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# A Characterisation of Tangent Subplanes of PG$(2, q^3)$

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**Abstract**

In [2], the authors determine the representation of order-$q$-subplanes and order-$q$-sublines of PG$(2, q^3)$ in the Bruck-Bose representation in PG$(6, q)$. In particular, they showed that an order-$q$-subplane of PG$(2, q^3)$ corresponds to a certain ruled surface in PG$(6, q)$. In this article we show that the converse holds, namely that any ruled surface satisfying the required properties corresponds to a tangent order-$q$-subplane of PG$(2, q^3)$.

## 1 Introduction

We begin with a brief introduction to 2-spreads in PG$(5, q)$, and the Bruck-Bose representation of PG$(2, q^3)$ in PG$(6, q)$, and introduce the notation we will use.

A 2-*spread* of PG$(5, q)$ is a set of $q^3 + 1$ planes that partition PG$(5, q)$. The following construction of a regular 2-spread of PG$(5, q)$ will be needed. Embed PG$(5, q)$ in PG$(5, q^3)$ and let $g$ be a line of PG$(5, q^3)$ disjoint from PG$(5, q)$. The Frobenius automorphism of GF$(q^3)$ where $x \mapsto x^q$ induces a collineation of PG$(5, q^3)$. Let $g^q$, $g^{q^2}$ be the conjugate lines of $g$; both of these are disjoint from $\mathrm{PG}(5,q)$. Let $P_i$ be a point on $g$; then the plane $\langle P_i, P_i^q, P_i^{q^2}\rangle$ meets $\mathrm{PG}(5,q)$ in a plane. As $P_i$ ranges over all the points of $g$, we get $q^3+1$ planes of $\mathrm{PG}(5,q)$ that partition $\mathrm{PG}(5,q)$. These planes form a regular spread $\mathcal{S}$ of $\mathrm{PG}(5,q)$. The lines $g$, $g^q$, $g^{q^2}$ are called the (conjugate skew) *transversal lines* of the spread $\mathcal{S}$. Conversely, given a regular 2-spread in $\mathrm{PG}(5,q)$, there is a unique set of three (conjugate skew) transversal lines in $\mathrm{PG}(5,q^3)$ that generate $\mathcal{S}$ in this way. See [6] for more information on 2-spreads.

We work in linear representation of a finite translation plane $\mathcal{P}$ of dimension at most three over its kernel, an idea which was developed independently by André [1] and Bruck and Bose [4, 5]. Let $\Sigma_\infty$ be a hyperplane of $\mathrm{PG}(6,q)$ and let $\mathcal{S}$ be a 2-spread of $\Sigma_\infty$. We use the phrase *a subspace of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ to mean a subspace of $\mathrm{PG}(6,q)$ that is not contained in $\Sigma_\infty$. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in $\mathrm{PG}(6,q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order $q^3$. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity $\ell_\infty$ have a natural correspondence to the elements of the 2-spread $\mathcal{S}$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane $\mathrm{PG}(2,q^3)$ if and only if $\mathcal{S}$ is a regular 2-spread of $\Sigma_\infty \cong \mathrm{PG}(5,q)$ (see [3]).

We will be using the cubic extension $\mathrm{PG}(6,q^3)$ of $\mathrm{PG}(6,q)$. If $K$ is a subspace or curve of $\mathrm{PG}(6,q)$, we use $K^*$ to denote the natural extension of $K$ to $\mathrm{PG}(6,q^3)$.

## 2 The characterisation

In [2], the authors prove the following result that an order-$q$-subplane of $\mathrm{PG}(2,q^3)$ corresponds to a certain ruled surface in $\mathrm{PG}(6,q)$. In this article we show that the converse holds, namely that any ruled surface satisfying the required properties corresponds to a tangent order-$q$-subplane of $\mathrm{PG}(2,q^3)$. We use the notation of Section 1 and recall the following result.

**Theorem 2.1** [2, Theorem 2.7] *Let $B$ be an order-$q$-subplane of* $\mathrm{PG}(2,q^3)$ *that is tangent to $\ell_\infty$ in the point $T$. Let $\pi_T$ be the spread element corresponding to $T$. Then $B$ determines a set $\mathcal{B}$ of points in* $\mathrm{PG}(6,q)$ *(where the affine points of $B$ correspond to the affine points of $\mathcal{B}$) such that:*

(a) *$\mathcal{B}$ is a ruled surface with conic directrix $\mathcal{C}$ contained in the plane $\pi_T \in \mathcal{S}$, and normal rational curve directrix $\mathcal{N}$ contained in a 3-space $\Sigma$ that meets $\Sigma_\infty$ in a spread element (distinct from $\pi_T$). The points of $\mathcal{B}$ lie on $q+1$ pairwise disjoint generator lines joining $\mathcal{C}$ to $\mathcal{N}$.*

(b) *The* $q+1$ *generator lines of* $\mathcal{B}$ *joining* $\mathcal{C}$ *to* $\mathcal{N}$ *are determined by a projectivity from* $\mathcal{C}$ *to* $\mathcal{N}$.

(c) *When we extend* $\mathcal{B}$ *to* $\mathrm{PG}(6, q^3)$, *it contains the conjugate transversal lines* $g, g^q, g^{q^2}$ *of the spread* $\mathcal{S}$.

In this article we prove the converse of this result.

**Theorem 2.2** *In* $\mathrm{PG}(6, q)$, *let* $\mathcal{C}$ *be a conic in a spread element* $\pi$ *such that in the cubic extension* $\mathrm{PG}(6, q^3)$, $\mathcal{C}^*$ *contains the three transversal points* $P = \pi^* \cap g$, $P^q = \pi^* \cap g^q$, $P^{q^2} = \pi^* \cap g^{q^2}$. *Let* $\Sigma$ *be a 3-space of* $\mathrm{PG}(6, q) \setminus \Sigma_\infty$ *about a spread element* $\alpha$ *distinct from* $\pi$. *Let* $\mathcal{N}$ *be a normal rational curve in* $\Sigma$ *that in the cubic extension contains the points* $Q = \alpha^* \cap g, Q^q = \alpha^* \cap g^q, Q^{q^2} = \alpha^* \cap g^{q^2}$. *In* $\mathrm{PG}(6, q^3)$, *let* $\mathcal{B}^*$ *be the unique ruled surface with directrices* $\mathcal{C}^*, \mathcal{N}^*$ *defined by the projectivity that maps* $P^{q^i} \mapsto Q^{q^i}$, $i = 1, 2, 3$. *Then the ruled surface* $\mathcal{B}$ *of* $\mathrm{PG}(6, q)$ *corresponds to an order-q-subplane of* $\mathrm{PG}(2, q^3)$ *that is tangent to* $\ell_\infty$.

To simplify the following statements, we define a **special conic** of a spread element $\pi$ to be a conic that in the cubic extension $\mathrm{PG}(6, q^3)$ contains the transversal points $P = \pi^* \cap g, P^q = \pi^* \cap g^q, P^{q^2} = \pi^* \cap g^{q^2}$. A **special normal rational curve** in a 3-space $\Sigma$ of $\mathrm{PG}(6, q) \setminus \Sigma_\infty$ through a spread element $\alpha \neq \pi$ is one which in the cubic extension $\mathrm{PG}(6, q^3)$ contains the three transversal points $Q = \alpha^* \cap g, Q^q = \alpha^* \cap g^q, Q^{q^2} = \alpha^* \cap g^{q^2}$. Note that a special normal rational curve is disjoint from $\Sigma_\infty$.

This allows us to make a compact statement that combines Theorems 2.1 and 2.2.

**Corollary 2.3** *Let* $\mathcal{B}$ *be a ruled surface of* $\mathrm{PG}(6, q)$ *defined by a projectivity from a conic directrix* $\mathcal{C}$ *to a normal rational curve directrix* $\mathcal{N}$. *Then* $\mathcal{B}$ *corresponds to an order-q-subplane of* $\mathrm{PG}(2, q^3)$ *if and only if* $\mathcal{C}$ *is a special conic in a spread element* $\pi$, $\mathcal{N}$ *is a special normal rational curve in a 3-space about a spread element distinct from* $\pi$, *and in the cubic extension* $\mathrm{PG}(6, q^3)$ *of* $\mathrm{PG}(6, q)$, $\mathcal{B}$ *contains the transversals of the regular spread* $\mathcal{S}$.

We will prove this result by counting. By Theorem 2.1, a tangent order-$q$-subplane of $\mathrm{PG}(2, q^3)$ corresponds in $\mathrm{PG}(6, q)$ to a ruled surface with a special conic directrix and a special normal rational curve directrix that when extended to $\mathrm{PG}(6, q^3)$ contains the transversals of the spread $\mathcal{S}$. We show the converse is true by counting the number of tangent order-$q$-subplanes of $\mathrm{PG}(2, q^3)$, and the number of such ruled surfaces in $\mathrm{PG}(6, q)$ and showing that the two sets have the same number of elements. We proceed with a series of lemmas.

**Lemma 2.4** *The number of tangent order-$q$-subplanes of* $\mathrm{PG}(2, q^3)$ *through a fixed point* $T$ *of* $\ell_\infty$ *is* $q^7(q^3-1)(q^2+q+1)$.

**Proof** We first count the total number of order-$q$-subplanes in $\mathrm{PG}(2, q^3)$, it is

$$\frac{(q^6+q^3+1)(q^6+q^3)q^6(q^6-2q^3+1)}{(q^2+q+1)(q^2+q)q^2(q^2-2q+1)} = q^6(q^6+q^3+1)(q^2-q+1)(q^2+q+1).$$

Next we count the number $x$ of order-$q$-subplanes tangent to $\ell_\infty$. We count in two ways the number of pairs $(m, \pi)$ where $m$ is a line of $\mathrm{PG}(2, q^3)$ tangent to an order-$q$-subplane $\pi$. We have

$$q^6(q^6+q^3+1)(q^2-q+1)(q^2+q+1) \times (q^2+q+1)(q^3-q) = (q^6+q^3+1)x$$

and so $x = q^7(q^2-q+1)(q^2+q+1)^2(q-1)(q+1)$. As the subgroup $\mathrm{PGL}(3, q^3)$ fixing the line $\ell_\infty$ is transitive on the points of $\ell_\infty$, the number of order-$q$-subplanes tangent to $\ell_\infty$ at the point $T \in \ell_\infty$ is $x/(q^3+1) = q^7(q^3-1)(q^2+q+1)$. □

**Lemma 2.5** *Let* $\Sigma$ *be a 3-space of* $\mathrm{PG}(6, q) \setminus \Sigma_\infty$ *about a spread element. The number of special normal rational curves in* $\Sigma$ *is* $q^3(q^3-1)$.

**Proof** By [2, Theorem 2.5], the number of special normal rational curves in $\Sigma$ is equal to the number of order-$q$-sublines of a line $\ell$ ($\ell \neq \ell_\infty$) that are disjoint from $\ell_\infty$. There are

$$\binom{q^3+1}{3} \Big/ \binom{q+1}{3} = q^2(q^2+q+1)(q^2-q+1)$$

sublines of $\ell$. Of these,

$$\binom{q^3}{2} \Big/ \binom{q}{2} = q^2(q^2+q+1)$$

contain the point $\ell \cap \ell_\infty$. Hence there are $q^2(q^2+q+1)(q^2-q) = q^3(q^3-1)$ order-$q$-sublines of $\ell$ that are disjoint from $\ell_\infty$. □

**Lemma 2.6** *Two points in a spread element* $\pi$ *lie in a unique special conic of* $\pi$. *Further, every special conic of* $\pi$ *is non-degenerate.*

**Proof** In the cubic extension $\mathrm{PG}(6, q^3)$, $\pi^*$ contain the three transversal points $P = \pi^* \cap g, P^q = \pi^* \cap g^q, P^{q^2} = \pi^* \cap g^{q^2}$. Let $A, B$ be two points of $\pi$. We first show that $A, B, P, P^q, P^{q^2}$ are five points, no three collinear.

If the line $PP^q$ meets $\pi$ in a point $X$, then $X^q \in (PP^q)^q = P^qP^{q^2}$. As $X \in \pi$, $X^q = X$, and so $X, P, P^q, P^{q^2}$ are collinear, a contradiction as $P, P^q, P^{q^2}$ generate a plane and so are not collinear. So the lines $PP^q$, $PP^{q^2}$, $P^qP^{q^2}$ are all disjoint from $\pi$, that is, no point of $\pi$ is on one of these lines.

Next we show that the line $m = AB$ does not contain any of $P, P^q, P^{q^2}$. As $m$ is a line of $\pi$, we have $m^q = m$. If $P \in m$, then $P^q \in m^q = m$, and similarly $P^{q^2} \in m$, a contradiction. So $m$ does not contain $P$, $P^q$ or $P^{q^2}$.

Hence we can pick any two points $A, B$ of $\pi$ and the five points $A, B, P, P^q, P^{q^2}$ are no three collinear, and so lie in a unique non-degenerate conic $\mathcal{C}^*$ of $\pi^*$. This conic is fixed by the Frobenius automorphism $x \mapsto x^q$, and so $\mathcal{C}$ is a conic of $\pi$. Note that this also means that any special conic of $\pi$ is non-degenerate. $\square$

**Lemma 2.7** *The number of special conics in a spread element $\pi$ is $q^2 + q + 1$.*

**Proof** We want to count the number of conics of $\pi$ that in the cubic extension $\pi^*$ contain the three transversal points $P = \pi^* \cap g$, $P^q = \pi^* \cap g^q$, $P^{q^2} = \pi^* \cap g^{q^2}$. By Lemma 2.6, two points $A, B$ of $\pi$ lie in a unique special conic of $\pi$. The number of ways to choose $A, B$, so that the conic is distinct is $(q^2 + q + 1)(q^2 + q)/(q + 1)q = q^2 + q + 1$ as required. $\square$

**Lemma 2.8** *The number of triples $(\mathcal{C}, \mathcal{N}, \mathcal{B})$ where $\mathcal{C}$ is a special conic in a fixed spread element $\pi$, $\mathcal{N}$ is a special normal rational curve in any 3-space of* $\mathrm{PG}(6, q) \setminus \Sigma_\infty$ *about a spread element $\alpha \neq \pi$, and $\mathcal{B}$ is the unique ruled surface with directrices $\mathcal{C}, \mathcal{N}$ such that in the cubic extension* $\mathrm{PG}(6, q^3)$, *$\mathcal{B}^*$ contains the transversal lines $g, g^q, g^{q^2}$ is $q^9(q^3 - 1)(q^2 + q + 1)$.*

**Proof** In Lemma 2.7 we show that the number of special conics in a fixed spread element $\pi$ is $q^2 + q + 1$. There are $q^3$ choices for the spread element $\alpha$, and each spread element lies in $q^3$ 3-spaces of $\mathrm{PG}(6, q) \setminus \Sigma_\infty$. In Lemma 2.5 we showed that the number of special normal rational curves in a 3-space is $q^3(q^3 - 1)$. Finally, as a projectivity is uniquely determined by the image of three points, in the cubic extension $\mathrm{PG}(6, q^3)$ there is a unique ruled surface $\mathcal{B}^*$ with directrices $\mathcal{C}^*$ and $\mathcal{N}^*$ that contains the transversal lines $g, g^q, g^{q^2}$ of the spread $\mathcal{S}$. We now show that $\mathcal{B}^*$ meets $\mathrm{PG}(6, q)$ in a ruled surface $\mathcal{B}$.

The Frobenius automorphism $\sigma\colon x \mapsto x^q$ fixes $\mathcal{C}$ and $\mathcal{N}$ pointwise, and also fixes the set $\{g, g^q, g^{q^2}\}$. As $q \geq 2$, $\mathcal{C}$ has at least 3 points in $\pi$, and so $\mathcal{C}^*, \mathcal{C}^{*q}$ have at least six common points, hence $\sigma$ fixes $\mathcal{C}^*$. Similarly, $\sigma$ fixes $\mathcal{N}^*$. Thus $\sigma$ fixes $\mathcal{B}^*$ since $\mathcal{B}^*$ is determined by a projectivity, and the three lines $\{g, g^q, g^{q^2}\}$ uniquely determine this projectivity. As $\sigma$ fixes exactly the points of $\mathrm{PG}(6, q)$, it follows that $\mathcal{B}^*$ meets $\mathrm{PG}(6, q)$ in a ruled surface $\mathcal{B}$ with directrices $\mathcal{C}, \mathcal{N}$. That is, $\mathcal{B}$ satisfies the conditions of the lemma. Hence the number of triples is $(q^2 + q + 1) \times (q^3 \times q^3 \times q^3(q^3 - 1)) \times 1$ as required. $\square$

**Proof of Theorem 2.2** To complete the proof of Theorem 2.2, we count the number of triples $(\mathcal{C}, \mathcal{N}, \mathcal{B})$ where $\mathcal{B}$ is a ruled surface of $\mathrm{PG}(6, q)$ that corresponds to a tangent order-$q$-subplane of $\mathrm{PG}(2, q^3)$ through a fixed point $T$ of $\ell_\infty$, and $\mathcal{B}$ has conic directrix $\mathcal{C}$ and normal rational curve directrix $\mathcal{N}$. In Lemma 2.4 we showed that the number of tangent order-$q$-subplanes through a fixed point $T$ is $q^7(q^3-1)(q^2+q+1)$.

Now a tangent order-$q$-subplane that meets $\ell_\infty$ in the point $T$ contains $q^2$ order-$q$-sublines that are not through $T$. By [2, Theorem 2.5], each of these sublines corresponds to a special normal rational curve in some 3-space about a spread element. Moreover, this correspondence is exact. Hence a ruled surface $\mathcal{B}$ that corresponds to a tangent order-$q$-subplane has a exactly one conic directrix, and $q^2$ normal rational curve directrices. Thus the number of triples is $q^2 \times q^7(q^3-1)(q^2+q+1)$. This is the same as the number of triples in Lemma 2.8.

Hence the number of ruled surfaces satisfying the conditions of Theorem 2.2 is equal to the number of ruled surfaces that correspond to tangent order-$q$-subplanes. Hence every ruled surface satisfying the conditions of Theorem 2.2 does indeed correspond to a tangent order-$q$-subplane as required. □

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