# On resolvable Steiner 2-designs and maximal arcs in projective planes

Vladimir D. Tonchev\* Michigan Technological University Houghton, Michigan 49931, USA

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#### Abstract

A combinatorial characterization of resolvable Steiner 2-(v, k, 1) designs embeddable as maximal arcs in a projective plane of order (v-k)/(k-1) is proved, and a generalization of a conjecture by Andries Brouwer [9] is formulated.

### 1 Introduction

We assume familiarity with basic facts and notions from combinatorial design theory [2], [4], [12], [27].

Let  $D = \{X, \mathcal{B}\}$  be a Steiner 2-(v, k, 1) design with point set X, collection of blocks  $\mathcal{B}$ , and let v be a multiple of k: v = nk. Since every point of X is contained in

$$r = (v-1)/(k-1) = (nk-1)/(k-1)$$

blocks, it follows that k-1 divides n-1. Thus, n-1=s(k-1) for some integer  $s \ge 1$ , and

$$v = nk = (sk - s + 1)k.$$

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A parallel class (or spread) is a set of v/k = n pairwise disjoint blocks, and a resolution of D is a partition of the collection of blocks  $\mathcal{B}$  into r = (v-1)/(k-1) = sk+1 parallel classes. A design is resolvable if it admits a resolution.

Any 2-((sk - s + 1)k, k, 1) design with s = 1 is equivalent to an affine plane of order k, and admits exactly one resolution. If s > 1, a resolvable 2-((sk - s + 1)k, k, 1) design may admit more than one resolution.

A property of resolvable Steiner 2-designs having several resolutions, that has attracted considerable attention, is *orthogonality* (see [4, page 31], [12, Section II.7.7], [14], [21], and the references within): two resolutions  $R_1$ ,  $R_2$ ,

$$R_1 = P_1^{(1)} \cup P_2^{(1)} \cup \cdots P_r^{(1)}, \ R_2 = P_1^{(2)} \cup P_2^{(2)} \cup \cdots P_r^{(2)}$$
 (1)

are called *orthogonal* if

$$|P_i^{(1)} \cap P_j^{(2)}| \le 1$$
, for all  $1 \le i, j \le r$ ,

that is, every two parallel classes  $P_i^{(1)},\ P_j^{(2)},$  one from each resolution, share at most one block.

The subject of this paper is a concept which is somewhat similar, but yet different from orthogonality. We call two resolutions  $R_1$ ,  $R_2$  (1) compatible if they share one parallel class,

$$P_i^{(1)} = P_i^{(2)},$$

and

$$|P_{i'}^{(1)} \cap P_{i'}^{(1)}| \le 1$$

for  $i' \neq i$  and  $j' \neq j$ .

More generally, a set of m resolutions  $R_1, \ldots, R_m$  is compatible if every two of these resolutions are compatible.

Sets of mutually compatible resolutions arise naturally in resolvable Steiner designs associated with maximal arcs in projective planes [2, Section 8.4], [10].

In this paper, we prove an upper bound on the maximum number of mutually compatible resolutions, and give a characterization of the case when this maximum is achieved.

## 2 An upper bound on the number of mutually compatible resolutions

Suppose that  $\mathcal{P}$  is a projective plane of order q = sk. A maximal  $\{(sk - s + 1)k; k\}$ -arc [16], [25, p. 558], is a set  $\mathcal{A}$  of (sk - s + 1)k points of  $\mathcal{P}$  such that every line of  $\mathcal{P}$  is ether disjoint from  $\mathcal{A}$  or meets  $\mathcal{A}$  in exactly k points. The set of lines of  $\mathcal{P}$  which have no points in common with  $\mathcal{A}$  determines a maximal  $\{(sk - k + 1)s; s\}$ -arc  $\mathcal{A}^{\perp}$  in the dual plane  $\mathcal{P}^{\perp}$ .

A maximal arc with k=2 is called a *hyperoval* (or oval). Maximal arcs, and hyperovals in particular, have been studied in connection with the construction of projective planes [2, Section 8.4], [10], [20], [26], and partial geometries [25].

Maximal arcs with 1 < k < q do not exist in any Desarguesian plane of odd order q [3], and are known to exist in every Desarguesian plane of even order q [13], for any  $k = 2^i$ , k < q, as well as in some non-Desarguesian planes of even order [15], [24].

If k > 1, the non-empty intersections of a maximal  $\{(sk - s + 1)k; k\}$ arc  $\mathcal{A}$  with the lines of a projective plane  $\mathcal{P}$  of order q = sk are the blocks
of a resolvable  $2 \cdot ((sk - s + 1)k, k, 1)$  design D. Similarly, if s > 1, the
corresponding  $\{(sk - k + 1)s; s\}$ -arc  $\mathcal{A}^{\perp}$  in the dual plane is the point set
of a resolvable  $2 \cdot ((sk - k + 1)s, s, 1)$  design  $D^{\perp}$ . We will refer to D (resp.  $D^{\perp}$ ) as a design embeddable in  $\mathcal{P}$  (resp.  $\mathcal{P}^{\perp}$ ) as a maximal arc. The points
of  $D^{\perp}$  determine a set of (sk - k + 1)s mutually compatible resolutions of D. Respectively, the points of D determine a set of (sk - s + 1)k mutually
compatible resolutions of  $D^{\perp}$ .

**Theorem 2.1** Let  $S = \{R_1, \ldots, R_m\}$  be a set of m mutually compatible resolutions of a 2-((sk - s + 1)k, k, 1) design  $D = \{X, \mathcal{B}\}$ . Then

$$m \le (sk - k + 1)s.$$

The equality

$$m = (sk - k + 1)s$$

holds if and only if there exists a projective plane  $\mathcal{P}$  of order sk such that D is embeddable in  $\mathcal{P}$  as a maximal  $\{(sk-s+1)k;k\}$ -arc.

**Proof.** It is straightforward to check that if  $\mathcal{P}$  is a projective plane of order sk in which D is embedded as a maximal  $\{(sk - s + 1)k; k\}$ -arc, then

the points of the corresponding maximal  $\{(sk - k + 1)s; s\}$ -arc in the dual plane  $\mathcal{P}^{\perp}$  define a set of (sk - k + 1)s mutually compatible resolutions of D.

To prove the converse, we consider the simple incidence structure  $\mathcal{I}$  having as "points" the blocks of D, that is, having  $\mathcal{B}$  as a point set, and having blocks of size

$$v/k = n = sk - s + 1$$

being the parallel classes of D which appear in resolutions of S. Let  $r_i$  denote the number of blocks of  $\mathcal{I}$  containing the ith point of  $\mathcal{I}$ .

A point of D is contained in

$$r = sk + 1$$

blocks, and the total number b of blocks of D is equal to

$$b = \frac{vr}{k} = (sk - s + 1)(sk + 1).$$

Any block of D is disjoint from exactly

$$b - 1 - k(r - 1) = s(sk - k + 1)(k - 1)$$

other blocks of D. It follows that every block of D is contained in at most

$$\frac{b-1-k(r-1)}{\frac{v}{k}-1} = sk - k + 1$$

parallel classes of D which appear in resolutions from S, that is,

$$r_i \leq sk - k + 1$$
.

Let  $b_I$  denote the number of blocks of  $\mathcal{I}$ . Counting in two ways the incident pairs of a block and a point of  $\mathcal{I}$  gives

$$b_I(\frac{v}{k}) = \sum_{i=1}^b r_i \le b(sk - k + 1) = (sk - s + 1)(sk + 1)(sk - k + 1),$$

hence

$$b_I \le (sk+1)(sk-k+1),$$
 (2)

and the equality  $b_I = (sk + 1)(sk - k + 1)$  holds if and only if

$$r_1 = r_2 = \dots = r_b = sk - k + 1.$$

Now we define an incidence structure  $D^{\perp}$  on a set  $X^{\perp}$  of m points labeled by the m compatible resolutions  $R_1, \ldots, R_m$  from  $S = \{R_1, \ldots, R_m\}$ . The "blocks" of  $D^{\perp}$  are labeled by the parallel classes of D which appear in resolutions from S, that is, by the blocks of  $\mathcal{I}$ . By definition, a point  $D^{\perp}$  labeled by  $R_i$  is incident with r blocks of  $D^{\perp}$  which are labeled by the r blocks of  $\mathcal{I}$  being the parallel classes of D appearing in the resolution  $R_i$ . Since every two resolutions  $R_i$ ,  $R_j$  ( $i \neq j$ ) of S share exactly one parallel class, every two distinct points of  $D^{\perp}$  appear together in exactly one block of  $D^{\perp}$ .

Let  $k_j$   $(1 \leq j \leq b_I)$  denote the size of the jth block of  $D^{\perp}$ . We have

$$\sum_{j=1}^{b_I} k_j = mr,$$

$$\sum_{j=1}^{b_I} k_j (k_j - 1) = m(m - 1),$$

whence

$$\sum_{j=1}^{b_I} k_j^2 = m(m-1+r),$$

$$(\sum_{j=1}^{b_I} k_j)^2 = m^2 r^2 \le b_I \sum_{j=1}^{b_I} k_j^2 = b_I m(m-1+r),$$

and

$$mr^2 \le b_I(m-1+r).$$

Applying the inequality (2), we have

$$mr^2 \le (sk+1)(sk-k+1)(m-1+r).$$
 (3)

After the substitution r = sk + 1, inequality (3) simplifies to

$$m \le (sk - k + 1)s. \tag{4}$$

Now assume that equality holds in (4), that is,

$$m = (sk - k + 1)s,$$

which is possible only if  $b_I$  meets the equality in (2), that is,

$$b_I = (sk+1)(sk-k+1) = r(sk-k+1) = rn.$$

Then

$$\sum_{j=1}^{b_I} (k_j - s)^2 = \sum_{j=1}^{b_I} k_j^2 - 2s \sum_{j=1}^{b_I} k_j + s^2 b_I = 0,$$

thus, all blocks of  $D^{\perp}$  are of size s, and  $D^{\perp}$  is a 2-((sk-k+1)s, s, 1) design. Suppose that m = (sk-k+1)s. We define an incidence structure  $\mathcal{P}$  with points labeled by the v = (sk-s+1)k points of D and the

$$b^{\perp} = b_I = (sk+1)(sk-k+1)$$

blocks of  $D^{\perp}$ . Thus,  $\mathcal{P}$  has

$$\bar{v} = v + b^{\perp} = (sk - s + 1)k + (sk + 1)(sk - k + 1) = (sk)^2 + sk + 1$$

points.

The blocks of  $\mathcal{P}$  are of two types. The blocks of the first type are

$$b = (s(k-1) + 1)(sk + 1)$$

blocks, each being a union of a block B of D with the block of the dual structure of  $\mathcal{I}$ , associated with the point of  $\mathcal{I}$  labeled by B. Each such block is of size

$$k + (sk - k + 1) = sk + 1 = r.$$

The blocks of the second type are also of size r = sk + 1 and are labeled by the m = (sk - k + 1)s points of  $D^{\perp}$ , and coincide with the blocks of the dual structure of  $D^{\perp}$ . Thus,  $\mathcal{P}$  has

$$\bar{b} = b + m = (sk)^2 + sk + 1$$

blocks.

Since every two points of  $D^{\perp}$  appear in exactly one block of  $D^{\perp}$ , every two blocks of  $\mathcal{P}$  of the second type intersect in exactly one point. Since the r blocks of  $D^{\perp}$  through a point of  $D^{\perp}$  correspond to a parallel class of blocks of  $\mathcal{I}$ , every block of  $\mathcal{P}$  of the first type meets every block of the second type in exactly one point.

To determine how pairs of blocks of the first type intersect, we consider the block graph  $\Gamma$  of D, that is, the vertices of  $\Gamma$  are labeled by the blocks of D, where two vertices are adjacent if and only if the corresponding blocks are not disjoint. The graph  $\Gamma$  is strongly regular with parameters

$$(b, a, \lambda, \mu),$$

where b = (s(k-1) + 1)(sk + 1) is the number of vertices,

$$a = k(r-1) = sk^2$$

is the degree of each vertex, and

$$\lambda = (r-2) + (k-1)^2 = k(k+s-2), \ \mu = k^2.$$

The complementary graph  $\bar{\Gamma}$  is strongly regular of degree

$$\bar{a} = b - 1 - a = s^2 k^2 + 2sk - s^2 k - k^2 s - s.$$

The  $b_I = (sk+1)(sk-k+1)$  blocks of  $\mathcal{I}$ , being parallel classes of D belonging to resolutions of S, are cocliques in  $\Gamma$  of size

$$\frac{v}{k} = n = sk - s + 1.$$

By the property of S, every two blocks of  $\mathcal{I}$  can share at most one point, hence the corresponding (sk - k + 1)-cocliques of  $\Gamma$ , can share at most one vertex. Viewed as (sk - k + 1)-cliques of  $\bar{\Gamma}$ , the blocks of  $\mathcal{I}$  cover

$$b_{I}\binom{sk-k+1}{2} = \frac{(sk+1)(sk-k+1)(sk-s+1)(sk-s)}{2}$$

edges of  $\bar{\Gamma}$ , which is equal to the total number of edges of  $\bar{\Gamma}$ . It follows that every two blocks of  $\mathcal{P}$  of the first type share exactly one point. Thus,  $\mathcal{P}$  is a projective plane of order sk in which D is embedded as a maximal  $\{(sk-s+1)k;k\}$ -arc.

This completes the proof.

**Remark 2.2** It is an interesting open question whether the results from this section can be generalized to designs having m mutually compatible resolutions, with m slightly smaller than (sk - k + 1)s, in the spirit of the results by A. Beutelspacher and K. Metsch on partial projective planes [8], [7].

### 3 On a conjecture by Andries Brouwer

It is conceivable that a resolvable 2-((sk - s + 1)k, k, 1) design admitting a set of mutually compatible resolutions that achieves the bound of Theorem 2.1 possesses a high degree of symmetry. One measure of symmetry is the p-rank of the incidence matrix over a finite field of characteristic p that divides r - 1 = sk, which is the order of the related projective plane.

The special case  $s=2,\ k=2^{t-1},\ t\geq 2$  corresponds to projective planes of order  $2^t$ . A 2- $(2^{2t-1}-2^{t-1},2^{t-1},1)$  design arising from a maximal  $(2^{2t-1}-2^{t-1};2^{t-1})$ -arc  $\mathcal{A}$  is called an *oval* design [2, 8.4], in reference to the fact that the corresponding maximal  $(2^t+2;2)$ -arc  $\mathcal{A}^{\perp}$  in the dual plane is a hyperoval. The 2-rank of oval designs in  $PG(2^t,2)$  has been studied extensively. In 1989 Mackenzie [22] (see also [2, Theorem 8.4.1], [18]) proved that the 2-rank of any oval design in  $PG(2^t,2)$  is bounded from above by  $3^t-2^t$ . It was conjectured by Assmus that this upper bound is always achievable. This conjecture was proved consequently by Carpenter [11], by using a result by Blokhuis and Moorhouse [8].

In the smallest case, t=2, a maximal (6; 2)-arc (s=k=2) in the plane of order 4 and its dual plane is a hyperoval, yielding the unique trivial 2-(6, 2, 1) design. The next case, t=3, corresponds to the projective plane of order 8, which contains only one class (up to projective equivalence) of hyperovals, or (10; 2)-arcs, and consequently, one (up to isomorphism) maximal (28; 4)-arc, yielding a 2-(28, 4, 1) oval design.

Designs with the latter parameters, 2-(28,4,1), have been the subject of several papers ([1], [5], [9], [17], [19], [23]). According to the Handbook of Combinatorial Designs [12, page 37], there are at least 4,747 known nonisomorphic designs with these parameters, and all designs possessing nontrivial automorphisms have been classified (Krćadinac [19]). The more recent publication by Al-Azemi, Anton Betten, and Dieter Betten [1] gives a much bigger number of nonisomorphic 2-(28,4,1) designs, namely, 68,806 such designs, all having a blocking set, and among those, 68,484 designs have a trivial automorphism group.

In [9], Andries Brouwer investigated the embeddability of 2-(28, 4, 1) designs as unitals in projective planes of order 9. The 2-ranks of the 138 designs examined by Brouwer in [9] ranged between 19 and 27, with the exception of 2-rank 20. The minimum 2-rank, 19, was achieved by a design being the smallest member of the family of Ree unitals, and one of the two 2-(28, 4, 1) designs having 2-transitive automorphism groups (the second being the clas-

sical Hermitian unital, having 2-rank 21). It was shown in [17] that there are no 2-(28, 4, 1) designs of 2-rank 20, and there are exactly four nonisomorphic designs of 2-rank 21, one being the classical Hermitian unital.

It turns out that the 2-(28, 4, 1) Ree unital is isomorphic to the oval design in the plane of order 8,  $PG(2^3, 2)$ .

Brouwer [9] made the conjecture that 19 is the minimum 2-rank of any 2-(28, 4, 1) design, and this minimum is achieved by the Ree unital only. This conjecture was proved to be true by McGuire, Tonchev and Ward [23].

Taking into account Carpenter's result about the 2-rank of oval designs in  $PG(2^t, 2)$  [11], it is tempting to believe that the following generalization of Brouwer's conjecture is true.

Conjecture 3.1 . If  $\mathcal{D}$  is a 2- $(2^{2t-1}-2^{t-1},2^{t-1},1)$  design  $(t \geq 2)$ , with an incidence matrix A, then

$$rank_2(A) \ge 3^t - 2^t,$$

and the equality

$$rank_2(A) = 3^t - 2^t$$

holds if and only if  $\mathcal{D}$  is embeddable as a maximal  $(2^{2t-1} - 2^{t-1}; 2^{t-1})$ -arc in  $PG(2^t, 2)$ .

The conjecture is trivially true for t = 2, and its validity for t = 3 follows from the results of [23].

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