

AFFINE CARTESIAN CODES

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ABSTRACT. We compute the basic parameters (dimension, length, minimum distance) of affine evaluation codes defined on a cartesian product of finite sets. Given a sequence of positive integers, we construct an evaluation code, over a degenerate torus, with prescribed parameters of a certain type. As an application of our results, we recover the formulas for the minimum distance of various families of evaluation codes.

1. INTRODUCTION

Let K be an arbitrary field and let A_1, \dots, A_n be a collection of non-empty subsets of K with a finite number of elements. Consider the following finite sets: (a) the *cartesian product*

$$X^* := A_1 \times \cdots \times A_n \subset \mathbb{A}^n,$$

where $\mathbb{A}^n = K^n$ is an affine space over the field K , and (b) the *projective closure* of X^*

$$Y := \{[(\gamma_1, \dots, \gamma_n, 1)] \mid \gamma_i \in A_i \text{ for all } i\} \subset \mathbb{P}^n,$$

where \mathbb{P}^n is a projective space over the field K . We also consider X , the image of $X^* \setminus \{0\}$ under the map $\mathbb{A}^n \setminus \{0\} \mapsto \mathbb{P}^{n-1}$, $\gamma \mapsto [\gamma]$. In what follows d_i denotes $|A_i|$, the cardinality of A_i for $i = 1, \dots, n$. We may always assume that $2 \leq d_i \leq d_{i+1}$ for all i (see Proposition 3.2). As usual, we denote the finite field with q elements by \mathbb{F}_q . The multiplicative group of the field K will be denoted by K^* .

Let $S = K[t_1, \dots, t_n]$ be a polynomial ring, let P_1, \dots, P_m be the points of X^* , and let $S_{\leq d}$ be the K -vector space of all polynomials of S of degree at most d . The *evaluation map*

$$\text{ev}_d: S_{\leq d} \longrightarrow K^{|X^*|}, \quad f \mapsto (f(P_1), \dots, f(P_m)),$$

defines a linear map of K -vector spaces. The image of ev_d , denoted by $C_{X^*}(d)$, defines a *linear code*. Permitting an abuse of language, we are referring to $C_{X^*}(d)$ as a *linear code*, even though the field K might not be finite. We call $C_{X^*}(d)$ the *affine cartesian evaluation code* (*cartesian code* for short) of degree d on the set X^* . If K is finite, cartesian codes are special types of affine Reed-Muller codes in the sense of [27, p. 37].

The *dimension* and the *length* are two of the basic parameters of $C_{X^*}(d)$, they are defined as $\dim_K C_{X^*}(d)$ and $|X^*|$, respectively. A third basic parameter of $C_{X^*}(d)$ is the *minimum distance*, which is given by

$$\delta_{X^*}(d) = \min\{\|\text{ev}_d(f)\| : \text{ev}_d(f) \neq 0; f \in S_{\leq d}\},$$

where $\|\text{ev}_d(f)\|$ is the number of non-zero entries of $\text{ev}_d(f)$. It is well known that the code $C_{X^*}(d)$ has the same parameters that $C_Y(d)$, the projective evaluation code of degree d on Y . We give a short proof of this fact by showing that these codes are equal (Proposition 2.9).

2010 *Mathematics Subject Classification*. Primary 13P25; Secondary 14G50, 94B27, 11T71.

Key words and phrases. Evaluation codes, minimum distance, complete intersections, vanishing ideals, degree, regularity, Hilbert function, algebraic invariants.

The second author was supported by COFAA-IPN and SNI. The third author was supported by SNI.

The main results of this paper describe the basic parameters of cartesian evaluation codes and show the existence of cartesian codes—over degenerate tori—with prescribed parameters of a certain type.

Some families of evaluation codes—including several variations of Reed-Muller codes—have been studied extensively using commutative algebra methods (e.g., Hilbert functions, resolutions, Gröbner bases), see [4, 5, 8, 11, 16, 18, 19, 20, 23, 26]. In this paper we use these methods to study the family of cartesian codes.

A key observation that allows us to use commutative algebra methods to study evaluation codes is that the kernel of the evaluation map ev_d is precisely $S_{\leq d} \cap I(X^*)$, where $I(X^*)$ is the *vanishing ideal* of X^* consisting of all polynomials of S that vanish on X^* . Thus, as is seen in the references given above, the algebra of $S/I(X^*)$ is related to the basic parameters of $C_{X^*}(d)$. Below we will clarify some more the role of commutative algebra in coding theory.

Let $S[u] = \bigoplus_{d=0}^{\infty} S[u]_d$ be a polynomial ring with the standard grading, where $u = t_{n+1}$ is a new variable. Recall that the *vanishing ideal* of Y , denoted by $I(Y)$, is the ideal of $S[u]$ generated by the homogeneous polynomials that vanish on Y . We use the algebraic invariants (regularity, degree, Hilbert function) of the graded ring $S[u]/I(Y)$ as a tool to study the described codes. It is a fact that this graded ring has the same invariants that the affine ring $S/I(X^*)$ [12, Remark 5.3.16]. The *Hilbert function* of $S[u]/I(Y)$ is given by

$$H_Y(d) := \dim_K(S[u]_d/I(Y) \cap S[u]_d).$$

According to [13, Lecture 13], we have that $H_Y(d) = |Y|$ for $d \geq |Y| - 1$. This means that $|Y|$ is the *degree* of $S[u]/I(Y)$ in the sense of algebraic geometry [13, p. 166]. The *regularity* of $S[u]/I(Y)$, denoted by $\text{reg } S[u]/I(Y)$, is the least integer $\ell \geq 0$ such that $H_Y(d) = |Y|$ for $d \geq \ell$.

The algebraic invariants of $S[u]/I(Y)$ occur in algebraic coding theory, as we now briefly explain. The code $C_{X^*}(d)$, has *length* $|Y|$ and *dimension* $H_Y(d)$. The knowledge of the regularity of $S[u]/I(Y)$ is important for applications to coding theory: for $d \geq \text{reg } S[u]/I(Y)$ the code $C_{X^*}(d)$ coincides with the underlying vector space $K^{|X^*|}$ and has, accordingly, minimum distance equal to 1. Thus, potentially good codes $C_{X^*}(d)$ can occur only if $1 \leq d < \text{reg}(S[u]/I(Y))$.

The contents of this paper are as follows. We show that the vanishing ideal $I(Y)$ is a complete intersection (Proposition 2.5). Then, one can use [5, Corollary 2.6] to compute the algebraic invariants of $I(Y)$ in terms of the sequence d_1, \dots, d_n . As a consequence, we compute the dimension of $C_{X^*}(d)$ and show that $\delta_{X^*}(d) = 1$ for $d \geq \sum_{i=1}^n (d_i - 1)$ (Theorem 3.1).

In Section 3, we show upper bounds in terms of d_1, \dots, d_n on the number of roots, over X^* , of polynomials in S which do not vanish at all points of X^* (Proposition 3.6, Corollary 3.7). The main theorem of Section 3 is a formula for the minimum distance of $C_{X^*}(d)$ (Theorem 3.8). In general, the problem of computing the minimum distance of a linear code is difficult because it is NP-hard [29]. The basic parameters of evaluation codes over finite fields have been computed in a number of cases. Our main results provide unifying tools to treat some of these cases. As an application, if Y is a projective torus in \mathbb{P}^n over a finite field K , we recover a formula of [21] for the minimum distance of $C_Y(d)$ (Corollary 3.10). If Y is the image of \mathbb{A}^n under the map $\mathbb{A}^n \rightarrow \mathbb{P}^n$, $x \mapsto [(x, 1)]$, we also recover a formula of [4] for the minimum distance of $C_Y(d)$ (Corollary 3.11). If $Y = \mathbb{P}^n$, the parameters of $C_Y(d)$ are described in [23, Theorem 1] (see also [15]), notice that in this case Y does not arise as the projective closure of some cartesian product X^* .

Finally, in Section 4, we consider cartesian codes over degenerate tori. Given a sequence d_1, \dots, d_n of positive integers, there exists a finite field \mathbb{F}_q such that d_i divides $q - 1$ for all i . We use this field to construct a cartesian code—over a degenerate torus—with previously

fixed parameters, expressed in terms of d_1, \dots, d_n (Theorem 4.2). As a byproduct, we obtain formulae for the basic parameters of any affine evaluation code over a degenerate torus (see Definition 4.1). Thus, we are also recovering the main results of [9, 10] (Remark 4.3).

It should be mentioned that we do not know of any efficient decoding algorithm for the family of cartesian codes. The reader is referred to [3, Chapter 9], [14, 28] and the references there for some available decoding algorithms for some families of linear codes.

For all unexplained terminology and additional information, we refer to [6, 13, 24] (for commutative algebra and the theory of Hilbert functions), and [17, 25, 27] (for the theory of linear codes).

2. COMPLETE INTERSECTIONS AND ALGEBRAIC INVARIANTS

We keep the same notations and definitions used in Section 1. In what follows d_i denotes $|A_i|$, the cardinality of A_i for $i = 1, \dots, n$. In this section we show that $I(Y)$ is a complete intersection and compute the algebraic invariants of $I(Y)$ in terms of d_1, \dots, d_n .

Theorem 2.1. (Combinatorial Nullstellensatz [2, Theorem 1.2]) *Let $S = K[t_1, \dots, t_n]$ be a polynomial ring over a field K , let $f \in S$, and let $a = (a_i) \in \mathbb{N}^n$. Suppose that the coefficient of t^a in f is non-zero and $\deg(f) = a_1 + \dots + a_n$. If A_1, \dots, A_n are subsets of K , with $|A_i| > a_i$ for all i , then there are $x_1 \in A_1, \dots, x_n \in A_n$ such that $f(x_1, \dots, x_n) \neq 0$.*

Lemma 2.2. (a) $|Y| = |X^*| = d_1 \cdots d_n$.

(b) If A_i is a subgroup of (K^*, \cdot) for all i , then $|X^*|/|A_1 \cap \dots \cap A_n| = |X|$.

(c) If $G \in I(X^*)$ and $\deg_{t_i}(G) < d_i$ for $i = 1, \dots, n$, then $G = 0$.

Proof. (a) The map $X^* \mapsto Y$, $x \mapsto [(x, 1)]$, is bijective. Thus, $|Y| = |X^*|$. (b) Since A_i is a group for all i , the sets X^* and X are also groups under componentwise multiplication. Thus, there is an epimorphism of groups $X^* \mapsto X$, $x \mapsto [x]$, whose kernel is equal to

$$\{(\gamma, \dots, \gamma) \in X^* : \gamma \in A_1 \cap \dots \cap A_n\}.$$

Thus, $|X^*|/|A_1 \cap \dots \cap A_n| = |X|$. To show (c) we proceed by contradiction. Assume that G is non-zero. Then, there is a monomial $t^a = t_1^{a_1} \cdots t_n^{a_n}$ of G with $\deg(G) = a_1 + \dots + a_n$, where $a = (a_1, \dots, a_n)$ and $a_i > 0$ for some i . As $\deg_{t_i}(G) < d_i$ for all i , then $a_i < |A_i| = d_i$ for all i . Thus, by Theorem 2.1, there are x_1, \dots, x_n with $x_i \in A_i$ for all i such that $G(x_1, \dots, x_n) \neq 0$, a contradiction to the assumption that G vanishes on X^* . \square

Lemma 2.3. *Let f_i be the polynomial $\prod_{\gamma \in A_i} (t_i - \gamma)$ for $1 \leq i \leq n$. Then*

$$I(X^*) = (f_1, \dots, f_n).$$

Proof. “ \supset ” This inclusion is clear because f_i vanishes on X^* by construction. “ \subset ” Take f in $I(X^*)$. Let \succ be the reverse lexicographical order on the monomials of S . By the division algorithm [1, Theorem 1.5.9, p. 30], we can write

$$f = g_1 f_1 + \dots + g_n f_n + G,$$

where each of the terms of G is not divisible by any of the leading monomials $t_1^{d_1}, \dots, t_n^{d_n}$, i.e., $\deg_{t_i}(G) < d_i$ for all i . As G belongs to $I(X^*)$, by Lemma 2.2, we get that $G = 0$. Thus, $f \in (f_1, \dots, f_n)$. \square

The degree and the regularity of $S[u]/I(Y)$ can be computed from its Hilbert series. Indeed, the Hilbert series can be written as

$$F_Y(t) := \sum_{i=0}^{\infty} H_Y(i)t^i = \sum_{i=0}^{\infty} \dim_K(S[u]/I(Y))_i t^i = \frac{h_0 + h_1 t + \cdots + h_r t^r}{1-t},$$

where h_0, \dots, h_r are positive integers. This follows from the fact that $I(Y)$ is a Cohen-Macaulay ideal of height n [7]. The number r is the regularity of $S[u]/I(Y)$ and $h_0 + \cdots + h_r$ is the degree of $S[u]/I(Y)$ (see [30, Corollary 4.1.12]).

Definition 2.4. A homogeneous ideal $I \subset S$ is called a *complete intersection* if there exists homogeneous polynomials g_1, \dots, g_r such that $I = (g_1, \dots, g_r)$, where r is the height of I .

Proposition 2.5. (a) $I(Y) = (\prod_{\gamma \in A_1} (t_1 - u\gamma), \dots, \prod_{\gamma \in A_n} (t_n - u\gamma))$.

(b) $I(Y)$ is a complete intersection.

(c) $F_Y(t) = \prod_{i=1}^n (1 + t + \cdots + t^{d_i-1}) / (1-t)$.

(d) $\text{reg } S[u]/I(Y) = \sum_{i=1}^n (d_i - 1)$ and $\text{deg}(S[u]/I(Y)) = |Y| = d_1 \cdots d_n$.

Proof. (a) For $i = 1, \dots, n$, we set $f_i = \prod_{\gamma \in A_i} (t_i - \gamma)$. Let \succ be the reverse lexicographical order on the monomials of $S[u]$. Since f_1, \dots, f_n form a Gröbner basis with respect to this order, by Lemma 2.3 and [16, Lemma 3.7], the vanishing ideal $I(Y)$ is equal to (f_1^h, \dots, f_n^h) , where $f_i^h = \prod_{\gamma \in A_i} (t_i - u\gamma)$ is the homogenization of f_i with respect to a new variable u . Part (b) follows from (a) because $I(Y)$ is an ideal of height n [7]. (c) This part follows using (a) and a well known formula for the Hilbert series of a complete intersection (see [30, p. 104]). (d) This part follows directly from [5, Corollary 2.6]. \square

Definition 2.6. Let $\{Q_i\}_{i=1}^m$ be a set of representatives for the points of Y . The map

$$\text{ev}'_d: S[u]_d \rightarrow K^{|Y|}, \quad f \mapsto (f(Q_i)/f_0(Q_i))_{i=1}^m,$$

where $f_0(t_1, \dots, t_n, u) = u^d$, defines a linear map of K -vector spaces. The image of ev'_d , denoted by $C_Y(d)$, is called a *projective evaluation code* of degree d on the set Y .

It is not hard to see that the map ev'_d is independent of the set of representatives that we choose for the points of Y .

Definition 2.7. The *affine Hilbert function* of $S/I(X^*)$ is given by

$$H_{X^*}(d) := \dim_K S_{\leq d}/I(X^*)_{\leq d}, \quad \text{where } I(X^*)_{\leq d} = S_{\leq d} \cap I(X^*).$$

As the evaluation map ev_d induces an isomorphism $S_{\leq d}/I(X^*)_{\leq d} \simeq C_{X^*}(d)$, as K -vector spaces, the dimension of $C_{X^*}(d)$ is $H_{X^*}(d)$.

Lemma 2.8. [12, Remark 5.3.16] $H_{X^*}(d) = H_Y(d)$ for $d \geq 0$.

In particular, from this lemma, the dimension and the length of the cartesian code $C_{X^*}(d)$ are $H_Y(d)$ and $\text{deg}(S[u]/I(Y))$, respectively.

Proposition 2.9. $C_{X^*}(d) = C_Y(d)$ for $d \geq 1$.

Proof. Since $S[u]_d/I(Y)_d \simeq C_Y(d)$ and $S_{\leq d}/I(X^*)_{\leq d} \simeq C_{X^*}(d)$, by Lemma 2.8, we get that the linear codes $C_{X^*}(d)$ and $C_Y(d)$ have the same dimension, and the same length. Thus, it suffices to show the inclusion “ \supset ”. Any point of $C_Y(d)$ has the form $W = (f(P_i, 1))_{i=1}^m$, where P_1, \dots, P_m are the points of X^* and $f \in S[u]_d$. If \tilde{f} is the polynomial $f(t_1, \dots, t_n, 1)$, then \tilde{f} is in $S_{\leq d}$ and $f(P_i, 1) = \tilde{f}(P_i)$ for all i . Thus, W is in $C_{X^*}(d)$, as required. \square

3. CARTESIAN EVALUATION CODES

In this section we compute the basic parameters of cartesian codes and give some applications. If d is at most $\sum_{i=1}^n (d_i - 1)$, we show an upper bound in terms of d_1, \dots, d_n on the number of roots, over X^* , of polynomials in $S_{\leq d}$ which do not vanish at all points of X^* .

We begin by computing some of the basic parameters of $C_{X^*}(d)$, the cartesian evaluation code of degree d on X^* .

Theorem 3.1. *The length of $C_{X^*}(d)$ is $d_1 \cdots d_n$, its minimum distance is 1 for $d \geq \sum_{i=1}^n (d_i - 1)$, and its dimension is*

$$H_{X^*}(d) = \binom{n+d}{d} - \sum_{1 \leq i \leq n} \binom{n+d-d_i}{d-d_i} + \sum_{i < j} \binom{n+d-(d_i+d_j)}{d-(d_i+d_j)} - \sum_{i < j < k} \binom{n+d-(d_i+d_j+d_k)}{d-(d_i+d_j+d_k)} + \cdots + (-1)^n \binom{n+d-(d_1+\cdots+d_n)}{d-(d_1+\cdots+d_n)}.$$

Proof. The length of $C_{X^*}(d)$ is $|X^*| = d_1 \cdots d_n$. We set $r = \sum_{i=1}^n (d_i - 1)$. By Proposition 2.5, the regularity of $S[u]/I(Y)$ is equal to r , i.e., $H_Y(d) = |Y|$ for $d \geq r$. Thus, by Lemmas 2.2 and 2.8, $H_{X^*}(d) = |X^*|$ for $d \geq r$, i.e., $C_{X^*}(d) = K^{|X^*|}$ for $d \geq r$. Hence $\delta_{X^*}(d) = 1$ for $d \geq r$. By Proposition 2.5, the ideal $I(Y)$ is a complete intersection generated by n homogeneous polynomials f_1, \dots, f_n of degrees d_1, \dots, d_n . Thus, applying [5, Corollary 2.6] and using the equality $H_{X^*}(d) = H_Y(d)$, we obtain the required formula for the dimension. \square

Proposition 3.2. *If $d_1 = 1$ and $X' = A_2 \times \cdots \times A_n$, then $C_{X^*}(d) = C_{X'}(d)$ for $d \geq 1$.*

Proof. Let α be the only element of A_1 and let Y' be the projective closure of X' . Then, by Proposition 2.5, we get

$$I(Y) = (t_1 - u\alpha, f_2^h, \dots, f_n^h) \quad \text{and} \quad I(Y') = (f_2^h, \dots, f_n^h),$$

where $f_i^h = \prod_{\gamma \in A_i} (t_i - u\gamma)$ for $i = 2, \dots, n$. Since $S[u]/I(Y)$ and $K[t_2, \dots, t_n, u]/I(Y')$ have the same Hilbert function, we get that the dimension and the length of $C_{X^*}(d)$ and $C_{X'}(d)$ are the same. Thus, to show the equality $C_{X^*}(d) = C_{X'}(d)$, it suffices to show the inclusion “ \subset ”. Any element of $C_{X^*}(d)$ has the form

$$W = (f(\alpha, Q_1), \dots, f(\alpha, Q_m)),$$

where Q_1, \dots, Q_m are the points of X' and $f \in S_{\leq d}$. If \tilde{f} is the polynomial $f(\alpha, t_2, \dots, t_n)$, then \tilde{f} is in $K[t_2, \dots, t_n]_{\leq d}$ and $f(\alpha, Q_i) = \tilde{f}(Q_i)$ for all i . Thus, W is in $C_{X'}(d)$, as required. \square

Since permuting the sets A_1, \dots, A_n does not affect neither the parameters of the corresponding cartesian evaluation codes, nor the invariants of the corresponding vanishing ideal, by Proposition 3.2 we may always assume that $2 \leq d_i \leq d_{i+1}$ for all i , where $d_i = |A_i|$.

For $G \in S$, we denote the zero set of G in X^* by $Z_{X^*}(G)$. We begin with a general bound that will be refined later in this section. The proof of [22, Lemma 3A, p. 147] can be easily adapted to obtain the following auxiliary result.

Lemma 3.3. *Let $0 \neq G = G(t_1, \dots, t_n) \in S$ be a polynomial of total degree d . If $d_i \leq d_{i+1}$ for all i , then*

$$|Z_{X^*}(G)| \leq \begin{cases} d_2 \cdots d_n d & \text{if } n \geq 2, \\ d & \text{if } n = 1. \end{cases}$$

Proof. By induction on $n + d \geq 1$. If $n + d = 1$, then $n = 1$, $d = 0$ and the result is obvious. If $n = 1$, then the result is clear because G has at most d roots in K . Thus, we may assume $d \geq 1$ and $n \geq 2$. We can write G as

$$(\dagger) \quad G = G(t_1, \dots, t_n) = G_0(t_1, \dots, t_{n-1}) + G_1(t_1, \dots, t_{n-1})t_n + \dots + G_r(t_1, \dots, t_{n-1})t_n^r,$$

where $G_r \neq 0$ and $0 \leq r \leq d$. Let $\beta_1, \dots, \beta_{d_1}$ be the elements of A_1 . We set

$$H_k = H_k(t_2, \dots, t_n) := G(\beta_k, t_2, \dots, t_n) \quad \text{for } 1 \leq k \leq d_1.$$

Case (I): $H_k(t_2, \dots, t_n) = 0$ for some $1 \leq k \leq d_1$. From Eq. (\dagger) we get

$$H_k(t_2, \dots, t_n) = G_0(\beta_k, t_2, \dots, t_{n-1}) + G_1(\beta_k, t_2, \dots, t_{n-1})t_n + \dots + G_r(\beta_k, t_2, \dots, t_{n-1})t_n^r = 0.$$

Therefore $G_i(\beta_k, t_2, \dots, t_{n-1}) = 0$ for $i = 0, \dots, r$. Hence $t_1 - \beta_k$ divides $G_i(t_1, \dots, t_{n-1})$ for all i . Thus, by Eq. (\dagger) , we can write

$$G(t_1, \dots, t_n) = (t_1 - \beta_k)G'(t_1, \dots, t_n)$$

for some $G' \in S$. Notice that $\deg(G') + n = d - 1 + n < d + n$. Hence, by induction, we get

$$|Z_{X^*}(G)| \leq |Z_{X^*}(t_1 - \beta_k)| + |Z_{X^*}(G'(t_1, \dots, t_n))| \leq d_2 \cdots d_n + d_2 \cdots d_n(d - 1) = d_2 \cdots d_n d.$$

Case (II): $H_k(t_2, \dots, t_n) \neq 0$ for $1 \leq k \leq d_1$. Observe the inclusion

$$Z_{X^*}(G) \subset \bigcup_{k=1}^{d_1} (\{\beta_k\} \times Z(H_k)),$$

where $Z(H_k) = \{a \in A_2 \times \dots \times A_n \mid H_k(a) = 0\}$. As $\deg(H_k) + n - 1 < d + n$ and $d_i \leq d_{i+1}$ for all i , then by induction

$$|Z_{X^*}(G)| \leq \sum_{k=1}^{d_1} |Z(H_k)| \leq d_1 d_3 \cdots d_n d \leq d_2 d_3 \cdots d_n d,$$

as required. \square

Lemma 3.4. *Let $d_1, \dots, d_{n-1}, d', d$ be positive integers such that $d = \sum_{i=1}^k (d_i - 1) + \ell$ and $d' = \sum_{i=1}^{k'} (d_i - 1) + \ell'$ for some integers k, k', ℓ, ℓ' satisfying that $0 \leq k, k' \leq n - 2$ and $1 \leq \ell \leq d_{k+1} - 1$, $1 \leq \ell' \leq d_{k'+1} - 1$. If $d' \leq d$ and $d_i \leq d_{i+1}$ for all i , then $k' \leq k$ and*

$$(*) \quad -d_{k'+1} \cdots d_{n-1} + \ell' d_{k'+2} \cdots d_{n-1} \leq -d_{k+1} \cdots d_{n-1} + \ell d_{k+2} \cdots d_{n-1},$$

where $d_{k+2} \cdots d_{n-1} = 1$ (resp., $d_{k'+2} \cdots d_{n-1} = 1$) if $k = n - 2$ (resp., $k' = n - 2$).

Proof. First we show that $k' \leq k$. If $k' > k$, from the equality

$$\ell = (d - d') + \ell' + [(d_{k+1} - 1) + \dots + (d_{k'+1} - 1)],$$

we obtain that $\ell \geq d_{k+1}$, a contradiction. Thus, $k' \leq k$. Since $d_{k+2} \cdots d_{n-1}$ is a common factor of each term of Eq. $(*)$, we need only show the equivalent inequality:

$$(**) \quad d_{k+1} - \ell \leq (d_{k'+1} - \ell') d_{k'+2} \cdots d_{k+1}.$$

If $k = k'$, then $d_{k'+2} \cdots d_{k+1} = 1$ and $d - d' = \ell - \ell' \geq 0$. Hence, $\ell \geq \ell'$ and Eq. $(**)$ holds. If $k \geq k' + 1$, then

$$d_{k+1} - \ell \leq d_{k+1} \leq d_{k'+2} \cdots d_{k+1} \leq d_{k'+2} \cdots d_{k+1} (d_{k'+1} - \ell').$$

Thus, Eq. $(**)$ holds. \square

Lemma 3.5. *If $0 \neq G \in S$. Then, there are $r \geq 0$ distinct elements β_1, \dots, β_r in A_n and $G' \in S$ such that*

$$G = (t_n - \beta_1)^{a_1} \cdots (t_n - \beta_r)^{a_r} G', \quad a_i \geq 1 \text{ for all } i,$$

and $G'(t_1, \dots, t_{n-1}, \gamma) \neq 0$ for any $\gamma \in A_n$.

Proof. Fix a monomial ordering in S . If the degree of G is zero, we set $r = 0$ and $G = G'$. Assume that $\deg(G) > 0$. If $G(t_1, \dots, t_{n-1}, \gamma) \neq 0$ for all $\gamma \in A_n$, we set $G = G'$ and $r = 0$. If $G(t_1, \dots, t_{n-1}, \gamma) = 0$ for some $\gamma \in A_n$, then by the division algorithm there are F and H in S such that $G = (t_n - \gamma)F + H$, where H is a polynomial whose terms are not divisible by the leading term of $t_n - \gamma$, i.e., H is a polynomial in $K[t_1, \dots, t_{n-1}]$. Thus, as $G(t_1, \dots, t_{n-1}, \gamma) = 0$, we get that $H = 0$ and $G = (t_n - \gamma)F$. Since $\deg(F) < \deg(G)$, the result follows using induction on the total degree of G . \square

Proposition 3.6. *Let $G = G(t_1, \dots, t_n) \in S$ be a polynomial of total degree $d \geq 1$ such that $\deg_{t_i}(G) \leq d_i - 1$ for $i = 1, \dots, n$. If $d_i \leq d_{i+1}$ for all i and $d = \sum_{i=1}^k (d_i - 1) + \ell$ for some integers k, ℓ such that $1 \leq \ell \leq d_{k+1} - 1$, $0 \leq k \leq n - 1$, then*

$$|Z_{X^*}(G)| \leq d_{k+2} \cdots d_n (d_1 \cdots d_{k+1} - d_{k+1} + \ell),$$

where we set $d_{k+2} \cdots d_n = 1$ if $k = n - 1$.

Proof. We proceed by induction on n . By Lemma 3.5, there are $r \geq 0$ distinct elements β_1, \dots, β_r in A_n and $G' \in S$ such that

$$G = (t_n - \beta_1)^{a_1} \cdots (t_n - \beta_r)^{a_r} G', \quad a_i \geq 1 \text{ for all } i,$$

and $G'(t_1, \dots, t_{n-1}, \gamma) \neq 0$ for any $\gamma \in A_n$. Notice that $r \leq \sum_{i=1}^r a_i \leq d_n - 1$ because the degree of G in t_n is at most $d_n - 1$. We may assume that $A_n = \{\beta_1, \dots, \beta_{d_n}\}$. Let d'_i be the degree of $G'(t_1, \dots, t_{n-1}, \beta_i)$ and let $d' = \max\{d'_i \mid r + 1 \leq i \leq d_n\}$.

Case (I): Assume $n = 1$. Then, $k = 0$ and $d = \ell$. Then $|Z_{X^*}(G)| \leq \ell$ because a non-zero polynomial in one variable of degree d has at most d roots.

Case (II): Assume $n \geq 2$ and $k = 0$. Then, $d = \ell \leq d_1 - 1$. Hence, by Lemma 3.3, we get

$$|Z_{X^*}(G)| \leq d_2 \cdots d_n d = d_2 \cdots d_n \ell = d_{k+2} \cdots d_n (d_1 \cdots d_{k+1} - d_{k+1} + \ell),$$

as required.

Case (III): Assume $n \geq 2$, $k \geq 1$ and $d' = 0$. Then, $|Z_{X^*}(G)| = r d_1 \cdots d_{n-1}$. Thus, it suffices to show the inequality

$$r d_1 \cdots d_{n-1} \leq d_1 \cdots d_n - d_{k+1} \cdots d_n + \ell d_{k+2} \cdots d_n.$$

All terms of this inequality have $d_{k+2} \cdots d_{n-1}$ as a common factor. Hence, this case reduces to showing the following equivalent inequality

$$r d_1 \cdots d_{k+1} \leq d_n (d_1 \cdots d_{k+1} - d_{k+1} + \ell).$$

We can write $d_n = r + 1 + \delta$ for some $\delta \geq 0$. If we substitute d_n by $r + 1 + \delta$, we get the equivalent inequality

$$d_{k+1}(r + 1) \leq \ell r + d_1 \cdots d_{k+1} + \ell + \delta d_1 \cdots d_{k+1} - \delta d_{k+1} + \delta \ell.$$

We can write $d = r + \delta_1$ for some $\delta_1 \geq 0$. Next, if we substitute r by $\sum_{i=1}^k (d_i - 1) + \ell - \delta_1$ on the left hand side of this inequality, we get

$$0 \leq \ell[r + 1 + \delta - d_{k+1}] + d_{k+1}[d_1 \cdots d_k - 1 - \sum_{i=1}^k (d_i - 1) + \delta_1] + \delta[d_1 \cdots d_{k+1} - d_{k+1}].$$

Since $r + 1 + \delta - d_{k+1} \geq r + 1 + \delta - d_n = 0$ and $k \geq 1$, this inequality holds. This completes the proof of this case.

Case (IV): Assume $n \geq 2$, $k \geq 1$ and $d' \geq 1$. We may assume that $\beta_{r+1}, \dots, \beta_m$ are the elements β_i of $\{\beta_{r+1}, \dots, \beta_{d_n}\}$ such that $G'(t_1, \dots, t_{n-1}, \beta_i)$ has positive degree. We set

$$G'_i = G'(t_1, \dots, t_{n-1}, \beta_i)$$

for $r + 1 \leq i \leq m$. Notice that $d = \sum_{i=1}^r a_i + \deg(G') \geq r + d' \geq d'_i$. The polynomial

$$H := (t_n - \beta_1)^{a_1} \cdots (t_n - \beta_r)^{a_r}$$

has exactly $rd_1 \cdots d_{n-1}$ roots in X^* . Hence, counting the roots of G' that are not in $Z_{X^*}(H)$, we obtain:

$$(\star) \quad |Z_{X^*}(G)| \leq rd_1 \cdots d_{n-1} + \sum_{i=r+1}^m |Z(G'_i)|,$$

where $Z(G'_i)$ is the set of zeros of G'_i in $A_1 \times \cdots \times A_{n-1}$. For each $r + 1 \leq i \leq m$, we can write $d'_i = \sum_{i=1}^{k'_i} (d_i - 1) + \ell'_i$, with $1 \leq \ell'_i \leq d_{k'_i+1} - 1$. The proof of this case will be divided in three subcases.

Subcase (IV.a): Assume $\ell \geq r$ and $k = n - 1$. The degree of G'_i in the variable t_j is at most $d_j - 1$ for $j = 1, \dots, n - 1$. Hence, by Lemma 2.2, the non-zero polynomial G'_i cannot be the zero-function on $A_1 \times \cdots \times A_{n-1}$. Therefore, $|Z(G'_i)| \leq d_1 \cdots d_{n-1} - 1$ for $r + 1 \leq i \leq m$. Thus, by Eq. (\star), we get the required inequality

$$|Z_{X^*}(G)| \leq rd_1 \cdots d_{n-1} + (d_n - r)(d_1 \cdots d_{n-1} - 1) \leq d_1 \cdots d_n - d_n + \ell,$$

because in this case $d_{k+2} \cdots d_n = 1$ and $\ell \geq r$.

Subcase (IV.b): Assume $\ell > r$ and $k \leq n - 2$. Then, we can write

$$d - r = \sum_{i=1}^k (d_i - 1) + (\ell - r)$$

with $1 \leq \ell - r \leq d_{k+1} - 1$. Since $d'_i \leq d - r$ for $i = r + 1, \dots, m$, by applying Lemma 3.4 to the sequence $d_1, \dots, d_{n-1}, d'_i, d - r$, we get $k'_i \leq k$ for $r + 1 \leq i \leq m$. By induction hypothesis we can bound $|Z(G'_i)|$. Then, using Eq. (\star) and Lemma 3.4, we obtain:

$$\begin{aligned} |Z_{X^*}(G)| &\leq rd_1 \cdots d_{n-1} + \sum_{i=r+1}^m d_{k'_i+2} \cdots d_{n-1} (d_1 \cdots d_{k'_i+1} - d_{k'_i+1} + \ell'_i) \\ &\leq rd_1 \cdots d_{n-1} + (d_n - r)[(d_{k+2} \cdots d_{n-1})(d_1 \cdots d_{k+1} - d_{k+1} + \ell - r)]. \end{aligned}$$

Thus, by factoring out the common term $d_{k+2} \cdots d_{n-1}$, we need only show the inequality:

$$\begin{aligned} rd_1 \cdots d_{k+1} + (d_n - r)(d_1 \cdots d_{k+1} - d_{k+1} + \ell - r) &\leq \\ d_n(d_1 \cdots d_{k+1} - d_{k+1} + \ell). & \end{aligned}$$

After simplification, we get that this inequality is equivalent to $r(d_n - d_{k+1} + \ell - r) \geq 0$. This inequality holds because $d_n \geq d_{k+1}$ and $\ell > r$.

Subcase (IV.c): Assume $\ell \leq r$. We can write $d - r = \sum_{i=1}^s (d_i - 1) + \tilde{\ell}$, where $1 \leq \tilde{\ell} \leq d_{s+1} - 1$ and $s \leq k$. Notice that $s < k$. Indeed, if $s = k$, then from the equality

$$(\star\star) \quad d - r = \sum_{i=1}^s (d_i - 1) + \tilde{\ell} = \sum_{i=1}^k (d_i - 1) + \ell - r$$

we get that $\tilde{\ell} = \ell - r \geq 1$, a contradiction. Thus, $s \leq n - 2$. As $d - r \geq d'_i$, by applying Lemma 3.4 to $d_1, \dots, d_{n-1}, d'_i, d - r$, we have $k'_i \leq s \leq n - 2$ for $i = r + 1, \dots, m$. By induction hypothesis we can bound $|Z(G'_i)|$. Therefore, using Eq. (★) and Lemma 3.4, we obtain:

$$\begin{aligned} |Z_{X^*}(G)| &\leq rd_1 \cdots d_{n-1} + \sum_{i=r+1}^m [d_1 \cdots d_{n-1} - d_{k'_i+1} \cdots d_{n-1} + d_{k'_i+2} \cdots d_{n-1} \ell'_i] \\ &\leq rd_1 \cdots d_{n-1} + (d_n - r)[d_1 \cdots d_{n-1} - d_{s+1} \cdots d_{n-1} + d_{s+2} \cdots d_{n-1} \tilde{\ell}]. \end{aligned}$$

Thus, we need only show the inequality

$$\begin{aligned} rd_1 \cdots d_{n-1} + (d_n - r)[d_1 \cdots d_{n-1} - d_{s+1} \cdots d_{n-1} + d_{s+2} \cdots d_{n-1} \tilde{\ell}] &\leq \\ d_1 \cdots d_n - d_{k+1} \cdots d_n + d_{k+2} \cdots d_n \ell. & \end{aligned}$$

After cancelling out some terms, we get the following equivalent inequality:

$$(\ddagger) \quad d_{k+1} \cdots d_n - d_{k+2} \cdots d_n \ell \leq (d_n - r)[d_{s+1} \cdots d_{n-1} - d_{s+2} \cdots d_{n-1} \tilde{\ell}].$$

The proof now reduces to show this inequality.

Subcase (IV.c.1): Assume $k = n - 1$. Then, Eq. (‡) simplifies to

$$d_n - \ell \leq (d_n - r)[d_{s+1} \cdots d_{n-1} - d_{s+2} \cdots d_{n-1} \tilde{\ell}].$$

Since $d_n \geq r + 1$, it suffices to show the inequality

$$r + 1 - \ell \leq d_{s+2} \cdots d_{n-1} (d_{s+1} - \tilde{\ell}).$$

From Eq. (★★), we get

$$r + (1 - \ell) = \ell - \tilde{\ell} + \sum_{i=s+1}^{n-1} (d_i - 1) + (1 - \ell) = -\tilde{\ell} + d_{s+1} + \sum_{i=s+2}^{n-1} (d_i - 1).$$

Hence, the last inequality is equivalent to

$$\sum_{i=s+2}^{n-1} (d_i - 1) \leq (d_{s+2} \cdots d_{n-1} - 1)(d_{s+1} - \tilde{\ell}).$$

This inequality holds because $d_{s+2} \cdots d_{n-1} \geq \sum_{i=s+2}^{n-1} (d_i - 1) + 1$.

Subcase (IV.c.2): Assume $k \leq n - 2$. By canceling out the common term $d_{k+2} \cdots d_{n-1}$ in Eq. (‡), we obtain the following equivalent inequality

$$d_{k+1} d_n - d_n \ell \leq (d_n - r)(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}).$$

We rewrite this inequality as

$$r(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) \leq d_n[(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) - d_{k+1}] + \ell d_n.$$

Since $d_n \geq r + 1$ it suffices to show the inequality

$$\begin{aligned} r(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) &\leq \\ r[(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) - d_{k+1}] &+ [(d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) - d_{k+1}] + \ell d_n. \end{aligned}$$

After a quick simplification, this inequality reduces to

$$(r + 1)d_{k+1} \leq (d_{s+2} \cdots d_{k+1})(d_{s+1} - \tilde{\ell}) + \ell d_n.$$

From Eq. (★★), we get $r + 1 = (-\tilde{\ell} + d_{s+1}) + (\ell + \sum_{i=s+2}^k (d_i - 1))$. Hence, the last inequality is equivalent to

$$d_{k+1} \sum_{i=s+2}^k (d_i - 1) \leq d_{k+1}(d_{s+2} \cdots d_k - 1)(d_{s+1} - \tilde{\ell}) + \ell(d_n - d_{k+1}).$$

This inequality holds because $d_{s+2} \cdots d_k \geq \sum_{i=s+2}^k (d_i - 1) + 1$. This completes the proof of the proposition. \square

Corollary 3.7. *Let $d \geq 1$ be an integer. If $d_i \leq d_{i+1}$ for all i and $d = \sum_{i=1}^k (d_i - 1) + \ell$ for some integers k, ℓ such that $1 \leq \ell \leq d_{k+1} - 1$ and $0 \leq k \leq n - 1$, then*

$$\max\{|Z_{X^*}(F)| : F \in S_{\leq d}; F \not\equiv 0\} \leq d_{k+2} \cdots d_n (d_1 \cdots d_{k+1} - d_{k+1} + \ell).$$

Proof. Let $F = F(t_1, \dots, t_n) \in S$ be an arbitrary polynomial of total degree $d' \leq d$ such that $F(P) \neq 0$ for some $P \in X^*$. We can write $d' = \sum_{i=1}^{k'} (d_i - 1) + \ell'$ with $1 \leq \ell' \leq d_{k'+1} - 1$ and $0 \leq k' \leq k$. Let \prec be the graded reverse lexicographical order on the monomials of S . In this order $t_1 \succ \cdots \succ t_n$. For $1 \leq i \leq n$, let f_i be the polynomial $\prod_{\gamma \in A_i} (t_i - \gamma)$. Recall that $d_i = |A_i|$, i.e., f_i has degree d_i . By the division algorithm [1, Theorem 1.5.9, p. 30], we can write

$$(††) \quad F = h_1 f_1 + \cdots + h_n f_n + G',$$

for some $G' \in S$ with $\deg_{t_i}(G') \leq d_i - 1$ for $i = 1, \dots, n$ and $\deg(G') = d'' \leq d'$. If G' is a constant, by Eq. (††) and using that $0 \neq F(P) = G'(P)$, we get $Z_{X^*}(F) = \emptyset$. Thus, we may assume that the polynomial G' has positive degree d'' . We can write $d'' = \sum_{i=1}^{k''} (d_i - 1) + \ell''$, where $1 \leq \ell'' \leq d_{k''+1}$ and $0 \leq k'' \leq k'$. Notice that $Z_{X^*}(F) = Z_{X^*}(G')$. By Proposition 3.6, and applying Lemma 3.4 to the sequences d_1, \dots, d_n, d'', d' and d_1, \dots, d_n, d', d , we obtain

$$\begin{aligned} |Z_{X^*}(F)| = |Z_{X^*}(G')| &\leq d_1 \cdots d_n - d_{k''+1} \cdots d_n + d_{k''+2} \cdots d_n \ell'' \\ &\leq d_1 \cdots d_n - d_{k'+1} \cdots d_n + d_{k'+2} \cdots d_n \ell' \\ &\leq d_1 \cdots d_n - d_{k+1} \cdots d_n + d_{k+2} \cdots d_n \ell. \end{aligned}$$

Thus, $|Z_{X^*}(F)| \leq d_1 \cdots d_n - d_{k+1} \cdots d_n + d_{k+2} \cdots d_n \ell$, as required. \square

We come to the main result of this section.

Theorem 3.8. *Let K be a field and let $C_{X^*}(d)$ be the cartesian evaluation code of degree d on the finite set $X^* = A_1 \times \cdots \times A_n \subset K^n$. If $2 \leq d_i \leq d_{i+1}$ for all i , with $d_i = |A_i|$, and $d \geq 1$, then the minimum distance of $C_{X^*}(d)$ is given by*

$$\delta_{X^*}(d) = \begin{cases} (d_{k+1} - \ell) d_{k+2} \cdots d_n & \text{if } d \leq \sum_{i=1}^n (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum_{i=1}^n (d_i - 1), \end{cases}$$

where $k \geq 0$, ℓ are the unique integers such that $d = \sum_{i=1}^k (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

Proof. If $d \geq \sum_{i=1}^n (d_i - 1)$, then the minimum distance of $C_{X^*}(d)$ is equal to 1 by Theorem 3.1. Assume that $1 \leq d \leq \sum_{i=1}^n (d_i - 1) - 1$. We can write

$$A_i = \{\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,d_i}\}, \quad i = 1, \dots, n.$$

For $1 \leq i \leq k + 1$, consider the polynomials

$$f_i = \begin{cases} (\beta_{i,1} - t_i)(\beta_{i,2} - t_i) \cdots (\beta_{i,d_i-1} - t_i) & \text{if } 1 \leq i \leq k, \\ (\beta_{k+1,1} - t_{k+1})(\beta_{k+1,2} - t_{k+1}) \cdots (\beta_{k+1,\ell} - t_{k+1}) & \text{if } i = k + 1. \end{cases}$$

The polynomial $G = f_1 \cdots f_{k+1}$ has degree d and $G(\beta_{1,d_1}, \beta_{2,d_2}, \dots, \beta_{n,d_n}) \neq 0$. From the equality

$$\begin{aligned} Z_{X^*}(G) &= [(A_1 \setminus \{\beta_{1,d_1}\}) \times A_2 \times \cdots \times A_n] \cup \\ &\quad [\{\beta_{1,d_1}\} \times (A_2 \setminus \{\beta_{2,d_2}\}) \times A_3 \times \cdots \times A_n] \cup \\ &\quad \vdots \\ &\quad [\{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k-1,d_{k-1}}\} \times (A_k \setminus \{\beta_{k,d_k}\}) \times A_{k+1} \times \cdots \times A_n] \cup \\ &\quad [\{\beta_{1,d_1}\} \times \cdots \times \{\beta_{k,d_k}\} \times \{\beta_{k+1,1}, \dots, \beta_{k+1,\ell}\} \times A_{k+2} \times \cdots \times A_n], \end{aligned}$$

we get that the number of zeros of G in X^* is given by:

$$|Z_{X^*}(G)| = \sum_{i=1}^k (d_i - 1)(d_{i+1} \cdots d_n) + \ell d_{k+2} \cdots d_n = d_1 \cdots d_n - d_{k+1} \cdots d_n + \ell d_{k+2} \cdots d_n.$$

By Lemma 2.2, one has $|X^*| = d_1 \cdots d_n$. Therefore

$$\begin{aligned} \delta_{X^*}(d) &= \min\{\|\text{ev}_d(F)\| : \text{ev}_d(F) \neq 0; F \in S_{\leq d}\} = |X| - \max\{|Z_{X^*}(F)| : F \in S_{\leq d}; F \neq 0\} \\ &\leq d_1 \cdots d_n - |Z_{X^*}(G)| = (d_{k+1} - \ell) d_{k+2} \cdots d_n, \end{aligned}$$

where $\|\text{ev}_d(F)\|$ is the number of non-zero entries of $\text{ev}_d(F)$ and $F \neq 0$ means that F is not the zero function on X^* . Thus

$$\delta_{X^*}(d) \leq (d_{k+1} - \ell) d_{k+2} \cdots d_n.$$

The reverse inequality follows at once from Corollary 3.7. \square

Definition 3.9. If K is a finite field, the set $\mathbb{T} = \{(x_1, \dots, x_{n+1}) \in \mathbb{P}^n \mid x_i \in K^* \text{ for all } i\}$ is called a *projective torus* in \mathbb{P}^n , where $K^* = K \setminus \{0\}$.

As a consequence of our main result, we recover the following formula for the minimum distance of a parameterized code over a projective torus.

Corollary 3.10. [21, Theorem 3.5] *Let $K = \mathbb{F}_q$ be a finite field with $q \neq 2$ elements. If \mathbb{T} is a projective torus in \mathbb{P}^n and $d \geq 1$, then the minimum distance of $C_{\mathbb{T}}(d)$ is given by*

$$\delta_{\mathbb{T}}(d) = \begin{cases} (q-1)^{n-k-1}(q-1-\ell) & \text{if } d \leq (q-2)n-1, \\ 1 & \text{if } d \geq (q-2)n, \end{cases}$$

where k and ℓ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-2$ and $d = k(q-2) + \ell$.

Proof. If $A_i = K^*$ for $i = 1, \dots, n$, then $X^* = (K^*)^n$, $Y = \mathbb{T}$, and $d_i = q-1$ for all i . Since $\delta_{X^*}(d) = \delta_Y(d)$, the result follows at once from Theorem 3.8. \square

As another consequence of our main result, we recover a formula for the minimum distance of an evaluation code over an affine space.

Corollary 3.11. [4, Theorem 2.6.2] *Let $K = \mathbb{F}_q$ be a finite field and let Y be the image of \mathbb{A}^n under the map $\mathbb{A}^n \rightarrow \mathbb{P}^n$, $x \mapsto [(x, 1)]$. If $d \geq 1$, the minimum distance of $C_Y(d)$ is given by:*

$$\delta_Y(d) = \begin{cases} (q-\ell)q^{n-k-1} & \text{if } d \leq n(q-1)-1, \\ 1 & \text{if } d \geq n(q-1), \end{cases}$$

where k and ℓ are the unique integers such that $k \geq 0$, $1 \leq \ell \leq q-1$ and $d = k(q-1) + \ell$.

Proof. If $A_i = K$ for $i = 1, \dots, n$, then $X^* = K^n = \mathbb{A}^n$ and $d_i = q$ for all i . Since $\delta_{X^*}(d) = \delta_Y(d)$, the result follows at once from Theorem 3.8. \square

Example 3.12. If $X^* = \mathbb{F}_2^n$, then the basic parameters of $C_{X^*}(d)$ are given by

$$|X^*| = 2^n, \quad \dim C_{X^*}(d) = \sum_{i=0}^d \binom{n}{i}, \quad \delta_{X^*}(d) = 2^{n-d}, \quad 1 \leq d \leq n.$$

Example 3.13. Let $K = \mathbb{F}_9$ be a field with 9 elements. Assume that $A_i = K$ for $i = 1, \dots, 4$. For certain values of d , the basic parameters of $C_{X^*}(d)$ are given in the following table:

d	1	2	3	4	5	10	16	20	28	31	32
$ X^* $	6561	6561	6561	6561	6561	6561	6561	6561	6561	6561	6561
$\dim C_{X^*}(d)$	5	15	35	70	126	981	3525	5256	6526	6560	6561
$\delta_{X^*}(d)$	5832	5103	4374	3645	2916	567	81	45	5	2	1

4. CARTESIAN CODES OVER DEGENERATE TORI

Given a non decreasing sequence of positive integers d_1, \dots, d_n , we construct a cartesian code, over a degenerate torus, with prescribed parameters in terms of d_1, \dots, d_n .

Definition 4.1. Let $K = \mathbb{F}_q$ be a finite field and let $v = (v_1, \dots, v_n)$ be a sequence of positive integers. The set

$$X^* = \{(x_1^{v_1}, \dots, x_n^{v_n}) \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{A}^n,$$

is called a *degenerate torus* of type v .

The main result of this section is:

Theorem 4.2. *Let $2 \leq d_1 \leq \dots \leq d_n$ be a sequence of integers. Then, there is a finite field $K = \mathbb{F}_q$ and a degenerate torus X^* such that the length of $C_{X^*}(d)$ is $d_1 \cdots d_n$, its dimension is*

$$\begin{aligned} \dim_K C_{X^*}(d) = & \binom{n+d}{d} - \sum_{1 \leq i \leq n} \binom{n+d-d_i}{d-d_i} + \sum_{i < j} \binom{n+d-(d_i+d_j)}{d-(d_i+d_j)} - \\ & \sum_{i < j < k} \binom{n+d-(d_i+d_j+d_k)}{d-(d_i+d_j+d_k)} + \dots + (-1)^n \binom{n+d-(d_1+\dots+d_n)}{d-(d_1+\dots+d_n)}, \end{aligned}$$

its minimum distance is 1 if $d \geq \sum_{i=1}^n (d_i - 1)$, and

$$\delta_{X^*}(d) = (d_{k+1} - \ell) d_{k+2} \cdots d_n \quad \text{if } d \leq \sum_{i=1}^n (d_i - 1) - 1,$$

where $k \geq 0$, ℓ are the unique integers such that $d = \sum_{i=1}^k (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

Proof. Pick a prime number p relatively prime to $m = d_1 \cdots d_n$. Then, by Euler formula, $p^{\varphi(m)} \equiv 1 \pmod{m}$, where φ is the Euler function. We set $q = p^{\varphi(m)}$. Hence, there exists a finite field \mathbb{F}_q with q elements such that d_i divides $q - 1$ for $i = 1, \dots, n$. We set $K = \mathbb{F}_q$.

Let β be a generator of the cyclic group (K^*, \cdot) . There are positive integers v_1, \dots, v_n such that $q - 1 = v_i d_i$ for $i = 1, \dots, n$. Notice that d_i is equal to $o(\beta^{v_i})$, the order of β^{v_i} for $i = 1, \dots, n$. We set $A_i = \langle \beta^{v_i} \rangle$, where $\langle \beta^{v_i} \rangle$ is the subgroup of K^* generated by β^{v_i} . If X^* is the cartesian product of A_1, \dots, A_n , it not hard to see that X^* is given by

$$X^* = \{(x_1^{v_1}, \dots, x_n^{v_n}) \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{A}^n,$$

i.e., X^* is a degenerate torus of type $v = (v_1, \dots, v_n)$. The length of $|X^*|$ is $d_1 \cdots d_n$ because $|A_i| = d_i$ for all i . The formulae for the dimension and the minimum distance of $C_{X^*}(d)$ follow from Theorems 3.1 and 3.8. \square

Remark 4.3. Let $K = \mathbb{F}_q$ be a finite field and let β be a generator of the cyclic group (K^*, \cdot) . If X^* is a degenerate torus of type $v = (v_1, \dots, v_n)$, then X^* is the cartesian product of A_1, \dots, A_n , where A_i is the cyclic group generated by β^{v_i} . Thus, if $d_i = |A_i|$ for $i = 1, \dots, n$, the affine evaluation code over X^* is a cartesian code. Hence, according to Theorem 3.1 and 3.8, the basic parameters of $C_{X^*}(d)$ can be computed in terms of d_1, \dots, d_n as in Theorem 4.2. Therefore, we are recovering the main results of [9, 10].

As an illustration of Theorem 4.2 consider the following example.

Example 4.4. Consider the sequence $d_1 = 2, d_2 = 5, d_3 = 9$. The prime number $q = 181$ satisfies that d_i divides $q - 1$ for all i . In this case $v_1 = 90, v_2 = 36, v_3 = 20$. The basic parameters of the cartesian codes $C_{X^*}(d)$, over the degenerate torus

$$X^* = \{(x_1^{90}, x_2^{36}, x_3^{20}) \mid x_i \in \mathbb{F}_{181}^* \text{ for } i = 1, 2, 3\},$$

are shown in the following table. Notice that the regularity of $S[u]/I(Y)$ is 13.

d	1	2	3	4	5	6	7	8	9	10	11	12	13
$ X^* $	90	90	90	90	90	90	90	90	90	90	90	90	90
$\dim C_{X^*}(d)$	4	9	16	25	35	45	55	65	74	81	86	89	90
$\delta_{X^*}(d)$	45	36	27	18	9	8	7	6	5	4	3	2	1

Notice that if $K' = \mathbb{F}_9$, and we pick subsets A_1, A_2, A_3 of K' with $|A_1| = 2, |A_2| = 5, |A_3| = 9$, the cartesian evaluation code $C_{X'}(d)$, over the set $X' = A_1 \times A_2 \times A_3$, has the same parameters that $C_{X^*}(d)$ for any $d \geq 1$.

Acknowledgments. We thank the referees for their careful reading of the paper and for the improvements that they suggested.

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