# PERFECT CODES IN THE DISCRETE SIMPLEX 

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#### Abstract

We study the problem of existence of (nontrivial) perfect codes in the discrete $n$-simplex $\Delta_{\ell}^{n}:=\left\{\left(x_{0}, \ldots, x_{n}\right): x_{i} \in \mathbb{Z}_{+}, \sum_{i} x_{i}=\ell\right\}$ under $\ell_{1}$ metric. The problem is motivated by the so-called multiset codes, which have recently been introduced by the authors as appropriate constructs for error correction in the permutation channels. It is shown that $e$-perfect codes in the 1 -simplex $\Delta_{\ell}^{1}$ exist for any $\ell \geq 2 e+1$, the 2 -simplex $\Delta_{\ell}^{2}$ admits an $e$-perfect code if and only if $\ell=3 e+1$, while there are no perfect codes in higherdimensional simplices. In other words, perfect multiset codes exist only over binary and ternary alphabets.


## 1. Introduction

The study of perfect codes is a classical, and perhaps one of the most attractive topics in coding theory. The best studied case are certainly codes in the Hamming metric spaces [33, 9, [28, 32, 36, 6, 16, as they are historically the first codes that were introduced and are most relevant in practice. There are various other interesting examples in the literature, however, such as perfect codes under the Lee metric [2, 3, 14, 20, 22, 23, 35], Levenshtein metric [27, 8, codes in projective spaces [17], Grassmanians [10, 29], etc. Delsarte's conjecture [11] on the non-existence of perfect constant-weight codes under the Johnson metric has also inspired a lot of research, and still remains unsolved [31, 12, 34, 15, 21, 13. Many of these problems can be regarded as particular instances of the general theory of perfect codes in distance-transitive graphs [7] (but not all cases of interest fit into this framework). In the present paper we investigate perfect codes in discrete simplices of arbitrary dimension. As discussed in Section 2, codes in such spaces arise naturally in the context of error correction in the so-called permutation channels.

The paper is organized as follows. The basic concepts used in the sequel are introduced in the following subsection. Subsection 1.2 summarizes the main contributions of the paper. Section 2 explains the motivation for studying codes in discrete simplices; the notion of multiset codes is introduced here and some of their properties are established. Proofs of the results are given in Section 3 .
1.1. Notation and terminology. Let $\mathbb{Z}_{+}=\{0,1,2, \ldots\}$ denote the set of nonnegative integers. Let $(S, d)$ be a finite metric space with an integer-valued metric $d$, and $\mathcal{C} \subseteq S$ an error-correcting code. Elements of $\mathcal{C}$ are called codewords in this context.

[^0]Definition 1.1. $\mathcal{C}$ is said to be $e$-perfect, $e \in \mathbb{Z}_{+}$, if balls of radius $e$ centered at codewords are disjoint and cover the entire space:

$$
\begin{equation*}
\mathcal{B}(x, e) \cap \mathcal{B}(y, e)=\emptyset \quad \text { for every } \quad x, y \in \mathcal{C}, x \neq y \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\bigcup_{x \in \mathcal{C}} \mathcal{B}(x, e)=S \tag{1.2}
\end{equation*}
$$

where $\mathcal{B}(x, e)=\{w \in S: d(x, w) \leq e\}$ is the decoding region of the codeword $x$. In other words, every element of $S$ is at distance $\leq e$ from exactly one codeword.

Clearly, every singleton $\mathcal{C}=\{x\}$ is $D$-perfect, with $D$ the diameter of the space $S$, and $S$ itself is 0-perfect. In the rest of the paper, we shall be interested only in nontrivial perfect codes - those with $|\mathcal{C}| \geq 2$ and $e \geq 1$.

Let $n, \ell \in \mathbb{Z}_{+}$. The space under consideration in this paper is the discrete version of the standard $n$-simplex:

$$
\begin{equation*}
\Delta_{\ell}^{n}:=\left\{\left(x_{0}, \ldots, x_{n}\right): x_{i} \in \mathbb{Z}_{+}, \quad \sum_{i=0}^{n} x_{i}=\ell\right\} \tag{1.3}
\end{equation*}
$$

endowed with the following metric:

$$
\begin{equation*}
d(x, y)=\frac{1}{2}\|x-y\|_{1}=\frac{1}{2} \sum_{i=0}^{n}\left|x_{i}-y_{i}\right| \tag{1.4}
\end{equation*}
$$

where $x=\left(x_{0}, \ldots, x_{n}\right), y=\left(y_{0}, \ldots, y_{n}\right)$. (The constant $1 / 2$ is taken for convenience because $\|x-y\|_{1}$ is always even for $x, y \in \Delta_{\ell}^{n}$.) The diameter of $\Delta_{\ell}^{n}$ under $d$ is clearly $\ell$. Note that for $x, y \in \Delta_{\ell}^{n}$ we can also write:

$$
\begin{equation*}
d(x, y)=\sum_{x_{i}>y_{i}}\left(x_{i}-y_{i}\right)=\sum_{x_{i}<y_{i}}\left(y_{i}-x_{i}\right) \tag{1.5}
\end{equation*}
$$

To our knowledge, codes in this space have not been analyzed before. Perfect codes under $\ell_{1}$ distance seem to have been studied only in the integer lattice $\mathbb{Z}^{n}$ (as periodic extensions of the codes under the Lee metric), see e.g. [20, 22, 14].

It is particularly useful to represent the metric space $\left(\Delta_{\ell}^{n}, d\right)$ as a graph ${ }^{1}$ with $\left|\Delta_{\ell}^{n}\right|=\binom{n+\ell}{\ell}$ vertices, and with edges connecting vertices at distance one. This representation allows one to visualize the space under study, as well as codes in this space, at least for $n=1,2$. Unfortunately, the resulting graph is not distancetransitive and the general methods developed for such graphs [7] cannot be applied.
1.2. Main results. The following theorem summarizes the main contributions of the paper. Its proof is deferred to Section 3.
Theorem 1.2. Let $e \geq 1$.
(1) Nontrivial e-perfect code in $\left(\Delta_{\ell}^{1}, d\right)$ exists for every $\ell \geq 2 e+1$. Such a code has $\left\lceil\frac{\ell+1}{2 e+1}\right\rceil$ codewords.
(2) Nontrivial e-perfect code in $\left(\Delta_{\ell}^{2}, d\right)$ exists if and only if $\ell=3 e+1$. Furthermore, there are exactly two such codes in $\Delta_{3 e+1}^{2}$, each having three codewords.
(3) Nontrivial e-perfect code in $\left(\Delta_{\ell}^{n}, d\right), n \geq 3$, does not exist for any $e$ and $\ell$.

[^1]In addition to the existence proofs, we shall also enumerate in Section 3 all perfect codes in one- and two-dimensional simplices.

## 2. Motivation - Multiset codes

2.1. Permutation channel. Let $\mathcal{A}=\{0,1, \ldots, n\}$ be a finite alphabet with $n+$ $1 \geq 2$ symbols. A permutation channel over $\mathcal{A}$ is a communication channel that takes sequences of symbols from $\mathcal{A}$ as inputs, and for any input sequence outputs a random permutation of this sequence. Such channels arise, for example, in some types of packet networks [5] in which the packets comprising a single message are routed separately and are frequently sent over different routes in the network. Consequently, the receiver cannot rely on them being delivered in any particular order.

In addition to random permutations, the channel is assumed to impose various types of "noise" on the transmitted sequence, such as insertions, deletions, and substitutions of symbols. For example, in a networking scenario mentioned above, packet deletions can be caused by network congestion and consequent buffer overflows in the routers, while packet substitutions (i.e., errors) are usually caused by noise or malfunctioning of network equipment. Therefore, the permutation channel with these types of impairments is indeed a relevant model.
2.2. Coding for the permutation channel. It is clear from the definition of the permutation channel that, when transmitting sequences through it, no information should be encoded in the order of symbols in the sequence because it is impossible to recover this information. The only carrier of information should be the multiset of the symbols sent, i.e., the number of occurrences of each symbol from $\mathcal{A}$ in the sequence. The appropriate space in which error-correcting codes for the permutation channel should be defined is therefore the set of all multisets over the channel alphabet [26].

Formally, a multiset $X$ is an ordered pair $\left(\mathcal{A}, \mathrm{m}_{X}\right)$ where $\mathcal{A}$ is the ground set (the channel alphabet in our case) and $\mathbb{m}_{X}: \mathcal{A} \rightarrow \mathbb{Z}_{+}$is a multiplicity function which encodes the numbers of occurrences of the elements of $\mathcal{A}$ in $X$. The cardinality of $X$ is the number of elements it contains, including repetitions, namely $|X|=\sum_{i=0}^{n} \mathrm{~m}_{X}(i)$. Since the alphabet $\mathcal{A}=\{0,1, \ldots, n\}$ is fixed, multisets can be identified with their multiplicity functions, and these can be identified with $(n+1)$ tuples $\left(\mathrm{m}_{X}(0), \mathrm{m}_{X}(1), \ldots, \mathbb{m}_{X}(n)\right) \in \mathbb{Z}_{+}^{n+1}$. The set of all multisets over $\mathcal{A}$ can therefore be identified with the space $\mathbb{Z}_{+}^{n+1}$, and the set of all multisets of given cardinality $\ell$ with

$$
\begin{equation*}
\left\{\left(x_{0}, x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}_{+}^{n+1}: \sum_{i=0}^{n} x_{i}=\ell\right\} \tag{2.1}
\end{equation*}
$$

which is precisely the discrete simplex $\Delta_{\ell}^{n}$. We shall focus here on the latter case only and study codes in $\Delta_{\ell}^{n}$, i.e., codes whose all codewords have the same cardinality. This convention is also practically motivated because it somewhat simplifies the communication protocol [25, 26]. In order to study codes in $\Delta_{\ell}^{n}$, it is convenient to introduce a metric on this space so that the minimum distance of the code can be defined, and guarantees on the number of correctable errors provided. A natural metric on the space of multisets is the so-called symmetric difference metric defined
by:

$$
\begin{equation*}
|X \triangle Y|=\sum_{i=0}^{n}\left|\mathrm{~m}_{X}(i)-\mathrm{m}_{Y}(i)\right| \tag{2.2}
\end{equation*}
$$

where $\triangle$ denotes the symmetric difference of (multi)sets. This is obviously the $\ell_{1}$ distance between the sequences $\left(\mathrm{m}_{X}(0), \ldots, \mathrm{m}_{X}(n)\right)$ and $\left(\mathrm{m}_{Y}(0), \ldots, \mathrm{m}_{Y}(n)\right)$. We have thus shown that the metric space $\left(\Delta_{\ell}^{n}, d\right)$ is an appropriate space for defining error-correcting codes in permutation channels.

It should be pointed out that, even though we have defined codes in $\Delta_{\ell}^{n}$, the codewords from $\Delta_{\ell}^{n}$ are not what is actually sent through the permutation channel, they only describe the multisets that are transmitted. Namely, if $\left(c_{0}, \ldots, c_{n}\right)$ is a codeword to be sent, then what is actually transmitted are $c_{0}$ copies of the symbol $0, c_{1}$ copies of the symbol 1 , etc. Therefore, for a multiset code defined in $\Delta_{\ell}^{n}, \ell$ represents the length of the code (the number of symbols in codewords $=$ the cardinality of the codewords) and $n+1$ is the size of the alphabet.

Remark 2.1. Note that in our setting the dimension of the code space depends on the size of the alphabet $(n+1)$, not on the length of the code $(\ell)$. This stands in sharp contrast with most other coding scenarios.

Remark 2.2. It is also interesting to observe that the Johnson space $J(n+1, \ell)$ (the set of all binary sequences of length $n+1$ and Hamming weight $\ell$ ) is a subset of $\Delta_{\ell}^{n}$. Furthermore, in the case of binary sequences, the metric $d$ from 1.4 reduces to the Johnson metric. However, (non)existence of perfect codes in $\Delta_{\ell}^{n}$ does not imply (non)existence of perfect codes in $J(n+1, \ell$ ), and the methods used in this paper do not seem to be sufficient to settle Delsarte's conjecture (except in some special cases that are already known, e.g., [15, Cor 2]).

Using the above terminology, we can now restate Theorem 1.2 as follows:
(1) Nontrivial e-perfect multiset code of length $\ell$ over a binary alphabet exists for every $\ell \geq 2 e+1$. Such a code has $\left\lceil\frac{\ell+1}{2 e+1}\right\rceil$ codewords.
(2) Nontrivial e-perfect multiset code of length $\ell$ over a ternary alphabet exists if and only if $\ell=3 e+1$. Furthermore, there are exactly two e-perfect multiset codes of length $3 e+1$, each having three codewords.
(3) Nontrivial e-perfect multiset code of length $\ell$ over a q-ary alphabet, $q>2$, does not exist for any $e$ and $\ell$.
Finally, we note that the framework presented in this section is a generalization of coding in power sets [18, 19, 25], where codewords are taken to be sets rather than multisets. Such approaches to coding for the permutation channel are somewhat analogous to the approach of Kötter and Kschischang [24] of using codes in projective spaces and Grassmanians for error correction in networks employing random linear network coding. In both cases, the guiding idea is to define codes in the space of objects invariant under the channel transformation - (multi)sets are invariant under permutations, whereas vector spaces are invariant (with high probability) under random linear combinations.

## 3. Proofs

We now proceed with the proof of our claim. To that end, it will be useful to represent the simplex $\Delta_{\ell}^{n}$ as the corresponding graph with $\binom{n+\ell}{\ell}$ vertices, and
with edges connecting vertices at distance one. As noted in Section 1.1, such a representation will allow us to visualize the spaces under study, at least in the case of binary and ternary alphabets.
3.1. Binary alphabet. One-dimensional case is simple to analyze. The space

$$
\begin{equation*}
\Delta_{\ell}^{1}=\{(\ell-t, t): t=0, \ldots, \ell\} \tag{3.1}
\end{equation*}
$$

can be represented as a path with $\left|\Delta_{\ell}^{1}\right|=\ell+1$ vertices, the leftmost vertex being $\left(\begin{array}{ll}\ell & 0\end{array}\right)$ and the rightmost $(0, \ell)$ for example (see Figure 1 ).

Since the diameter of $\left(\Delta_{\ell}^{1}, d\right)$ is $\ell$ and any two codewords of an $e$-perfect code must be at distance $\geq 2 e+1$, nontrivial code can exist only if $\ell \geq 2 e+1$. It is not hard to conclude that a perfect code exists for any such $\ell$ (see also [4] for the case $e=1$ ). Figure 1 provides an illustration of such a code, and Proposition 3.1 lists all perfect codes in $\Delta_{\ell}^{1}$.


Figure 1. 1-perfect code in $\Delta_{8}^{1}(n=1, \ell=8, e=1)$. Black dots represent codewords; dots belonging to a gray region comprise the decoding region of the corresponding codeword.

Proposition 3.1. Let $\ell=q(2 e+1)+r$ for some $q \geq 1,0 \leq r<2 e+1$. Then there are exactly $M=\min \{r+1,2 e+1-r\}>0$ perfect codes in $\Delta_{\ell}^{1}$, each having $q+1=\left\lceil\frac{\ell+1}{2 e+1}\right\rceil$ codewords. Let also $s=\min \{r, e\}$. Then all perfect codes in $\Delta_{\ell}^{1}$ can be enumerated as
$(3.2) \mathcal{C}_{1}^{(m)}=\{(\ell-s+m-1-i(2 e+1), s-m+1+i(2 e+1)): \quad i=0, \ldots, q\}$,
for $m=1, \ldots, M$.
Proof. Considering the geometry of the space $\Delta_{\ell}^{1}$ and the corresponding graph, it is clear that a perfect code has to be of the form

$$
\begin{equation*}
\{(\ell-j-i(2 e+1), j+i(2 e+1))\} \tag{3.3}
\end{equation*}
$$

for some fixed $j$, and for $i$ ranging from 0 to some largest value. Namely, once we have fixed the "leftmost" codeword $(\ell-j, j)$, all the other codewords are determined by the fact that neighboring codewords have to be at distance $2 e+1$ from each other. In that way we ensure that the decoding regions are disjoint and that all intermediate points are covered. Therefore, to prove that $\mathcal{C}_{1}^{(m)}$ are perfect, i.e., that the entire $\Delta_{\ell}^{1}$ is covered, it is enough to show that the endpoints $(\ell, 0)$ and $(0, \ell)$ are covered. Assume that $r \leq e$, in which case $M=r+1$ and $s=r$. Then $0 \leq s-m+1 \leq r \leq e$, and hence the vertex $(\ell, 0)$ is at distance $\leq e$ from the codeword $(\ell-s+m-1, s-m+1)$. Similarly, $0 \leq r-s+m-1 \leq r \leq e$ and therefore the vertex $(0, \ell)$ is at distance $\leq e$ from the codeword $(r-s+m-1, \quad \ell-r+s-m+1)$ (obtained for $i=q$ in (3.2). Similar analysis applies when $r>e$. This proves that the codes $\mathcal{C}_{1}^{(m)}$ are perfect.

It is left to prove that 3.2 lists all perfect codes in $\Delta_{\ell}^{1}$. Assume that $r \leq e$. In that case the "leftmost" codeword of $\mathcal{C}_{1}^{(m)}$ is $(\ell-r+m-1, r-m+1)$, $m=1, \ldots, r+1$. Therefore, we have found $r+1$ codes with "leftmost" codewords
$(\ell, 0), \ldots,(\ell-r, r)$. Suppose that we try to construct another perfect code by specifying $(\ell-r-k, r+k), k>0$, as its "leftmost" codeword. Since the end point $(\ell, 0)$ has to be covered, we can assume that $k \leq e-r$. Then its "rightmost" codeword is obtained by shifting for $i(2 e+1)$ and is therefore either $(2 e+1-k, \quad \ell-$ $2 e-1+k)$ (for $i=q-1$ ) or ( $-k, \ell+k$ ) (for $i=q$ ). The second case is clearly impossible, and the first fails to give a perfect code because the point ( $0, \ell$ ) does not belong to a decoding region of some codeword (its distance from the "rightmost" codeword is $2 e+1-k>e$ ). Again, the proof is similar for $r>e$.
3.2. Ternary alphabet. Consider now the two-dimensional simplex $\Delta_{\ell}^{2}$. The graph representation of this space is a triangular grid graph, as illustrated in Figure 4 (we assume that the leftmost vertex corresponds to $(\ell, 0,0)$, the rightmost to $(0, \ell, 0)$, and the top to $(0,0, \ell))$. Balls under the metric $d$ in this graph are "hexagons", perhaps clipped if the center of the ball is too close to the edge (in fact, this space is easily seen to be a "triangle" cut out from the hexagonal lattice, see Figure 4). Hence, we need to examine whether a perfect packing of hexagons is possible within this graph, i.e., whether there is a configuration of hexagons covering the entire graph without overlapping. We first briefly discuss some properties of $\Delta_{\ell}^{2}$ that will be useful.

Observe that, given some $x \in \Delta_{\ell}^{2}$, we can express any point $y \in \Delta_{\ell}^{2}$ by specifying a path from $x$ to $y$ in the corresponding graph. The first node on this path, call it $x^{\prime}$, is a neighbor of $x$, the second node is a neighbor of $x^{\prime}$, etc. The neighbors of $x=\left(x_{0}, x_{1}, x_{2}\right)$, i.e., points that are at distance 1 from it, are obtained by adding 1 to some coordinate of $x$, and -1 to some other coordinate. A convenient way of describing neighbors and paths in $\Delta_{\ell}^{2}$ is as follows. Define the vector $f_{i, j}$, $i, j \in\{1,2,3\}$, to have a 1 at the $i$ 'th position, a -1 at the $j$ 'th position, and a 0 at the remaining position. For example, $f_{1,2}=(1,-1,0)$. Clearly, $f_{i, j}=-f_{j, i}$ and by convention we take $f_{i, i}=(0,0,0)$. These vectors describe all possible directions of moving from some point, and hence any neighbor $x^{\prime}$ of $x$ can be described by specifying the direction, namely $x^{\prime}=x+f_{i, j}$ (see Figure 2). Therefore, any $y \in \Delta_{\ell}^{2}$ can be expressed as

$$
\begin{equation*}
y=x+\sum_{i, j} \alpha_{i, j} f_{i, j} \tag{3.4}
\end{equation*}
$$

for some integers $\alpha_{i, j} \geq 0$. If $d(x, y)=\delta$, then clearly there exists a representation


Figure 2. Neighbors of $x$ in $\Delta_{\ell}^{2}$.
of this form with $\sum_{i, j} \alpha_{i, j}=\delta$. Another way to write this is

$$
\begin{equation*}
y=x+\left(s_{0}, s_{1}, s_{2}\right) \tag{3.5}
\end{equation*}
$$

where $\sum_{i} s_{i}=0$ and $\sum_{i}\left|s_{i}\right|=2 \delta$.
The following lemma will also be used in the sequel. The statement is illustrated in Figure 3, and the proof (of the more general version) is given in the following subsection (see Lemma 3.8 and Remark 3.9).

Lemma 3.2. Let $x, y, w \in \Delta_{\ell}^{2}$ be such that $d(x, w)=d(y, w)=e+1, d(x, w+$ $\left.f_{1,2}\right)=e$, and $d\left(y, w+f_{2,1}\right)=e$. Then there can be no $z \in \Delta_{\ell}^{2}$ such that $w \in \mathcal{B}(z, e)$, $\mathcal{B}(x, e) \cap \mathcal{B}(z, e)=\emptyset$ and $\mathcal{B}(y, e) \cap \mathcal{B}(z, e)=\emptyset$.

Let us elaborate on the meaning of this lemma. Suppose we have two codewords $(x, y)$ and a point $w$ lying outside their decoding regions. Since we are trying to build a perfect code, the point $w$ has to belong to a decoding region of a third codeword $z$. The lemma asserts that if $w$ is bounded by $\mathcal{B}(x, e)$ and $\mathcal{B}(y, e)$ in some direction, say $f_{1,2}$ (recall that $f_{2,1}=-f_{1,2}$ ), then such a codeword cannot exist, and therefore $x$ and $y$ cannot be codewords of a perfect code.


Figure 3. Illustration of Lemma 3.2 and Lemma 3.6.
We now proceed with proof of the main claim, namely the (non)existence of perfect codes. If $\ell=3 e+1$, then it is not hard to exhibit a perfect code (see Figure 44. In fact, there are exactly two such codes:

$$
\begin{align*}
\mathcal{C}_{2}^{(1)} & =\{(2 e+1, e, 0),(0,2 e+1, e),(e, 0,2 e+1)\} \\
\mathcal{C}_{2}^{(2)} & =\{(2 e+1,0, e),(e, 2 e+1,0),(0, e, 2 e+1)\} \tag{3.6}
\end{align*}
$$

Proposition 3.3. Codes $\mathcal{C}_{2}^{(1)}$ and $\mathcal{C}_{2}^{(2)}$ are e-perfect in $\Delta_{3 e+1}^{2}$.
The proof of the proposition is straightforward and is omitted. In the following we prove that these are the only two perfect codes when $\ell=3 e+1$, and that there are no perfect codes for $\ell \neq 3 e+1$.

We start by observing the vertex $(\ell, 0,0)$. For this vertex to be covered there must exist a codeword of the form

$$
\begin{equation*}
x=\left(\ell-t, x_{1}, x_{2}\right) \tag{3.7}
\end{equation*}
$$



Figure 4. 2-perfect code $\left(\mathcal{C}_{2}^{(2)}\right)$ in $\Delta_{7}^{2}(n=2, \ell=7, e=2)$.
with $x_{1}+x_{2}=t \leq e$. Observe now the point

$$
\begin{equation*}
v=\left(\ell-x_{1}-e-1, x_{1}+e+1,0\right) . \tag{3.8}
\end{equation*}
$$

(Needless to say, we assume that $v \in \Delta_{\ell}^{2}$, i.e., that $v_{0}=\ell-x_{1}-e-1 \geq 0$; otherwise, the diameter of $\Delta_{\ell}^{2}$ would be $\ell \leq 2 e$ and no nontrivial perfect code could exist.) We have $d(x, v)=e+1$ and so the point $v$ is not covered by $\mathcal{B}(x, e)$. To cover it we need another codeword $y$ with $d(v, y)=e$ and $d(x, y)=2 e+1$.
Lemma 3.4. Let $x, v \in \Delta_{\ell}^{2}$ be given by (3.7) and (3.8), respectively. Then the point $y \in \Delta_{\ell}^{2}$ satisfying $d(v, y)=e, d(x, y)=2 e+1$ is of the form

$$
\begin{equation*}
y=\left(\ell-x_{1}-2 e-1, x_{1}+e+1+u, e-u\right) \tag{3.9}
\end{equation*}
$$

with $0 \leq u \leq e$, and with the property that

$$
\begin{equation*}
x_{2}>0 \Rightarrow u=e \tag{3.10}
\end{equation*}
$$

Proof. Let $y=\left(\ell-x_{1}-2 e-1+s, y_{1}, y_{2}\right)$ for some $s \in \mathbb{Z}$. If $s<0$ we have $d(v, y) \geq v_{0}-y_{0}=e-s>e$ which contradicts one of the assumptions of the lemma. We next show that the assumption $s>0$ also leads to a contradiction. We can assume that $x_{0}>y_{0}$; otherwise, the vertex $(\ell, 0,0)$ would be covered by both $x$ and $y$. We can also assume that $s \leq x_{1}$, for otherwise we would have $x_{0}-y_{0} \leq 2 e-t$, and since the sum of the remaining $x_{i}$ 's is $t$ it would follow that

$$
\begin{align*}
d(x, y) & =\sum_{x_{i}>y_{i}}\left(x_{i}-y_{i}\right)=x_{0}-y_{0}+\sum_{i>0, x_{i}>y_{i}}\left(x_{i}-y_{i}\right) \\
& \leq x_{0}-y_{0}+\sum_{i>0} x_{i} \leq 2 e . \tag{3.11}
\end{align*}
$$

Now, since $v_{0}-y_{0}=e-s<e$ and $y_{2} \geq v_{2}=0$, we must have $v_{1}-y_{1}=$ $x_{1}+e+1-y_{1}=s$ in order to achieve $d(v, y)=e($ see 1.5), and hence

$$
\begin{equation*}
y_{1}=x_{1}-s+e+1 \geq e+1>x_{1} \tag{3.12}
\end{equation*}
$$

where the first inequality follows from the above assumption that $s \leq x_{1}$. Since $y_{0}<x_{0}$ and $y_{1}-x_{1}=e+1-s$, in order to have $d(x, y)=2 e+1$ we must have $y_{2}-x_{2}=e+s$. But this is impossible because

$$
\begin{equation*}
y_{2}-x_{2} \leq y_{2}=\ell-y_{0}-y_{1}=e<e+s \tag{3.13}
\end{equation*}
$$

where we have used (3.12). We thus conclude that $s$ must be zero. In that case we have $v_{0}-y_{0}=e$, and since $d(v, y)=e$, we must also have $y_{1} \geq v_{1}=x_{1}+e+1$. This shows that $y$ is necessarily of the form 3.9. To prove the last part of the claim observe that $y_{0}<x_{0}, y_{1}-x_{1}=e+1+u$, and $d(x, y)=2 e+1$ imply that $y_{2}-x_{2}=e-u$ when $u<e$. But since $y_{2}=e-u$, this can only hold if $x_{2}=0$ whenever $y_{2}>0$.

Assume therefore that we have two codewords of the form $\sqrt{3.7}$ and $\sqrt{3.9}$, and observe the point

$$
\begin{equation*}
w=\left(\ell-t-e-1, x_{1}+u, \quad \max \left\{x_{2}, y_{2}\right\}+1\right) \tag{3.14}
\end{equation*}
$$

where $y_{2}=e-u$. (Here again we assume that $w_{0} \geq 0$ because otherwise the diameter of $\Delta_{\ell}^{2}$ would be $\ell \leq 2 e$.) To show that $w \in \Delta_{\ell}^{2}$, consider two cases: 1.) $x_{2}>0$; by (3.10) this implies that $y_{2}=e-u=0$ and $\max \left\{x_{2}, y_{2}\right\}=x_{2}$, wherefrom $\sum_{i} w_{i}=\ell, 2$.) $x_{2}=0$; in this case $t=x_{1}$ and $\max \left\{x_{2}, y_{2}\right\}=y_{2}=e-u$, so we again have $\sum_{i} w_{i}=\ell$. Furthermore, we have that $d(x, w)=d(y, w)=e+1$. This is shown easily by considering the above two cases. Namely, if $x_{2}>0$, then $y_{2}=e-u=0$ and so $y=\left(\ell-x_{1}-2 e-1, x_{1}+2 e+1,0\right), w=\left(\ell-t-e-1, x_{1}+e, x_{2}+1\right)$, and by (1.5) the statement follows. The case $x_{2}=0$ is similar.

We shall need the following claim in the sequel (we omit the proof because the statement is geometrically quite clear).
Lemma 3.5. Let $x, y, w \in \Delta_{\ell}^{2}$ be such that $d(x, w)=d(y, w)=e+1, d\left(x, w+f_{k, l}\right)=$ $d\left(x, w+f_{m, l}\right)=d\left(y, w+f_{k, m}\right)=e$. In words, $w$ is outside the decoding regions of $x$ and $y$, but its neighbors along three consecutive directions (see Figure 2) are not. Then the point $z$ such that $w \in \mathcal{B}(z, e), \mathcal{B}(x, e) \cap \mathcal{B}(z, e)=\mathcal{B}(y, e) \cap \mathcal{B}(z, e)=\emptyset$ lies on the direction $f_{l, k}=-f_{k, l}$, i.e., $z=w+e f_{l, k}$.
Lemma 3.6. Let $x, y \in \Delta_{\ell}^{2}$ be given by (3.7) and (3.9), respectively. Let also either a.) $t<e$, or b.) $t=e$ but $0<x_{1}<e$. Then $x$ and $y$ cannot be codewords of an e-perfect code.

Proof. Assume first that $x_{2}>0$. Then, as noted above, $y=\left(\ell-x_{1}-2 e-1, x_{1}+\right.$ $2 e+1,0), w=\left(\ell-t-e-1, x_{1}+e, x_{2}+1\right)$. Furthermore, $w+f_{1,2}=(\ell-t-$ $\left.e, x_{1}+e-1, x_{2}+1\right)$ and $w+f_{2,1}=\left(\ell-t-e-2, x_{1}+e+1, x_{2}+1\right)$. By using (1.5) we easily find that $d(x, w)=d(y, w)=e+1$ and $d\left(x, w+f_{1,2}\right)=d\left(y, w+f_{2,1}\right)=e$ (for the last equality we need the fact that either $t<e$, or $t=e$ but $x_{1}>0$ ). Hence, by Lemma 3.2, we conclude that there exists no codeword $z$ whose decoding region contains $w$ and is disjoint from the decoding regions of $x$ and $y$.

Assume now that $x_{2}=0$. If $u>0$, then $x=\left(\ell-x_{1}, x_{1}, 0\right) y=\left(\ell-x_{1}-2 e-\right.$ $\left.1, x_{1}+e+u+1, e-u\right), w=\left(\ell-x_{1}-e-1, x_{1}+u, e-u+1\right), w+f_{1,2}=\left(\ell-x_{1}-\right.$ $\left.e, x_{1}+u-1, e-u+1\right)$, and $w+f_{2,1}=\left(\ell-x_{1}-e-2, x_{1}+u+1, e-u+1\right)$. We therefore again have $d(x, w)=d(y, w)=e+1$ and $d\left(x, w+f_{1,2}\right)=d\left(y, w+f_{2,1}\right)=e$, and by Lemma 3.2 the conclusion follows.

Finally, if $x_{2}=0$ and $u=0$, then $y=\left(\ell-x_{1}-2 e-1, x_{1}+e+1, e\right)$, $w=\left(\ell-x_{1}-e-1, x_{1}, e+1\right), w+f_{1,3}=\left(\ell-x_{1}-e, x_{1}, e\right), w+f_{2,3}=$ $\left(\ell-x_{1}-e-1, x_{1}+1, e\right)$, and $w+f_{2,1}=\left(\ell-x_{1}-e-2, x_{1}+1, e+1\right)$. Therefore, we have $d(x, w)=d(y, w)=e+1, d\left(x, w+f_{1,3}\right)=e$, and $d\left(y, w+f_{2,3}\right)=$ $d\left(y, w+f_{2,1}\right)=e$. By Lemma 3.5 we conclude that the codeword $z$ covering $w$ has to be $z=w+e f_{3,2}=\left(\ell-x_{1}-e-1, x_{1}-e, 2 e+1\right)$, but this is impossible
because we have assumed that $x_{1}<e$ and therefore the second coordinate of $z$ is negative.

The previous lemma shows that either $(\ell-e, e, 0)$ or $(\ell-e, 0, e)$ must be a codeword if the vertex $(\ell, 0,0)$ is to be covered, and similarly for the other two vertices $(0, \ell, 0)$ and $(0,0, \ell)$. This proves that the codes given by (3.6) are the only perfect codes in $\Delta_{3 e+1}^{2}$. It is left to prove that for $\ell \neq 3 e+1$ perfect codes do not exist.
Proposition 3.7. There are no e-perfect codes in $\Delta_{\ell}^{2}$ for $\ell \neq 3 e+1$.
Proof. The proof is illustrated in Figure 5, but we also give here a more formal version. By the above arguments, we can assume that $x=(\ell-e, 0, e)$ is a codeword. Observe the point $v=(\ell-e-1, e+1,0)$. By Lemma 3.4 we conclude that for $v$ to be covered we must take $y=(\ell-2 e-1,2 e+1,0)$ to be a codeword. Hence, we must have $\ell \geq 2 e+1$ for the perfect code to exist. Now observe $w=(\ell-2 e-1, e, e+1)$. We have $d(x, w)=d(y, w)=e+1$ and so there must exist a third codeword $z$ covering $w$. Note also that $d\left(x, w+f_{1,2}\right)=$ $d\left(x, w+f_{1,3}\right)=d\left(y, w+f_{2,3}\right)=e$ and so by Lemma 3.5 we conclude that $z$ has to be of the form $w+e f_{3,1}$, i.e., $z=(\ell-3 e-1, e, 2 e+1)$. Therefore, we must have $\ell \geq 3 e+1$ for the perfect code to exist. The case $\ell=3 e+1$ has been settled, so assume that $\ell>3 e+1$. Next, observe the point $u=(\ell-3 e-2,2 e+1, e+1)$. We have $d(z, u)=d(y, u)=e+1$ and $d(x, u)=2 e+2$. Therefore, to cover $u$ we need a fourth codeword $q$. Since $d\left(z, u+f_{1,2}\right)=d\left(z, u+f_{3,2}\right)=d\left(y, u+f_{1,3}\right)=e$, by Lemma 3.5 we conclude that $q=(\ell-4 e-2,3 e+1, e+1$ ) (and so we must have $\ell>4 e+1)$. Finally, observe the point $p=(\ell-3 e-2,3 e+2,0)$. Its distance from the codewords $x, y, z, q$ is easily seen to be $>e$, and therefore we need another codeword to cover it. However, since $d(q, p)=d(y, p)=e+1$ and $d\left(q, p+f_{3,1}\right)=d\left(q, p+f_{3,2}\right)=d\left(y, p+f_{1,2}\right)=e$, this codeword would (by Lemma 3.5 have to be $p+e f_{2,3}=(\ell-3 e-2,4 e+2,-e)$ which is impossible.


Figure 5. Proof of Proposition 3.7 .
3.3. Larger alphabets. We now turn to the higher-dimensional case.

As in two dimensions, given some $x \in \Delta_{\ell}^{n}$, we can always express the point $y \in \Delta_{\ell}^{n}$ by specifying a path from $x$ to $y$. This is formalized by using vectors $f_{i, j}$, as before (the $n$-dimensional vector $f_{i, j}$ has a 1 at the $i$ 'th position, a -1 at the $j$ 'th position, and zeros elsewhere, e.g., $\left.f_{1,2}=(1,-1,0, \ldots, 0)\right)$. Namely, for any $y \in \Delta_{\ell}^{n}$ we can write

$$
\begin{equation*}
y=x+\sum_{i, j} \alpha_{i, j} f_{i, j} \tag{3.15}
\end{equation*}
$$

for some integers $\alpha_{i, j} \geq 0$. If $d(x, y)=\delta$, then there exists such a representation of $y$ with $\sum_{i, j} \alpha_{i, j}=\delta$. We call two directions $f_{i, j}$ and $f_{k, l}$ orthogonal if $\{i, j\} \cap\{k, l\}=$ $\emptyset$, i.e., if there is no coordinate at which both of them are nonzero.

The following claim is a generalization of Lemma 3.2 to higher dimensions. Suppose we have two codewords $(x, y)$ and a point $w$ lying outside their decoding regions. The lemma asserts that if $w$ is bounded by $\mathcal{B}(x, e)$ and $\mathcal{B}(y, e)$ in some direction, say $f_{1,2}$, then the codeword $z$ covering $w$ has to lie in the subspace orthogonal to $f_{1,2}$, i.e., it must be of the form

$$
\begin{equation*}
z=w+\left(0,0, s_{2}, \ldots, s_{n}\right) \tag{3.16}
\end{equation*}
$$

where $\sum_{i} s_{i}=0$ and $\sum_{i}\left|s_{i}\right|=2 e$.
Lemma 3.8. Let $x, y, w \in \Delta_{\ell}^{n}$ be such that $d(x, w)=d(y, w)=e+1, d(x, w+$ $\left.f_{1,2}\right)=e$, and $d\left(y, w+f_{2,1}\right)=e$. Then the point $z$ such that $w \in \mathcal{B}(z, e), \mathcal{B}(x, e) \cap$ $\mathcal{B}(z, e)=\emptyset$ and $\mathcal{B}(y, e) \cap \mathcal{B}(z, e)=\emptyset$ must have a representation of the form:

$$
\begin{equation*}
z=w+\sum_{i, j \notin\{1,2\}} \alpha_{i, j} f_{i, j} \tag{3.17}
\end{equation*}
$$

with $\alpha_{i, j} \geq 0, \sum_{i, j \notin\{1,2\}} \alpha_{i, j}=e$.
Proof. The point $z$ has to be at distance $e$ from $w$. (If the distance were larger, the ball $\mathcal{B}(z, e)$ would not contain $w$, and if it were smaller this ball would intersect $\mathcal{B}(x, e)$ and $\mathcal{B}(y, e)$.) We can therefore write

$$
\begin{equation*}
z=w+\sum_{i, j} \alpha_{i, j} f_{i, j} \tag{3.18}
\end{equation*}
$$

where $\alpha_{i, j} \geq 0, \sum_{i, j} \alpha_{i, j}=e$. We need to show that in such a representation we necessarily have $\alpha_{i, j}=0$ whenever $i \in\{1,2\}$ or $j \in\{1,2\}$. Suppose that this is not true, and that $\alpha_{1,3}>0$ for example (the proof is similar if any other $\alpha_{i, j}$ with $i \in\{1,2\}$ or $j \in\{1,2\}$ is assumed positive). Since $f_{1,3}=f_{1,2}+f_{2,3}$, we can write

$$
\begin{align*}
z & =w+f_{1,2}+f_{2,3}+\left(\alpha_{1,3}-1\right) f_{1,3}+\sum_{(i, j) \neq(1,3)} \alpha_{i, j} f_{i, j}  \tag{3.19}\\
& =w+f_{1,2}+\sum_{i, j} \beta_{i, j} f_{i, j}
\end{align*}
$$

where $\beta_{i, j} \geq 0, \sum_{i, j} \beta_{i, j}=e$, which implies that $d\left(z, w+f_{1,2}\right)=e$. But we have assumed that also $d\left(x, w+f_{1,2}\right)=e$, which means that $\mathcal{B}(x, e) \cap \mathcal{B}(z, e) \neq \emptyset$, a contradiction.

Remark 3.9. Since there are no orthogonal directions in the two-dimensional simplex $\Delta_{\ell}^{2}$, the above lemma implies that if $w$ is "trapped" between $\mathcal{B}(x, e)$ and $\mathcal{B}(y, e)$,
then there exists no $z$ with $w \in \mathcal{B}(z, e)$ and $\mathcal{B}(z, e) \cap \mathcal{B}(x, e)=\mathcal{B}(z, e) \cap \mathcal{B}(y, e)=\emptyset$. This is precisely the statement of Lemma 3.2

Let us now continue with the proof of nonexistence of perfect codes. As in the two-dimensional case, we start by observing the vertex $(\ell, 0, \ldots, 0)$. For this vertex to be covered there must exist a codeword of the form

$$
\begin{equation*}
x=\left(\ell-t, x_{1}, \ldots, x_{n}\right) \tag{3.20}
\end{equation*}
$$

with $x_{1}+\ldots+x_{n}=t \leq e$. Without loss of generality, we assume that $x_{1}>0$ whenever $t>0$. Observe now the point

$$
\begin{equation*}
v=\left(\ell-x_{1}-e-1, x_{1}+e+1,0, \ldots, 0\right) \tag{3.21}
\end{equation*}
$$

We have $d(x, v)=e+1$ and so the point $v$ is not covered by $\mathcal{B}(x, e)$. To cover it we need another codeword $y$ with $d(v, y)=e$ and $d(x, y)=2 e+1$.

Lemma 3.10. The point $y$ satisfying $d(v, y)=e, d(x, y)=2 e+1$ is of the form

$$
\begin{equation*}
y=\left(\ell-x_{1}-2 e-1, x_{1}+e+1+u, y_{2}, \ldots, y_{n}\right) \tag{3.22}
\end{equation*}
$$

with $0 \leq u \leq e, y_{2}+\cdots+y_{n}=e-u$, and with the property that

$$
\begin{equation*}
x_{i}>0 \Rightarrow y_{i}=0 \quad \text { for } \quad i=2, \ldots, n \tag{3.23}
\end{equation*}
$$

Proof. Let $y=\left(\ell-x_{1}-2 e-1+s, y_{1}, \ldots, y_{n}\right)$ for some $s \in \mathbb{Z}$. If $s<0$ we have $d(v, y) \geq v_{0}-y_{0}=e-s>e$ which contradicts one of the assumptions of the lemma. Let us show that the case $s>0$ is also impossible. We can assume that $x_{0}>y_{0}$; otherwise, the vertex $(\ell, 0, \ldots, 0)$ would be covered by both $x$ and $y$. We can also assume that $s \leq x_{1}$, for otherwise we would have $x_{0}-y_{0} \leq 2 e-t$, and since the sum of the remaining $x_{i}$ 's is $t$ it would follow that

$$
\begin{align*}
d(x, y) & =\sum_{x_{i}>y_{i}}\left(x_{i}-y_{i}\right)=x_{0}-y_{0}+\sum_{i>0, x_{i}>y_{i}}\left(x_{i}-y_{i}\right) \\
& \leq x_{0}-y_{0}+\sum_{i>0} x_{i} \leq 2 e \tag{3.24}
\end{align*}
$$

Since $v_{0}-y_{0}=e-s<e$ and $y_{i} \geq v_{i}=0$ for $i \geq 2$, we must have $v_{1}-y_{1}=$ $x_{1}+e+1-y_{1}=s$ in order to achieve $d(v, y)=e$, and hence

$$
\begin{equation*}
y_{1}=x_{1}-s+e+1 \geq e+1>x_{1} \tag{3.25}
\end{equation*}
$$

where the first inequality follows from the above assumption that $s \leq x_{1}$. Since $y_{0}<x_{0}$ and $y_{1}-x_{1}=e+1-s$, in order to have $d(x, y)=2 e+1$ some of the remaining $y_{i}$ 's, $i \geq 2$, have to be greater than the corresponding $x_{i}$ 's for exactly $\sum_{i \geq 2, y_{i}>x_{i}}\left(y_{i}-x_{i}\right)=e+s$. But this is impossible because

$$
\begin{equation*}
\sum_{i \geq 2, y_{i}>x_{i}}\left(y_{i}-x_{i}\right) \leq \sum_{i \geq 2} y_{i}=\ell-y_{0}-y_{1}=e<e+s \tag{3.26}
\end{equation*}
$$

where we have used 3.25. We thus conclude that $s$ must be zero. In that case we have $v_{0}-y_{0}=e$, and since $d(v, y)=e$, we must also have $y_{1} \geq v_{1}=x_{1}+e+1$. This shows that $y$ is necessarily of the form 3.22 . To prove the last part of the claim observe that $y_{0}<x_{0}, y_{1}-x_{1}=e+1+u$, and $d(x, y)=2 e+1$ imply that $\sum_{i \geq 2, y_{i}>x_{i}}\left(y_{i}-x_{i}\right)=e-u$. But since $\sum_{i \geq 2} y_{i}=e-u$, this can only hold if $x_{i}=0$ whenever $y_{i}>0, i \geq 2$.

Assume therefore that we have two codewords of the form 3.20 and 3.22, and observe the point

$$
\begin{align*}
w= & \left(\ell-t-e-1, x_{1}+u\right. \\
& \left.\max \left\{x_{2}, y_{2}\right\}+1, \max \left\{x_{3}, y_{3}\right\}, \ldots, \max \left\{x_{n}, y_{n}\right\}\right) \tag{3.27}
\end{align*}
$$

By using (3.23) it is not hard to conclude that $w \in \Delta_{\ell}^{n}$ and that $d(x, w)=d(y, w)=$ $e+1$, and hence we need a third codeword $z$ to cover $w$. Such a codeword, however, cannot exist, as shown below.

Assume first that $u>0$. Then we have that $d\left(x, w+f_{1,2}\right)=e$ and $d\left(y, w+f_{2,1}\right)=$ $e$. By using Lemma 3.8 we then conclude that the codeword $z$ which covers $w$ must be of the form $z=w+\left(0,0, s_{2}, \ldots, s_{n}\right)$ with $\sum_{i} s_{i}=0$ and $\sum_{i}\left|s_{i}\right|=2 e$ (the second condition is needed in order to have $d(z, w)=e)$. Therefore

$$
\begin{align*}
z= & \left(\ell-t-e-1, x_{1}+u, \quad \max \left\{x_{2}, y_{2}\right\}+1+s_{2}\right. \\
& \left.\max \left\{x_{3}, y_{3}\right\}+s_{3}, \ldots, \max \left\{x_{n}, y_{n}\right\}+s_{n}\right) \tag{3.28}
\end{align*}
$$

Now, since $x_{0}-z_{0}=e+1$ and $x_{1}<z_{1}$ we must have

$$
\begin{equation*}
\sum_{i \geq 2, x_{i}>z_{i}}\left(x_{i}-z_{i}\right)=e \tag{3.29}
\end{equation*}
$$

in order for $d(x, z)=2 e+1$ to hold. Similarly, from $z_{0}>y_{0}$ and $y_{1}-z_{1}=e+1$ we conclude that

$$
\begin{equation*}
\sum_{i \geq 2, y_{i}>z_{i}}\left(y_{i}-z_{i}\right)=e \tag{3.30}
\end{equation*}
$$

But it is not hard to conclude that we cannot simultaneously have $\sqrt{3.29}$ and 3.30 because $x_{i}$ 's and $y_{i}$ 's, $i \geq 2$, are never simultaneously positive (3.23). Namely, since $\sum_{s_{i}<0}\left|s_{i}\right|=e$, even if we achieve $d(y, z)=2 e+1$ (by letting $s_{i}$ 's to be negative on the coordinates where $y_{i}$ 's are positive), we would have $d(x, z)=e+1$ because there are no more negative $s_{i}$ 's to obtain 3.29 . We thus conclude that it is not possible to find a codeword $z$ which covers $w$, and whose decoding region is disjoint from those of the codewords $x$ and $y$.

It is left to consider the case when $u=0$. In that case

$$
\begin{equation*}
y=\left(\ell-x_{1}-2 e-1, x_{1}+e+1, y_{2}, \ldots, y_{n}\right) . \tag{3.31}
\end{equation*}
$$

Note that now $y_{2}+\cdots+y_{n}=e$ and hence we can assume that $y_{2}>0$. Observe the point

$$
\begin{align*}
w^{\prime}= & \left(\ell-t-e-1, x_{1}+1, y_{2}-1\right. \\
& \left.\max \left\{x_{3}, y_{3}\right\}+1, \max \left\{x_{4}, y_{4}\right\}, \ldots, \max \left\{x_{n}, y_{n}\right\}\right) \tag{3.32}
\end{align*}
$$

We again have $d\left(x, w^{\prime}\right)=d\left(y, w^{\prime}\right)=e+1$, and $d\left(x, w^{\prime}+f_{1,2}\right)=d\left(y, w^{\prime}+f_{2,1}\right)=$ $e$. Therefore, the codeword $z^{\prime}$ covering $w^{\prime}$ has to be of the form $z^{\prime}=w^{\prime}+$ $\left(0,0, r_{2}, \ldots, r_{n}\right)$ with $\sum_{i} r_{i}=0$ and $\sum_{i}\left|r_{i}\right|=2 e$. By the same reasoning as above we conclude that we cannot simultaneously achieve that $d\left(x, z^{\prime}\right)=2 e+1$ and $d\left(y, z^{\prime}\right)=2 e+1$, and hence the codeword $z$ whose decoding region contains $w^{\prime}$ and is disjoint from the decoding regions of $x$ and $y$ does not exist.

The proof of the claim is now complete - nontrivial perfect codes in $\Delta_{\ell}^{n}, n>2$, do not exist.

## Acknowledgments

The authors are very grateful to the reviewers for a detailed reading and many useful comments on the original version of the manuscript. This work was supported by the Ministry of Science and Technological Development of the Republic of Serbia (grants TR32040 and III44003). Part of the work was done while M. Kovačević was visiting Aalborg University, Denmark, under the support of the COST action IC1104. He is very grateful to the Department of Electronic Systems, and in particular to Čedomir Stefanović and Petar Popovski, for their hospitality.

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[^0]:    Date: October 10, 2013.
    2010 Mathematics Subject Classification. 94B25, 05B40, 52C17, 05C12, 68R99.
    Key words and phrases. Multiset codes, permutation channel, discrete simplex, perfect codes, sphere packing, integer codes, Manhattan metric.

[^1]:    ${ }^{1}$ In the graph theoretic literature, 1-perfect codes are also known as efficient dominating sets (see, e.g., 4]).

