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$\mathbb{Z}_2\mathbb{Z}_4$-Additive Formally Self-Dual Codes

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Abstract We study odd and even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes. The images of these codes are binary codes whose weight enumerators are that of a formally self-dual code but may not be linear. Three constructions are given for formally self-dual codes and existence theorems are given for codes of each type defined in the paper.

Keywords Formally self-dual codes · Type I codes · Type II codes · $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

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1 Introduction

Self-dual codes over fields have been a widely studied object since the beginning of coding theory. These codes are important in coding theory and have interesting relations with finite designs and unimodular lattices. Formally self-dual codes have weight enumerators that satisfy many of the same algebraic conditions of self-dual codes without being self-dual themselves. It has been a long standing question to classify formally self-dual codes and self-dual codes.

Following the major results in [23], where it was shown that interesting binary codes were found as the Gray image of quaternary codes, a great deal of study was given to codes over $\mathbb{Z}_4$. By the results of Delsarte decades earlier, this study was extended to consider $\mathbb{Z}_2\mathbb{Z}_4$ codes [6], [7]. Specifically, because these codes had binary images that were propelinear codes [29].

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One of the major tools in the constructions of self-dual codes is the building up construction. Essentially, it is a technique for constructing larger self-dual codes from smaller ones. Building up constructions were first given for binary codes in [9]. They were extended in many places, for example in [15], [18], [19], and [5]. Recently, these ideas were extended to formally self-dual codes in [24] and [16]. In this work, we study formally self-dual $\mathbb{Z}_2\mathbb{Z}_4$ codes and find various building up constructions from formally self-dual $\mathbb{Z}_2\mathbb{Z}_4$ codes.

1.1 $\mathbb{Z}_2\mathbb{Z}_4$-Additive Codes

Denote by $\mathbb{Z}_2$ and $\mathbb{Z}_4$ the rings of integers modulo 2 and modulo 4, respectively. Let $\mathbb{Z}_2^n$ and $\mathbb{Z}_4^n$ denote the space of n-tuples over these rings. We say that a binary code is any non-empty subset C of $\mathbb{Z}_2^n$, and if that subcode is a vector space then we say that it is a linear code. Similarly, any non-empty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a submodule of $\mathbb{Z}_4^n$ is called a quaternary linear code.

In 1973, Delsarte (see [13]), defined additive codes as subgroups of the underlying abelian group in a translation association scheme. For the binary Hamming scheme, namely, when the underlying abelian group is of order 2^n, the only structures for the abelian group are those of the form $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, with $\alpha + 2\beta = n$ ([14]). This means that the subgroups $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ are the only additive codes in a binary Hamming scheme. We distinguish them from additive codes over finite fields (see [2,3,25]), by calling them $\mathbb{Z}_2\mathbb{Z}_4$-additive codes (see [4,6,7,21,28,29]).

The structure of these codes is given as follows. We write $\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ as $\mathbf{v} = (\mathbf{v}_1|\mathbf{v}_2)$ where $\mathbf{v}_1 = (x_1, \ldots, x_\alpha) \in \mathbb{Z}_2^\alpha$ and $\mathbf{v}_2 = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta$.

We can now define a Gray map for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, by taking an extension of the usual Gray map. Recall that the usual Gray map is $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, $\phi(3) = (1,0)$. We define $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, by $\Phi(\mathbf{v}_1|\mathbf{v}_2) = (\mathbf{v}_1|\phi(y_1), \ldots, \phi(y_\beta))$. In general, we denote a $\mathbb{Z}_2\mathbb{Z}_4$-additive code by $\mathcal{C}$ and a binary code by C. Moreover, for any $\mathbb{Z}_2\mathbb{Z}_4$ code in $\mathbb{Z}_2^\alpha\mathbb{Z}_4^\beta$ we shall say that $n = \alpha + 2\beta$.

For vectors $\mathbf{v}_1 \in \mathbb{Z}_2^n$ and $\mathbf{v}_2 \in \mathbb{Z}_4^\beta$ we denote by $wt_H(\mathbf{v}_1)$ the Hamming weight of $\mathbf{v}_1$ and by $wt_L(\mathbf{v}_2)$ the Lee weight of $\mathbf{v}_2$. Then, for a vector $\mathbf{v} = (\mathbf{v}_1|\mathbf{v}_2) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we define the weight of $\mathbf{v}$, denoted by $wt(\mathbf{v})$, as $wt_H(\mathbf{v}_1) + wt_L(\mathbf{v}_2)$. Note that since $wt(\mathbf{v}) = wt_H(\Phi(\mathbf{v}))$, the map Φ is an isometry.

Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then this code is also isomorphic to an abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. We say $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group. It follows then that it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords and the number of order two codewords in $\mathcal{C}$ is $2^{\gamma+\delta}$. Denote by X and Y the set of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ coordinate positions respectively. It is immediate that $|X| = \alpha$ and $|Y| = \beta$. We make the convention that X corresponds to the first α coordinates and Y corresponds to the last β coordinates.

Let $\mathcal{C}_X$ be the punctured code of $\mathcal{C}$ by deleting the coordinates outside X and let $\mathcal{C}_Y$ be the punctured code of $\mathcal{C}$ by deleting the coordinates outside Y. Denote by $\mathcal{C}_b$ the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$. The code $\mathcal{C}_b$ is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$.

For a $\mathbb{Z}_2\mathbb{Z}_4$ code we say that $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \delta; \kappa)$, where γ, δ and κ are defined as above. Additionally, we extend this notation to the binary images as follows. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, which is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. We say that the binary image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*linear code* of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$.

We see that $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are a generalization of both binary linear codes and quaternary linear codes. When $\beta = 0$, the binary code $C = \mathcal{C}$ corresponds to a binary linear code and when $\alpha = 0$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a quaternary linear code.

In [7], it is shown that a $\mathbb{Z}_2\mathbb{Z}_4$-addtive code is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with standard generator matrix of the form:

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right), \tag{1}$$

where I_k is the identity matrix of size $k \times k$; T_b, S_b are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$.

We denote by $\mathbf{1}$ the all ones vector, and by $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ the vector with the first α coordinates equal 1 and the last β coordinates equal 2.

Definition 1 *We define a binary code C to be antipodal if for any codeword $\mathbf{z} \in C$, we have that $\mathbf{z} + \mathbf{1} \in C$. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we say that $\mathcal{C}$ is antipodal if $\Phi(\mathcal{C})$ is antipodal.*

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is antipodal if and only if $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in \mathcal{C}$. Notice also that the vector $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$ corresponds to the all one binary vector under the Gray map.

As in [7], we define the inner product for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ as

$$[\mathbf{u}, \mathbf{v}] = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4,$$

where the computations are made taking the zeros and ones in the α binary coordinates as quaternary zeros and ones, respectively.

Note that the inner product is an element of $\mathbb{Z}_4$ and not $\mathbb{Z}_2$. We denote by $\mathcal{C}^\perp$ the $\mathbb{Z}_2\mathbb{Z}_4$-*additive dual code* of $\mathcal{C}$; i.e.,

$$\mathcal{C}^\perp = \{\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid [\mathbf{u}, \mathbf{v}] = 0 \text{ for all } \mathbf{u} \in \mathcal{C}\}.$$

The code $\mathcal{C}$ is called self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and self-dual if $\mathcal{C}^\perp = \mathcal{C}$. For a complete description of self-dual codes and an extensive bibliography see [30].

If $C = \phi(\mathcal{C})$, the binary code $\Phi(\mathcal{C}^\perp)$ is denoted by $C_\perp$ and called the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C.

If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$ code with parameters $(\alpha, \beta; \gamma, \delta; \kappa)$ then $\mathcal{C}^\perp$ has parameters $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$ where

$$\begin{aligned}\bar{\gamma} &= \alpha + \gamma - 2\kappa,\\ \bar{\delta} &= \beta - \gamma - \delta + \kappa,\\ \bar{\kappa} &= \alpha - \kappa.\end{aligned}$$

1.2 $\mathbb{Z}_2\mathbb{Z}_4$ Formally Self-Dual Codes

In general, a code C over any ring is said to be formally self-dual if its weight enumerator is the same as the weight enumerator of its orthogonal. For example, any self-dual code is necessarily formally self-dual but, of course, there are formally self-dual codes that are not self-dual. For quaternary codes, a code can be formally self-dual with respect to the Lee weight enumerator but not with respect to the Hamming weight enumerator and vice versa. We shall define a weight enumerator for $\mathbb{Z}_2\mathbb{Z}_4$ codes and give the MacWilliams relations, and then define what we mean for formally self-dual codes in this case.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Define the weight enumerator of $\mathcal{C}$ to be

$$W_{\mathcal{C}}(x, y) = \sum_{\mathbf{c} \in \mathcal{C}} x^{n - wt(\mathbf{c})} y^{wt(\mathbf{c})}, \tag{2}$$

where $n = \alpha + 2\beta$. We know from [6,14,29] that for the weight enumerator defined in (2) we have

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(x + y, x - y).$$

For binary codes we have the usual Hamming weight enumerator; i.e.,

$$W_C(x, y) = \sum_{\mathbf{c} \in C} x^{n - wt_H(\mathbf{c})} y^{wt_H(\mathbf{c})}.$$

Definition 2 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $W_{\mathcal{C}^\perp}(x, y) = W_{\mathcal{C}}(x, y)$, then C is a $\mathbb{Z}_2\mathbb{Z}_4$-additive, or shortly $\mathbb{Z}_2\mathbb{Z}_4$, formally self-dual code.*

Example 1 *Consider the codes generated by the following matrices:*

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right).$$

Let $\mathcal{C}$ be the code generated by the first matrix and $\mathcal{D}$ be the code generated by the second matrix. It is clear that $\mathcal{C}^\perp = \mathcal{D}$ and that the weight enumerator of both is $W_{\mathcal{C}}(x, y) = x^4 + 2x^3y + x^2y^2$. Hence, these codes are $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual. The code $\mathcal{C}$ has parameters $(2, 1; 2, 0; 2)$ whereas the code $\mathcal{D}$ has parameters $(2, 1; 0, 1; 0)$. Note that $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes do not have necessarily the same parameters.

Following the terminology of [5] and [17], where the definitions were made for self-dual codes, we define the following terms.

Definition 3 *If a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code $\mathcal{C}$ has odd weights, then $\mathcal{C}$ is said to be Type 0. If $\mathcal{C}$ has only even weights, but not all weights are doubly-even, then the code $\mathcal{C}$ is said to be Type I. If all the codewords in $\mathcal{C}$ have doubly-even weight, then it is said to be Type II.*

In general, if all the codewords of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ have even weights, then $\mathcal{C}$ is an even code; otherwise, $\mathcal{C}$ is an odd code. Notice, that this is different than the definition first given in [1] where Type I and Type II for self-dual codes are defined in terms of Euclidean weight with an eye towards the constructed lattice. In our definition we are strictly concerned with the Lee weights.

We shall given an example of an odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code.

Example 2 *Consider the following generator matrices:*

$$G = \left(\begin{array}{cc|cc} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right), G' = \left(\begin{array}{cc|cc} 1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 2 \end{array}\right).$$

The codes $\mathcal{C}$ and $\mathcal{C}'$ generated by G and G' respectively are odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes of length 6 that are orthogonals of each other with weight enumerator: $W_{\mathcal{C}}(x,y) = W_{\mathcal{C}^\perp}(x,y) = x^6 + 4x^3y^3 + 3x^2y^4$. Notice that its Gray image has minimum weight 3 which is higher than any self-dual code at that length.

Definition 4 *Let C be a binary code. Then we say that C is a formally self-dual code if the weight enumerator of C is held invariant by the action of the MacWilliams relations.*

Note that in this case we are not assuming that C is a linear code. If C is a non-linear binary code then we do not have that $W_C(w,y) = W_{C^\perp}(x,y)$ since $C^\perp = \langle C \rangle^\perp$ and $\langle C \rangle$ is larger than C when C is non-linear. In general, what we are seeking are binary codes that are images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes under the Gray map, since they correspond to the structures defined by Delsarte, and have weight enumerators held invariant by the MacWilliams relations.

The following is immediate given that Φ is an isomotery.

Theorem 1 *If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code, then $\Phi(\mathcal{C})$ is a binary formally self-dual code.*

It follows immediately from this theorem that if $\mathcal{C}$ is a formally self-dual code then α must be even, since n is even for a binary formally self-dual code and $n = \alpha + 2\beta$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. The weight enumerator is held invariant by the action of the MacWilliams relations and hence the invariant theory for binary self-dual codes described in [26, Chapter 19] also applies to $\mathcal{C}$. Therefore, $\mathcal{C}$ is held invariant by the action of the matrix

$$M_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

If the code is Type I then it is also held invariant by the action of the matrix

$$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

since n must be even. Additionally, if the code is Type II then it is held invariant by

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

where i is the complex number with $i^2 = -1$. Hence, the standard Gleason's Theorem applies.

Theorem 2 (Gleason [22],[26]) *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code. Then,*

$$\begin{cases} W_{\mathcal{C}}(x,y) \in \mathbb{C}[x^2+y^2, y(x-y)], & \text{if } \mathcal{C} \text{ is Type 0,} \\ W_{\mathcal{C}}(x,y) \in \mathbb{C}[x^2+y^2, x^8+14x^4y^4+y^8], & \text{if } \mathcal{C} \text{ is Type I,} \\ W_{\mathcal{C}}(x,y) \in \mathbb{C}[x^8+14x^4y^4+y^8, x^4y^4(x^4-y^4)^4], & \text{if } \mathcal{C} \text{ is Type II.} \end{cases} \quad (3)$$

It follows from Theorem 2 that a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code that is Type II must have length a multiple of 8.

2 Separability and Existence of $\mathbb{Z}_2\mathbb{Z}_4$ Formally Self-Dual Codes of Certain Types

In this section, we shall give some conditions related to the separability and antipodality in terms of the type of the code. In particular, we compare these conditions to the behavior in the case of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes given in [7]. Also we shall prove when codes of each type exist deppending on the parameters α and β.

Lemma 1 *Let $\mathcal{C}$ and $\mathcal{C}'$ be $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes. Then the code $\mathcal{C} \times \mathcal{C}' = \{(\mathbf{v}_X, \mathbf{w}_X|\mathbf{v}_Y, \mathbf{w}_Y)\}, \mathbf{v} = (\mathbf{v}_X|\mathbf{v}_Y) \in \mathcal{C}, \mathbf{w} = (\mathbf{w}_X|\mathbf{w}_Y) \in \mathcal{C}'\}$ is a formally self-dual code and $W_{\mathcal{C}\times\mathcal{C}'}(x,y) = W_{\mathcal{C}}(x,y)W'_{\mathcal{C}}(x,y)$.*

Proof: It is immediate that $(\mathcal{C}\times\mathcal{C}')^\perp = \mathcal{C}^\perp \times \mathcal{C}'^\perp$ and that $W_{\mathcal{C}\times\mathcal{C}'}(x,y) = W_{\mathcal{C}}(x,y)W'_{\mathcal{C}}(x,y)$.

We have that

$$W_{\mathcal{C}\times\mathcal{C}'}(x,y) = W_{\mathcal{C}}(x,y)W_{\mathcal{C}'}(x,y) = W_{\mathcal{C}^\perp}(x,y)W_{(\mathcal{C}')^\perp}(x,y) = W_{(\mathcal{C}\times\mathcal{C}')^\perp}(x,y).$$

This gives that $\mathcal{C}\times\mathcal{C}'$ is a formally self-dual code. $\Box$

Note that if the generator matrices of $\mathcal{C}$ and $\mathcal{C}'$ are $G = \begin{pmatrix} G_b | G_q \end{pmatrix}$ and $G' = \begin{pmatrix} G'_b | G'_q \end{pmatrix}$ respectively, then the generator matrix of $\mathcal{C}\times\mathcal{C}'$ is

$$G = \left(\begin{array}{cc|cc} G_b & \mathbf{0} & G_q & \mathbf{0} \\ \mathbf{0} & G'_b & \mathbf{0} & G'_q \end{array}\right).$$

Example 3 *Consider the code $\mathcal{C}$ in Example 1 and the code $\mathcal{C}'$ in Example 2. Then a generator matrix of $\mathcal{C} \times \mathcal{C}'$ is*

$$G = \left(\begin{array}{cccc|ccc} 0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&0&1&1&0&3&1 \\ 0&0&1&0&0&0&2 \end{array}\right),$$

and the weight enumerator is $W_{\mathcal{C}\times\mathcal{C}'}(x,y) = x^{10} + 2x^9y + 2x^8y^2 + 4x^7y^3 + 11x^6y^4 + 10x^5y^5 + 3x^4y^6$.

We shall now define separable codes and relate them to the lemma above.

Definition 5 *A $\mathbb{Z}_2\mathbb{Z}_4$-additive code is said to be separable if the generator matrix can be written in the form* $\left(\begin{array}{c|c} A & \mathbf{0} \\ \mathbf{0} & B \end{array}\right)$, *where A is a binary matrix and B is a quaternary matrix. Equivalently, we can say that a $\mathbb{Z}_2\mathbb{Z}_4$ code $\mathcal{C}$ is separable if every vector in $\mathcal{C}$ is o f the form $(\mathbf{v}, \mathbf{w})$ where $\mathbf{v} \in C$ and $\mathbf{w} \in D$ where C is a binary code and D is a quaternary code. In this case we say $\mathcal{C} = C \times D$.*

Note that we can think of binary and quaternary codes as $\mathbb{Z}_2\mathbb{Z}_4$ codes so that the cross product given in Lemma 1 applies here as the cross product of two $\mathbb{Z}_2\mathbb{Z}_4$ codes. This gives the following.

Corollary 1 *If C is a binary formally self-dual code and D is a quaternary formally self-dual code then $C \times D$ is a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code.*

Proof: The proof follows from Lemma 1. $\Box$

For a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code $\mathcal{C}$, there are some conditions that relates separability, antipodality and the Type of the code as it was proved in [5]. If $\mathcal{C}$ is an antipodal code, then $\mathcal{C}$ is of Type I or Type II. Also if $\mathcal{C}$ is separable then necessarily $\mathcal{C}$ is antipodal. As a result, if $\mathcal{C}$ is Type 0, then $\mathcal{C}$ is non-separable and non-antipodal. These properties are not satisfied in the case of $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes.

Unlike for self-dual codes, a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual can be separable. For example, consider the code $\langle(10|)\rangle \times \langle|2\rangle \subseteq \mathbb{Z}_2^2 \times \mathbb{Z}_4$. Its orthogonal is $\langle(01|)\rangle \times \langle|2\rangle$. These codes have vectors of weight 1 and hence are odd. Therefore, the code is an odd separable formally self-dual code. Also, unlike the case for self-dual codes [5], separability does not imply that the code is antipodal. For example, in the separable code given above, $\langle(10|)\rangle \times \langle|2\rangle$, the code is separable but not antipodal; i.e., $(11|2)$ is not in the code.

Theorem 3 *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{Z}_2\mathbb{Z}_4$ codes with $\mathcal{C}^\perp = \mathcal{D}$. Then $(\mathcal{C}_b)_X^\perp = (\mathcal{D}_b)_X$.*

Proof: Let $\mathbf{v}$ and $\mathbf{w}$ be order 2 codewords of $\mathcal{C}$ and $\mathcal{D}$ respectively. Since the quaternary components contain only elements from $\{0, 2\}$ we have that $[\mathbf{v}_X, \mathbf{w}_X] = 0$ where the inner product is the binary inner product. Hence, every vector in $(\mathcal{D}_b)_X$ is orthogonal to every vector in $(\mathcal{C}_b)_X$. It is also true that every vector in $(\mathcal{C}_b)_X$ is orthogonal to every vector in $(\mathcal{D}_b)_X$. This gives the result. $\square$

For self-dual codes, this result implies that $(\mathcal{C}_b)_X$ was a self-dual code. However, it is not true that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code then $(\mathcal{C}_b)_X$ is a formally self-dual code. Considering the $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes $\mathcal{C}$ and $\mathcal{D}$ in Example 1, we have that $(\mathcal{C}_b)_X = \mathbb{Z}_2^2$ and $(\mathcal{D}_b)_X = \{00|0\}$. While they are duals of each other, they are not formally self-dual.

The following theorem characterizes even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes.

Theorem 4 *A $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\mathcal{C}$ is even if and only if $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in \mathcal{C}$.*

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. If $\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] \equiv 2wt(\mathbf{v}_X) + 2wt(\mathbf{v}_Y) \pmod 4$. Therefore, if a vector $\mathbf{v}$ has even weight then $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 0$ and if $\mathbf{v}$ has odd weight $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 2$. If $\mathcal{C}$ is an even code then $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in \mathcal{C}^\perp$. Notice that $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$ is the only vector with Lee weight n, hence since the weight enumerators of $\mathcal{C}$ and $\mathcal{C}^\perp$ are the same we have that $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in C$. If $\mathcal{C}$ is an odd code then $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \notin \mathcal{C}^\perp$ and by the same reasoning $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$ is not in $\mathcal{C}$. $\square$

Note that from the previous theorem, we have that every Type I or Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code is antipodal.

Now we will study the existence of Type 0, Type I and Type II codes. We shall give conditions for the values of α and β and also examples of codes having each possible α and β by using the direct product of codes. We start with the case of Type 0 codes.

The code $\mathcal{C}_1 = \langle(10|)\rangle$ is a Type 0 code with $\alpha = 2$ and $\beta = 0$. The code $\mathcal{C}_2 = \langle(00|1)\rangle$ is a Type 0 code with $\alpha = 2$ and $\beta = 1$. Notice that the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are not self-dual, but they are formally self-dual. Moreover, given any Type 0 formally self-dual code we can take the direct product with $\mathcal{C}_3 = \langle(|2)\rangle$ and still have a Type 0 formally self-dual code. Notice that $\mathcal{C}_3$ is not Type 0, but the direct product of a Type 0 with any other type will result in a Type 0 code. Taking direct products of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ we have the following theorem.

Theorem 5 *There exists a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code for all α, β satisfying $\alpha = 2a$, $\beta = b$ with $a \geq 1$ and $b \geq 0$ or $a = 0$ and $b \geq 2$.*

Proof: The only case that remains is to find Type 0 codes for $\alpha = 0$. In this case, take $(|1, 0)$ which is a Type 0 code. $\square$

The code $\mathcal{C}_4 = \langle(11|)\rangle$ is a Type I code with $\alpha = 2$ and $\beta = 0$. The code $\mathcal{C}_3 = \langle(|2)\rangle$, given above, is a Type I code with $\alpha = 0$ and $\beta = 1$. The codes $\mathcal{C}_3$ and $\mathcal{C}_4$ are self-dual codes. Notice that taking the direct product of a Type I code with a Type I or Type II code results in a Type I code. This gives the following theorem.

Theorem 6 *There exists a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code for all α, β satisfying $\alpha = 2a$, $\beta = b$ with $a \geq 0, b \geq 0$.*

Finally, we shall now examine when Type II codes exist.

Theorem 7 *Let $\mathcal{C}$ be a Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code, then $\alpha+2\beta$ must be a multiple of 8 and $\alpha \equiv 0 \pmod 4$.*

Proof: Let $\mathcal{C}$ be a Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. Since $\mathcal{C}$ and $\mathcal{C}^\perp$ have the same weight enumerator, we have that $\mathcal{C}^\perp$ is Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual. By Theorem 4, $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$ belongs to $\mathcal{C}$ and $\mathcal{C}^\perp$.

By Gleason's Theorem (Theorem 2), we have that $n = \alpha + 2\beta$ must be a multiple of 8, and hence α must be even.

If $(\mathbf{1}_\alpha \mid \mathbf{0}_\beta)$ is not in $\mathcal{C}^\perp$ then there exists a vector $\mathbf{v} \in \mathcal{C}$ whose binary part has oddly many ones. Since the weight have to be even, the quaternary part of $\mathbf{v}$ must then have an odd number of ± 1. Then, the vector $\mathbf{v} + \mathbf{v} \in \mathcal{C}$ has an all zero binary part and an odd number of coordinates with a 2 in them. Hence, it has weight congruent to 2 (mod 4). This is a contradiction since the code is Type II. Hence, $(\mathbf{1}_\alpha \mid \mathbf{0}_\beta)$ must be in $\mathcal{C}^\perp$. Since $\mathcal{C}^\perp$ is Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual, $(\mathbf{1}_\alpha \mid \mathbf{0}_\beta)$ must have doubly-even weight. Therefore, $\alpha = wt(\mathbf{1}_\alpha \mid \mathbf{0}_\beta) \equiv 0 \pmod 4$. $\square$

Example 4 *Consider the code generated by the following matrix:*

$$\left(\begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

Notice that the third row is not self-orthogonal, so the code is not self-dual. The weight enumerator of the code is $x^8 + 14x^4y^4 + y^8$, so the code is formally self-dual. Specifically, this code is a Type II code that is formally self-dual but is not self-dual. Here n is 8, so this is the smallest possible example of a Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code.

We know that the $[8, 4, 4]$ binary Hamming code is a doubly-even binary self-dual code. The code $\mathcal{D}_4^\oplus$ in [11] with generator matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

is a quaternary code with the weight enumerator of the Hamming code. Hence, both can be viewed as $\mathbb{Z}_2\mathbb{Z}_4$ Type II codes. We note that the direct product of two Type II codes results in a Type II code. Using these two codes and the $\mathbb{Z}_2\mathbb{Z}_4$-additive code in Example 4 together with the results of Lemma 1 and Theorem 7 we get the following.

Theorem 8 *There exists a Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code for all α, β satisfying $\alpha = 8a + 4b$, $\beta = 2b + 4c$, $a, b, c \geq 0$.*

3 Subcodes of $\mathbb{Z}_2\mathbb{Z}_4$ Formally Self-Dual Codes

When $\mathcal{C}$ is a binary self-dual code, then the subset of double-even vectors of $\mathcal{C}$ is a linear subcode. This subcode is used in [10] to obtain the shadow of a self-dual code. In this section, we study the properties of the subset of all double-even vectors in a Type I code and also the subset of even vectors in an odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code.

It is well known that for binary vectors we have

$$wt(\mathbf{v}+\mathbf{w}) = wt(\mathbf{v}) + wt(\mathbf{w}) - 2|\mathbf{v} * \mathbf{w}|, \tag{4}$$

where $*$ is the component-wise product. Then it follows immediately that if $\mathbf{v}$ and $\mathbf{w}$ are binary doubly-even vectors and $\mathbf{v}+\mathbf{w}$ is a binary doubly-even vector then the two vectors must be orthogonal. This gives that any binary Type II formally self-dual code must be self-dual. Example 4 showed that this is not true for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

For $\mathbb{Z}_4$ vectors, Equation 4 is not true. Consider for example the two quaternary vectors (1120) and (1111). These two vectors are orthogonal and both have doubly-even weight but their sum is (2231) which has weight $6 \equiv 2 \pmod 4$. Moreover, we can take the sum of two doubly-even vectors which is also doubly-even but where the vectors are not orthogonal. For example, take the quaternary vectors (1111000) and (0001111). Their sum is (1112111) which has weight 8, yet the vectors have inner product $1 \neq 0$. Hence, the usual techniques do not apply in $\mathbb{Z}_4$ codes or in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

We do however, have the following elementary lemma.

Lemma 2 *Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code and let $\mathcal{C}'$ be the subcode of doubly even vectors then*

$$W_{\mathcal{C}'}(x,y) = \frac{1}{2}(W_{\mathcal{C}}(x,y) + W_{\mathcal{C}}(x,iy)), \tag{5}$$

where i is the complex number with $i^2 = -1$.

Proof: Doubly-even vectors are counted twice in $(W_{\mathcal{C}}(x,y) + W_{\mathcal{C}}(x,iy))$ and singly-even vectors are counted once in $W_{\mathcal{C}}(x,y)$ and negatively in $W_{\mathcal{C}}(x,iy)$. $\square$

Unlike in the binary self-dual case, it is not immediate that the subcode of doubly-even vectors is a linear subcode. However, we can use this lemma to show that the subcode of doubly-even vectors is precisely half the code.

Theorem 9 *Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code and let $\mathcal{C}'$ be the subcode of doubly-even vectors then $|\mathcal{C}'| = \frac{|\mathcal{C}|}{2}$.*

Proof: Let $\mathcal{C}$ be a Type I code. Let $p(x,y) = x^2 + y^2$ and $q(x,y) = x^8 + 14x^4y^4 + y^8$. We know from Theorem 2 that the weight enumerator of $\mathcal{C}$ can be written as

$$W_{\mathcal{C}}(x,y) = \sum_{j=0}^{\frac{n}{2}} A_j p(x,y)^j q(x,y)^{\frac{n-2j}{8}}.$$

Additionally, we know that for any $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{D}$, $W_{\mathcal{D}}(1,1) = |\mathcal{D}|$. Now,

$$\begin{aligned} W_{\mathcal{C}'}(x,y) &= \frac{1}{2}\Big(\sum_{j=0}^{\frac{n}{2}} A_j p(x,y)^j q(x,y)^{\frac{n-2j}{8}} + \sum_{j=0}^{\frac{n}{2}} A_j p(x,iy)^j q(x,iy)^{\frac{n-2j}{8}}\Big) \\ &= \frac{1}{2}\sum_{j=0}^{\frac{n}{2}} A_j q(x,y)^{\frac{n-2j}{8}} (p(x,y)^j - p(x,iy)^j) \\ &= \frac{1}{2}\sum_{j=0}^{\frac{n}{2}} A_j q(x,y)^{\frac{n-2j}{8}} ((x^2+y^2)^j - (x^2-y^2)^j). \end{aligned}$$

The second step follows from the fact that $q(x,y) = q(x,iy)$. Then, we have

$$|\mathcal{C}'| = W_{\mathcal{C}'}(1,1) = \frac{1}{2}\sum_{j=0}^{\frac{n}{2}} A_j q(1,1)^{\frac{n-2j}{8}} (p(1,1)^j - (0)^j) = \frac{1}{2} W_{\mathcal{C}}(1,1) = \frac{1}{2}|\mathcal{C}|. \tag{6}$$

This gives the result. $\square$

Theorem 10 *Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $\mathcal{C}'$ the subcode of doubly-even vectors, then either $\mathcal{C}'$ is a subgroup of index 2 or $\langle \mathcal{C}' \rangle = \mathcal{C}$.*

Proof: We have seen in Theorem 9 that $|\mathcal{C}'| = \frac{|\mathcal{C}|}{2}$. If $\mathcal{C}'$ is linear then we are done. If it is not then it generates a subcode of $\mathcal{C}'$ but since the order of a subgroup must divide the order of a group we have that it must be $\mathcal{C}$. $\square$

Moreover, we shall show in the following examples that both situations can occur. The following example is of a Type I code where the subcode of doubly-even vectors is not linear.

Example 5 *Consider the quaternary linear code $\mathcal{E}_7^+$ in [11] and $\mathcal{C}'$ its subcode of doubly-even vectors. The code has generator matrix:*

$$\begin{pmatrix} 1\,0\,0\,3\,1\,1\,0 \\ 1\,0\,1\,0\,0\,3\,1 \\ 1\,1\,0\,1\,0\,0\,3 \\ 2\,2\,2\,2\,2\,2\,2 \end{pmatrix}.$$

The code is a Type I self-dual code with weight enumerator:

$$W_{\mathcal{C}}(x,y) = x^{14} + 14x^{10}y^4 + 49x^8y^6 + 49x^6y^8 + 14x^4y^{10} + y^{14}.$$

The weight enumerator for $\mathcal{C}'$ is

$$W_{\mathcal{C}'}(x,y) = x^{14} + 14x^{10}y^4 + 49x^6y^8.$$

Notice that summing the first two vectors which both have doubly-even weight give the vector (2013101) *which has weight 6 which is not doubly-even. In fact, cyclic shifts of of* (3110100) *generate the code. In other words the subcode of doubly-even vectors generates the entire code. The binary image of this code is a non-linear* $(14, 2^7, 6)$ *code.*

We give now two examples of codes where the subcode of doubly-even vectors is linear.

Example 6 *Consider the code $\mathcal{C}$ generated by* $\begin{pmatrix} 1\ 1 & \vline & 0 \\ 0\ 0 & \vline & 2 \end{pmatrix}$. *Here the subcode $\mathcal{C}'$ is $\{(00|0),(11|2)\}$ which is a linear subcode. A purely quaternary linear code can have $\mathcal{C}'$ as a linear subcode as well. Consider the octacode, $\mathcal{O}_8$ in [11]. It has generator matrix:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{pmatrix}.$$

The weight enumerator for this code is:

$$x^{16} + 112x^{10}y^6 + 30x^8y^8 + 112x^6y^{10} + y^{16}.$$

The subcode $\mathcal{C}'$ is linear and has generator matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \end{pmatrix}.$$

In [10], and earlier in [31], the shadow of binary codes was considered. Namely, given a Type I binary self-dual code, the weight enumerator of the subcode C' of doubly-even vectors is computed by the formula in (5). Then, the weight enumerator of the shadow $C'^{\perp} - C$ is computed using the MacWilliams relations. If this weight enumerator had non-acceptable coefficients, that is either negative or non-integer, then there could be no code with this weight enumerator. This powerful technique relied on the fact that the subcode of doubly-even vectors was linear. This means that it does not necessarily apply in our situation. Hence, weight enumerators which are eliminated by investigating the shadow are still in play here. Namely, if C' is not linear then the computation of the weight enumerator of the shadow does not necessarily give the weight enumerator of a code. Then, if this weight enumerator is not acceptable it does not imply that there cannot be a code with this weight enumerator.

We now give some structural results about Type 0 codes taking into account the subcodes of doubly-even weight vectors.

Let $\mathcal{C}$ be a Type 0 code. Let $\mathcal{C}_0 = \{\mathbf{v} \in \mathcal{C} \mid wt(\mathbf{c}) \equiv 0 \bmod 2\}$.

Lemma 3 *If $\mathcal{C}$ is an odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code, then $\mathcal{C}_0$ is an additive subcode of $\mathcal{C}$ with $[\mathcal{C} : \mathcal{C}_0] = 2$. Moreover,*

$$W_{\mathcal{C}_0}(x,y) = \frac{1}{2}(W_{\mathcal{C}}(x,y) + W_{\mathcal{C}}(x,-y)). \qquad (7)$$

Proof: First we shall show that it is an additive subcode. If $\mathcal{C}$ is a Type 0 code, then by Theorem 4 we see that $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \notin \mathcal{C}$. Note that $\mathcal{C}_0 = \{\mathbf{v} \mid [\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 0\}$ and hence it is an additive subcode. Then, if a monomial represents even weight, the coefficient is the same in $W_{\mathcal{C}}(x, y)$ and $W_{\mathcal{C}}(x, -y))$. If the monomial represents odd weight, then the coefficients are negatives of each other. Hence, even weights are counted twice and odd weights are not counted in this sum. Dividing by 2 then gives the weight enumerator of $\mathcal{C}_0$. $\Box$

4 Constructions

In this section, we shall give different constructions of $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes starting from a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\mathcal{C}$. The first construction is the neighbor construction; we obtain an even code with the same length of $\mathcal{C}$, a Type 0 code. Constructions A, B and C gives new codes, $\bar{\mathcal{C}}, \tilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$, by increasing the length of a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\mathcal{C}$.

The first construction is the analog of the neighbor construction for $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes [5].

We say that $\mathcal{C}$ is a neighbor of $\mathcal{C}'$ if there is a vector $\mathbf{v}$ not in $\mathcal{C}$ such that $\mathcal{C}' = \langle \mathcal{C}_{\mathbf{v}}, \mathbf{v} \rangle$, where $\mathcal{C}_{\mathbf{v}} = \{\mathbf{w} \mid \mathbf{w} \in \mathcal{C}, [\mathbf{v}, \mathbf{w}] = 0\}$.

Theorem 11 *(***Neighbor Construction***) Let $\mathcal{C}$ be a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code, then $\mathcal{C}$ is a neighbor of an even formally self-dual code.*

Proof: Let $\mathcal{C}$ be an odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code and let $\mathcal{C}_0$ be the subcode of even vectors. Let $\mathcal{D} = \mathcal{C}^\perp$ and let $\mathcal{D}_0$ be the subcode of $\mathcal{D}$ of even vectors. Note that if $\mathbf{v} = (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$, then $\mathcal{C}_0 = \mathcal{C}_{\mathbf{v}}$ and $\mathcal{D}_0 = \mathcal{D}_{\mathbf{v}}$. We know that $W_{\mathcal{C}_0}(x, y) = W_{\mathcal{D}_0}(x, y)$. Let $\mathcal{C}' = \langle \mathcal{C}_0, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \rangle$ and $\mathcal{D}' = \langle \mathcal{D}_0, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \rangle$. Then, $W_{\mathcal{C}'}(x, y) = W_{\mathcal{C}}(x, y) + W_{\mathcal{C}}(y, x) = W_{\mathcal{D}'}(x, y)$. If $\mathbf{c} \in \mathcal{C}'$ and $\mathbf{d} \in \mathcal{D}'$ then $[\mathbf{c}, \mathbf{d}] = [\mathbf{c}_0 + \lambda(\mathbf{1}_\alpha \mid \mathbf{2}_\beta), \mathbf{d}_0 + \mu(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 0$, for $\lambda, \mu \in \{0, 1\}$. Since $|\mathcal{C}'| = |\mathcal{D}'|$ then $\mathcal{C}'^\perp = \mathcal{D}'$. Then, $\mathcal{C}' = \langle \mathcal{C}_{\mathbf{v}}, \mathbf{v} \rangle$, for $\mathbf{v} = (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)$, is an even formally self-dual code. $\Box$

In [24], a construction of larger binary formally self-dual codes was given from existing odd binary formally self-dual codes, which we shall refer to as the building up construction. That is, if C is the binary code then a larger code $\bar{C}$ is constructed from it. A constructive, simpler proof of this result was given in [16]. We shall generalize this construction, in the same setting as was done in [16], to $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. If $C = \phi(\mathcal{C})$ is linear, then we can consider the code $\bar{\mathcal{C}}$ constructed as $\bar{\mathcal{C}} = \phi^{-1}(\bar{C})$, where $\bar{C}$ is obtained from C by applying the binary building up construction, as in [16]. However, usually and most interestingly, $\phi(\mathcal{C})$ is not linear. The next theorem shows that we can generalize the building up construction for $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes whose Gray map image is not linear.

Let $\mathcal{C}$ be a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. Let $\mathcal{C}_0$ be the subcode of $\mathcal{C}$ of even vectors. By Lemma 3, we have that $\mathcal{C}_0$ is of index 2 in $\mathcal{C}$ and that there exists a vector $\mathbf{t} \in \mathcal{C} - \mathcal{C}_0$ such that $\mathcal{C} = \langle \mathcal{C}_0, \mathbf{t} \rangle$ and $\mathbf{t} + \mathbf{t} \in \mathcal{C}_0$. It also follows that $\mathcal{C}_0^\perp = \langle \mathcal{C}^\perp, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \rangle$. Define $\mathcal{C}_{\lambda,\mu}$ by

$$\mathcal{C}_{\lambda,\mu} = \mathcal{C}_0 + \lambda \mathbf{t} + \mu(\mathbf{1}_\alpha \mid \mathbf{2}_\beta).$$

Notice that λ and μ can be either 0 or 1.

Theorem 12 *(***Construction A***) Let $\mathcal{C}$ be a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code of length n and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Let $\mathcal{C}_0$ be the subcode of even vectors. The code*

$$\bar{\mathcal{C}} = \langle \{(0,0,c) \mid c \in \mathcal{C}_0\} \cup \{(1,0,c) \mid c \in \mathcal{C} - \mathcal{C}_0\}, (1,1,(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)) \rangle \qquad (8)$$

is an even formally self-dual code of length $n+2$ and type $(\alpha+2, \beta; \gamma+1, \delta; \bar{\kappa})$ with weight enumerator

$$W_{\mathcal{C}}(x,y) = x^2 W_{\mathcal{C}_{0,0}}(x,y) + xyW_{\mathcal{C}_{1,0}}(x,y) + y^2 W_{\mathcal{C}_{0,0}}(y,x) + xyW_{\mathcal{C}_{1,0}}(y,x).$$

The code

$$\bar{\mathcal{C}} = \langle \{(0,0,c) \mid c \in \mathcal{C}_0\} \cup \{(1,1,c) \mid c \in \mathcal{C} - \mathcal{C}_0\}, (1,0,((\mathbf{1}_\alpha \mid \mathbf{2}_\beta))) \rangle \qquad (9)$$

is an odd formally self-dual code of length $n+2$ any type $(\alpha+2, \beta; \gamma+1, \delta; \bar{\kappa})$ with weight enumerator

$$W_{\bar{\mathcal{C}}}(x,y) = x^2 W_{\mathcal{C}_{0,0}}(x,y) + y^2 W_{\mathcal{C}_{1,0}}(x,y) + xyW_{\mathcal{C}_{0,0}}(y,x) + xyW_{\mathcal{C}_{1,0}}(y,x).$$

Moreover, any code with these weight enumerators is a formally self-dual code. We refer to the first construction as the even Construction A and the second one as the odd Construction A.

Proof: Let $\mathcal{C}$ be a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $\mathcal{C}_0$ and $\mathcal{C}_{\lambda,\mu}$ be defined as above. Set $\mathcal{D} = \mathcal{C}^\perp$. Since the weight enumerator of $\mathcal{D}$ is the same as the weight enumerator of $\mathcal{C}$, $\mathcal{D}$ is also and odd formally self-dual code. Therefore, we have the same situation for $\mathcal{D}$. Namely, we define $\mathcal{D}_{\lambda,\mu}$ by

$$\mathcal{D}_{\lambda,\mu} = \mathcal{D}_0 + \lambda \mathbf{t}' + \mu(\mathbf{1}_\alpha \mid \mathbf{2}_\beta),$$

where $\mathcal{D}_0$ is the subcode of even vectors and $\mathbf{t}'$ is the vector such that $\langle \mathcal{D}_0, \mathbf{t}' \rangle = \mathcal{D}$. Notice that the weight enumerator of $\mathcal{C}_0$ and $\mathcal{D}_0$ are identical. Since the weight enumerators of $\mathcal{C}$ and $\mathcal{D}$ were identical then the weight enumerators of $\mathcal{C}_{1,0} = \mathcal{C} - \mathcal{C}_0$ and $\mathcal{D}_{1,0} = \mathcal{D} - \mathcal{D}_0$ are identical. Then, we see that $W_{\mathcal{C}_{0,1}}(x,y) = W_{\mathcal{C}_{0,0}}(y,x)$ and $W_{\mathcal{D}_{0,1}}(x,y) = W_{\mathcal{D}_{0,0}}(y,x)$. This gives that $\mathcal{C}_{0,1}$ and $\mathcal{D}_{0,1}$ have identical weight enumerators. Similarly,

$$W_{\mathcal{C}_{1,1}}(x,y) = W_{\mathcal{C}_{1,0}}(y,x) \text{ and } W_{\mathcal{D}_{1,1}}(x,y) = W_{\mathcal{D}_{1,0}}(y,x),$$

so $\mathcal{C}_{1,1}$ and $\mathcal{D}_{1,1}$ have identical weight enumerators. Thus, the weight enumerators of the corresponding cosets are identical.

We shall form two codes $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ that are orthogonals of each other and that have the same weight enumerator. Let $\bar{\mathcal{C}} = \cup(v_{\lambda,\mu}, \mathcal{C}_{\lambda,\mu})$ and $\bar{\mathcal{D}} =$

$\cup(w_{\lambda,\mu}, \mathcal{D}_{\lambda,\mu})$, where $v_{\lambda,\mu}, w_{\lambda,\mu} \in \mathbb{Z}_2^2$, for $\lambda, \mu \in \{0,1\}$. To ensure that the codes are additive we need $v_{\lambda,\mu} = \lambda v_{1,0} + \mu v_{0,1}$ and $w_{\lambda,\mu} = \lambda w_{1,0} + \mu w_{0,1}$. For $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ to be orthogonal we need

$$[(v_{\lambda,\mu}, w_{\lambda',\mu'})] = -[\mathcal{C}_{\lambda,\mu}, \mathcal{D}_{\lambda',\mu'}],$$

where $[\mathcal{C}_{\lambda,\mu}, \mathcal{D}_{\lambda',\mu'}]$ is the inner product of any two vectors in those cosets. Then, we have

$$\begin{aligned}[\mathcal{C}_{\lambda,\mu}, \mathcal{D}_{\lambda',\mu'}] &= [\mathbf{c}_0 + \lambda\mathbf{t} + \mu(\mathbf{1}_\alpha \mid \mathbf{2}_\beta), \mathbf{d}_0 + \lambda'\mathbf{t}' + \mu'(\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] \\ &= \lambda\mu'[\mathbf{t}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] + \mu\lambda'[\mathbf{t}', (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 2(\lambda\mu' + \mu\lambda).\end{aligned}$$

To ensure that the weight enumerators of $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ are identical, we also need that the weight of $v_{\lambda,\mu}$ is equal to the weight of $w_{\lambda,\mu}$. Let $v_{1,0} = (1,0)$, $v_{0,1} = (1,1)$, $w_{1,0} = (0,1)$, and $w_{0,1} = (1,1)$. It is a simple computation to verify that these choices satisfy all of the conditions and that it gives the code in (8). Moreover, since the odd vectors have an odd vector attached and the even vectors have an even vector attached the new code is even.

For the code in (9), let $v_{1,0} = (1,1)$, $v_{0,1} = (0,1)$, $w_{1,0} = (1,1)$ and $w_{0,1} = (1,0)$. It is a simple computation to verify that these choices satisfy all of the conditions and that it gives the code in (9). Moreover, since the odd vectors have an even vector attached and the even vectors have an odd vector attached the new code is odd.

The weight enumerators follow simply from the construction. Then, since these weight enumerators are held invariant by the MacWilliams relations any code with these weight enumerators are formally self-dual.

In both cases, the new constructions add 2 binary coordinates and one vector of order 2. Therefore, the new codes are of type $(\alpha+2, \beta; \gamma+1, \delta, \bar{\kappa})$. $\square$

Example 7 *We can continue the example given in Example 2. Applying the even Construction A we get an even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $n = 8$, generator matrix*

$$\left(\begin{array}{cccc|cc} 1&1&1&1&2&2 \\ 1&0&0&1&2&0 \\ 0&0&1&1&1&1 \end{array}\right),$$

and weight enumerator

$$W_{\bar{\mathcal{C}}}(x,y) = x^8 + 14x^4y^4 + y^8.$$

So the image of this code under the Gray map is the binary Hamming code of length 8. Applying the odd construction we get an odd $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $n = 8$, generator matrix

$$\left(\begin{array}{cccc|cc} 1&0&1&1&2&2 \\ 1&1&0&1&2&0 \\ 0&0&1&1&1&1 \end{array}\right),$$

and weight enumerator

$$W_{\bar{\mathcal{C}}}(x,y) = x^8 + 3x^5y^3 + 7x^4y^4 + 4x^3y^5 + xy^7.$$

Theorem 13 *Let $\mathcal{C}$ be an even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. The generator matrix of $\mathcal{C}$ can be written as*

$$\left(\begin{array}{cl|c} 11 & 1\dots1 & 2\dots2 \\ \hline 10 & w_X & w_Y \\ \hline 01 & & \\ \vdots & A_1 & B_1 \\ 01 & & \\ \hline 00 & & \\ \vdots & A_2 & B_2 \\ 00 & & \end{array}\right),$$

for some vector $(1, 0, w_X|w_Y) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and some binary matrices A_1 and A_2, and quaternary matrices B_1 and B_2 if and only if the code generated by

$$\left(\begin{array}{c|c} w_X & w_Y \\ A_1 & B_1 \\ A_2 & B_2 \end{array}\right)$$

is a Type 0 code $\mathcal{C}_1$ and $\bar{\mathcal{C}}_1 = \mathcal{C}$, that is $\mathcal{C}$ is the result of even Construction A applied to $\mathcal{C}_1$.

Corollary 2 *Let $\mathcal{C}$ be an even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. Then, there exists a Type 0 $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\mathcal{C}_1$ with $\bar{\mathcal{C}}_1 = \mathcal{C}$, where $\bar{\mathcal{C}}_1$ is the result of even Construction A applied to $\mathcal{C}_1$.*

Example 8 *Let $\mathcal{C}$ be the separable $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\langle(01|)\rangle \times \langle|2\rangle$. The code $\bar{\mathcal{C}}$, with the even Construction A, has generator matrix:*

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right).$$

Therefore, this is a case when $\mathcal{C}$ is separable and $\bar{\mathcal{C}}$ is also separable.

Let $\mathcal{C}$ be the separable $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\langle(01|)\rangle \times \langle(|01)\rangle$. Using the even Construction A we have that $\bar{\mathcal{C}}$ is generated by

$$\left(\begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

This code is non-separable. Therefore, we have an example of a separable code $\mathcal{C}$ where using the even construction we obtain $\bar{\mathcal{C}}$ which is non-separable.

Lemma 4 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code and $\bar{\mathcal{C}}$ the code obtained using the even or the odd Construction A. Let $\mathbf{v} = (\mathbf{v}_X|\mathbf{v}_Y) \in \mathcal{C}$ and $(\mathbf{a}, \mathbf{v}) \in \bar{\mathcal{C}}$, for some $\mathbf{a} \in \mathbb{Z}_2^2$. Then, $(\mathbf{a}, \mathbf{v}_X|\mathbf{0}) \in \bar{\mathcal{C}}$ if and only if $(\mathbf{v}_X|\mathbf{0}) \in \mathcal{C}$.*

Proof: The result follows from the construction of $\bar{\mathcal{C}}$. $\square$

Corollary 3 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code and $\bar{\mathcal{C}}$ the code obtained using the even or the odd Construction A. If $\mathcal{C}$ is non-separable, then $\bar{\mathcal{C}}$ is non-separable.*

The next construction shows how to obtain a formally self-dual code from a self-dual code using any self-orthogonal vector $\mathbf{v} \notin \mathcal{C}$ with $(\mathbf{v}+\mathbf{v}) \in \mathcal{C}$.

Theorem 14 *(***Construction B***) Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $\mathbf{v}$ be a self-orthogonal vector $\mathbf{v} \notin \mathcal{C}$ with $\mathbf{v}+\mathbf{v} \in \mathcal{C}$. Let $\mathcal{C}_{\mathbf{v}} = \{\mathbf{w} \mid \mathbf{w} \in \mathcal{C}, [\mathbf{w},\mathbf{v}] = 0\}$ and $\mathbf{t} \in \mathcal{C}$ such that $\langle \mathcal{C}_{\mathbf{v}}, \mathbf{t}\rangle = \mathcal{C}$. Let $\mathcal{C}_{\lambda,\mu} = \mathcal{C}_{\mathbf{v}} + \lambda\mathbf{t} + \mu\mathbf{v}$, $\lambda, \mu \in \{0,1\}$. Then the code*

$$\tilde{\mathcal{C}} = \langle \{(0,0,c) \mid c \in \mathcal{C}_{\mathbf{v}}\} \cup \{(0,1,c) \mid c \in \mathcal{C} - \mathcal{C}_{\mathbf{v}}\}, (1,1,\mathbf{v})\rangle \tag{10}$$

is a formally self-dual code of type $(\alpha+2, \beta; \tilde{\gamma}, \tilde{\delta}; \tilde{\kappa})$ which is not self-dual.

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ self-dual code with $\mathbf{v}$, $\mathbf{t}$, $\mathcal{C}_{\mathbf{v}}$ and $\mathcal{C}_{\lambda,\mu}$ defined as above.

We shall construct two codes $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ that are orthogonal to each other with the same weight enumerator. Let $\tilde{\mathcal{C}} = \cup(v_{\lambda,\mu}, \mathcal{C}_{\lambda,\mu})$ and $\tilde{\mathcal{D}} = \cup(w_{\lambda,\mu}, \mathcal{C}_{\lambda,\mu})$, where $v_{\lambda,\mu}, w_{\lambda,\mu} \in \mathbb{Z}_2^2$, for $\lambda, \mu \in \{0,1\}$. Note that $\tilde{\mathcal{C}}$ is of type $(\tilde{\alpha}, \tilde{\beta}; \tilde{\gamma}, \tilde{\delta}; \tilde{\kappa})$, where $\tilde{\alpha} = \alpha + 2$ and $\tilde{\beta} = \beta$.

On one hand, we need $[(v_{\lambda,\mu}, w_{\lambda',\mu'})] = -[\mathcal{C}_{\lambda,\mu}, \mathcal{C}_{\lambda',\mu'}]$ so that the codes are orthogonal each other. On the other hand, we need $v_{\lambda,\mu} = \lambda v_{1,0} + \mu v_{0,1}$ and $w_{\lambda,\mu} = \lambda w_{1,0} + \mu w_{0,1}$ to ensure that the codes are additive.

Note that $[\mathcal{C}_{\lambda,\mu}, \mathcal{C}_{\lambda',\mu'}] = [c_0 + \lambda\mathbf{t} + \mu\mathbf{v}, c_0 + \lambda'\mathbf{t} + \mu'\mathbf{v}] = 2(\lambda\mu' + \lambda'\mu)$. Consider $v_{0,1} = (1,1), v_{1,0} = (0,1)$, and $w_{0,1} = (1,1), w_{1,0} = (1,0)$. This gives $[(v_{\lambda,\mu}, \mathcal{C}_{\lambda,\mu}), (w_{\lambda',\mu'}, \mathcal{C}_{\lambda',\mu'})] = 0$.

Hence, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ are orthogonals and it is immediate that they have the same weight enumerator. Moreover, it is easy to see that $\tilde{\mathcal{C}} \neq \tilde{\mathcal{D}}$ so the code is not self-dual. Hence, $\tilde{C}$ and $\tilde{\mathcal{D}}$ are $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual codes that are not self-dual. □

Corollary 4 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ self-dual code with $\mathbf{v}$, $\mathbf{t}$ and $\mathcal{C}_{\mathbf{v}}$ as above. If G is the generator matrix for $\mathcal{C}_{\mathbf{v}}$, then $\begin{pmatrix} \mathbf{t} \\ G \end{pmatrix}$ is a generator matrix for $\mathcal{C}$ and*

$$\begin{pmatrix} 1 & 1 & \mathbf{v} \\ 0 & 1 & \mathbf{t} \\ \mathbf{0} & \mathbf{0} & G \end{pmatrix}$$

is a generator matrix for $\tilde{\mathcal{C}}$.

We shall now give a construction of Type II codes. It is in the spirit of the construction given in [9] for self-dual binary codes but because they are $\mathbb{Z}_2\mathbb{Z}_4$ codes and only formally self-dual there are numerous differences. Let $\mathcal{C}$ be a Type I code with $n \equiv 6 \pmod 8$. The following lemma appears in [10] in a binary setting but the proof can be modified to fit here.

Lemma 5 *Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $n = \alpha + 2\beta$ where the subcode of doubly-even vectors is linear. Then, the weights of the vectors in $\mathcal{C}_0^\perp - \mathcal{C}$ are congruent to $\frac{n}{2} \pmod 4$.*

Proof: In [10] they prove this result for binary codes by showing that the weight enumerator of a Type I code has a doubly-even subcode computed as in (11). Then, the MacWilliams relations give the weight enumerator for $\mathcal{C}_0^\perp$. Then, an algebraic argument gives the result, so the same situation is true for the present case. $\square$

Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with $n \equiv 6 \pmod 8$ and let $\mathcal{C}_0 = \mathcal{C}'$ be the subcode of vectors that have doubly-even weight. We have seen in Theorem 10 that this is either a subgroup of index 2 if it is linear or half the code which generates the whole code. We shall assume that $\mathcal{C}_0$ is linear, which we have seen does occur. Notice that $(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) \in \mathcal{C} - \mathcal{C}_0$ since $6 \equiv 2 \pmod 4$. This gives that $\mathcal{C} = \langle \mathcal{C}_0, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)\rangle$. Similarly to (7), we note that

$$W_{\mathcal{C}_0}(x,y) = \frac{1}{2}(W_{\mathcal{C}}(x,y) + W_{\mathcal{C}}(x,iy)). \tag{11}$$

Let $\mathbf{v}$ be the vector such that $\langle \mathcal{C}, \mathbf{v}\rangle = \mathcal{C}_0^\perp$. Define

$$\mathcal{C}_{\lambda,\mu} = \mathcal{C}_0 + \lambda(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) + \mu\mathbf{v}.$$

Let $\mathcal{D} = \mathcal{C}^\perp$ and define $\mathcal{D}_{\lambda,\mu} = \mathcal{D}_0 + \lambda(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) + \mu\mathbf{w}$ where $\mathbf{w}$ is the vector such that $D_0^\perp = \langle \mathcal{D}, \mathbf{w}\rangle$.

Lemma 6 *Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code where the subcode of doubly-even vectors is linear. Let $\mathbf{v}, \mathbf{w}$ be defined as above. Then, $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = [\mathbf{w}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 2$.*

Proof: If a vector has weight $3 \pmod 4$ then the number of units must be $1 \pmod 2$. Lemma 5 gives that the weights of $\mathbf{v}$ and $\mathbf{w}$ are $3 \pmod 4$. This gives that $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = [\mathbf{w}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 2$. $\square$

The next construction is different from the other construction techniques in that in certain cases we extend in the quaternary coordinates and in certain cases we extend in the binary coordinates.

Theorem 15 *(***Construction C***) Let $\mathcal{C}$ be a Type I $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $n = \alpha + 2\beta \equiv 6 \pmod 8$ such that the subcode $\mathcal{C}_0$ of doubly-even vectors is linear. Let $\mathbf{v}, \mathbf{w}$ be defined as above.*

– *If $[\mathbf{v}, \mathbf{w}] = a \in \{1, 3\}$ then define the code*

$$\widehat{\mathcal{C}} = \langle \{(\mathbf{c}, 0) \mid \mathbf{c} \in \mathcal{C}_0\} \cup \{(\mathbf{c}, 2) \mid \mathbf{c} \in \mathcal{C} - \mathcal{C}_0\}, (\mathbf{v}, a)\rangle. \tag{12}$$

– *If $[\mathbf{v}, \mathbf{w}] = 2$ then define the code*

$$\widehat{\mathcal{C}} = \langle \{(0, 0, \mathbf{c}) \mid \mathbf{c} \in \mathcal{C}_0\} \cup \{(1, 1, \mathbf{c}) \mid \mathbf{c} \in \mathcal{C} - \mathcal{C}_0\}, (1, 0, \mathbf{v})\rangle. \tag{13}$$

In either case, $\widehat{\mathcal{C}}$ is a Type II $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code with weight enumerator

$$W_{\widehat{\mathcal{C}}}(x,y) = x^2 W_{\mathcal{C}_{0,0}}(x,y) + y^2 W_{\mathcal{C}_{1,0}}(x,y) + xyW_{\mathcal{C}_{0,0}}(y,x) + xyW_{\mathcal{C}_{1,0}}(y,x).$$

In the first case, the code is of type $(\alpha, \beta+1, \gamma-1, \delta+1, \widehat{\kappa})$ and in the second one it is of type $(\alpha+2, \beta, \widehat{\gamma}, \widehat{\delta}, \widehat{\kappa})$.

Proof: We prove this in a similar way to other building up constructions except that we either extend the binary or quaternary coordinates depending on the case. Namely, let $\mathcal{C}_{\lambda,\mu}$ be defined as above and set $\mathcal{D} = \mathcal{C}^\perp$ with $\mathcal{D}_{\lambda,\mu}$ being defined similarly. Let $\mathbf{v}, \mathbf{w}$ be defined as above. Now we shall construct the extended code $\widehat{\mathcal{C}}$ depending on the value of $[\mathbf{v},\mathbf{w}]$.

If $[\mathbf{v},\mathbf{w}] \in \{1,3\}$, then we define $\widehat{\mathcal{C}} = \cup(\mathcal{C}_{\lambda,\mu}, v_{\lambda,\mu})$ and $\widehat{\mathcal{D}} = \cup(\mathcal{D}_{\lambda,\mu}, w_{\lambda,\mu})$, where $v_{\lambda,\mu}, w_{\lambda,\mu} \in \mathbb{Z}_4$. In the case $[\mathbf{v},\mathbf{w}] = 2$, we define $\widehat{\mathcal{C}} = \cup(v_{\lambda,\mu}, \mathcal{C}_{\lambda,\mu})$ and $\widehat{\mathcal{D}} = \cup(w_{\lambda,\mu}, \mathcal{D}_{\lambda,\mu})$, where $v_{\lambda,\mu}, w_{\lambda,\mu} \in \mathbb{Z}_2^2$. In both cases, we let $v_{\lambda,\mu} = \lambda v_{1,0} + \mu v_{0,1}$ and $w_{\lambda,\mu} = \lambda w_{1,0} + \mu w_{0,1}$ so that the codes are linear by design.

We shall construct $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ orthogonal, so we need

$$[v_{\lambda,\mu}, w_{\lambda',\mu'}] = -[\mathcal{C}_{\lambda,\mu}, \mathcal{D}_{\lambda',\mu'}].$$

By Lemma 6 we have that $[\mathbf{v}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = [\mathbf{w}, (\mathbf{1}_\alpha \mid \mathbf{2}_\beta)] = 2$ and, considering $\mathbf{c}_0 \in \mathcal{C}_0, \mathbf{d}_0 \in \mathcal{D}_0$, we obtain

$$\begin{aligned}[\mathcal{C}_{\lambda,\mu}, \mathcal{D}_{\lambda',\mu'}] &= [\mathbf{c}_0 + \lambda(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) + \mu\mathbf{v}, \mathbf{d}_0 + \lambda'(\mathbf{1}_\alpha \mid \mathbf{2}_\beta) + \mu'\mathbf{w}] \\ &= 2(\lambda\mu' + \lambda'\mu) + \mu\mu'[\mathbf{v},\mathbf{w}].\end{aligned}$$

Therefore, $\widehat{\mathcal{C}}$ is a Type II formally self-dual code by taking

$$\begin{cases} v_{1,0} = (2), v_{0,1} = (1), w_{1,0} = (2), w_{0,1} = (3), & \text{if } [\mathbf{v},\mathbf{w}] = 1, \\ v_{1,0} = (2), v_{0,1} = (3), w_{1,0} = (2), w_{0,1} = (1), & \text{if } [\mathbf{v},\mathbf{w}] = 3, \\ v_{1,0} = (1,1), v_{0,1} = (1,0), w_{1,0} = (1,1), w_{0,1} = (0,1), & \text{if } [\mathbf{v},\mathbf{w}] = 2. \end{cases}$$

Since $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, then, $|\mathcal{C}| = 2^{\gamma+2\delta}$. Moreover, since $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code, we have that $|\mathcal{C}| = |\mathcal{C}^\perp| = 2^{\frac{n}{2}}$, where $n = \alpha + 2\beta$. Therefore,

$$2(\gamma + 2\delta) = \alpha + 2\beta. \tag{14}$$

Similarly, if $\widehat{\mathcal{C}}$ is of type $(\widehat{\alpha}, \widehat{\beta}; \widehat{\gamma}, \widehat{\delta}; \widehat{\kappa})$, we have that

$$2(\widehat{\gamma} + 2\widehat{\delta}) = \widehat{\alpha} + 2\widehat{\beta}. \tag{15}$$

In the case $[\mathbf{v},\mathbf{w}] = 2$, we have $\widehat{\alpha} = \alpha + 2$ and $\widehat{\beta} = \beta$. In the case $[\mathbf{v},\mathbf{w}] \in \{1,3\}$, we have $\widehat{\alpha} = \alpha$, $\widehat{\beta} = \beta + 1$, $\widehat{\delta} = \delta + 1$, and therefore, $\widehat{\gamma} = \gamma - 1$ by (14) and (15). $\square$

5 Lattices

We shall describe some of the lattices that can be constructed by the Gray image of our codes.

For a binary code C define the kernel of C to be $ker(C) = \{\mathbf{v} \in C \mid \mathbf{v}+C = C\}$. If C contains the all zero vector then $ker(C)$ is a linear code. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$ code then $ker(\mathcal{C}) = \phi^{-1}(ker(\phi(C)))$; i.e., it is all the vectors that get mapped to the kernel of its image.

Lemma 7 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code. Then, $ker(\mathcal{C})$ is a non-trivial linear subcode of $\mathcal{C}$.*

Proof: By [21], we have that it is linear and that $ker(\mathcal{C}) = \{\mathbf{v} \in \mathcal{C} \mid 2\mathbf{v} * \mathbf{w} \in \mathcal{C}, \forall \mathbf{w} \in \mathcal{C}\}$. This implies that if $\mathbf{v}$ has order 2 then $\mathbf{v} \in ker(\mathcal{C})$. We know that if a vector $\mathbf{w} \in \mathcal{C}$ has order 4, then $\mathbf{w}+\mathbf{w}$ has order 2. Hence, the kernel is non-trivial and it is straightforward that it is linear. □

By Lemma 7, we have that a $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code $\mathcal{C}$ is the union of translates of its kernel. Moreover, we have that $\phi(C)$ is the union of translates of its kernel.

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the inner product $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$ for $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. An n-dimensional lattice Λ in $\mathbb{R}^n$ is a free $\mathbb{Z}$-module spanned by n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. An n by n matrix whose rows are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called a generator matrix G of Λ. The fundamental volume $V(\Lambda)$ of Λ is $|\det G|$. The dual lattice Λ^* is given by $\Lambda^* = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}\cdot\mathbf{w} \in \mathbb{Z} \text{ for all } \mathbf{w} \in \Lambda\}$. A lattice Λ is integral if $\Lambda \subseteq \Lambda^*$. An integral lattice with $\det(G) = 1$ (or $\Lambda = \Lambda^*$) is called *unimodular*. If the norm $\mathbf{v}\cdot\mathbf{v}$ is an even integer for all $\mathbf{v} \in \Lambda$, then Λ is called Type II. Unimodular lattices which are not even are called Type I.

Recall the standard construction of lattices from codes. (This code is often called Construction A but we will avoid this since we have already labeled a different construction with this name.) Namely, for a binary code C,

$$\Lambda(C) = \frac{1}{2}\{\mathbf{v} + 2\mathbb{Z}_2^n \mid \mathbf{v} \in C\}.$$

See [12],[8] for a complete description of this construction.

It is well known that if C is a binary self-dual code of Type I (Type II) then $\Lambda(C)$ is a unimodular lattice of Type I (Type II). See Theorem 2 on page 183 of [12].

The theta series of a lattice Λ is defined to be $\Theta_\Lambda(z) = \sum_{x\in\Lambda} q^{x\cdot x}$, with $q = e^{\pi i z}$. It is well known (see [12], [8]) that the theta series of a Type I lattice is an element of $\mathbb{C}[\theta_3(z), \Delta_8(z)]$ and that the theta series of a Type II lattice is an element of $\mathbb{C}[E_4(z), \Delta_{24}(z)]$. See Theorem 7, page 187 of [12] and Theorem 17, page 192 of [12] respectively for a complete description of these results. Moreover, if C is a binary linear code with weight enumerator $W_C(x, y)$ then the theta series of $\Lambda(C)$ is $\Theta_{\Lambda(C)}(z) = W_C(\theta_3(2z)), \theta_2(2z))$ where $\theta_3(2z)$ and $\theta_2(2z)$ are Jacobi theta functions. See Theorem 3, page 183 of [12].

Theorem 16 *Let $\mathcal{C}$ be an even $\mathbb{Z}_2\mathbb{Z}_4$ formally self-dual code.*

- *If $\mathcal{C}$ is a Type I code with $\phi(\mathcal{C})$ self-dual then $\Lambda(\phi(C))$ is a Type I lattice.*
- *If $\mathcal{C}$ is a Type II code with $\phi(\mathcal{C})$ self-dual then $\Lambda(\phi(C))$ is a Type II lattice.*
- *If $\mathcal{C}$ is a Type I code with $\phi(\mathcal{C})$ linear then $\Lambda(\phi(C))$ is a lattice whose theta series is an element of $\mathbb{C}[\theta_3(z), \Delta_8(z)]$.*
- *If $\mathcal{C}$ is a Type II code with $\phi(\mathcal{C})$ linear then $\Lambda(\phi(C))$ is a lattice whose theta series is an element of $\mathbb{C}[E_4(z), \Delta_{24}(z)]$.*
- *If $\mathcal{C}$ is a Type I code with $\phi(\mathcal{C})$ not linear then $\Lambda(\phi(C))$ is a sphere packing whose theta series satisfies the MacWilliams type relation in Theorem 1 of [27].*

Proof: The first two items are well known.

To prove the third, recall that the weight enumerator of a $\mathbb{Z}_2\mathbb{Z}_4$ Type I code is an element of $\mathbb{C}[x^2 + y^2, x^8 + 14x^4y^4 + y^8]$. Then, since $\Theta_{\Lambda(C)}(z) = W_C(\theta_3(2z)), \theta_2(2z))$ we have that the theta series is an element of $\mathbb{C}[\theta_3(z), \Delta_8(z)]$.

The fourth is similar, except that the weight enumerator is an element of $\mathbb{C}[x^8 + 14x^4y^4 + y^8, x^4y^4(x^4 - y^4)^4]$.

The fifth requires that the code C is the union of translates of a linear code. We have proven in Lemma 7 that the kernel is a non-trivial subcode. Then, the kernel of $\phi(\mathcal{C})$ is a non-trivial linear subcode. Hence, the image of the binary kernel is a lattice and the sphere packing is made up of the union of the translates of this lattice. Hence, Theorem 1 in [27] applies. □

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