in E$ and $\ell \in Pa$, then $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 1 \pmod{4}$, and $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 3$ (mod 4).
- (iv) If $\mathbf{P} \in E$ and $\ell \in Se$, then $\mathbf{P}^{\perp} \cap \ell \in E$ if $q \equiv 1 \pmod{4}$, and $\mathbf{P}^{\perp} \cap \ell \in I$ if $q \equiv 3$ (mod 4).

Next we define $\Box_q - 1 := \{s - 1 \mid s \in \Box_q\}$ and $\not\Box_q - 1 := \{s - 1 \mid s \in \not\Box_q\}$.

Lemma 2.7. [18] Using the above notation,

- (i) if $q \equiv 1 \pmod{4}$, then $|(\square_q 1) \cap \square_q| = \frac{q-5}{4}$ and $|(\square_q 1) \cap \square_q| = |(\square_q 1) \cap \square_q|$ $= |(\not\square_q - 1) \cap \not\square_q| = \frac{q-1}{4};$
- (ii) if $q \equiv 3 \pmod{4}$, then $|(\not \square_q 1) \cap \square_q| = \frac{q+1}{4}$ and $|(\square_q 1) \cap \square_q| = |(\square_q 1) \cap \not \square_q| = |(\square_q 1) \cap \not \square_q| = |(\square_q 1) \cap \not \square_q|$ = $|(\not \square_q 1) \cap \not \square_q| = \frac{q-3}{4}$.

Definition 2.8. Let **P** be a point not on \mathcal{O} and ℓ a line. We define E_{ℓ} (respectively, I_{ℓ}) to be the set of external (respectively, internal) points on ℓ , $Pa_{\mathbf{P}}$ (respectively, $Se_{\mathbf{P}}$) the set of passant (respectively, secant) lines through \mathbf{P} , and $T_{\mathbf{P}}$ the set of tangent lines through **P.** Also, $N_{Pa,E}(\mathbf{P})$ (respectively, $N_{Se,E}(\mathbf{P})$) is defined to be the set of external points on the passant (respectively, secant) lines through **P**.

In the following lemma, we list the sizes of the above defined sets as well as the action of G on these sets. Also, we adopt standard notation from permutation group theory. For instance, if $W \subseteq I$, then $W^g := \{w^g \mid w \in W\}$, $G_{\mathbf{P}}$ is the stabilizer of **P** in G, and for $M \subseteq G, M^g$ is the conjugate of M under g.

Lemma 2.9. Using the above notation, if $\mathbf{P} \in I$, we have

- (i) $|E_{\mathbf{P}^{\perp}}| = |Se_{\mathbf{P}}| = \frac{q+1}{2}$, (ii) $|I_{\mathbf{P}^{\perp}}| = |Pa_{\mathbf{P}}| = \frac{q+1}{2}$,
- (iii) $|N_{Pa,E}(\mathbf{P})| = |N_{Se,E}(\mathbf{P})| = \frac{(q+1)^2}{4};$

moreover, if **P** is not a point on \mathcal{O} , ℓ is a non-tangent line, and $g \in G$, we have

- (iv) $I_{\ell}^g = I_{\ell g}$ and $Pa_{\mathbf{P}}^g = Pa_{\mathbf{P}^g}$,
- (v) $\check{E}^g_\ell = E_{\ell^g}$ and $Se^g_{\mathbf{P}} = Se_{\mathbf{P}^g}$,
- (vi) $H^g_{\mathbf{P}} = H_{\mathbf{P}^g}$,
- (vii) $N_{Pa,E}^{\hat{g}}(\mathbf{P}) = N_{Pa,E}(\mathbf{P}^g)$ and $N_{Se,E}^g(\mathbf{P}) = N_{Se,E}(\mathbf{P}^g)$,
- (viii) $(\mathbf{P}^{\perp})^g = (\mathbf{P}^g)^{\perp}$, where \perp is the polarity of $\mathrm{PG}(2,q)$ defined as above.

Proof: The above (i) - (iii) follow from from Tables 1 and 2 and simple counting, and (iv) - (vii) follow from the fact that G preserves incidence.

By the definition of G, it is clear that the following two lemmas are true.

Lemma 2.10. Let **P** be a point of PG(2,q). Then the polarity \perp defines a bijection between $I_{\mathbf{P}^{\perp}}$ and $Pa_{\mathbf{P}}$, and also a bijection between $E_{\mathbf{P}^{\perp}}$ and $Se_{\mathbf{P}}$.

Lemma 2.11. Let W be a subgroup of G. Suppose that $g \in G$ and **P** is a point of PG(2,q). Then

$$(W^g)_{\mathbf{P}^g} = W^g_{\mathbf{P}}.$$

Proposition 2.12. Let \mathbf{P} be a point not on \mathcal{O} and set $K = G_{\mathbf{P}}$. Then K is transitive on each of $I_{\mathbf{P}^{\perp}}$, $E_{\mathbf{P}^{\perp}}$, $Pa_{\mathbf{P}}$, and $Se_{\mathbf{P}}$. Moreover, if $\mathbf{P} \in E$, then K is also transitive on $T_{\mathbf{P}}$.

Proof: The case where $\mathbf{P} \in I$ is Proposition 2.11 in [19]; the case where $\mathbf{P} \in E$ or \mathcal{O} is Lemma 2.11(iii) in [17].

Lemma 2.13. [17, Corollary 2.16] Let \mathbf{P} be a point of $\mathrm{PG}(2,q)$ and let \perp be the polarity of $\mathrm{PG}(2,q)$ defined above. Then for $g \in G_{\mathbf{P}}$ we have $\mathbf{P}^{\perp} = (\mathbf{P}^{\perp})^g$. Consequently, \mathbf{P}^{\perp} is fixed setwise by $G_{\mathbf{P}}$. Moreover, $G_{\mathbf{P}^{\perp}} = G_{\mathbf{P}}$.

Lemma 2.14. Assume that $\mathbf{P} \in I$ and $\ell = \mathbf{P}^{\perp}$. Let $\mathbf{Q} \in E_{\ell}$ and $\ell^* \in T_{\mathbf{Q}}$. Suppose that \mathbf{P}_1 and \mathbf{P}_2 are two distinct external points on ℓ^* and let ℓ_1 and ℓ_2 be the tangent lines different from ℓ^* through \mathbf{P}_1 and \mathbf{P}_2 , respectively. Then ℓ_1 and ℓ_2 meet in an external point on a secant line through \mathbf{P} if and only if one of the following two cases occurs:

- (i) \mathbf{P}_1 and \mathbf{P}_2 are on two passant lines through \mathbf{P}_2 ;
- (ii) \mathbf{P}_1 and \mathbf{P}_2 are on two secant lines through \mathbf{P} .

Proof: Since G is transitive on I and preserves incidence, without loss of generality, we may assume that $\mathbf{P} = (1, 0, -\xi)$, and thus $\ell = [1, 0, -\xi^{-1}]$. Since $K := G_{\mathbf{P}}$ is transitive on E_{ℓ} by Proposition 2.12, we can assume that $\mathbf{Q} = (0, 1, 0)$. Let $\ell^* = [1, 0, 0]$ be a tangent line through \mathbf{Q} . It is clear that

$$E_{\ell^*} = \{ (0, 1, m) \mid m \in \mathbb{F}_q \}.$$

Let $\mathbf{P}_1 = (0, 1, m_1)$ and $\mathbf{P}_2 = (0, 1, m_2)$ be two distinct external points on ℓ^* . Then the tangent lines through \mathbf{P}_1 and \mathbf{P}_2 different from ℓ^* are $\ell_1 = [m_1^2, -4m_1, 4]$ and $\ell_2 = [m_2^2, -4m_2, 4]$, respectively. So we have that $\mathbf{P}_3 := \ell_1 \cap \ell_2 = (1, \frac{m_1 + m_2}{4}, \frac{m_1 m_2}{4}) \in E$. Thus the line through \mathbf{P} and \mathbf{P}_3 is

$$\ell_{\mathbf{P},\mathbf{P}_3} = \left[m_1 + m_2, -4\left(\frac{m_1m_2}{4\xi} + 1\right), \frac{m_1 + m_2}{\xi} \right],$$

which is a secant line if and only if

$$16\left(\frac{m_1m_2}{4\xi}+1\right)^2 - \frac{4(m_1+m_2)^2}{\xi} = \frac{(m_1^2-4\xi)(m_2^2-4\xi)}{\xi^2} \in \Box_q$$

if and only if either $m_i^2 - 4\xi \in \mathbb{P}_q$ for i = 1, 2 or $m_i^2 - 4\xi \in \Box_q$ for i = 1, 2. Since the line through **P** and **P**_i (i = 1 or 2) is $\ell_{\mathbf{P},\mathbf{P}_i} = [1, -\frac{m_i}{\xi}, \frac{1}{\xi}]$, and its discriminant is $\frac{m_i^2 - 4\xi}{\xi^2}$, we

conclude that $\ell_{\mathbf{P},\mathbf{P}_3}$ is a secant line if and only if either (i) \mathbf{P}_1 and \mathbf{P}_2 are on two passant lines through \mathbf{P} or (ii) \mathbf{P}_1 and \mathbf{P}_2 are on two secant lines through \mathbf{P} .

Definition 2.15. Let $N \subseteq E$. We define χ_N to be the characteristic (column) vector of N with respect to E; that is, χ_N is a column vector of length |E| whose entries are indexed by the external points such that if $\mathbf{P} \in N$ then the entry of χ_N indexed by \mathbf{P} is 1, 0 otherwise. For a line ℓ , if no confusion occurs, we should use χ_ℓ to replace ℓ_{E_ℓ} . Also, if $N = {\mathbf{P}}$ is a singleton set, we will frequently use $\chi_{\mathbf{P}}$ to replace $\chi_{{\mathbf{P}}}$.

Remark 2.16. In the rest of this section, χ_N for $N \subseteq E$ will be always viewed as a column vector over \mathbb{Z} , the ring of integer.

Corollary 2.17. Let $\mathbf{P} \in I$. Using the above notation, we have

$$\chi_{N_{Pa,E}(\mathbf{P})} \equiv \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell} \pmod{2},$$

where $\ell(\mathbf{P})$ is a tangent line through an external point on \mathbf{P}^{\perp} , $T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines distinct from $\ell(\mathbf{P})$ through the external points that are on both $\ell(\mathbf{P})$ and the passant lines through \mathbf{P} , and the congruence means entrywise congruence.

Proof: It is clear that $|T(\mathbf{P}, \ell(\mathbf{P}))| = \frac{q+1}{2}$ since there are $\frac{q+1}{2}$ passant lines through \mathbf{P} and each of them meets $\ell(\mathbf{P})$ in an external point. Let $\ell \in T(\mathbf{P}, \ell(\mathbf{P}))$. Then by Lemma 2.14, any tangent line other than ℓ in $T(\mathbf{P}, \ell(\mathbf{P}))$ meets ℓ in an external point on a secant line through \mathbf{P} , and if we use $IE(\ell, \ell(\mathbf{P}))$ to denote their intersections with ℓ then the points in $E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))$ must be on the passant lines through \mathbf{P} . Since

$$(E_{\ell_1} \setminus IE(\ell_1, \ell(\mathbf{P}))) \cap (E_{\ell_2} \setminus IE(\ell_2, \ell(\mathbf{P}))) = \emptyset$$

for two distinct lines $\ell_1, \ell_2 \in T(\mathbf{P}, \ell(\mathbf{P}))$ and

$$|E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))| = q - \frac{q-1}{2} = \frac{q+1}{2},$$

it follows that

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$$\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} |E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))| = \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \frac{q+1}{2} = \frac{(q+1)^2}{4}$$

which is the same as the size of $N_{Pa,E}(\mathbf{P})$ by Lemma 2.9(iii). Consequently, we must have

$$\bigcup_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} E_{\ell} \setminus IE(\mathbf{P}, \ell(\mathbf{P})) = \bigcup_{\ell \in Pa_{\mathbf{P}}} E_{\ell} = N_{Pa, E}(\mathbf{P}).$$

Moreover, since each point in $IE(\ell, \ell(\mathbf{P}))$ lies on exactly two lines in $T(\mathbf{P}, \ell(\mathbf{P}))$ and each point in $E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))$ doesn't lie on any line other than ℓ in $T(\mathbf{P}, \ell(\mathbf{P}))$, we obtain

$$\sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell}} = \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))} + \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \sum_{\mathbf{Q} \in IE(\ell, \ell(\mathbf{P}))} \chi_{\mathbf{Q}}$$
$$= \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))} + 2 \sum_{\mathbf{Q} \in M} \chi_{\mathbf{Q}}$$
$$\equiv \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{E_{\ell} \setminus IE(\ell, \ell(\mathbf{P}))}$$
$$= \sum_{\ell \in Pa_{\mathbf{P}}} \chi_{\ell}$$
$$= \chi_{N_{Pa,E}(\mathbf{P})} \pmod{2}$$
where $M = \{\ell_{1} \cap \ell_{2} \mid \ell_{1}, \ell_{2} \in T(\mathbf{P}, \ell(\mathbf{P})), \ell_{1} \neq \ell_{2}\}.$

Lemma 2.18. Assume that $q \equiv 1 \pmod{4}$. Let $\mathbf{P} \in \mathcal{O}$. Then there exits a set $\mathcal{M}(\mathbf{P})$ consisting of an odd number of internal points such that, for each external point $\mathbf{Q} \in \mathbf{P}^{\perp}$, the number of passant lines through \mathbf{Q} and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplicity, is odd.

Remark 2.19. In this lemma, it is possible that $\mathbf{Q}, \mathbf{Q}_1, ..., \mathbf{Q}_k$ are on the same passant line ℓ , where $\mathbf{Q} \in E_{\mathbf{P}^{\perp}}$ and $\mathbf{Q}_i \in \mathcal{M}(\mathbf{P})$ for $1 \leq i \leq k$. If this circumstance occurs, then the line ℓ should be counted k times.

Proof: Without loss of generality, we may assume that $\mathbf{P} = (0, 0, 1)$, and so $\ell := \mathbf{P}^{\perp} = [1, 0, 0]$. Using Lemma 2.1 and (2.5), we have

$$K := H_{\ell} = \left\{ \begin{pmatrix} d^2 & -bd & b^2 \\ 0 & 1 & -\frac{2b}{d} \\ 0 & 0 & \frac{1}{d^2} \end{pmatrix} \middle| d \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}.$$
 (2.7)

Since

$$\begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & -kb & (kb)^2 \\ 0 & 1 & -2kb \\ 0 & 0 & 1 \end{pmatrix}$$
(2.8)

for any positive integer k, it is obvious that

$$\left\{ \left(\begin{array}{ccc} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{array} \right) \middle| b \in \mathbb{F}_q \right\}$$
(2.9)

is a collineation subgroup of order q in K, which we denote by T.

For $(0, 1, u_1)$, $(0, 1, u_2)$ where $u_1, u_2 \in \mathbb{F}_q$ and $u_1 \neq u_2$, we have

$$(0,1,u_1) \begin{pmatrix} 1 & -\frac{u_1-u_2}{2} & \left(\frac{u_1-u_2}{2}\right)^2 \\ 0 & 1 & -(u_1-u_2) \\ 0 & 0 & 1 \end{pmatrix} = (0,1,u_2);$$

this implies that T is transitive on $E_{\ell} = \{(0, 1, u) \mid u \in \mathbb{F}_q\}.$

Now let $\mathbf{P}_1 = (1, 0, -\xi) \in I$, set $\mathcal{M}(\mathbf{P}) := {\mathbf{P}_1^g \mid g \in T}$ which is the *T*-orbit of \mathbf{P}_1 , and let $\mathbf{Q} = (0, 1, u) \in \ell$. Then

$$\mathcal{M}(\mathbf{P}) = \{ (1, -b, b^2 - \xi) \mid b \in \mathbb{F}_q \}$$

and the lines through both \mathbf{Q} and the points in $\mathcal{M}(\mathbf{P})$ form the multiset

$$L(\mathbf{Q}) = \{ [b^2 + ub - \xi, u, -1] \mid b \in \mathbb{F}_q \}.$$

Note that a line $[b^2 + ub - \xi, u, -1] \in L(\mathbf{Q})$ is passant if and only if $\frac{(u+2b)^2}{4\xi} - 1 \in \Box_q$. Since the number of $t \in \not{\square}_q$ satisfying $t - 1 \in \Box_q$ is $|(\not{\square}_q - 1) \cap \Box_q| = \frac{q-1}{4}$ by Lemma 2.7(i), it follows that the number of $b \in \mathbb{F}_q \setminus \{-\frac{u}{2}\}$ satisfying $\frac{(u+2b)^2}{4\xi} - 1 \in \Box_q$ is $2(\frac{q-1}{4}) = \frac{q-1}{2}$. Moreover, when $b = -\frac{u}{2}, \frac{(u+2b)^2}{4\xi} - 1 = -1 \in \Box_q$ as $q \equiv 1 \pmod{4}$. Hence, the number of $b \in \mathbb{F}_q$ satisfying $\frac{(u+2b)^2}{4\xi} - 1 \in \Box_q$ is $\frac{q-1}{2} + 1 = \frac{q+1}{2}$. Thus, counted with multiplicity, there are $\frac{q+1}{2}$ passant lines in $L(\mathbf{Q})$. Therefore, there are an odd number of internal points (precisely $\frac{q+1}{2}$) in $\mathcal{M}(\mathbf{P})$ connecting \mathbf{Q} by a passant line as $q \equiv 1 \pmod{4}$. Since T is transitive on both $\mathcal{M}(\mathbf{P})$ and E_ℓ and preserves incidence, we conclude that the number of passant lines through an external point on \mathbf{P}^{\perp} and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplicity, must be odd. WU

Corollary 2.21. Assume that $q \equiv 1 \pmod{4}$. Let ℓ be a tangent line. Then

$$\chi_{\ell} = \sum_{\mathbf{P} \in \mathcal{M}(\ell^{\perp})} \sum_{\ell' \in Pa_{\mathbf{P}}} \chi_{\ell'}$$
$$= \sum_{\mathbf{P} \in \mathcal{M}(\ell^{\perp})} \chi_{N_{Pa,E}(\mathbf{P})} \pmod{2}$$

where the congruence is entrywise congruence.

Proof: Let $\mathbf{P} \in \mathcal{M}(\ell^{\perp})$. Then from Corollary 2.17, it follows that

$$\chi_{N_{Pa,E}(\mathbf{P})} = \sum_{\substack{\ell' \in Pa_{\mathbf{P}} \\ \\ \equiv \sum_{\ell' \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell'} \pmod{2},$$
(2.10)

where $\ell(\mathbf{P})$ is a tangent line through an external point on \mathbf{P}^{\perp} and $T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines different from $\ell(\mathbf{P})$ through the external points that are both $\ell(\mathbf{P})$ and the passant lines through \mathbf{P} .

Further, if we take $\ell(\mathbf{P}) = \ell$ for each $\mathbf{P} \in \mathcal{M}(\ell^{\perp})$ and set $W(\mathbf{P}) := \{\ell \cap \ell_1 \mid \ell_1 \in Pa_{\mathbf{P}}\},$ then

$$\sum_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})} \sum_{\ell'\in Pa_{\mathbf{P}}} \chi_{\ell'} \equiv \sum_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})} \sum_{\ell'\in T(\mathbf{P},\ell(\mathbf{P}))} \chi_{\ell'}$$

$$= \sum_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})} \sum_{\mathbf{Q}\in W(\mathbf{P})} \sum_{\ell'\in T_{\mathbf{Q}}\setminus\{\ell\}} \chi_{\ell'}$$

$$= \sum_{\mathbf{Q}\in E_{\ell}} \sum_{\ell'\in T\setminus\{\ell\}} a_{\ell'}\chi_{\ell'}$$

$$\equiv \sum_{\ell'\in T\setminus\{\ell\}} \chi_{\ell'} \pmod{2},$$
(2.11)

where $a_{\ell'}$ for $\ell' \in T \setminus \{\ell\}$ are odd. In (2.11), the second equality follows from the definition of $T(\ell, \ell(\mathbf{P}))$ and the third equality holds since the multiset

$$\bigcup_{\mathbf{Q}\in E_{\ell}}\bigcup_{L(\mathbf{Q})}T_{\mathbf{Q}}\setminus\{\ell\},\tag{2.12}$$

where $L(\mathbf{Q}) := \{ \ell_{\mathbf{P}_1, \mathbf{Q}} \in Pa \mid \mathbf{P}_1 \in \mathcal{M}(\ell^{\perp}) \}$, is the same as the multiset

$$\bigcup_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})}\bigcup_{\mathbf{Q}\in W(\mathbf{P})}T_{\mathbf{Q}}\setminus\{\ell\},$$

and the tangent line ℓ' other than ℓ through an external point **Q** on ℓ occurs an odd number of times in (2.12) by Lemma 2.18.

Since $\sum_{\ell' \in T} \chi_{\ell'} \equiv 0 \pmod{2}$, it follows that

$$\sum_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})} \sum_{\ell'\in Pa_{\mathbf{P}}} \chi_{\ell'} \equiv \sum_{\mathbf{P}\in\mathcal{M}(\ell^{\perp})} \chi_{N_{Pa,E}(\mathbf{P})}$$
$$\equiv \sum_{\ell'\in T\setminus\{\ell\}} \chi_{\ell'}$$
$$\equiv \chi_{\ell} + \sum_{\ell'\in T} \chi_{\ell'}$$
$$\equiv \chi_{\ell} \pmod{2}.$$

Lemma 2.22. Let $\mathbf{P} \in E$ and let T_1 and T_2 be the two tangent lines through \mathbf{P} . Assume that $Z \subseteq (E_{T_1} \cup E_{T_2}) \setminus \{\mathbf{P}\}$. Then there is a set $\mathcal{M}'(\mathbf{P})$ consisting of an even number of internal points such that, for any point $\mathbf{Q} \in Z$, the number of passant lines through \mathbf{Q} and the points in $\mathcal{M}'(\mathbf{P})$, counted with multiplicity, is odd, and the number of passant lines through \mathbf{P} and the points in $\mathcal{M}'(\mathbf{P})$, counted with multiplicity, is even.

Proof: Since G is transitive on E, without loss of generality, we may assume that $\mathbf{P} = (0, 1, 0)$, and thus $T_1 = [1, 0, 0]$ and $T_2 = [0, 0, 1]$ are two tangent lines through \mathbf{P} . Let $K := G_{\mathbf{P}}$ be the stabilizer of \mathbf{P} in G. Using (2.4), we have

$$K = \left\{ \mathbf{d} \left(d^2, 1, \frac{1}{d^2} \right) \middle| d^2 \in \Box_q \right\} \cup \left\{ \mathbf{ad} \left(\frac{1}{c^2}, -1, c^2 \right) \middle| c^2 \in \Box_q \right\} \\ \cup \left\{ \mathbf{d} \left(d^2, \frac{1}{\xi}, \frac{1}{d^2 \xi^2} \right) \middle| d^2 \in \Box_q \right\} \cup \left\{ \mathbf{ad} \left(\frac{1}{c^2}, -\frac{1}{\xi}, \frac{c^2}{\xi^2} \right) \middle| c^2 \in \Box_q \right\}.$$
(2.13)

Let $\mathbf{P}_1 = (1, 1, x)$, where $x \in \mathbf{P}_q$ (respectively, $x \in \mathbf{P}_q$) and $1 - x \in \mathbf{P}_q$, be an internal point for $q \equiv 3 \pmod{4}$ (respectively, $q \equiv 1 \pmod{4}$). (Note that such an x in the last coordinate of \mathbf{P}_1 exists in \mathbb{F}_q .) Then the K-orbit of \mathbf{P}_1 is

$$\mathcal{O}_{\mathbf{P}_1} = \left\{ \left(1, \frac{1}{d^2}, \frac{x}{d^4}\right) \middle| d^2 \in \Box_q \right\} \cup \left\{ \left(1, \frac{1}{\xi d^2}, \frac{x}{\xi^2 d^4}\right) \middle| d^2 \in \Box_q \right\}.$$

To prove the first part of the lemma, we need only show that it holds for

$$Z = (E_{T_1} \cup E_{T_2}) \setminus \{\mathbf{P}\}.$$

Let $\mathbf{Q} = (0, 1, 1) \in \mathbb{Z}$. Using (2.13), we have that $K_{\mathbf{Q}}$ only contains the identity collineation. So K is transitive on \mathbb{Z} as $|\mathbb{Z}| = |K| = 2(q-1)$. The lines through \mathbf{Q} and the points in $\mathcal{O}_{\mathbf{P}_1}$ form the multiset

$$L(\mathbf{Q}) = \{ [x - d^2, d^4, -d^4] \mid d^2 \in \Box_q \} \cup \{ [x - d^2\xi, d^4\xi^2, -d^4\xi^2] \mid d^2 \in \Box_q \}.$$

A line in $L(\mathbf{Q})$ is passant if and only if

$$\frac{(d^2-2)^2}{4(1-x)} - 1 \in \Box_q$$

or

$$\frac{(d^2\xi - 2)^2}{4(1-x)} - 1 \in \Box_q,$$

where $d^2 \in \Box_q$. The number of d^2 satisfying either of the above two equations is equal to that of $t \in \mathbb{F}_q^*$ satisfying $\frac{(t-2)^2}{4(1-x)} - 1 \in \Box_q$ since $\mathbb{F}_q^* = \Box_q \cup \Box_q \xi$, where $\Box_q \xi = \{d^2\xi \mid d^2 \in \Box_q\}$. For the case where $q \equiv 3 \pmod{4}$, since the number of $t \in \mathbb{F}_q$ satisfying $\frac{(t-2)^2}{4(1-x)} - 1 \in \Box_q$ is equal to $2|(\not \Box_q - 1) \cap \Box_q| = 2(\frac{q+1}{4}) = \frac{q+1}{2}$ by Lemma 2.9(ii) and t = 0 is one of them, we see that the number of passant lines in $L(\mathbf{Q})$, counted with multiplicity, is $\frac{q-1}{2}$ which is odd since $q \equiv 3 \pmod{4}$. For the case where $q \equiv 1 \pmod{4}$, since the number of $t \in \mathbb{F}_q \setminus \{2\}$ satisfying $\frac{(t-2)^2}{4(1-x)} - 1 \in \Box_q$ is equal to $2|(\Box_q - 1) \cap \Box_q| = 2(\frac{q-1}{4}) = \frac{q-1}{2}$ by Lemma 2.9(i), t = 0 is not one of the solutions and t = 2 also satisfies $\frac{(t-2)^2}{4(1-x)} - 1 \in \Box_q$, we see that the number of passant lines in $L(\mathbf{Q})$, counted with multiplicity, is $\frac{q+1}{2}$, which is odd as $q \equiv 1 \pmod{4}$. Now we set $\mathcal{M}'(\mathbf{P}) := \mathcal{O}_{\mathbf{P}_1}$, and so $|\mathcal{M}'(\mathbf{P})| = q - 1$ is even. Since K is transitive on both $Z = (E_{T_1} \cup E_{T_2}) \setminus \{\mathbf{P}\}$ and the points in $\mathcal{M}'(\mathbf{P})$, the number of passant lines through a point in Z and the points in $\mathcal{M}'(\mathbf{P})$, counted with multiplicity, must be odd.

The lines through **P** and the points in $\mathcal{M}'(\mathbf{P})$ form the multiset

$$\left\{ \left[1, 0, -\frac{d^4}{x}\right] \middle| d^2 \in \Box_q \right\} \cup \left\{ \left[1, 0, -\frac{d^4\xi^2}{x}\right] \middle| d^2 \in \Box_q \right\}.$$

each or none of which is a passant line accordingly as $q \equiv 3 \pmod{4}$ or $q \equiv 1 \pmod{4}$. Hence, we conclude that the number of passant lines through **P** and the points in $\mathcal{M}(\mathbf{P})$, counted with multiplicity, is even.

Remark 2.23. Let $\mathbf{P} \in E$. In the following discussion, without being further mentioned, $\mathcal{M}'(\mathbf{P})$ will always denote a set consisting of an even number of internal points that satisfy the conditions with $Z = E_{T_1} \setminus \{\mathbf{P}\}$ in the above lemma, where T_1 is one of the two tangent lines through \mathbf{P} .

Corollary 2.24. Let $\mathbf{P} \in E$ and let T_1 and T_2 be the two tangent lines through \mathbf{P} . Then

$$\begin{split} \chi_{T_1} + \chi_{T_2} &\equiv \sum_{\mathbf{Q} \in \mathcal{M}'(\mathbf{P})} \sum_{\ell \in Se_{\mathbf{Q}}} \chi_{\ell} \\ &\equiv \sum_{\mathbf{Q} \in \mathcal{M}'(\mathbf{P})} \chi_{N_{Se,E}(\mathbf{Q})} \pmod{2}, \end{split}$$

where the congruence means entrywise congruence.

Proof: Let $\mathbf{Q} \in \mathcal{M}'(\mathbf{P})$. Then Corollary 2.17 gives

$$\chi_{N_{Pa,E}(\mathbf{Q})} \equiv \sum_{\substack{\ell' \in Pa_{\mathbf{Q}} \\ \ell' \in T(\mathbf{Q}, \ell(\mathbf{Q}))}} \chi_{\ell'} \pmod{2},$$

$$(2.14)$$

where $\ell(\mathbf{Q})$ is a tangent line through an external point on \mathbf{Q}^{\perp} and $T(\mathbf{Q}, \ell(\mathbf{Q}))$ is the set tangent lines through the external points that are on both $\ell(\mathbf{Q})$ and the passant lines through \mathbf{Q} . Let $\mathbf{1}$ be the all-one column vector of length |E|. Since

$$\mathbf{l} + \chi_{N_{Pa,E}(\mathbf{Q})} \equiv \chi_{N_{Se,E}(\mathbf{Q})} \pmod{2} \tag{2.15}$$

and $|\mathcal{M}'(\mathbf{P})|$ is even, we have

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$$\sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \sum_{\ell\in Se_{\mathbf{Q}}} \chi_{\ell} \equiv \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} (\mathbf{1} + \chi_{N_{Pa,E}(\mathbf{Q})})$$

$$\equiv \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \mathbf{1} + \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \chi_{N_{Pa,E}(\mathbf{Q})}$$

$$\equiv \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \chi_{N_{Pa,E}(\mathbf{Q})}$$

$$\equiv \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \sum_{\ell\in T(\mathbf{Q},\ell(\mathbf{Q}))} \chi_{\ell} \pmod{2}.$$
(2.16)

Further, if we set $\ell(\mathbf{Q}) := T_1$ for each $\mathbf{Q} \in \mathcal{M}'(\mathbf{P})$ and set $W'(\mathbf{Q}) := \{T_1 \cap \ell_1 \mid \ell_1 \in Pa_{\mathbf{Q}}\},\$ since the multiset

$$\bigcup_{\mathbf{P}_1 \in E_{T_1}} \bigcup_{L'(\mathbf{P}_1)} T_{\mathbf{P}_1} \setminus \{T_1\}$$

where $L'(\mathbf{P}_1) = \{\ell_{\mathbf{P}_1,\mathbf{P}_2} \in Pa \mid \mathbf{P}_2 \in \mathcal{M}'(\mathbf{P})\}$, is the same as the multiset

$$\bigcup_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})}\bigcup_{\mathbf{P}_1\in W'(\mathbf{Q})}T_{\mathbf{P}_1}\setminus\{T_1\},$$
(2.17)

and the tangent line ℓ other than T_1 through an external point $\mathbf{P}_1 \neq \mathbf{P}$ (respectively, $\mathbf{P}_1 = \mathbf{P}$) on T_1 occurs an odd (respectively, even) number of times in (2.17) by Lemma 2.22, we obtain

$$\sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \sum_{\ell\in T(\mathbf{Q},\ell(\mathbf{Q}))} \chi_{\ell} \equiv \sum_{\mathbf{Q}\in\mathcal{M}'(\mathbf{P})} \sum_{\mathbf{P}_{1}\in W'(\mathbf{Q})} \sum_{\ell\in T_{\mathbf{P}_{1}}\setminus\{T_{1}\}} \chi_{\ell}$$

$$= \sum_{\mathbf{P}_{1}\in E_{T_{1}}} \sum_{\ell\in T_{\mathbf{P}_{1}}\setminus\{T_{1}\}} b_{\ell}\chi_{\ell}$$

$$= b_{T_{2}}\chi_{T_{2}} + \sum_{\ell\in T\setminus\{T_{1},T_{2}\}} b_{\ell}\chi_{\ell}$$

$$\equiv \sum_{\ell\in T\setminus\{T_{1},T_{2}\}} \chi_{\ell} \pmod{2},$$
(2.18)

where b_{ℓ} for $\ell \in T \setminus \{T_1, T_2\}$ are all odd integers and b_{T_2} is an even integer. Using (2.16), (2.18), and the fact that $\sum_{\ell \in T} \chi_{\ell} = \mathbf{0} \pmod{2}$, we have

$$\chi_{T_1} + \chi_{T_2} \equiv \sum_{\substack{\chi \in T \setminus \{T_1, T_2\} \\ \mathbf{Q} \in M'(\mathbf{P})}} \chi_{\ell}$$
$$\equiv \sum_{\mathbf{Q} \in \mathcal{M}'(\mathbf{P})} \chi_{N_{Pa,E}(\mathbf{Q})} \pmod{2}.$$

3. The Conjugacy Classes and Intersection Parity

In this section, we review the conjugacy classes of H and study their intersections with some special subsets of H.

3.1. Conjugacy classes. Recall that

$$H = \left\{ \left(\begin{array}{ccc} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{array} \right) \middle| a, b, c, d \in \mathbb{F}_q, \ ad-bc = 1 \right\}$$

is the subgroup of G that is isomorphic to PSL(2,q). If we define T = tr(q) + 1, where $g \in H$ and tr(g) is the trace of g, then the conjugacy classes of H can be read as follows.

Lemma 3.1. [17, Lemma 3.2] The conjugacy classes of H are given as follows.

- (i) $D = \{ \mathbf{d}(1, 1, 1) \};$
- (i) F^+ and F^- , where $F^+ \cup F^- = \{g \in H \mid T(g) = 4, g \neq \mathbf{d}(1,1,1)\};$ (ii) $[\theta_i] = \{g \in H \mid T(g) = \theta_i\}, 1 \le i \le \frac{q-5}{4} \text{ if } q \equiv 1 \pmod{4}, \text{ or } 1 \le i \le \frac{q-3}{4} \text{ if } q \equiv 3 \pmod{4}, \text{ where } \theta_i \in \Box_q, \theta_i \ne 4, \text{ and } \theta_i 4 \in \Box_q;$
- (iv) $[0] = \{g \in H \mid T(g) = 0\};$

(v) $[\pi_k] = \{g \in H \mid T(g) = \pi_k\}, 1 \le k \le \frac{q-1}{4} \text{ if } q \equiv 1 \pmod{4}, \text{ or } 1 \le k \le \frac{q-3}{4} \text{ if } q \equiv 3 \pmod{4}, \text{ where } \pi_i \in \Box_q, \pi_k \neq 4, \text{ and } \pi_k - 4 \in \not\square_q.$

Remark 3.2. The set $F^+ \cup F^-$ forms one conjugacy class of G, and splits into two equalsized classes F^+ and F^- of H. For our purpose, we denote $F^+ \cup F^-$ by [4]. Also, each of D, $[\theta_i]$, [0], and $[\pi_k]$ forms a single conjugacy class of G. The class [0] consists of all the elements of order 2 in H.

In the following, for convenience, we frequently use C to denote any one of D, [0], [4], $|\theta_i|$, or $|\pi_k|$. That is,

$$C = D, [0], [4], [\theta_i], \text{ or } [\pi_k].$$
 (3.1)

3.2. Intersection properties.

Definition 3.3. Let $\mathbf{P} \in I$, $\mathbf{Q} \in E$, $\ell \in Pa$. We define

$$\mathcal{H}_{\mathbf{P},\mathbf{Q}} = \{h \in H \mid (\mathbf{P}^{\perp})^h \in Pa_{\mathbf{Q}}\} \text{ and } \mathcal{S}_{\mathbf{P},\ell} = \{h \in H \mid (\mathbf{P}^{\perp})^h = \ell\}.$$

That is, $\mathcal{H}_{\mathbf{P},\mathbf{Q}}$ consists of all the elements of H that map the passant line \mathbf{P}^{\perp} to a passant line through \mathbf{Q} and $\mathcal{S}_{\mathbf{P},\ell}$ is the set of elements in H that map \mathbf{P}^{\perp} to the passant line ℓ .

Using the above notation, since G preserves incidence, for $q \in G$, $\mathbf{P} \in I$, and $\ell \in Pa$, we have

$$\mathcal{H}^{g}_{\mathbf{P},\mathbf{Q}} = \mathcal{H}_{\mathbf{P}^{g},\mathbf{Q}^{g}}, \ \mathcal{S}^{g}_{\mathbf{P},\ell} = \mathcal{S}_{\mathbf{P}^{g},\ell^{g}}.$$
(3.2)

The following corollary is apparent.

Corollary 3.4. Let $g \in G$ and C be given in (3.1) and let **P** and **Q** be two external points. Then $(C \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}})^g = C \cap \mathcal{H}_{\mathbf{P}^g,\mathbf{Q}^g}.$

Next the size of the intersection of each conjugacy class of H with K which stabilizes an element of I in H is calculated.

Corollary 3.5. Let $\mathbf{P} \in I$ and $K = H_{\mathbf{P}}$. Then we have

- (i) $|K \cap D| = 1$;
- (ii) $|K \cap [4]| = 0;$
- (iii) $|K \cap [\pi_k]| = 2$ for each k;
- (iv) $|K \cap [\theta_i]| = 0$ for each i; (v) $|K \cap [0]| = \frac{q+1}{2}$ or $\frac{q-1}{2}$ accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$.

Proof: Let $\mathbf{Q} = (1, 0, -\xi)$ and $K_1 = H_{\mathbf{Q}}$. Since H is transitive on I, it follows $\mathbf{Q}^g = \mathbf{P}$ for some $g \in H$. By Lemma 2.11, we have $K_1^g = K$. Consequently,

$$|K \cap C| = |(K_1 \cap C)^g|.$$

Therefore, to prove the corollary, it is enough to consider $\mathbf{P} = \mathbf{Q}$. It is clear that $|D \cap K| =$ 1. Let $g \in K \cap C$. Then the quadruples (a, b, c, d) determining g satisfy the following equations

$$\begin{aligned}
bd - ac\xi &= 0 \\
b^2 - a^2\xi &= -\xi(d^2 - c^2\xi) \\
ad - bc &= 1 \\
a + d &= s,
\end{aligned}$$
(3.3)

where $s^2 = 0, 4, \pi_k, \theta_i$. The equations in (3.3) give (1) $a = d = \frac{s}{2}, c^2 = \frac{s^2 - 4}{4\xi}, b^2 = \frac{(s^2 - 4)\xi}{4}$ and (2) $a = -d, s = 0, c^2 \xi - 1 = a^2$. From Case (1), we see that $|K \cap [\pi_k]| = 2$ for each $[\pi_k]$ and $|K \cap C| = 0$ for $C = [\theta_i]$, [4]; moreover, if $q \equiv 3 \pmod{4}$, we obtain one group element $\mathbf{ad}(-\xi, -1, \xi^{-1}) \in K \cap [0]$ in Case (1). Since the number of $t \in \mathbb{Z}_q$ satisfying $t-1 \in \Box_q$ is $\frac{q-1}{4}$ or $\frac{q-3}{4}$ accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$ by Lemma 2.9, the number of $c \in \mathbb{F}_q^*$ satisfying $c^2\xi - 1 \in \Box_q$ is $2|(\ \not\Box_q - 1) \cap \Box_q|$ which is $\frac{q-1}{2}$ or $\frac{q-3}{2}$ accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$. When $q \equiv 1 \pmod{4}$, c = 0 also satisfies $c^2\xi - 1 \in \Box_q$. Therefore, Case (1) and Case (2) give $\frac{q+1}{2}$ or $\frac{q-1}{2}$ different group elements in $K \cap [0]$ depending on q. Now the corollary is proved. \Box

In the following lemma, we investigate the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ for each $C \neq [0]$ and $\mathbf{P} \in I$, $\mathbf{Q} \in E$. Recall that $\ell_{\mathbf{P},\mathbf{Q}}$ is the line through \mathbf{P} and \mathbf{Q} .

Lemma 3.6. Assume that $q \equiv 1 \pmod{4}$. Let $\mathbf{P} \in I$ and $\mathbf{Q} \in E$. Suppose that C = D, [4], $[\pi_k] (1 \leq k \leq \frac{q-1}{4}), [\theta_i] (1 \leq i \leq \frac{q-5}{4}).$

- (i) If $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$, then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is even for each C.
- (ii) If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is odd if and only if C = D.
- (iii) If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, for each class $[\pi_k]$ with $1 \leq k \leq \frac{q-1}{4}$, there are two different points \mathbf{Q}_1 , $\mathbf{Q}_2 \in E_{\ell_{\mathbf{P},\mathbf{Q}}}$ such that $|[\pi_k] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}_j}|$ is odd for j = 1, 2; moreover, the two points associated with one class $[\pi_{k_1}]$ are different from those associated with the other class $[\pi_{k_2}]$, where $[\pi_{k_1}] \neq [\pi_{k_2}]$.

Proof: Since G acts transitively on I and preserves incidence, without loss of generality, we may assume that $\mathbf{P} = (1, 0, -\xi)$. From (2.4), it follow that

$$K := G_{\mathbf{P}} = \begin{cases} \begin{pmatrix} d^{2} & cd\xi & c^{2}\xi^{2} \\ 2cd & d^{2} + c^{2}\xi & 2cd\xi \\ c^{2} & cd & d^{2} \end{pmatrix} \middle| d, c \in \mathbb{F}_{q}, d^{2} - c^{2}\xi = 1 \end{cases}$$

$$\bigcup \begin{cases} \begin{pmatrix} d^{2} & -cd\xi & c^{2}\xi^{2} \\ 2cd & -d^{2} - c^{2}\xi & 2cd\xi \\ c^{2} & -cd & d^{2} \end{pmatrix} \middle| d, c \in \mathbb{F}_{q}, -d^{2} + c^{2}\xi = 1 \end{cases}$$

$$\bigcup \begin{cases} \begin{pmatrix} d^{2} & cd & c^{2} \\ 2cd\xi^{-1} & d^{2} + c^{2}\xi^{-1} & 2cd \\ c^{2}\xi^{-2} & cd\xi^{-1} & d^{2} \end{pmatrix} \middle| d, c \in \mathbb{F}_{q}, d^{2}\xi - c^{2} = 1 \end{cases}$$

$$\bigcup \begin{cases} \begin{pmatrix} d^{2} & -cd & c^{2} \\ 2cd\xi^{-1} & d^{2} + c^{2}\xi^{-1} & 2cd \\ c^{2}\xi^{-2} & cd\xi^{-1} & d^{2} \end{pmatrix} \middle| d, c \in \mathbb{F}_{q}, -d^{2}\xi + c^{2} = 1 \end{cases}.$$

$$(3.4)$$

Since K is transitive on both $Pa_{\mathbf{P}}$ and $Se_{\mathbf{P}}$ by Proposition 2.12 and

$$|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = |(\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C)^g| = |\mathcal{H}_{\mathbf{P},\mathbf{Q}^g} \cap C|$$

by Corollary 3.4, we may assume that \mathbf{Q} is on either ℓ_1 or ℓ_2 , where $\ell_1 = [1, 0, \xi^{-1}] \in Pa_{\mathbf{P}}$ and $\ell_2 = [0, 1, 0] \in Se_{\mathbf{P}}$.

Case I. $\mathbf{Q} \in \ell_1$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$.

In this case, $\mathbf{Q} = (1, x, -\xi)$ for some $x \in \mathbb{F}_q^*$ and $x^2 + \xi \in \Box_q$, and

$$Pa_{\mathbf{Q}} = \{ [1, s, (1+sx)\xi^{-1}] \mid s \in \mathbb{F}_q, s^2 - 4(1+sx)\xi^{-1} \in \mathcal{Q}_q \}.$$

Using (3.4), we obtain that

$$K_{\mathbf{Q}} = \{ \mathbf{d}(1, 1, 1), \mathbf{ad}(1, -\xi^{-1}, \xi^{-2}) \}.$$

It is apparent that $\mathbf{d}(1,1,1)$ fixes each line in $Pa_{\mathbf{Q}}$. From

$$\mathbf{ad}(1, -\xi^{-1}, \xi^{-2})^{-1}(1, s, (1+sx)\xi^{-1})^{\top} = ((1+sx)\xi, -s\xi, 1)^{\top},$$

it follows that $[1, s, (1 + sx)\xi^{-1}] \in Pa_{\mathbf{Q}}$ is fixed by $K_{\mathbf{Q}}$ if and only if s = 0 or $s = -2x^{-1}$. Therefore, $K_{\mathbf{Q}}$ has two orbits of length 1 on $Pa_{\mathbf{Q}}$, i.e. $\{\ell_1 = [1, 0, \xi^{-1}]\}$ and

 $\{\ell_3 = [1, -2x^{-1}, -\xi^{-1}]\}$, and all other orbits, whose representatives are \mathcal{R}_1 , have length 2. From

$$\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = |\mathcal{S}_{\mathbf{P},\ell_1} \cap C| + |\mathcal{S}_{\mathbf{P},\ell_3} \cap C| + 2\sum_{\ell \in \mathcal{R}_1} |\mathcal{S}_{\mathbf{P},\ell} \cap C|,$$

it follows that the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is determined by that of $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C| + |\mathcal{S}_{\mathbf{P},\ell_3} \cap C|$. Here we used the fact that $|\mathcal{S}_{\mathbf{P},\ell} \cap C| = |\mathcal{S}_{\mathbf{P},\ell'} \cap C|$ if $\{\ell,\ell'\}$ is an orbit of $K_{\mathbf{P}}$ on $Pa_{\mathbf{Q}}$. It is clear that $|\mathcal{S}_{\mathbf{P},\ell} \cap D| = |\mathcal{S}_{\mathbf{P},\ell_3} \cap D| = 0$.

Note that the quadruples (a, b, c, d) that determine group elements in $S_{\mathbf{P}, \ell_1} \cap C$ satisfy the following equations

$$\begin{array}{rcl} -2cd + 2ab\xi^{-1} &= & 0\\ c^2 - a^2\xi^{-1} &= & (d^2 - b^2\xi^{-1})\xi^{-1}\\ a + d &= & s\\ ad - bc &= & 1 \end{array}$$
(3.5)

where $s^2 = 4$, π_k , θ_i . The first two equations in (3.5)give $c = \pm \sqrt{-1}c\xi^{-1}$ and $a = \pm \sqrt{-1}d$. Combining them with the last two equations (3.5), we obtain 0, 4 or 8 quadruples (a, b, c, d) satisfying the above equations, among which, both (a, b, c, d) and (-a, -b, -c, -d) do appear at the same time. Therefore, $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C|$ is 0, 2, or 4. Particularly, in [0], there might be only 2 elements satisfying the above conditions.

Similarly, the quadruples (a, b, c, d) that determine a group element in $S_{\mathbf{P},\ell_3} \cap C$ satisfy the following equations

$$\begin{array}{rcl} -2cd + 2ab\xi^{-1} &=& -2x^{-1}(d^2 - b^2\xi^{-1}) \\ c^2 - a^2\xi^{-1} &=& -\xi^{-1}(d^2 - b^2\xi^{-1}) \\ a + d &=& s \\ ad - bc &=& 1, \end{array}$$
(3.6)

where $s^2 = 4$, π_k , θ_i . The first two equations in (3.6) give

$$d^2 - b^2 \xi^{-1} = \pm A, \tag{3.7}$$

where

$$A = \sqrt{\frac{1}{(x^2 + \xi^{-1})\xi}}.$$
(3.8)

From (3.7), $c^2 - d^2 = \mp A\xi^{-1}$ and $a^2 + d^2 = s^2 - 2 - 2bc$, it follows that $(b\xi^{-1} + c)^2 = -(\pm 2A \pm 2 - s^2)\xi^{-1}$

$$(b\xi^{-1} + c)^2 = -(\pm 2A + 2 - s^2)\xi^{-1}.$$
(3.9)

Hence, if (3.6) determines an odd number of group elements, then

$$-(\pm 2A+2-s^2)\xi^{-1}\notin \square_q.$$

If $-(\pm 2A + 2 - s^2)\xi^{-1} \in \Box_q$ and we set $B_{(\pm)} := \sqrt{-(\pm 2A + 2 - s^2)\xi^{-1}}$, by $c^2 - d^2\xi^{-1} = \pm A\xi^{-1}$ and $a^2 - b^2\xi^{-1} = \mp A\xi^{-1}$, we have

$$d = \frac{1}{2s} [s^2 + (\xi B_{(\pm)}^2 - 2B_{(\pm)}b)] \left(\text{or } d = \frac{1}{2s} [s^2 + (\xi B_{(\pm)}^2 + 2B_{(\pm)}b)] \right)$$
(3.10)

and thus

$$a = \frac{1}{2s} [s^2 - (\xi B_{(\pm)}^2 - 2B_{(\pm)}b)] \left(\text{or } a = \frac{1}{2s} [s^2 - (\xi B_{(\pm)}^2 + 2B_{(\pm)}b)] \right).$$
(3.11)

Combining $b = (\pm B_{(\pm)} - c)\xi^{-1}$ and ad - bc = 1, we have

$$\left(\xi - \frac{B_{(\pm)}^2 \xi^2}{s^2}\right) c^2 + \left(\frac{\xi B_{(\pm)}^3 \xi^2}{s^2} - B_{(\pm)} \xi\right) c + \left(\frac{s^2}{4} - \frac{B_{(\pm)}^4 \xi^2}{4s^2} - 1\right) = 0$$
(3.12)

or

$$\left(\xi - \frac{B_{(\pm)}^2 \xi^2}{s^2}\right) c^2 - \left(\frac{\xi B_{(\pm)}^3 \xi^2}{s^2} - B_{(\pm)} \xi\right) c + \left(\frac{s^2}{4} - \frac{B_{(\pm)}^4 \xi^2}{4s^2} - 1\right) = 0.$$
(3.13)

The discriminant of (3.12) or (3.13) is

$$\Delta = \xi^2 \left(\frac{\xi B_{(\pm)}^2 \xi}{s^2} - 1 \right) \left(\frac{s^2}{\xi} - \frac{4}{\xi} - B_{(\pm)}^2 \right) = \frac{4x^2 \xi}{(x^{-2} + \xi^{-1})s^2} \in \Box_q.$$
(3.14)

From (3.10), (3.11), (3.12), (3.13), and (3.14), it follows that (3.6) produces 2 or 4 group elements; that is, $|S_{\mathbf{P},\ell_3} \cap C| = 2$ or 4.

If $-(\pm 2A + 2 - s^2)\xi^{-1} = 0$, then s^2 is one of 2A + 2 and -2A + 2 since

$$(2A+2)(-2A+2) = \frac{4x^2}{x^2+\xi^{-1}} \in \square_q.$$

Therefore, in this case, we have $|[s^2] \cap S_{\mathbf{P},\ell_3}| = 1$. It is also clear that, for the same $[s^2]$,

$$|[s^2] \cap \mathcal{S}_{\mathbf{P},\mathbf{P}_1^{\perp}}|$$

is odd, where $\mathbf{P}_1 = (1, -x, -\xi) \in E_{\ell_3}$. Moreover, when x runs over

$$L := \{ x \in \mathbb{F}_q^* \mid x^2 + \xi \in \Box_q \}$$

once, each $[\pi_k]$ with $1 \le k \le \frac{q-1}{4}$ appears exactly twice in the multiset

$$\left\{2\sqrt{\frac{1}{(x^{-2}+\xi^{-1})\xi}}+2\right\} \bigcup \left\{-2\sqrt{\frac{1}{(x^{-2}+\xi^{-1})\xi}}+2\right\}.$$

Note that

$$\pm \frac{2}{\sqrt{(x_1^{-2} + \xi^{-1})\xi}} + 2 = \pm \frac{2}{\sqrt{(x_2^{-2} + \xi^{-1})\xi}} + 2$$

if and only if $x_1 = \pm x_2$. Therefore, for each class $[\pi_k]$ with $1 \le k \le \frac{q-1}{4}$, there are two different points \mathbf{Q}_1 , $\mathbf{Q}_2 \in E_{\ell_{\mathbf{P},\mathbf{Q}}}$ such that $|[\pi_k] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}_j}|$ is odd for j = 1, 2; further, the two points associated with one class $[\pi_{k_1}]$ are different from those associated with the other class $[\pi_{k_2}]$, where $[\pi_{k_1}] \ne [\pi_{k_2}]$. The proof of (iii) is completed.

Case II. $\mathbf{Q} = \ell_1 \cap \mathbf{P}^{\perp}$

In this case, $\mathbf{Q} = (0, 1, 0)$. From (3.4), it follows that

$$K_{\mathbf{Q}} = \{ \mathbf{d}(1,1,1), \mathbf{ad}(-1,1,-1), \mathbf{d}(-1,-\xi^{-1},-\xi^{-2}), \mathbf{ad}(1,-\xi^{-1},\xi^{-2}) \}.$$

Since $Pa_{\mathbf{Q}} = \{[1, 0, -x] \mid x \in \not\square_q\}$, it follows that the passant lines through \mathbf{Q} that are fixed by $K_{\mathbf{Q}}$ are $\ell_1 = [1, 0, \xi^{-1}]$ and $\ell_4 = [1, 0, -\xi^{-1}]$. Thus, $K_{\mathbf{Q}}$ has two orbits of length 1 on $Pa_{\mathbf{Q}}$ and all the other orbits, whose representatives are \mathcal{R}_2 , have length 2. By

$$|\mathcal{H}_{\mathbf{P},\mathbf{Q}}\cap C| = |\mathcal{S}_{\mathbf{P},\ell_1}\cap C| + |\mathcal{S}_{\mathbf{P},\ell_4}\cap C| + 2\sum_{\ell\in\mathcal{R}_2}|\mathcal{S}_{\mathbf{P},\ell}\cap C|,$$

we obtain that the parity of $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is determined by that of $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C| + |\mathcal{S}_{\mathbf{P},\ell_4} \cap C|$. From the discussions in **Case I**, we know that $|\mathcal{S}_{\mathbf{P},\ell_1} \cap C|$ always even. Since $\ell_4 = \mathbf{P}^{\perp}$ and $G_{\mathbf{P}} = G_{\mathbf{P}^{\perp}}$ by Lemma 2.13, it follows from Corollary 3.5 that $|\mathcal{S}_{\mathbf{P},\ell_4} \cap C|$ is odd if and only if C = D. The proof of (ii) is completed.

Case III. $\mathbf{Q} \in \ell_2$.

In this case, $\mathbf{Q} = (1, 0, -y)$ for some $y \in \Box_q$. Using (3.4), we see that

$$K_{\mathbf{Q}} = \{\mathbf{d}(1,1,1), \mathbf{d}(-1,1,-1)\}$$

Moreover, all the orbits of $K_{\mathbf{Q}}$ on $Pa_{\mathbf{Q}} = \{[1, s, y^{-1}] \mid s \in \mathbb{F}_q^*, s^2 - 4y^{-1} \in \mathbb{P}_q\}$ have length 2, then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is even for each C. Part (i) is proved.

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Lemma 3.7. Assume that $q \equiv 3 \pmod{4}$. Let $\mathbf{P} \in I$ and $\mathbf{Q} \in E$. Suppose that C = D, [4], $[\pi_k] (1 \leq k \leq \frac{q-3}{4}), [\theta_i] (1 \leq i \leq \frac{q-3}{4}).$

- (i) If $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, for each class $[\theta_i]$ with $1 \leq i \leq \frac{q-3}{4}$, there are two different points \mathbf{Q}_1 , $\mathbf{Q}_2 \in E_{\ell_{\mathbf{P},\mathbf{Q}}}$ such that $|[\theta_i] \cap \mathcal{H}_{\mathbf{P},\mathbf{Q}_j}|$ is odd for j = 1, 2; moreover, the two points associated with one class $[\theta_{i_1}]$ are different from those associated with the other class $[\theta_{i_2}]$, where $[\theta_{i_1}] \neq [\theta_{i_2}]$.
- (ii) If $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is odd if and only if C = D.
- (iii) If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$, then $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C|$ is even for each C.

Proof: The proof is essentially the same as the one of Lemma 3.6. We omit the details. \Box

4. Group Algebra FH

4.1. **2-Blocks of H.** In this section we recall several results on the 2-blocks of $H \cong PSL(2,q)$. We refer the reader to [14] or [2] for a general introduction on this subject.

Let **R** be the ring of algebraic integers in the complex field \mathbb{C} . We choose a maximal ideal **M** of **R** containing 2**R**. Let $F = \mathbf{R}/\mathbf{M}$ be the residue field of characteristic 2, and let $*: \mathbf{R} \to F$ be the natural ring homomorphism. Define

$$\mathbf{S} = \{ \frac{r}{s} \mid r \in \mathbf{R}, s \in \mathbf{R} \setminus \mathbf{M} \}.$$

$$(4.1)$$

Then it is clear that the map $*: \mathbf{S} \to F$ defined by $(\frac{r}{s})^* = r^*(s^*)^{-1}$ is a ring homomorphism with kernel $\mathcal{P} = \{\frac{r}{s} \mid r \in \mathbf{M}, s \in \mathbf{R} \setminus \mathbf{M}\}$. In the rest of this article, F will always be the field of characteristic 2 constructed as above. Note that F is an algebraic closure of \mathbb{F}_2 .

Let Irr(H) and IBr(H) be the set of irreducible ordinary characters and the set of irreducible Brauer characters of H, respectively. In the following, we deduce the 2-blocks of H from the known results on the 2-blocks of PSL(2,q). For basic results on blocks of finite groups, we refer the reader to Chapter 3 of [14].

The character tables of PSL(2, q) were obtained by Jordan and Schur independently; see[11], [12], or [15] for the detailed discussions. The irreducible characters of H can be read off from the character tables of PSL(2, q) as follows.

Lemma 4.1. ([11], [12], [15]) The irreducible ordinary characters of H are:

- (i) $1 = \chi_0, \gamma, \chi_1, ..., \chi_{\frac{q-1}{4}}, \beta_1, \beta_2, \phi_1, ..., \phi_{\frac{q-5}{4}}$ if $q \equiv 1 \pmod{4}$, where $1 = \chi_0$ is the trivial character, γ is the character of degree q, χ_s for $1 \leq s \leq \frac{q-1}{4}$ are the characters of degree q - 1, ϕ_r for $1 \leq r \leq \frac{q-5}{4}$ are the characters of degree q + 1, and β_i for i = 1, 2 are the characters of degree $\frac{q+1}{2}$:
- and β_i for i = 1, 2 are the characters of degree $\frac{q+1}{2}$; (ii) $1 = \chi_0, \chi_1, ..., \chi_{\frac{q-3}{4}}, \beta_1, \eta_2, \eta_1, ..., \phi_{\frac{q-3}{4}}$ if $q \equiv 3 \pmod{4}$, where $1 = \chi_0$ is the trivial character, γ is the character of degree q, χ_s for $1 \leq s \leq \frac{q-1}{3}$ are the characters of degree q - 1, ϕ_r for $1 \leq r \leq \frac{q-3}{4}$ are the characters of degree q + 1, and η_i for i = 1, 2 are the characters of degree $\frac{q-1}{2}$;

The following lemma tells us how the irreducible ordinary characters of H are partitioned into 2-blocks.

Lemma 4.2. [17, Lemma 4.1] First assume that $q \equiv 1 \pmod{4}$ and $q - 1 = m2^n$, where $2 \nmid m$.

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(i) The principal block B_0 of H contains $2^{n-2} + 3$ irreducible characters

 $\chi_0 = 1, \ \gamma, \ \beta_1, \ \beta_2, \ \phi_{i_1}, ..., \phi_{i_{(2^{n-2}-1)}},$

where $\chi_0 = 1$ is the trivial character of H, γ is the irreducible character of degree q of H, β_1 and β_2 are the two irreducible characters of degree $\frac{q+1}{2}$, and ϕ_{i_k} for $1 \le k \le 2^{n-2} - 1$ are distinct irreducible characters of degree q + 1 of H.

- (ii) H has $\frac{q-1}{4}$ blocks B_s of defect 0 for $1 \leq s \leq \frac{q-1}{4}$, each of which contains an irreducible ordinary character χ_s of degree q-1.
- (iii) If $m \ge 3$, then H has $\frac{m-1}{2}$ blocks B'_t of defect n-1 for $1 \le t \le \frac{m-1}{2}$, each of which contains 2^{n-1} irreducible ordinary characters ϕ_{t_i} for $1 \le i \le 2^{n-1}$.
- Now assume that $q \equiv 3 \pmod{4}$ and $q+1 = m2^n$, where $2 \nmid m$.
- (iv) The principal block B_0 of H contains $2^{n-2} + 3$ irreducible characters

$$\chi_0 = 1, \ \gamma, \ \eta_1, \ \eta_2, \ \chi_{i_1}, ..., \chi_{i_{(2^{n-2}-1)}},$$

where $\chi_0 = 1$ is the trivial character of H, γ is the irreducible character of degree q of H, η_1 and η_2 are the two irreducible characters of degree $\frac{q-1}{2}$, and χ_{i_k} for $1 \leq k \leq 2^{n-2} - 1$ are distinct irreducible characters of degree $q - \overline{1}$ of H.

- (v) H has $\frac{q-3}{4}$ blocks B_r of defect 0 for $1 \leq r \leq \frac{q-3}{4}$, each of which contains an
- irreducible ordinary character ϕ_r of degree q + 1. (vi) If $m \ge 3$, then H has $\frac{m-1}{2}$ blocks B'_t of defect n-1 for $1 \le t \le \frac{m-1}{2}$, each of which contains 2^{n-1} irreducible ordinary characters χ_{t_i} for $1 \le i \le 2^{n-1}$.

Moreover, the above blocks form all the 2-blocks of H.

Remark 4.3. Parts (i) and (iv) are from Theorem 1.3 in [13] and their proofs can be found in Chapter 7 of III in [2]. Parts (ii) and (v) are special cases of Theorem 3.18 in [14]. Parts (iii) and (vi) are proved in Sections II and VIII of [3].

4.2. Block Idempotents. Let Bl(H) be the set of 2-blocks of H. If $B \in Bl(H)$, we write

$$f_B = \sum_{\chi \in Irr(B)} e_{\chi},$$

where $e_{\chi} = \frac{\chi(1)}{|H|} \sum_{g \in H} \chi(g^{-1})g$ is a central primitive idempotent of $\mathbf{Z}(\mathbb{C}H)$ and Irr(B) = $Irr(H) \cap B$. For future use, we define $IBr(B) = IBr(H) \cap B$. Since f_B is an element of $\mathbf{Z}(\mathbb{C}H)$, we may write

$$f_B = \sum_{C \in cl(H)} f_B(\widehat{C})\widehat{C},$$

where cl(H) is the set of conjugacy classes of H, \hat{C} is the sum of elements in the class C, and

$$f_B(\widehat{C}) = \frac{1}{|H|} \sum_{\chi \in Irr(B)} \chi(1)\chi(x_C^{-1})$$

$$(4.2)$$

with a fixed element $x_C \in C$.

Theorem 4.4. Let $B \in Bl(H)$. Then $f_B \in \mathbf{Z}(\mathbf{S}H)$. In other words, $f_B(\widehat{C}) \in \mathbf{S}$ for each block of H.

Proof: It follows from Corollary 3.8 in [14]. We extend the ring homomorphism $*: \mathbf{S} \to F$ to a ring homomorphism $*: \mathbf{S}H \to F$ FH by setting $(\sum_{g \in H} s_g g)^* = \sum_{g \in H} s_g^* g$. Note that * maps $\mathbf{Z}(\mathbf{S}H)$ onto $\mathbf{Z}(FH)$ via $(\sum_{C \in cl(H)} s_C \widehat{C})^* = \sum_{C \in cl(H)} s_C^* \widehat{C}$. Now we define

$$e_B = (f_B)^* \in \mathbf{Z}(FH),$$

which is the *block idempotent* of *B*. Note that $e_B e_{B'} = \delta_{BB'} e_B$ for *B*, $B' \in Bl(H)$, where $\delta_{BB'}$ equals 1 if B = B', 0 otherwise. Also $1 = \sum_{B \in Bl(H)} e_B$.

All the block idempotents of the 2-blocks of H are given in the following lemma; see [17] for the detailed calculations.

Lemma 4.5. [17, Lemma 4.4] First assume that $q \equiv 1 \pmod{4}$ and $q - 1 = m2^n$ with $2 \nmid m$.

- 1. Let B_0 be the principal block of H. Then
 - (a) $e_{B_0}(D) = 1.$

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- (b) $e_{B_0}(\widehat{F^+}) = e_{B_0}(\widehat{F^-}) \in F.$
- (c) $e_{B_0}([\widehat{\theta_i}]) \in F, e_{B_0}([\widehat{0}]) = 0.$
- (d) $e_{B_0}([\pi_k]) = 1.$
- 2. Let B_s be any block of defect 0 of H. Then
 - (a) $e_{B_s}(\widehat{D}) = 0.$
 - (b) $e_{B_s}(\widehat{F^+}) = e_{B_s}(\widehat{F^-}) = 1.$
 - (c) $e_{B_s}(\widehat{[0]}) = e_{B_s}(\widehat{[\theta_i]}) = 0.$
 - (d) $e_{B_s}([\pi_k]) \in F$.
- 3. Suppose m ≥ 3 and let B'_t be any block of defect n − 1 of H. Then
 (a) e_{B'}(D
 = 0.
 - (b) $e_{B'_t}(\widehat{F^+}) = e_{B'_t}(\widehat{F^-}) = 1.$ (c) $e_{B'_t}([\widehat{\theta_i}]) \in F, e_{B'_t}([\widehat{0}]) = 0.$ (d) $e_{B'_t}([\widehat{\pi_k}]) = 0.$

Now assume that $q \equiv 3 \pmod{4}$. Suppose that $q + 1 = m2^n$ with $2 \nmid m$.

- 4. Let B_0 be the principal block of H. Then
 - (a) $e_{B_0}(\widehat{D}) = 1.$ (b) $e_{B_0}(\widehat{F^+}) = e_{B_0}(\widehat{F^-}) \in F.$
 - (c) $e_{B_0}([\widehat{\theta_i}]) = 1.$

(d)
$$e_{B_0}([0]) = 0, \ e_{B_0}([\pi_k]) \in F$$

- 5. Let B_r be any block of defect 0 of H. Then
 - (a) $e_{B_r}(\widehat{D}) = 0.$ (b) $e_{B_r}(\widehat{F^+}) = e_{B_r}(\widehat{F^-}) = 1.$ (c) $e_{B_r}(\widehat{[0]}) = e_{B_r}(\widehat{[\pi_k]}) = 0.$
 - $(\mathbf{d}) = (\widehat{\mathbf{d}}) = (\widehat{\mathbf{d}})$
 - (d) $e_{B_r}([\theta_i]) \in F.$
- 6. Suppose that $m \ge 3$ and let B'_t be any block of defect n-1 of H. Then (a) $e_{B'_t}(\widehat{D}) = 0$.

(b)
$$e_{B'_t}(F^+) = e_{B'_t}(F^-) = 1.$$

(c)
$$e_{B'_t}([\theta_i]) = 0.$$

(d)
$$e_{B'_{4}}([0]) = 0, \ e_{B'_{4}}([\pi_{k}]) \in F.$$

The following corollary will be used in the proof of Lemma 6.2.

Corollary 4.6. Let B_s $(1 \le s \le \frac{q-1}{4})$ or B_r $(1 \le r \le \frac{q-3}{4})$ be the blocks of defect 0 of H depending on whether $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$. Using the above notation,

- (i) if $q \equiv 1 \pmod{4}$, for each B_s , there is a class $[\pi_k]$ such that $e_{B_s}([\pi_k]) \neq 0$;
- (ii) if $q \equiv 3 \pmod{4}$, for each B_r , there is a class $[\theta_i]$ such that $e_{B_r}([\theta_i]) \neq 0$.

Proof: First we assume that $q \equiv 1 \pmod{4}$. From Theorem 8.9 in [12], we have $\chi_s(g_k) = -\delta^{(2k)s} - \delta^{-(2k)s}$ for $1 \leq k \leq \frac{q-1}{4}$, where χ_s is the irreducible ordinary character lying in $B_s, g_k \in [\pi_k]$, and δ is a primitive (q+1)-th root of unit in \mathbb{C} . Note that

$$f_{B_s}(\widehat{[\pi_k]}) = \frac{1}{|H|} \sum_{\substack{\chi_s \in B_s \\ q = -\frac{q-1}{|H|}}} \chi_s(1) \chi_s(g_k^{-1})$$

Since

$$\sum_{k=1}^{(q-1)/4} e_{B_s}(\widehat{[\pi_k]}) = \left(-\frac{q-1}{|H|} \sum_{\substack{k=1\\ k=1}}^{(q-1)/4} \delta^{(2k)s} + \delta^{-(2k)s}\right)^* \\ = \left(\frac{\delta^{2s} - \delta^{\frac{q+3}{2}s}}{1 - \delta^{2s}} + \frac{\delta^{-2s} - \delta^{-\frac{q+3}{2}s}}{1 - \delta^{-2s}}\right)^* \\ = \left(\frac{\delta^{2s} - \delta^{\frac{q+3}{2}s}}{1 - \delta^{2s}} + \frac{\delta^{-2s} - \delta^{\frac{q-1}{2}s}}{1 - \delta^{-2s}}\right)^* \\ = \left(\frac{\delta^{2s} - \delta^{\frac{q+3}{2}s}}{1 - \delta^{2s}} + \frac{1 - \delta^{\frac{q+3}{2}s}}{\delta^{2s} - 1}\right)^* \\ = 1,$$

we conclude that $e_{B_s}([\widehat{\pi_k}]) \neq 0$ for some k. Part (i) is proved.

Part (ii) can be proved in the same fashion using Theorem 8.11 in [12]; we omit the details. $\hfill \Box$

Let M be an **S**H-module. We denote the reduction $M/\mathcal{P}M$, which is an FH-module, by \overline{M} . Then the following lemma is apparent.

Lemma 4.7. Let M be an SH-module and $B \in Bl(H)$. Using the above notation, we have

$$\overline{f_B M} = e_B \overline{M}$$

i.e. reduction commutes with projection onto a block B.

5. Linear Maps and Their Matrices

Let F be the algebraic closure of \mathbb{F}_2 defined in Section 4. From now on, χ_N for $N \subseteq E$ will be always regarded as a vector over F. Recall that for $\mathbf{P} \in I$, $N_{Pa,E}(\mathbf{P})$ (respectively, $N_{Se,E}(\mathbf{P})$) is the set of external points on the passant (respectively, secant) lines through \mathbf{P} . We define \mathbf{D} (respectively, \mathbf{D}') to be the incidence matrix of E and $N_{Pa,E}(\mathbf{P})$ (respectively, $N_{Se,E}(\mathbf{P})$) for $\mathbf{P} \in I$. Namely, the columns of \mathbf{D} and \mathbf{D}' can be viewed as the characteristic vectors of $N_{Pa,E}(\mathbf{P})$ and $N_{Se,E}(\mathbf{P})$, respectively. In the following, we always regard both \mathbf{D} and \mathbf{D}' as matrices over F.

Definition 5.1. For $\mathbf{P} \in I$, we define $\mathcal{G}_{\mathbf{P}}$ to be the column characteristic vector of \mathbf{P} with respect to I, i.e. $\mathcal{G}_{\mathbf{P}}$ is a 0-1 column vector of length |I| with entries indexed by the internal points; the entry of $\mathcal{G}_{\mathbf{P}}$ is 1 if and only if it is indexed by \mathbf{P} .

Let k be the complex field \mathbb{C} , the algebraic closure F of \mathbb{F}_2 , or the ring **S** in (4.1). Let k^I and k^E be the free k-modules with the bases $\{\mathcal{G}_{\mathbf{P}} \mid \mathbf{P} \in I\}$ and $\{\chi_{\mathbf{P}} \mid \mathbf{P} \in E\}$, respectively. If we extend the actions of H on the bases of k^I and k^E , which are defined by $\chi_{\mathbf{P}} \cdot h = \chi_{\mathbf{P}^h}$ and $\mathcal{G}_{\mathbf{Q}} \cdot h = \mathcal{G}_{\mathbf{Q}^h}$ for $\mathbf{P} \in I$, $\mathbf{Q} \in E$, and $h \in H$, linearly to k^I and k^E respectively, then both k^I and k^E are kH-permutation modules. Since H is transitive on I, we have

$$k^I = \operatorname{Ind}_K^H(1_k),$$

where K is the stabilizer of an element of I in H and $\operatorname{Ind}_{K}^{H}(1_{k})$ is the kH-module induced by 1_{k} .

Lemma 5.2. [19, Lemma 5.2] Let K be the stabilizer of an internal point in H.

Assume that $q \equiv 1 \pmod{4}$. Let χ_s , $1 \leq s \leq \frac{q-1}{4}$, be the irreducible ordinary characters of degree q - 1, ϕ_r , $1 \leq r \leq \frac{q-5}{4}$, irreducible ordinary characters of degree q + 1, γ the irreducible of degree q, and β_j , $1 \leq j \leq 2$, irreducible ordinary characters of degree $\frac{q+1}{2}$. (i) If $q \equiv 1 \pmod{8}$, then

$$1\uparrow_{K}^{H} = 1 + \sum_{s=1}^{(q-1)/4} \chi_{s} + \gamma + \beta_{1} + \beta_{2} + \sum_{j=1}^{(q-9)/4} \phi_{r_{j}},$$

where ϕ_{r_j} , $1 \le j \le \frac{q-9}{4}$, may not be distinct. (ii) If $q \equiv 5 \pmod{8}$, then

$$1\uparrow_{K}^{H} = 1 + \sum_{s=1}^{(q-1)/4} \chi_{s} + \gamma + \sum_{j=1}^{(q-5)/4} \phi_{r_{j}},$$

where ϕ_{r_j} , $1 \leq j \leq \frac{q-5}{4}$, may not be distinct.

Next assume that $q \equiv 3 \pmod{4}$. Let χ_s , $1 \leq s \leq \frac{q-3}{4}$, be the irreducible ordinary characters of degree q - 1, ϕ_r , $1 \leq r \leq \frac{q-3}{4}$, the irreducible ordinary characters of degree q + 1, γ the irreducible character of degree q, and η_j , $1 \leq j \leq 2$, the irreducible ordinary characters of degree $\frac{q-1}{2}$.

(iii) If $q \equiv 3 \pmod{8}$, then

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$$1\uparrow_{K}^{H} = 1 + \sum_{r=1}^{(q-3)/4} \phi_{r} + \eta_{1} + \eta_{2} + \sum_{j=1}^{(q-3)/4} \chi_{s_{j}},$$

where χ_{s_j} , $1 \leq j \leq \frac{q-3}{4}$, may not be distinct. (iv) If $q \equiv 7 \pmod{8}$, then

$$1\uparrow_{K}^{H}=1+\sum_{r=1}^{(q-3)/4}\phi_{r}+\sum_{j=1}^{(q+1)/4}\chi_{s_{j}},$$

where χ_{s_j} , $1 \leq j \leq \frac{q+1}{4}$, may not be distinct.

Corollary 5.3. Using the above notation,

- (i) if $q \equiv 1 \pmod{4}$, then the character of $Ind_K^H(1_{\mathbb{C}}) \cdot f_{B_s}$ is χ_s for each block B_s of defect 0;
- (ii) if $q \equiv 3 \pmod{4}$, then the character of $Ind_K^H(1_{\mathbb{C}}) \cdot f_{B_r}$ is ϕ_r for each block B_r of defect 0.

Proof: The corollary follows from Lemma 4.2 and Lemma 5.2.

Since H preserves incidence, the following corollary is obvious.

Corollary 5.4. Let $\mathbf{P} \in I$. Using the above notation, we have

 $\chi_{N_{Pa,E}(\mathbf{P})} \cdot h = \chi_{N_{Pa,E}(\mathbf{P}^h)}, \chi_{N_{Se,E}(\mathbf{P})} \cdot h = \chi_{N_{Se,E}(\mathbf{P}^h)}$

for $h \in H$.

In the rest of the article, we always view $\mathcal{G}_{\mathbf{P}}$ as a vector over F. Consider the maps $\phi_{\mathbf{B}}$, $\phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}'}$ from F^I to F^E defined by extending

$$\mathcal{G}_{\mathbf{P}} \mapsto \chi_{\mathbf{P}^{\perp}}, \mathcal{G}_{\mathbf{P}} \mapsto \chi_{N_{Pa,E}(\mathbf{P})}, \mathcal{G}_{\mathbf{P}} \mapsto \chi_{N_{Se,E}(\mathbf{P})}$$

linearly to F^{I} , respectively. Then it is clear that as F-linear maps, the marices of $\phi_{\mathbf{B}}, \phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}'}$ are \mathbf{B}, \mathbf{D} , and \mathbf{D}' , respectively, and for $\mathbf{x} \in F^{I}, \phi_{\mathbf{B}}(\mathbf{x}) = \mathbf{B}\mathbf{x}, \phi_{\mathbf{D}}(\mathbf{x}) = \mathbf{D}\mathbf{x}$ and $\phi_{\mathbf{D}'}(\mathbf{x}) = \mathbf{D}'\mathbf{x}$. Moreover, we have the following result.

Lemma 5.5. The maps $\phi_{\mathbf{B}}$, $\phi_{\mathbf{D}}$, and $\phi_{\mathbf{D}'}$ are all FH-module homomorphisms from F^I to F^E .

Proof: Let $\mathcal{G}_{\mathbf{P}}$ be a basis element of F^{I} . Then $\phi(\mathcal{G}_{\mathbf{P}} \cdot h) = \phi(\mathcal{G}_{\mathbf{P}}) \cdot h$ since

$$\phi_{\mathbf{B}}(\mathcal{G}_{\mathbf{P}} \cdot h) = \chi_{(\mathbf{P}^{h})^{\perp}} = \chi_{(\mathbf{P}^{\perp})^{h}} = \chi_{\mathbf{P}^{\perp}} \cdot h = \phi_{\mathbf{B}}(\mathcal{G}_{\mathbf{P}}) \cdot h.$$

By linearity of $\phi_{\mathbf{B}}$, we have $\phi_{\mathbf{B}}(\mathbf{x}) \cdot h = \phi_{\mathbf{B}}(\mathbf{x} \cdot h)$ for each $\mathbf{x} \in F^{I}$. The proof of the map $\phi_{\mathbf{B}}$ being *FH*-homomorphism is completed.

The proofs of the other two maps being homomorphisms are similar since

$$\chi_{N_{Pa,E}(\mathbf{P})} \cdot h = \chi_{N_{Pa,E}(\mathbf{P}^h)}, \chi_{N_{Se,E}(\mathbf{P})} \cdot h = \chi_{N_{Se,E}(\mathbf{P}^h)}$$

for $h \in H$ and $\mathbf{P} \in I$ by Corollary 5.4. We omit the details.

For convenience, we use $\operatorname{col}_F(\mathbf{C})$ to denote the column space of the matrix \mathbf{C} over F.

Corollary 5.6. Using the above notation, we have $\operatorname{Im}(\phi_{\mathbf{B}}) = \operatorname{col}_F(\mathbf{B})$, $\operatorname{Im}(\phi_{\mathbf{D}}) = \operatorname{col}_F(\mathbf{D})$, and $\operatorname{Im}(\phi_{\mathbf{D}'}) = \operatorname{col}_F(\mathbf{D}')$.

Now we define $\mathcal{M}_1 := \langle \chi_\ell \mid \ell \in T \rangle_F$ and $\mathcal{M}_2 := \langle \chi_{\ell_i} + \chi_{\ell_j} \mid \ell_i \neq \ell_j \in T \rangle_F$ to be the spans of the corresponding characteristic vectors over F.

Lemma 5.7. The dimensions of \mathcal{M}_1 and \mathcal{M}_2 over F are $\dim_F(\mathcal{M}_1) = q$ and $\dim_F(\mathcal{M}_2) = q - 1$, respectively. Moreover, the all-one column vector **1** of length |E| is neither in \mathcal{M}_1 nor in \mathcal{M}_2 .

Proof: Since $\sum_{\ell \in T} \chi_{\ell} = \mathbf{0}$, where **0** is the zero column vector of |E|, it follows that $\{\chi_{\ell} \mid \ell \in T\}$ is linearly dependent over F, i.e. $\dim_F(\mathcal{M}_1) \leq q$. Now let $T' \subset T$ with |T'| = q and suppose that $\{\chi_{\ell} \mid \ell \in T'\}$ is linearly dependent over F. Then $\sum_{\ell \in T'} a_{\ell}\chi_{\ell} = \mathbf{0}$, where $a_{\ell} \in F$ and $a_{\ell_1} \neq 0$ for some $\ell_1 \in T'$. Since there are q external points on ℓ_1 and there are only q-1 tangent lines other than ℓ_1 in T', some external point on ℓ_1 must be passed only by ℓ_1 among the tangent lines in T', which forces $a_{\ell_1} = 0$, a contradiction. This shows that T' must be linearly independent over F, and so $\dim_F(\mathcal{M}_1) = q$. Moreover, if $T' \subset T$ and |T'| = q, then $\{\chi_{\ell} \mid \ell \in T'\}$ must be a basis for \mathcal{M}_1 .

Next if ℓ_1 is a tangent line, then $\mathcal{M}_2 = \langle \chi_{\ell_1} + \chi_{\ell} \mid \ell \in T \setminus \{\ell_1\}\rangle_F$ since $\chi_{\ell_i} + \chi_{\ell_j} = (\chi_{\ell_1} + \chi_{\ell_i}) + (\chi_{\ell_1} + \chi_{\ell_j})$. As $\sum_{\ell \in T \setminus \{\ell_1\}} (\chi_{\ell_1} + \chi_{\ell}) = \mathbf{0}$, $\dim_F(\mathcal{M}_2) \leq q-1$. Let $T' \subset T \setminus \{\ell_1\}$ with |T'| = q-1 and suppose that $\{\chi_{\ell_1} + \chi_{\ell} \mid \ell \in T'\}$ is linearly dependent over F. Then $\sum_{\ell \in T'} a_\ell(\chi_{\ell_1} + \chi_{\ell}) = \sum_{\ell \in T'} a_\ell\chi_\ell = \mathbf{0}$ since |T'| is even, where $a_\ell \in F$ and $a_{\ell_2} \neq 0$ for some $\ell_2 \in T'$. By applying the same argument in the first paragraph of this proof, again, we obtain that $a_{\ell_2} = 0$ which is a contradiction. Therefore, $\{\chi_{\ell_1} + \chi_{\ell} \mid \ell \in T'\}$ is linearly independent over F, and so $\dim_F(\mathcal{M}_2) = q-1$. Moreover, if $T' \subset T \setminus \{\ell_1\}$ and |T'| = q-1, then $\{\chi_{\ell_1} + \chi_{\ell} \mid \ell \in T'\}$ must be a basis for \mathcal{M}_2 .

Now we assume that $\mathbf{1} \in \mathcal{M}_1$ and $\{\chi_\ell \mid \ell \in T'\}$ with $T' \subset T$ and |T'| = q is a basis for \mathcal{M}_1 . Then $\sum_{\ell \in T'} a_\ell \chi_\ell = \mathbf{1}$, where $a_\ell \in F$ for $\ell \in T'$ and $a_{\ell_k} \neq 0$ for some $\ell_k \in T'$. Since $|T' \setminus \{\ell_k\}| = q - 1$, some external point on ℓ_k must be only passed by ℓ_k among all

Lemma 5.8. If $q \equiv 1 \pmod{4}$, then $\operatorname{col}_F(\mathbf{D}) = \mathcal{M}_1$; if $q \equiv 3 \pmod{4}$, then $\operatorname{col}_F(\mathbf{D}) =$ \mathcal{M}_2 .

Proof: Assume that $q \equiv 1 \pmod{4}$. Let $\chi_{N_{Pa,E}(\mathbf{P})}$ be the column of **D** indexed by **P**. Then $\chi_{N_{Pa,E}(\mathbf{P})}$ is an *F*-linear combination of the generating elements of \mathcal{M}_1 by Corollary 2.17. Now if χ_{ℓ} is a generating element of \mathcal{M}_1 , then it is an *F*-linear combination of the columns of **D** by Corollary 2.21. Therefore, $\operatorname{col}_F(\mathbf{D}) = \mathcal{M}_1$.

Now we assume that $q \equiv 3 \pmod{4}$. Let $\chi_{N_{Pa,E}(\mathbf{P})}$ be the column of **D** indexed by **P**. Suppose that $\ell(\mathbf{P})$ is a tangent line through an external point on \mathbf{P}^{\perp} and $T(\mathbf{P}, \ell(\mathbf{P}))$ is the set of tangent lines through the external points on $\ell(\mathbf{P})$ that are also on the passant lines through **P**. Then by Corollary 2.17 and the fact that $|T(\mathbf{P}, \ell(\mathbf{P}))| = \frac{q+1}{2}$ is even, we have

$$\chi_{N_{Pa,E}(\mathbf{P})} = \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} \chi_{\ell}$$
$$= \sum_{\ell \in T(\mathbf{P}, \ell(\mathbf{P}))} (\chi_{\ell} + \chi_{\ell(\mathbf{P})});$$

that is, $\chi_{N_{Pa,E}(\mathbf{P})} \in \mathcal{M}_2$. Now let $\chi_{\ell_1} + \chi_{\ell_2}$ be a generating element of \mathcal{M}_2 . Then we have

$$\chi_{\ell_1} + \chi_{\ell_2} = \sum_{\mathbf{Q} \in \mathcal{M}'(\mathbf{P})} \chi_{N_{Pa,E}(\mathbf{Q})}$$

by Corollary 2.24, where $\mathbf{P} = \ell_1 \cap \ell_2$. Hence, $\operatorname{col}_F(\mathbf{D}) = \mathcal{M}_2$.

Corollary 5.9. If $q \equiv 1 \pmod{4}$, $\operatorname{rank}_2(\mathbf{D}) = q$; if $q \equiv 3 \pmod{4}$, $\operatorname{rank}_2(\mathbf{D}) = q - 1$.

It follows from Lemmas 5.7 and 5.8. **Proof:**

Further, we have the following decomposition of $\operatorname{col}_F(\mathbf{D}')$.

Lemma 5.10. If $q \equiv 3 \pmod{4}$, then $\operatorname{col}_F(\mathbf{D}') = \langle \mathbf{1} \rangle \oplus \operatorname{col}_F(\mathbf{D})$ as FH-modules, where $\langle 1 \rangle$ is the trivial FH-module generated by the all-one column vector 1.

Since each row of \mathbf{D}' has $\frac{(q-1)^2}{4}$ 1s, then **Proof:**

$$\sum_{\mathbf{P}\in I}\chi_{N_{Se,E}(\mathbf{P})}=\mathbf{1}.$$

For $h \in H$,

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$$\mathbf{1} \cdot h = \left(\sum_{\mathbf{P} \in I} \chi_{N_{Se,E}(\mathbf{P})}\right) \cdot h = \sum_{\mathbf{P} \in I} \chi_{N_{Se,E}(\mathbf{P}^h)} = \sum_{\mathbf{P} \in I} \chi_{N_{Se,E}(\mathbf{P})} = \mathbf{1} \in \operatorname{col}_F(\mathbf{D}).$$

Consequently, $\langle \mathbf{1} \rangle$ is indeed a trivial submodule of $\operatorname{col}_F(\mathbf{D}')$.

It is clear that $\operatorname{col}_F(\mathbf{D}') = \langle \mathbf{1} \rangle + \operatorname{col}_F(\mathbf{D})$ since $\chi_{N_{Se,E}(\mathbf{P})} \in \operatorname{col}_F(\mathbf{D}')$ if and only if $\chi_{N_{Se,E}(\mathbf{P})} = \mathbf{1} + \chi_{N_{Pa,E}(\mathbf{P})} \in \langle \mathbf{1} \rangle + \operatorname{col}_F(\mathbf{D}).$ Further, $\langle \mathbf{1} \rangle \cap \operatorname{col}_F(\mathbf{D}) = \mathbf{0}$ since $\operatorname{col}_F(\mathbf{D}) = \mathcal{M}_2$ and $\mathbf{1} \notin \mathcal{M}_2$ by Lemmas 5.7 and 5.8. Therefore, $\operatorname{col}_F(\mathbf{D}') = \langle \mathbf{1} \rangle \oplus \operatorname{col}_F(\mathbf{D})$.

 \square

6. Statement and Proof of Main Theorem

The main theorem is given as follows.

Theorem 6.1. Let $Im(\phi_{\mathbf{B}})$ and $Im(\phi_{\mathbf{D}})$ be defined as above. As FH-modules,

(i) if $q \equiv 1 \pmod{4}$, then

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \operatorname{Im}(\phi_{\mathbf{D}}) \oplus (\bigoplus_{s=1}^{(q-1)/4} M_s),$$

where M_s for $1 \leq s \leq \frac{q-1}{4}$ are pairwise non-isomorphic simple FH-modules of dimension q-1;

(ii) if $q \equiv 3 \pmod{4}$, then

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \langle \mathbf{1} \rangle \oplus \operatorname{Im}(\phi_{\mathbf{D}}) \oplus (\bigoplus_{r=1}^{(q-3)/4} M_r),$$

where M_r for $1 \leq s \leq \frac{q-3}{4}$ are pairwise non-isomorphic simple FH-modules of dimension q+1 and $\langle 1 \rangle$ is the trivial FH-module generated by the all-one column vector of length |E|.

To prove the main theorem, we need the following lemma.

Lemma 6.2. Let $q - 1 = 2^n m$ or $q + 1 = 2^n m$ with $2 \nmid m$ depending on whether $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$. Using the above notation,

- (i) if $q \equiv 1 \pmod{4}$, then $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_0} = \operatorname{Im}(\phi_{\mathbf{D}})$, $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_s} \neq \mathbf{0}$ for $1 \leq s \leq \frac{q-1}{4}$, and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_t} = \mathbf{0}$ for $m \geq 3$ and $1 \leq t \leq \frac{m-1}{2}$;
- (ii) if $q \equiv 3 \pmod{4}$, then $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_0} = \operatorname{Im}(\phi_{\mathbf{D}'})$, $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_r} \neq \mathbf{0}$ for $1 \leq r \leq \frac{q-3}{4}$, and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_t} = \mathbf{0}$ for $m \geq 3$ and $1 \leq t \leq \frac{m-1}{2}$.

Proof: It is clear that $\operatorname{Im}(\phi_{\mathbf{B}})$ is generated by $\{\chi_{\mathbf{P}^{\perp}} \mid \mathbf{P} \in I\}$ over F. Let $B \in Bl(H)$. Since

$$\begin{split} \chi_{\mathbf{P}^{\perp}} \cdot e_B &= \sum_{C \in cl(H)} e_B(\widehat{C}) \sum_{h \in C} \chi_{\mathbf{P}^{\perp}} \cdot h \\ &= \sum_{C \in cl(H)} e_B(\widehat{C}) \sum_{h \in C} \chi_{(\mathbf{P}^{\perp})^h}, \\ &= \sum_{C \in cl(H)} e_B(\widehat{C}) \sum_{h \in C} \sum_{\mathbf{Q} \in (\mathbf{P}^{\perp})^h \cap E} \chi_{\mathbf{Q}}, \end{split}$$

we have

$$\chi_{\mathbf{P}^{\perp}} \cdot e_B = \sum_{\mathbf{Q} \in I} \mathcal{S}(B, \mathbf{P}, \mathbf{Q}) \chi_{\mathbf{Q}},$$

where

$$S(B, \mathbf{P}, \mathbf{Q}) := \sum_{C \in cl(H)} |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap C| e_B(\widehat{C}).$$

Assume first that $q \equiv 1 \pmod{4}$. If $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$, then $S(B,\mathbf{P},\mathbf{Q}) = 0$ for each $B \in Bl(H)$ since $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = 0$ in F for each $C \neq [0]$ by Lemma 3.6(i) and $e_{B_0}(\widehat{[0]}) = e_{B_s}(\widehat{[0]}) = e_{B_s'}(\widehat{[0]}) = 0$ by 1(c), 2(c), 3(c) of Lemma 4.5.

If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then by Lemma 3.6(ii) and 1(a), 1(c), 2(a), 2(c), 3(a), 3(c) of Lemma 4.5,

$$\begin{aligned} S(B_0, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_0}([0]) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B_0}(\widehat{D}) &= 0 + 1 = 1, \\ S(B_s, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_s}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B_s}(\widehat{D}) &= 0 + 0 = 0, \\ S(B'_t, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B'_t}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B'_t}(\widehat{D}) &= 0 + 0 = 0. \end{aligned}$$

If **Q** is on a passant line ℓ through **P** and **Q** $\notin \mathbf{P}^{\perp}$, then by Lemma 3.6(iii) and 1(c), 1(d), 2(c), 2(d), 3(c), 3(d) of Lemma 4.5,

By Lemma 3.6(iii) and the fact that there are $\frac{q-1}{4}$ classes of the form $[\pi_k]$ and there are $\frac{q-1}{2}$ points on ℓ that are not on \mathbf{P}^{\perp} , we have that for each $[\pi_k]$ there exist two external points \mathbf{Q}_1 and \mathbf{Q}_2 on ℓ such that $|\mathcal{H}_{\mathbf{P},\mathbf{Q}_j} \cap [\pi_k]|$ (j = 1 or 2) is odd and for each $\mathbf{Q} \in \ell$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$ there is a class $[\pi_k]$ such that $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap [\pi_k]|$ is odd. Combining the above analysis with Lemma 4.6, we obtain that for each B_s , there is a Q and a class $[\pi_k]$ such that $S(B_s, \mathbf{P}, \mathbf{Q}) = e_{B_s}(\widehat{[\pi_k]}) \neq 0$.

Therefore, we have shown that $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_0} = \operatorname{Im}(\phi_{\mathbf{D}})$ by definition, $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_s} \neq \mathbf{0}$ for each s, and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_s} = \mathbf{0}$. The proof of (i) is completed.

Now assume that $q \equiv 3 \pmod{4}$. If $\ell_{\mathbf{P},\mathbf{Q}} \in Pa_{\mathbf{P}}$, then $S(B,\mathbf{P},\mathbf{Q}) = 0$ for each $B \in Bl(H)$ since $|\mathcal{H}_{\mathbf{P},\mathbf{Q}} \cap C| = 0$ by 4(d), 5(c), 6(d) of Lemma 4.5.

Let $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ and $\mathbf{Q} \in \mathbf{P}^{\perp}$, then by Lemma 3.7(ii) and 4(a), 4(d), 5(a), 5(c), 5(d) of Lemma 4.5

$$\begin{split} S(B_0, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_0}([0]) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B_0}(\widehat{D}) &= 0 + 1 = 1, \\ S(B_s, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_s}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B_s}(\widehat{D}) &= 0 + 0 = 0, \\ S(B'_t, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B'_t}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap D|e_{B'_t}(\widehat{D}) &= 0 + 0 = 0. \end{split}$$

If $\ell_{\mathbf{P},\mathbf{Q}} \in Se_{\mathbf{P}}$ and $\mathbf{Q} \notin \mathbf{P}^{\perp}$, then by Lemma 3.7(i), 4(c), 4(d), 5(c), 5(d), 6(c), 6(d) of Lemma 4.5,

$$\begin{aligned} S(B_0, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_0}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [\theta_i]|e_{B_0}(\widehat{[\theta_i]}) &= 1, \\ S(B_s, \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_s}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [\theta_i]|e_{B_s}(\widehat{[\pi_k]}) &= e_{B_s}(\widehat{[\theta_i]}) \\ S(B_t', \mathbf{P}, \mathbf{Q}) &= |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [0]|e_{B_t'}(\widehat{[0]}) + |\mathcal{H}_{\mathbf{P}, \mathbf{Q}} \cap [\theta_i]|e_{B_t'}(\widehat{[\theta_i]}) &= 0. \end{aligned}$$

From Lemma 3.7(i) and Lemma 4.6, we have that for each B_s , there is a **Q** and a class $[\theta_i]$ such that $S(B_s, \mathbf{P}, \mathbf{Q}) = e_{B_s}(\widehat{[\theta_i]}) \neq 0$.

Therefore, we have shown that $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_0} = \operatorname{Im}(\phi_{\mathbf{D}'})$ by definition, $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_s} \neq \mathbf{0}$ for each s, and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_t} = \mathbf{0}$. The proof of (ii) is completed.

Proof of Theorem 6.1: Let B be a 2-block of defect 0 of H. Then by Lemma 4.7, we have

$$F^I \cdot e_B = \overline{\mathbf{S}^I \cdot f_B}.$$

Therefore, by Corollary 5.3, $F^I \cdot e_b = N$, where N is the simple FH-module of dimension q-1 or q+1 lying in B accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$. It is clear that $\phi_{\mathbf{B}}(F^I) = \operatorname{Im}(\phi_{\mathbf{B}})$.

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Assume that $q \equiv 1 \pmod{4}$ and $q - 1 = m2^n$ with $2 \nmid m$. Since

$$1 = e_{B_0} + \sum_{s=1}^{(q-1)/4} e_{B_s} + \sum_{t=1}^{(m-1)/2} e_{B'_t},$$

we have

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_{0}} \oplus \left(\bigoplus_{s=1}^{(q-1)/4} \operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_{s}} \right) \oplus \left(\bigoplus_{t=1}^{(m-1)/2} \operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_{t}'} \right) \\
= \operatorname{Im}(\phi_{\mathbf{D}}) \oplus \left(\bigoplus_{s=1}^{(q-1)/4} \phi_{\mathbf{B}}(F^{I}) \cdot e_{B_{s}} \right) \\
= \operatorname{Im}(\phi_{\mathbf{D}}) \oplus \left(\bigoplus_{s=1}^{s=1/4} \phi_{\mathbf{B}}(F^{I} \cdot e_{B_{s}}) \right) \tag{6.1}$$

$$= \operatorname{Im}(\phi_{\mathbf{D}}) \oplus \left(\bigoplus_{s=1}^{(q-1)/4} \phi_{\mathbf{B}}(N_{s}) \right) \\
= \operatorname{Im}(\phi_{\mathbf{D}}) \oplus \left(\bigoplus_{s=1}^{s=1/4} M_{s} \right), \tag{6.1}$$

where N_s is the simple module of dimension q-1 lying in B_s for each s by the discussion in the first paragraph and $M_s := \phi_{\mathbf{B}}(N_s)$ for each s. In (6.1), the terms $e_{B'_t}$ for $1 \le t \le \frac{m-1}{2}$ and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_t}$ for $1 \le t \le \frac{m-1}{2}$ appear only when $m \ge 3$; the second equality holds since $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B'_t} = \mathbf{0}$ for each t and $\operatorname{Im}(\phi_{\mathbf{B}}) \cdot e_{B_0} = \operatorname{Im}(\phi_{\mathbf{D}})$ by Lemma 6.2(i); and the third equality holds since $\phi_{\mathbf{B}}$ is an FH-homomorphism by Lemma 5.5 and $e_{B_s} \in FH$. Consider the map

$$\lambda_S: N_S \to \phi_{\mathbf{B}}(N_s)$$

defined by $\lambda_s(n) = \phi_{\mathbf{B}}(n)$ for $n \in N_s$, where $1 \leq s \leq \frac{q-1}{4}$. It is clear that λ_s is the same as the restriction of $\phi_{\mathbf{B}}$ to N_s . Consequently, λ_s is a surjective *FH*-homomorphism. Moreover, Ker (λ_s) is either **0** or N_s since, otherwise, Ker (λ_s) would be a non-trivial submodule of N_s which is impossible. If Ker $(\lambda_s) = N_s$, then $\phi_{\mathbf{B}}(N_s) = \phi_{\mathbf{B}}(F^I) \cdot e_{B_s} = \mathbf{0}$, which is not the case by Lemma 6.2(i). Thus, we must have Ker $(\lambda_s) = \mathbf{0}$; that is, λ_s is an *FH*-isomorphism. So we have shown that $M_s := \text{Im}(N_s) \cong N_s$ and thus M_s for $1 \leq s \leq \frac{q-1}{4}$ are pairwise non-isomorphic simple modules of dimension q - 1. The proof of (i) is finished.

Now assume that $q \equiv 3 \pmod{4}$. Applying the same argument as above, we have

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \operatorname{Im}(\phi_{\mathbf{D}'}) \oplus (\bigoplus_{r=1}^{(q-3)/4} M_r),$$

where M_r for $1 \le r \le \frac{q-3}{4}$ are pairwise non-isomorphic simple *FH*-modules of dimension q+1. Since $\operatorname{Im}(\phi_{\mathbf{D}'}) = \langle \mathbf{1} \rangle \oplus \operatorname{Im}(\phi_{\mathbf{D}})$ by Lemma 5.10, it follows that

$$\operatorname{Im}(\phi_{\mathbf{B}}) = \langle \mathbf{1} \rangle \oplus \operatorname{Im}(\phi_{\mathbf{D}}) \oplus (\bigoplus_{r=1}^{(q-3)/4} M_r).$$

Now Conjecture 1.1 follows as a corollary.

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Corollary 6.3. Let \mathcal{L} and \mathcal{L}_0 be the \mathbb{F}_2 -null spaces of \mathbf{B} and \mathbf{B}_0 , respectively. Then

$$\dim_{\mathbb{F}_2}(\mathcal{L}) = \begin{cases} \frac{q^2 - 1}{4} - q, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q^2 - 1}{4} - q + 1, & \text{if } q \equiv 3 \pmod{4} \end{cases}$$

and

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$$\dim_{\mathbb{F}_2}(\mathcal{L}_0) = \begin{cases} \frac{q^2 - 1}{4}, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q^2 - 1}{4} + 1, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Proof: From Theorem 6.1 and Corollary 5.9, it follows that the 2-rank of **B** is

$$\operatorname{rank}_2(\mathbf{B}) = q + \frac{(q-1)^2}{4}$$

or

rank₂(**B**) = 1 + (q - 1) +
$$\frac{(q - 1)^2}{4}$$

accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$. Therefore, the dimension of the \mathbb{F}_2 -null space of **B** is

$$\dim_{\mathbb{F}_2}(\mathcal{L}) = \frac{q(q-1)}{2} - (q + \frac{(q-1)^2}{4}) = \frac{q^2 - 1}{4} - q$$

or

$$\dim_{\mathbb{F}_2}(\mathcal{L}) = \frac{q(q-1)}{2} - (1 + (q-1) + \frac{(q-1)(q-3)}{4}) = \frac{q^2 - 1}{4} - q + 1$$

accordingly as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$.

Since $\operatorname{rank}_2(\mathbf{B}) = \operatorname{rank}_2(\mathbf{B}_0)$, the dimension of \mathcal{L}_0 can be calculated in the same way. We omit the details.

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