# On the Geil-Matsumoto Bound and the Length of AG Codes 

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#### Abstract

The Geil-Matsumoto bound conditions the number of rational places of a function field in terms of the Weierstrass semigroup of any of the places. Lewittes' bound preceded the Geil-Matsumoto bound and it only considers the smallest generator of the numerical semigroup. It can be derived from the Geil-Matsumoto bound and so it is weaker. However, for general semigroups the Geil-Matsumoto bound does not have a closed formula and it may be hard to compute, while Lewittes' bound is very simple. We give a closed formula for the Geil-Matsumoto bound for the case when the Weierstrass semigroup has two generators. We first find a solution to the membership problem for semigroups generated by two integers and then apply it to find the above formula. We also study the semigroups for which Lewittes's bound and the Geil-Matsumoto bound coincide. We finally investigate on some simplifications for the computation of the GeilMatsumoto bound.


Key words: Algebraic function field, Weierstrass semigroup, Geil-Matsumoto bound, gonality bound, Lewittes' bound.

## 1 Introduction

Given $n$ pairwise distinct places $P_{1}, \ldots, P_{n}$ of degree one of an algebraic function field $F / \mathbb{F}_{q}$, and a divisor $G$ with support disjoint from $\left\{P_{1}, \ldots, P_{n}\right\}$, the geometric Goppa code $C_{P_{1}, \ldots, P_{n}}(G)$ is defined by $\left\{\left(f\left(P_{1}\right), \ldots, f\left(P_{n}\right)\right): f \in L(G)\right\}$. See $\left[9\right.$ for a general reference. Then, the length of $C_{P_{1}, \ldots, P_{n}}(G)$ is $n$ and it is bounded by the number of places of degree one of $F / \mathbb{F}_{q}$. Thus, an important problem of algebraic coding theory is bounding the number of places of degree one of function fields.

The Hasse-Weil bound for the number of places of degree one of a function field as well as Serre's improvement use only the genus of the function field and the field size. Geil and Matsumoto give in 4] a bound in terms of the Weierstrass semigroup of a rational place (i.e. the set of pole orders of rational functions having only poles in that place). It is a neat formula although it is
not closed and it may be computationally hard to calculate. Lewittes' bound [7, also called the gonality bound, preceded the Geil-Matsumoto bound and it only considers the smallest generator of the numerical semigroup. It can be derived from the Geil-Matsumoto bound and so it is weaker. The advantage of Lewittes' bound with respect to the Geil-Matsumoto bound is that Lewittes' bound is very simple to compute.

Important curves such as hyperelliptic curves, Hermitian curves or Geil's norm-trace curves [3] have Weierstrass semigroups generated by two integers. Also, for any numerical semigroup $\Lambda$ generated by two coprime integers, one can get the equation of a curve having a place whose Weierstrass semigroup is $\Lambda$ 5].

In Section 2, we give some notions on numerical semigroups and solve the membership problem for numerical semigroups generated by two coprime integers. Then in Section 3 we use the result in Section 2 to deduce a closed formula for the Geil-Matsumoto bound when the Weierstrass semigroup is generated by two integers. In Section 4 we return to semigroups generated by any number of integers and study in which cases Lewittes' bound and the Geil-Matsumoto bound coincide. In Section 5we give a result that may simplify the computation of the Geil-Matsumoto bound.

## 2 Membership for semigroups with two generators

Let $\mathbb{N}_{0}$ be the set of non-negative integers. A numerical semigroup is a subset of $\mathbb{N}_{0}$ containing 0 , closed under addition and with finite complement in $\mathbb{N}_{0}$. A general reference for numerical semigroups is [8]. For a numerical semigroup $\Lambda$ define the genus of $\Lambda$ as the number $g=\#\left(\mathbb{N}_{0} \backslash \Lambda\right)$. The elements in $\Lambda$ are called the non-gaps of $\Lambda$ while the elements in $\mathbb{N}_{0} \backslash \Lambda$ are called the gaps of $\Lambda$.

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps. If $a_{1}, \ldots, a_{l}$ are the generators of a semigroup $\Lambda$ then $\Lambda=\left\{n_{1} a_{1}+\cdots+n_{l} a_{l}: n_{1}, \ldots, n_{l} \in \mathbb{N}_{0}\right\}$ and so $a_{1}, \ldots, a_{l}$ are necessarily coprime. If $a_{1}, \ldots, a_{l}$ are coprime, we call $\left\{n_{1} a_{1}+\cdots+n_{l} a_{l}: n_{1}, \ldots, n_{l} \in \mathbb{N}_{0}\right\}$ the semigroup generated by $a_{1}, \ldots, a_{l}$ and denote it by $\left\langle a_{1}, \ldots, a_{l}\right\rangle$.

Among numerical semigroups, those generated by two integers, that is, numerical semigroups of the form $\left\{m a+n b: m, n \in \mathbb{N}_{0}\right\}$ for some coprime integers $a, b$, have a particular interest. Important curves such as hyperelliptic curves, Hermitian curves or Geil's norm-trace curves [3] have Weierstrass semigroups generated by two integers. Properties of semigroups generated by two coprime integers can be found in [6]. For instance, the semigroup generated by $a$ and $b$ has genus $\frac{(a-1)(b-1)}{2}$, and any element $i \in \Lambda$ can be uniquely written as $i=m a+n b$ with $m, n$ integers such that $0 \leqslant n \leqslant a-1$. From the results in [5. Section 3.2] one can get, for any numerical semigroup $\Lambda$ generated by two coprime integers the equation of a curve having a point whose Weierstrass
semigroup is $\Lambda$.
For a numerical semigroup, the membership problem is that of determining, for any integer $i$ whether it belongs or not to the numerical semigroup. In the next lemma we first state a result already proved in [6] and then we give a solution to the membership problem for semigroups generated by two coprime integers. By $x \bmod a$ with $x, a$ integers we mean the smallest positive integer congruent with $x$ modulo $a$.

Lemma 2.1. Suppose $\Lambda$ is generated by $a, b$ with $a<b$. Let $c$ be the inverse of $b$ modulo $a$.

1. Any $i \in \Lambda$ can be uniquely written as $i=m a+n b$ for some $m, n \geqslant 0$ with $n \leqslant a-1$.
2. $i \in \Lambda$ if and only if $b(i \cdot c \bmod a) \leqslant i$.

Proof. 1. Suppose $i \in \Lambda$. Then $i=\tilde{m} a+\tilde{n} b$ for some non-negative integers $\tilde{m}, \tilde{n}$. Let $n=\tilde{n} \bmod a$ and $m=\tilde{m}+b\left\lfloor\frac{\tilde{n}}{a}\right\rfloor$. Then $i=\tilde{m} a+\tilde{n} b=$ $\tilde{m} a+\left(a\left\lfloor\frac{\tilde{n}}{a}\right\rfloor+(\tilde{n} \bmod a)\right) b=m a+n b$ with obviously $m, n \geqslant 0$ and $n \leqslant a-1$.

For uniqueness, suppose $i=m a+n b$ for some $m, n \geqslant 0$ and $n \leqslant a-1$, and simultaneously, $i=m^{\prime} a+n^{\prime} b$ for some $m^{\prime}, n^{\prime} \geqslant 0$ and $n^{\prime} \leqslant a-1$. Then $\left(m-m^{\prime}\right) a=\left(n^{\prime}-n\right) b$. Since $a$ and $b$ are coprime, $a$ must divide $n-n^{\prime}$ which can only happen if $n=n^{\prime}$ and so $m=m^{\prime}$.
2. If $i \in \Lambda$ then by the previous statement there exist unique integers $m, n \geqslant$ 0 with $n \leqslant a-1$ such that $i=m a+n b$. In this case, $i \cdot c \bmod a=$ $(m a+n b) \cdot c \bmod a=n$ and then it is obvious that $b(i \cdot c \bmod a) \leqslant i$.
On the other hand, suppose $i \in \mathbb{N}_{0}$ and define $n=i \cdot c \bmod a$. Then $i-n b$ is a multiple of $a$ since $(i-n b) \bmod a=((i \bmod a)-(n b \bmod a))$ $\bmod a=0$. If $n b \leqslant i$ then $i-n b$ is a positive multiple of $a$, say $m a$, and $i=m a+n b$, so $i \in \Lambda$.

Remark 2.2. Notice that for the case $b=a+1$ the condition $b(i \cdot c \bmod a) \leqslant i$ is equivalent to $(a+1)(i \bmod a) \leqslant i$ and to $a(i \bmod a) \leqslant i-(i \bmod a)$ and so $i \bmod a \leqslant\left\lfloor\frac{i}{a}\right\rfloor$. Therefore, $i \in\langle a, a+1\rangle$ if and only if the remainder of the division of $i$ by $a$ is at most its quotient. This was already proved in [2].

## 3 The Geil-Matsumoto bound

Let $N_{q}(g)$ be the maximum number of rational places of degree one of a function field over $\mathbb{F}_{q}$ with genus $g$. The Hasse-Weil bound [9, Theorem V.2.3] states $\left|N_{q}(g)-(q+1)\right| \leqslant 2 g \sqrt{q}$. Serre's refinement [9, Theorem V.3.1] states $\mid N_{q}(g)-$ $(q+1) \mid \leqslant g\lfloor 2 \sqrt{q}\rfloor$. This means that either $N_{q}(g) \leqslant q+1$ or

$$
\begin{equation*}
N_{q}(g) \leqslant S_{q}(g):=q+1+g\lfloor 2 \sqrt{q}\rfloor . \tag{1}
\end{equation*}
$$

Since $g\lfloor 2 \sqrt{q}\rfloor \geqslant 0$ it is enough to state equation 1 .
If we consider the Weierstrass semigroup $\Lambda$ of any such places then we can define $N_{q}(\Lambda)$ as the maximum number of rational places of degree one of a function field over $\mathbb{F}_{q}$ such that the Weierstrass semigroup at one of the places is $\Lambda$. Lewittes' bound [7] states, if $\lambda_{1}$ is the first non-zero element in $\Lambda$,

$$
N_{q}(\Lambda) \leqslant L_{q}(\Lambda):=q \lambda_{1}+1
$$

and the Geil-Matsumoto bound [4] is

$$
\begin{equation*}
N_{q}(\Lambda) \leqslant G M_{q}(\Lambda):=\#\left(\Lambda \backslash \cup_{\lambda_{i}} \text { generator of } \Lambda\left(q \lambda_{i}+\Lambda\right)\right)+1 \tag{2}
\end{equation*}
$$

In [4, 5] the next result is proved, from which Lewittes' bound can be deduced from the Geil-Matsumoto bound.

Lemma 3.1. $\#\left(\Lambda \backslash\left(q \lambda_{1}+\Lambda\right)\right)=q \lambda_{1}$.
Here, for a numerical semigroup generated by two coprime integers $a, b$ we describe the Geil-Matsumoto bound in terms of $a, b$ giving a formula which is simpler to compute than (2).

Theorem 3.2. The Geil-Matsumoto bound for the semigroup generated by a and $b$ with $a<b$ is

$$
\begin{align*}
G M_{q}(\langle a, b\rangle) & =1+\sum_{n=0}^{a-1} \min \left(q,\left\lceil\frac{q-n}{a}\right\rceil \cdot b\right)  \tag{3}\\
& = \begin{cases}1+q a & \text { if } q \leqslant\left\lfloor\frac{q}{a}\right\rfloor b \\
1+(q \bmod a) q+(a-(q \bmod a))\left\lfloor\frac{q}{a}\right\rfloor b & \text { if }\left\lfloor\frac{q}{a}\right\rfloor b<q \leqslant\left\lceil\frac{q}{a}\right\rceil(b 4) \\
1+a b\left\lceil\frac{q}{a}\right\rceil-(a-(q \bmod a)) b & \text { if } q>\left\lceil\frac{q}{a}\right\rceil b\end{cases}
\end{align*}
$$

Proof. The Geil-Matsumoto bound for the semigroup generated by $a$ and $b$ with $a<b$ is $1+\#\left\{i \in \Lambda: \begin{array}{l}i-q a \notin \Lambda \\ i-q b \notin \Lambda\end{array}\right\}$. By Lemma 2.1] $i \in \Lambda$ if and only if $b(i c$ $\bmod a) \leqslant i$, where $c$ is the inverse of $b$ modulo $a$. Now, suppose that $i \in \Lambda$ can be expressed as $i=m a+n b$ for some integers $m, n \geqslant 0, n \leqslant a-1$. Then

$$
\begin{aligned}
i-q a \notin \Lambda & \Longleftrightarrow b((i-q a) c \bmod a)>i-q a \\
& \Longleftrightarrow b((m a+n b-q a) c \bmod a)>i-q a \\
& \Longleftrightarrow b(n b c \bmod a)>i-q a \\
& \Longleftrightarrow b n>i-q a \\
& \Longleftrightarrow b n>(m-q) a+n b \\
& \Longleftrightarrow(m-q) a<0 \\
& \Longleftrightarrow m<q
\end{aligned}
$$

$$
\begin{aligned}
i-q b \notin \Lambda & \Longleftrightarrow b((i-q b) c \bmod a)>i-q b \\
& \Longleftrightarrow b((\operatorname{ma}+n b-q b) c \bmod a)>i-q b \\
& \Longleftrightarrow b((n-q) b c \bmod a)>i-q b \\
& \Longleftrightarrow b((n-q) \bmod a)>i-q b \\
& \Longleftrightarrow b((n-q) \bmod a)>m a+(n-q) b \\
& \Longleftrightarrow b[((n-q) \bmod a)-(n-q)]>m a \\
& \left.\Longleftrightarrow b\left(-\left\lvert\, \frac{n-q}{a}\right.\right] a\right)>m a \\
& \Longleftrightarrow b\left(\left\lceil-\frac{n-q}{a}\right\rceil\right)>m \\
& \Longleftrightarrow b\left\lceil\frac{q-n}{a}\right\rceil>m
\end{aligned}
$$

Consequently, the Geil-Matsumoto bound is

$$
1+\sum_{n=0}^{a-1} \min \left(q,\left\lceil\frac{q-n}{a}\right\rceil \cdot b\right)
$$

Now some technical steps lead to the next formula.
$G M_{q}(\langle a, b\rangle)= \begin{cases}1+q a & \text { if } q \leqslant\left\lceil\frac{q-a+1}{a}\right\rceil b \\ 1+(q \bmod a) q+(a-(q \bmod a))\left\lceil\frac{q-a+1}{a}\right\rceil b & \text { if }\left\lceil\frac{q-a+1}{a}\right\rceil b<q \leqslant\left\lceil\frac{q}{a}\right\rceil b \\ 1+a b\left\lceil\frac{q}{a}\right\rceil-(a-(q \bmod a)) b & \text { if } q>\left\lceil\frac{q}{a}\right\rceil b\end{cases}$
Since $\left\lceil\frac{q-a+1}{a}\right\rceil$ is the unique integer between $\frac{q-a+1}{a}$ and $\frac{q}{a}$, one has $\left\lceil\frac{q-a+1}{a}\right\rceil=$ $\left\lfloor\frac{q}{a}\right\rfloor$, and the formula in (5) coincides with that in (4).

## 4 Coincidences of Lewittes's and the Geil-Matsumoto bound

We are interested now in the coincidences of Lewittes's and the Geil-Matsumoto bound. To get an idea, one can see in Table 1 the portion of semigroups for which they coincide for several values of the genus and the field size.

Beelen and Ruano proved in [1, Proposition 9] that if $q \in \Lambda$ then the bounds coincide. For the case of two generators, from equation (3) we deduce that $G M_{q}(\langle a, b\rangle)=L_{q}(\langle a, b\rangle)$ if and only if $q \leqslant\left\lfloor\frac{q}{a}\right\rfloor b$. Otherwise, the Geil-Matsumoto bound always gives an improvement with respect to Lewittes's bound. We want to generalize these results to semigroups with any number of generators.

Theorem 4.1. Let $\Lambda=\left\langle\lambda_{1}, \ldots, \lambda_{n}\right\rangle$ with $\lambda_{1}<\lambda_{i}$ for all $i>1$. The next statements are equivalent

1. $G M_{q}(\Lambda)=L_{q}(\Lambda)$,
2. $\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)=\Lambda \backslash\left(q \lambda_{1}+\Lambda\right)$,
3. $q\left(\lambda_{i}-\lambda_{1}\right) \in \Lambda$ for all $i>1$.

Proof. By Lemma 3.1 it is obvious that 2 implies 1 . The converse follows from the inclusion $\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right) \subseteq \Lambda \backslash\left(q \lambda_{1}+\Lambda\right)$ and the equality $G M_{q}(\Lambda)=L_{q}(\Lambda)$ which, by Lemma 3.1 implies that $\#\left(\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)\right)=\#\left(\Lambda \backslash\left(q \lambda_{1}+\Lambda\right)\right)$.

For the equivalence of the last two statements notice that $q\left(\lambda_{i}-\lambda_{1}\right) \in \Lambda$ for all $i>1 \Longleftrightarrow q \lambda_{i} \in q \lambda_{1}+\Lambda$ for all $i>1 \Longleftrightarrow q \lambda_{i}+\Lambda \subseteq q \lambda_{1}+\Lambda$ for all $i>1$ $\Longleftrightarrow \Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)=\Lambda \backslash\left(q \lambda_{1}+\Lambda\right)$.

Notice that under the hypothesis $q \in \Lambda$ then $q\left(\lambda-\lambda_{1}\right) \in \Lambda$ is satisfied by all $\lambda \in \Lambda$. So, Theorem 4.1 generalizes Beelen-Ruano's result.

Theorem 4.1 suggests to analyze under what conditions $q\left(\lambda_{i}-\lambda_{1}\right) \in \Lambda$ for some $i>1$. Let us first see in what cases $q\left(\lambda_{i}-\lambda_{1}\right) \in\left\{x \lambda_{1}+y \lambda_{i}: x, y \in \mathbb{N}_{0}\right\}$. Notice that if $\operatorname{gcd}\left(\lambda_{1}, \lambda_{i}\right)=d$ then $\left\{x \lambda_{1}+y \lambda_{i}: x, y \in \mathbb{N}_{0}\right\}=d\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$, where by $d\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$ we mean the set $\left\{d \lambda: \lambda \in\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle\right\}$. Obviously, $d\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle \subseteq \Lambda$.
Lemma 4.2. Let $\operatorname{gcd}\left(\lambda_{1}, \lambda_{i}\right)=d$. Then $q\left(\lambda_{i}-\lambda_{1}\right) \in d\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$ if and only if $q d \leqslant\left\lfloor\frac{q d}{\lambda_{1}}\right\rfloor \lambda_{i}$. In particular, if $q \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{i}$ then $q\left(\lambda_{i}-\lambda_{1}\right) \in d\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$.
Proof. We need to prove that $q\left(\frac{\lambda_{i}}{d}-\frac{\lambda_{1}}{d}\right) \in\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$ if and only if $q d \leqslant\left\lfloor\frac{q d}{\lambda_{1}}\right\rfloor \lambda_{i}$. Suppose that $c$ is the inverse of $\frac{\lambda_{i}}{d}$ modulo $\frac{\lambda_{1}}{d}$. By Lemma $2.1 q\left(\frac{\lambda_{i}}{d}-\frac{\lambda_{1}}{d}\right) \in$ $\left\langle\frac{\lambda_{1}}{d}, \frac{\lambda_{i}}{d}\right\rangle$ if and only if $\frac{\lambda_{i}}{d}\left(q\left(\frac{\lambda_{i}}{d}-\frac{\lambda_{1}}{d}\right) c \bmod \frac{\lambda_{1}}{d}\right) \leqslant q\left(\frac{\lambda_{i}}{d}-\frac{\lambda_{1}}{d}\right)$, that is, $\frac{\lambda_{i}}{d}(q$ $\left.\bmod \frac{\lambda_{1}}{d}\right) \leqslant q\left(\frac{\lambda_{i}}{d}-\frac{\lambda_{1}}{d}\right)$ which is equivalent to $q d \leqslant\left\lfloor\frac{q d}{\lambda_{1}}\right\rfloor \lambda_{i}$.

Now, if $q \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{i}$, then $q d \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor d \lambda_{i} \leqslant\left\lfloor\frac{q d}{\lambda_{1}}\right\rfloor \lambda_{i}$ and the last statement follows.

Proposition 4.3. Suppose $\lambda_{1}<\lambda_{2}<\cdots<\lambda_{n}$ and let $\Lambda=\left\langle\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right\rangle$. If $q \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{2}$ then $G M_{q}(\Lambda)=L_{q}(\Lambda)$.
Proof. By hypothesis, $q \leqslant\left\lfloor\frac{q}{\lambda}\right\rfloor \lambda_{i}$ for all $i>1$. By Lemma 4.2, $q\left(\lambda_{i}-\lambda_{1}\right) \in \Lambda$ for all $i>1$ and by Theorem 4.1 $G M_{q}(\Lambda)=L_{q}(\Lambda)$.

Remark 4.4. As mentioned, the converse is true when restricted to semigroups with two generators. Otherwise the converse is not true in general. For instance, consider $\Lambda=\langle 5,7,18\rangle$ with $q=9$. We have $\Lambda=\{0,5,7,10,12,14,15,17,18, \ldots\}$ and $\Lambda \backslash \cup_{\lambda_{i}}$ generator of $\Lambda\left(q \lambda_{i}+\Lambda\right)=\{0,5,7,10,12,14,15,17,18,19,20,21,22,23$, $24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,46,47,48$, $49,51,53,54,56,58,61\}=\Lambda \backslash\left(q \lambda_{1}+\Lambda\right)$. So $G M_{q}(\langle 5,7,18\rangle)=L_{q}(\langle 5,7,18\rangle)=$ 46. However, $q(=9)>\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{2}(=7)$. The reason is that although $q\left(\lambda_{2}-\lambda_{1}\right) \notin$ $\left\langle\lambda_{1}, \lambda_{2}\right\rangle$, it holds that $q\left(\lambda_{2}-\lambda_{1}\right) \in\left\langle\lambda_{1}, \lambda_{2}, \lambda_{3}\right\rangle=\Lambda$.

In Table together with the portion of semigroups for which the Lewittes and the Geil-Matsumoto bounds coincide, we give the portion of semigroups satisfying the hypothesis in Proposition 4.3. From that table it is easy to check again that in general the converse of Proposition 4.3 is not true.

|  | Lewittes $=$ Geil-Matsumoto |  |  |  |  | $q \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{2}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Genus | q=2 | q=3 | q=9 | q=16 | q=256 | q=2 | $\mathrm{q}=3$ | $\mathrm{q}=9$ | $\mathrm{q}=16$ | $\mathrm{q}=256$ |
| 2 | $50.00 \%$ | $100 \%$ | $100 \%$ | $100 \%$ | $100 \%$ | $50.00 \%$ | $100 \%$ | $100 \%$ | $100 \%$ | $100 \%$ |
| 3 | $25.00 \%$ | $75.00 \%$ | $100 \%$ | $100 \%$ | $100 \%$ | $25.00 \%$ | $75.00 \%$ | $100 \%$ | $100 \%$ | $100 \%$ |
| 4 | $42.86 \%$ | $57.14 \%$ | $100 \%$ | $100 \%$ | $100 \%$ | $14.29 \%$ | $42.86 \%$ | $85.71 \%$ | $100 \%$ | $100 \%$ |
| 5 | $33.33 \%$ | $41.67 \%$ | $91.67 \%$ | $100 \%$ | $100 \%$ | $8.33 \%$ | $25.00 \%$ | $58.33 \%$ | $91.67 \%$ | $100 \%$ |
| 6 | $21.74 \%$ | $43.48 \%$ | $86.96 \%$ | $100 \%$ | $100 \%$ | $4.35 \%$ | $17.39 \%$ | $43.48 \%$ | $82.61 \%$ | $100 \%$ |
| 7 | $17.95 \%$ | $41.03 \%$ | $87.18 \%$ | $100 \%$ | $100 \%$ | $2.56 \%$ | $10.26 \%$ | $38.46 \%$ | $84.62 \%$ | $100 \%$ |
| 8 | $14.93 \%$ | $37.31 \%$ | $85.07 \%$ | $100 \%$ | $100 \%$ | $1.49 \%$ | $5.97 \%$ | $53.73 \%$ | $91.04 \%$ | $100 \%$ |
| 9 | $11.02 \%$ | $33.05 \%$ | $88.14 \%$ | $98.31 \%$ | $100 \%$ | $0.85 \%$ | $4.24 \%$ | $72.03 \%$ | $87.29 \%$ | $100 \%$ |
| 10 | $8.82 \%$ | $29.90 \%$ | $88.24 \%$ | $95.59 \%$ | $100 \%$ | $0.49 \%$ | $2.45 \%$ | $79.90 \%$ | $78.92 \%$ | $100 \%$ |
| 11 | $7.58 \%$ | $25.95 \%$ | $84.55 \%$ | $92.71 \%$ | $100 \%$ | $0.29 \%$ | $1.46 \%$ | $78.13 \%$ | $65.89 \%$ | $100 \%$ |
| 12 | $6.59 \%$ | $23.48 \%$ | $78.89 \%$ | $90.88 \%$ | $100 \%$ | $0.17 \%$ | $1.01 \%$ | $69.93 \%$ | $54.05 \%$ | $100 \%$ |
| 13 | $5.69 \%$ | $21.48 \%$ | $73.73 \%$ | $89.81 \%$ | $100 \%$ | $0.10 \%$ | $0.60 \%$ | $59.64 \%$ | $42.76 \%$ | $100 \%$ |
| 14 | $5.02 \%$ | $18.90 \%$ | $69.76 \%$ | $88.66 \%$ | $100 \%$ | $0.06 \%$ | $0.35 \%$ | $49.26 \%$ | $33.73 \%$ | $100 \%$ |
| 15 | $4.10 \%$ | $16.63 \%$ | $66.26 \%$ | $87.68 \%$ | $100 \%$ | $0.04 \%$ | $0.25 \%$ | $39.38 \%$ | $28.35 \%$ | $100 \%$ |
| 16 | $3.45 \%$ | $14.77 \%$ | $63.23 \%$ | $87.22 \%$ | $100 \%$ | $0.02 \%$ | $0.15 \%$ | $30.86 \%$ | $28.67 \%$ | $100 \%$ |
| 17 | $2.92 \%$ | $13.10 \%$ | $60.66 \%$ | $87.00 \%$ | $100 \%$ | $0.01 \%$ | $0.09 \%$ | $23.79 \%$ | $35.23 \%$ | $100 \%$ |
| 18 | $2.38 \%$ | $11.66 \%$ | $58.74 \%$ | $87.03 \%$ | $100 \%$ | $0.01 \%$ | $0.06 \%$ | $18.33 \%$ | $45.70 \%$ | $100 \%$ |
| 19 | $1.93 \%$ | $10.40 \%$ | $57.06 \%$ | $86.71 \%$ | $100 \%$ | $0.00 \%$ | $0.04 \%$ | $13.93 \%$ | $55.89 \%$ | $100 \%$ |
| 20 | $1.60 \%$ | $9.28 \%$ | $55.71 \%$ | $85.43 \%$ | $100 \%$ | $0.00 \%$ | $0.02 \%$ | $10.55 \%$ | $62.47 \%$ | $99.95 \%$ |
| 21 | $1.31 \%$ | $8.34 \%$ | $54.67 \%$ | $83.03 \%$ | $100 \%$ | $0.00 \%$ | $0.01 \%$ | $7.93 \%$ | $64.51 \%$ | $99.75 \%$ |
| 22 | $1.09 \%$ | $7.48 \%$ | $53.95 \%$ | $80.14 \%$ | $100 \%$ | $0.00 \%$ | $0.01 \%$ | $5.93 \%$ | $62.93 \%$ | $99.19 \%$ |
| 23 | $0.90 \%$ | $6.70 \%$ | $53.29 \%$ | $77.41 \%$ | $100 \%$ | $0.00 \%$ | $0.01 \%$ | $4.39 \%$ | $59.00 \%$ | $98.09 \%$ |
| 24 | $0.75 \%$ | $6.02 \%$ | $52.46 \%$ | $75.16 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $3.25 \%$ | $53.67 \%$ | $96.50 \%$ |
| 25 | $0.63 \%$ | $5.42 \%$ | $51.33 \%$ | $73.37 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $2.38 \%$ | $47.63 \%$ | $94.73 \%$ |
| 26 | $0.53 \%$ | $4.90 \%$ | $49.94 \%$ | $71.94 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $1.74 \%$ | $41.35 \%$ | $93.12 \%$ |
| 27 | $0.45 \%$ | $4.45 \%$ | $48.39 \%$ | $70.75 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $1.27 \%$ | $35.24 \%$ | $91.84 \%$ |
| 28 | $0.38 \%$ | $4.07 \%$ | $46.81 \%$ | $69.73 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $0.92 \%$ | $29.58 \%$ | $90.87 \%$ |
| 29 | $0.32 \%$ | $3.74 \%$ | $45.25 \%$ | $68.76 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $0.67 \%$ | $24.52 \%$ | $90.06 \%$ |
| 30 | $0.27 \%$ | $3.44 \%$ | $43.76 \%$ | $67.80 \%$ | $100 \%$ | $0.00 \%$ | $0.00 \%$ | $0.48 \%$ | $20.12 \%$ | $89.25 \%$ |

Table 1: Portion of semigroups for which the Lewittes and the Geil-Matsumoto bounds coincide and portion of semigroups satisfying the hypothesis in Proposition 4.3, that is $q \leqslant\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor \lambda_{2}$, where $\lambda_{1}, \lambda_{2}$ are the first and second smallest generators.

## 5 Simplifying the computation

Next we investigate in which cases the computation of $\Lambda \backslash \cup_{\lambda_{i}}$ generator of $\Lambda_{\Lambda}\left(q \lambda_{i}+\right.$ $\Lambda$ ) can be simplified to the computation of $\Lambda \backslash \cup_{i \in I}\left(q \lambda_{i}+\Lambda\right)$ for some index set $I$ smaller than the number of generators of $\Lambda$. The next proposition can be proved very similarly as we proved Theorem 4.1.

Proposition 5.1. Let $\Lambda=\left\langle\lambda_{1}, \ldots, \lambda_{n}\right\rangle$ and let $I$ be an index set included in $\{1, \ldots, n\}$. The next statements are equivalent.

1. $\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)=\Lambda \backslash \cup_{i \in I}\left(q \lambda_{i}+\Lambda\right)$.
2. For all $i \notin I$ there exists $1 \leqslant j \leqslant n, j \in I$ such that $q\left(\lambda_{i}-\lambda_{j}\right) \in \Lambda$.

One consequence of Proposition 5.1 is the next proposition.
Proposition 5.2. Let $\Lambda=\left\langle\lambda_{1}, \ldots, \lambda_{n}\right\rangle$ with $\lambda_{1}<\lambda_{2}<\cdots<\lambda_{n}$ and $\lambda_{1}<q$.

1. Let $\lambda_{j}$ be the maximum generator strictly smaller than $\frac{q}{\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor}$ then $\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)=\Lambda \backslash \cup_{i=1}^{j}\left(q \lambda_{i}+\Lambda\right)$.
2. Let $\lambda_{j}$ be the maximum generator strictly smaller than $2 \lambda_{1}-1$ then $\Lambda \backslash \cup_{i=1}^{n}\left(q \lambda_{i}+\Lambda\right)=\Lambda \backslash \cup_{i=1}^{j}\left(q \lambda_{i}+\Lambda\right)$.

Proof. The first statement is a consequence of Lemma 4.2 together with Proposition 5.1. For the second statement suppose that $q=x \lambda_{1}+y$ with $x, y$ integers and $x \geqslant 1$. Then $\frac{q}{\left\lfloor\frac{q}{\lambda_{1}}\right\rfloor}=\lambda_{1}+\frac{y}{x}$. The result follows from the inequalities $x \geqslant 1$ and $y \leqslant \lambda_{1}-1$.

We will call the generators that are strictly smaller than $2 \lambda_{1}-1$ GeilMatsumoto generators. What the last statement of the previous proposition says is that for computing the Geil-Matsumoto bound we only need to subtract from $\Lambda$ the sets $q \mu+\Lambda$ for $\mu$ a Geil-Matsumoto generator. Since in general we need to subtract these sets for all generators, this constitutes an improvement in terms of computation. In Table 2, we give the mean of the number of GeilMatsumoto generators and non-Geil-Matsumoto generators per semigroup for different genera. In Table 3, we give the portion of Geil-Matsumoto generators (and non-Geil-Matsumoto generators) with respect to the total number of generators for different genera. We observe that, although the portion of non-Geil-Matsumoto generators decreases with the genus, it remains still significant, with a portion of more than $30 \%$ for genus 25 .

Proposition 5.2 is a first consequence of Proposition 5.1 and it can be used to simplify the computation of the Geil-Matsumoto bound. We leave it as a problem for future research to find other consequences of Proposition 5.1 to get further simplifications.

| Genus | Mean of the number of GM generators per semigroup | Mean of the number of non-GM generators per semigroup |
| :---: | :---: | :---: |
| 2 | 1.50 | 1.00 |
| 3 | 1.75 | 1.00 |
| 4 | 2.00 | 1.14 |
| 5 | 2.33 | 1.42 |
| 6 | 2.52 | 1.43 |
| 7 | 2.79 | 1.62 |
| 8 | 3.07 | 1.76 |
| 9 | 3.32 | 1.89 |
| 10 | 3.57 | 2.00 |
| 11 | 3.85 | 2.17 |
| 12 | 4.10 | 2.27 |
| 13 | 4.38 | 2.41 |
| 14 | 4.65 | 2.53 |
| 15 | 4.92 | 2.65 |
| 16 | 5.20 | 2.76 |
| 17 | 5.48 | 2.88 |
| 18 | 5.76 | 2.98 |
| 19 | 6.05 | 3.09 |
| 20 | 6.35 | 3.20 |
| 21 | 6.64 | 3.30 |
| 22 | 6.94 | 3.40 |
| 23 | 7.24 | 3.50 |
| 24 | 7.55 | 3.59 |
| 25 | 7.86 | 3.68 |
| 26 | 8.17 | 3.77 |
| 27 | 8.49 | 3.86 |
| 28 | 8.81 | 3.94 |
| 29 | 9.13 | 4.03 |
| 30 | 9.46 | 4.10 |

Table 2: Mean of the number of Geil-Matsumoto generators and non-GeilMatsumoto generators per semigroup

| Genus | Total number of GM generators divided by the total number of generators | Total number of non-GM generators divided by the total number of generators | Mean of the portion of non-GM generators per semigroup |
| :---: | :---: | :---: | :---: |
| 2 | 60.00\% | 40.00\% | 41.67\% |
| 3 | 63.64\% | $36.36 \%$ | 35.42\% |
| 4 | 63.64\% | $36.36 \%$ | 38.57\% |
| 5 | 62.22\% | 37.78\% | 40.14\% |
| 6 | 63.74\% | 36.26\% | 37.43\% |
| 7 | 63.37\% | 36.63\% | 39.13\% |
| 8 | 63.58\% | 36.42\% | 39.03\% |
| 9 | 63.74\% | 36.26\% | 38.58\% |
| 10 | 64.03\% | 35.97\% | 38.39\% |
| 11 | 63.96\% | 36.04\% | 38.76\% |
| 12 | 64.34\% | 35.66\% | 38.26\% |
| 13 | 64.54\% | 35.46\% | 38.17\% |
| 14 | 64.75\% | 35.25\% | 37.99\% |
| 15 | 65.01\% | 34.99\% | 37.73\% |
| 16 | 65.30\% | 34.70\% | 37.45\% |
| 17 | 65.56\% | 34.44\% | 37.21\% |
| 18 | 65.88\% | 34.12\% | 36.87\% |
| 19 | 66.19\% | 33.81\% | 36.55\% |
| 20 | 66.49\% | 33.51\% | 36.25\% |
| 21 | 66.79\% | 33.21\% | 35.93\% |
| 22 | 67.11\% | 32.89\% | 35.59\% |
| 23 | 67.43\% | 32.57\% | 35.26\% |
| 24 | 67.76\% | 32.24\% | 34.91\% |
| 25 | 68.08\% | 31.92\% | 34.56\% |
| 26 | 68.41\% | 31.59\% | 34.21\% |
| 27 | 68.74\% | 31.26\% | 33.86\% |
| 28 | 69.07\% | 30.93\% | 33.50\% |
| 29 | 69.40\% | 30.60\% | 33.14\% |
| 30 | 69.74\% | 30.26\% | 32.77\% |

Table 3: Portion of Geil-Matsumoto generators

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