

Rank distribution of Delsarte codes

Javier de la Cruz¹, Elisa Gorla², Hiram H. López³, and Alberto Ravagnani^{2,*}

¹Universidad del Norte, Colombia

²Université de Neuchâtel, Switzerland

³Centro de Investigación y de Estudios Avanzados del IPN, México

Abstract

In analogy with the Singleton defect for classical codes, we propose a definition of rank defect for Delsarte rank-metric codes. We characterize codes whose rank defect and dual rank defect are both zero, and prove that the rank distribution of such codes is determined by their parameters. This extends a result by Delsarte on the rank distribution of MRD codes. In the general case of codes of positive defect, we show that the rank distribution is determined by the parameters of the code, together the number of codewords of small rank. Moreover, we prove that if the rank defect of a code and its dual are both one, and the dimension satisfies a divisibility condition, then the number of minimum-rank codewords and dual minimum-rank codewords is the same. Finally, we discuss how our results specialize to Gabidulin codes.

Introduction

Rank-metric codes were first introduced in coding theory by Delsarte in [5]. They are sets of matrices of fixed size, endowed with the rank distance. Rank-metric codes are of interest within network coding, public-key cryptography, and distributed storage, where they stimulated a series of works aimed at better understanding their properties. In this paper, we study the rank distribution of rank-metric codes. We always assume that the codes are linear, and often refer to them as Delsarte codes.

The study of the weight distribution of a code is a topic of current interest in coding theory, where several authors have studied the case of linear codes endowed with the Hamming distance. In particular, it is a classical result that the weight distribution of an MDS code is determined by its parameters. The so-called MRD codes are the analogue of MDS codes in the context of Delsarte codes. They were introduced in [5] by Delsarte, who also showed that

¹Part of the work was done while J. de la Cruz was visiting the University of Zurich. The first author thanks Joachim Rosenthal for the invitation.

²E. Gorla and A. Ravagnani were partially supported by the Swiss National Science Foundation through grant no. 200021-150207 and by the ESF COST Action IC1104.

³H. López was partially supported by CONACyT and by the Swiss Confederation through the Swiss Government Excellence Scholarship no. 2014.0432.

*Email addresses: jdelacruz@uninorte.edu.co, elisa.gorla@unine.ch, hlopez@math.cinvestav.mx, alberto.ravagnani@unine.ch.

the weight distribution of an MRD code is determined by its parameters. However, MRD codes only exist if the size of the matrix divides the dimension of the code. More precisely, for $n \times m$ matrices with $n \leq m$, m must divide the dimension of the code.

In this paper, we study the rank distribution of Delsarte codes. In Section 2 we define Quasi-MRD (or QMRD) codes as codes which have the largest possible minimum distance for their parameters, but are not MRD. We regard them as the best alternative to MRD codes, for dimensions for which MRD codes do not exist. While the dual of an MRD code is MRD, the dual of a QMRD code is not necessarily QMRD. When both \mathcal{C} and its dual are QMRD, we say that \mathcal{C} is dually QMRD. In Proposition 17 we provide a characterization of dually QMRD codes in terms of the number of codewords of minimum weight. Moreover, we show that QMRD codes exist for all choices of the parameters, and give examples of codes which are QMRD, but not dually QMRD. In analogy with the Singleton defect for classical codes, we propose a definition of rank defect for Delsarte codes. According to our definition, a code has rank defect zero if and only if it is either MRD or QMRD.

Using the MacWilliams identities, in Section 3 we derive formulas that relate the numbers $A_i(\mathcal{C})$ of codewords of \mathcal{C} of weight i for all values of i , showing that a few of the $A_i(\mathcal{C})$'s determine the others (see Theorem 25). In analogy with the classical case of MDS codes, our result implies that the rank distribution of a code which is MRD or dually QMRD is completely determined by its parameters. Notice that this is not the case in general for codes which are QMRD but not dually QMRD (so for codes of rank defect zero), as we show in Example 27.

In Section 4 we analyze a family of codes such that both the code and its dual have rank defect one. In Theorem 31 we show that they have the property that the code and its dual have the same number of minimum rank codewords.

Finally, in Section 5 we consider Gabidulin codes and discuss how our results specialize to this case.

1 Preliminaries

In this section we briefly recall the main definitions and results of the theory of Delsarte rank-metric codes.

Notation 1. Throughout the paper, q denotes a fixed prime power, and \mathbb{F}_q the finite field with q elements. We also work with positive integers $1 \leq n \leq m$, and denote by Mat the vector space of $n \times m$ matrices with entries in \mathbb{F}_q . Given a positive integer a , we denote by $[a]$ the set $\{1, \dots, a\}$ and by I_a the identity matrix of size a . The trace of a square matrix M is denoted and defined by $\text{Tr}(M) = \sum_{i=1}^n M_{ii}$. The rank of a matrix M is denoted by $\text{rk}(M)$. All dimensions are computed over \mathbb{F}_q , unless otherwise specified.

Definition 2. A **(Delsarte rank-metric) code** is an \mathbb{F}_q -linear subspace $\mathcal{C} \subseteq \text{Mat}$. The **minimum distance** of a code $\mathcal{C} \neq \{0\}$ is $d(\mathcal{C}) := \min\{\text{rk}(M) : M \in \mathcal{C}, M \neq 0\}$. The **rank distribution** of \mathcal{C} is the collection $(A_i(\mathcal{C}))_{i \in \mathbb{N}}$, where $A_i(\mathcal{C}) := |\{M \in \mathcal{C} : \text{rk}(M) = i\}|$ for $i \in \mathbb{N}$. The **dual** of a Delsarte code $\mathcal{C} \subseteq \text{Mat}$ is the code

$$\mathcal{C}^\perp := \{M \in \text{Mat} : \text{Tr}(MN^t) = 0 \text{ for all } N \in \mathcal{C}\} \subseteq \text{Mat}.$$

A code \mathcal{C} is **trivial** if $\mathcal{C} = \{0\}$ or $\mathcal{C} = \text{Mat}$.

Throughout the paper, \mathcal{C} denotes a non-trivial Delsarte code with minimum distance d and dimension t . We let d^\perp be the minimum distance of its dual \mathcal{C}^\perp .

Remark 3. The trace-product of matrices $(M, N) \mapsto \text{Tr}(MN^t)$ is a symmetric and non-degenerate bilinear form. In particular, the dual of a Delsarte code \mathcal{C} is a Delsarte code of dimension $\dim(\mathcal{C}^\perp) = mn - t$. Moreover, given Delsarte codes $\mathcal{C}, \mathcal{D} \subseteq \text{Mat}$ we have $(\mathcal{C} + \mathcal{D})^\perp = \mathcal{C}^\perp \cap \mathcal{D}^\perp$ and $(\mathcal{C} \cap \mathcal{D})^\perp = \mathcal{C}^\perp + \mathcal{D}^\perp$. Finally, $(\mathcal{C}^\perp)^\perp = \mathcal{C}$ for any Delsarte code \mathcal{C} .

The following result by Delsarte is well known. It is the analogue of the Singleton bound in the context of rank-metric codes.

Theorem 4 ([5], Theorem 5.4). Let \mathcal{C} be a Delsarte code with minimum distance d and dimension t . We have $t \leq m(n - d + 1)$.

Definition 5. A Delsarte code \mathcal{C} is **MRD** if $t = m(n - d + 1)$.

Theorem 6 ([5], Theorem 5.6). A Delsarte code \mathcal{C} is MRD if and only if the dual code \mathcal{C}^\perp is MRD.

We briefly recall the definition and the main algebraic properties of q -ary Gaussian coefficients, for the convenience of the reader. A standard reference is [1].

Definition 7. Let q be a prime power, and let a and b be integers. The q -ary Gaussian coefficient of a and b is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} := \begin{cases} 0 & \text{if } a < 0, b < 0, \text{ or } b > a, \\ 1 & \text{if } b = 0 \text{ and } a \geq 0, \\ \frac{(q^a - 1)(q^{a-1} - 1) \cdots (q^{a-b+1} - 1)}{(q^b - 1)(q^{b-1} - 1) \cdots (q - 1)} & \text{otherwise.} \end{cases}$$

Lemma 8. Let a, b, r be integers. The following hold:

1. $\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = 1$ for $a \geq 0$,
2. $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a-b \end{bmatrix}$,
3. $\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ r \end{bmatrix} = \begin{bmatrix} a \\ r \end{bmatrix} \begin{bmatrix} a-r \\ a-b \end{bmatrix}$,
4. $\begin{bmatrix} a \\ b \end{bmatrix} = q^b \begin{bmatrix} a-1 \\ b \end{bmatrix} + \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}$ for $a, b \geq 1$,
5. $\sum_{i=0}^a (-1)^i q^{\binom{i}{2}} \begin{bmatrix} a \\ i \end{bmatrix} = 0$ for $a \geq 1$.

In [5] Delsarte proved that the rank distribution of a code and of its dual satisfy MacWilliams identities. In particular, they determine each other. In this paper we use the following form of the MacWilliams identities for Delsarte codes, which was given in [8, Corollary 1 and Proposition 3]. An elementary combinatorial proof can be found in [16].

Theorem 9. Let \mathcal{C} be a Delsarte code. For any integer $0 \leq r \leq n$ we have

$$\sum_{i=0}^{n-r} A_i(\mathcal{C}) \begin{bmatrix} n-i \\ r \end{bmatrix} = \frac{|\mathcal{C}|}{q^{mr}} \sum_{j=0}^r A_j(\mathcal{C}^\perp) \begin{bmatrix} n-j \\ r-j \end{bmatrix}.$$

In particular, if \mathcal{C} is non-trivial then

$$\sum_{i=d}^{n-r} \binom{n-i}{r} A_i(\mathcal{C}) = \left(\frac{|\mathcal{C}|}{q^{mr}} - 1 \right) \binom{n}{r}$$

for $r = 0, \dots, d^\perp - 1$, where d and d^\perp denote the minimum distance of \mathcal{C} and \mathcal{C}^\perp , respectively.

2 Rank defect, MRD, and quasi-MRD codes

In this paper we study the rank distribution of codes which are MRD, or close to being MRD. Being MRD is the analogue of being MDS for codes endowed with the Hamming distance. A Hamming code is MDS if its minimum distance meets the Singleton bound. The natural analogue for rank-metric codes of the Singleton bound was given in Theorem 4. In terms of minimum distance, Theorem 4 may be stated as follows.

Corollary 10. Let $\mathcal{C} \subset \text{Mat}$ be a Delsarte code with minimum distance d and dimension t . Then

$$d \leq n - \left\lceil \frac{t}{m} \right\rceil + 1.$$

This motivates the definition of rank defect for Delsarte codes.

Definition 11. The **rank defect** of \mathcal{C} is

$$\text{Rdef}(\mathcal{C}) = n - \left\lceil \frac{t}{m} \right\rceil - d + 1.$$

For codes endowed with the Hamming distance, many authors regard the Singleton defect as a measure of how far the code is from being MDS (see e.g. [6]). The situation is slightly more complicated in the case of rank-metric codes. In fact, if \mathcal{C} is MRD then $\text{Rdef}(\mathcal{C}) = 0$. However, $\text{Rdef}(\mathcal{C})$ may be zero also for codes \mathcal{C} which are not MRD. This has a simple explanation: A code \mathcal{C} has $\text{Rdef}(\mathcal{C}) = 0$ if and only if its minimum distance has the largest possible value, for the given m, n , and t . However, if $m \nmid t$, then \mathcal{C} is not MRD. This observation motivates the definition of Quasi-MRD code.

Definition 12. A code \mathcal{C} of dimension t is **Quasi-MRD**, or **QMRD**, if $m \nmid t$ and \mathcal{C} attains the bound of Corollary 10. Equivalently, \mathcal{C} is QMRD if and only if $\text{Rdef}(\mathcal{C}) = 0$ and \mathcal{C} is not MRD.

Existence of MRD codes was established in [5] and [7] for all $1 \leq n \leq m$ and $1 \leq d \leq n$. Recently, constructions of MRD codes which are not equivalent to the previous ones appeared in [17] and [4]. We complete the picture by showing that QMRD codes exist for all choices of the parameters.

Example 13 (Existence of QMRD codes). For any $1 \leq n \leq m$ and $1 \leq t < nm$ such that $m \nmid t$, we can construct a QMRD Delsarte code of dimension t as follows. Let \mathcal{D} be an MRD code of dimension $\dim(\mathcal{D}) = m \lceil \frac{t}{m} \rceil$. \mathcal{D} has minimum distance

$$d = n - \frac{\dim \mathcal{D}}{m} + 1 = n - \left\lceil \frac{t}{m} \right\rceil + 1.$$

Let $\mathcal{C} \subseteq \mathcal{D}$ be a subspace of dimension t containing a codeword of \mathcal{D} of minimum weight. Then \mathcal{C} has minimum distance d , hence it is QMRD with the chosen parameters.

While the dual of an MRD code is MRD, the dual of a QMRD code is not necessarily QMRD, as the following example shows.

Example 14. Let $q = 2$, $n = m = 3$. Let \mathcal{C} be the code generated over \mathbb{F}_2 by the four matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

\mathcal{C} has dimension 4 and minimum distance 2, therefore it is QMRD. On the other hand,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{C}^\perp,$$

so the minimum distance of \mathcal{C}^\perp is 1. In particular, \mathcal{C}^\perp is not QMRD.

The existence of such examples motivates the definition of dually MRD code.

Definition 15. A code \mathcal{C} is **dually QMRD** if both \mathcal{C} and \mathcal{C}^\perp are QMRD.

One can also find examples over any ground field and for matrices of size $n \times m$, if $m \geq 2$.

Example 16. Let $2 \leq n \leq m$, $0 < \rho < m$, and let \mathcal{C} be a code of dimension ρ and minimum distance $d < n$. \mathcal{C} is not QMRD, since

$$\text{Rdef}(\mathcal{C}) = n - \left\lceil \frac{\rho}{m} \right\rceil - d + 1 \neq 0.$$

Its dual \mathcal{C}^\perp has dimension $mn - \rho$ and minimum distance

$$d^\perp \leq n - \left\lceil n - \frac{\rho}{m} \right\rceil + 1 = 1,$$

by Corollary 10. Hence $d^\perp = 1$ and \mathcal{C}^\perp is QMRD.

We can characterize dually QMRD codes in terms of their number of minimum-rank codewords.

Proposition 17. Let \mathcal{C} be QMRD of dimension t , and let $0 < \rho < m$ be the remainder obtained dividing t by m . Then \mathcal{C}^\perp is QMRD if and only if $A_d(\mathcal{C}) = \binom{n}{d} (q^\rho - 1)$.

Proof. Write $t = \alpha m + \rho$ with $\alpha \in \mathbb{N}$. Since \mathcal{C} is QMRD, we have $d = n - \alpha$. Moreover, since $\dim(\mathcal{C}^\perp) = nm - t$, the code \mathcal{C}^\perp is QMRD if and only if it has minimum distance $d^\perp = \alpha + 1$. Theorem 9 with $r = \alpha$ gives

$$\begin{bmatrix} n \\ \alpha \end{bmatrix} + A_d(\mathcal{C}) = q^\rho \left(\begin{bmatrix} n \\ \alpha \end{bmatrix} + \sum_{j=1}^{\alpha} A_j(\mathcal{C}^\perp) \begin{bmatrix} n-j \\ \alpha-j \end{bmatrix} \right). \quad (1)$$

Since $\alpha \leq n$, we have $\begin{bmatrix} n-j \\ \alpha-j \end{bmatrix} > 0$ for $j \in \{1, \dots, \alpha\}$. Therefore \mathcal{C}^\perp is QMRD if and only if $\sum_{j=1}^{\alpha} A_j(\mathcal{C}^\perp) \begin{bmatrix} n-j \\ \alpha-j \end{bmatrix} = 0$. Hence \mathcal{C}^\perp is QMRD if and only if

$$A_d(\mathcal{C}) = (q^\rho - 1) \begin{bmatrix} n \\ \alpha \end{bmatrix}.$$

By Lemma 8 we have $\begin{bmatrix} n \\ \alpha \end{bmatrix} = \begin{bmatrix} n \\ d \end{bmatrix}$, and the result follows. \square

Remark 18. Following the notation of Proposition 17, the code \mathcal{C} of Example 14 has $\rho = 1$. One can check that $A_d(\mathcal{C}) = 9 \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix} (2^1 - 1) = 7$, in accordance with the fact that \mathcal{C}^\perp is not QMRD.

The following is a simple consequence of Theorem 4.

Corollary 19. Let \mathcal{C} be a Delsarte code of dimension t , with minimum distance d and dual minimum distance d^\perp . The following hold:

1. If $m \mid t$, then either $d + d^\perp = n + 2$ or $d + d^\perp \leq n$.
2. If $m \nmid t$, then $d + d^\perp \leq n + 1$.

Proof. Theorem 4 applied to \mathcal{C} and \mathcal{C}^\perp gives

$$t \leq m(n - d + 1) \quad \text{and} \quad nm - t \leq m(n - d^\perp + 1). \quad (2)$$

Summing the two inequalities in (2) we obtain $n \leq 2n - d - d^\perp + 2$, i.e. $d + d^\perp \leq n + 2$. In addition, $d + d^\perp = n + 2$ if and only if both inequalities are equalities, which implies that \mathcal{C} is MRD, hence $m \mid t$. This proves part 2. If $t = km$, then the inequalities (2) become $k \leq n - d + 1$ and $n - k \leq n - d^\perp + 1$. Hence if \mathcal{C} is not MRD we have $k \leq n - d$ and $n - k \leq n - d^\perp$. It follows that $d + d^\perp \leq n$. \square

It is now easy to characterize the codes \mathcal{C} such that both \mathcal{C} and \mathcal{C}^\perp have rank defect zero as those which are MRD or dually QMRD. In Corollary 26 we will show that the rank distribution of such codes is determined by n , m and d .

Proposition 20. Let \mathcal{C} be a Delsarte code with minimum distance d and dual minimum distance d^\perp . The following hold:

1. \mathcal{C} is MRD iff \mathcal{C}^\perp is MRD iff $d + d^\perp = n + 2$.
2. \mathcal{C} is dually QMRD iff $d + d^\perp = n + 1$.

Proof. 1. In the proof of Corollary 19 we showed that $d + d^\perp = n + 2$ implies \mathcal{C} MRD hence, by symmetry, \mathcal{C}^\perp MRD. Conversely, if one of \mathcal{C} and \mathcal{C}^\perp is MRD, then the dual is also MRD by Theorem 6. Therefore

$$d + d^\perp = n - \frac{t}{m} + 1 + n - \frac{nm - t}{m} + 1 = n + 2.$$

2. If $d + d^\perp = n + 1$, then $m \nmid t$. The bound of Corollary 10 yields

$$d \leq n - \left\lceil \frac{t}{m} \right\rceil + 1 \quad \text{and} \quad d^\perp \leq \left\lfloor \frac{t}{m} \right\rfloor + 1.$$

Therefore both inequalities are equalities and \mathcal{C} and \mathcal{C}^\perp are QMRD. Conversely, if \mathcal{C} and \mathcal{C}^\perp are QMRD, then $m \nmid t$ and

$$d + d^\perp = n - \left\lceil \frac{t}{m} \right\rceil + 1 + \left\lfloor \frac{t}{m} \right\rfloor + 1 = n + 1.$$

\square

Generalized weights for Delsarte codes were introduced in [15], refining previous definitions for Gabidulin codes. We conclude this section with some observations on the connection between rank defect and generalized weights.

Definition 21. An **optimal anticode** $\mathcal{A} \subseteq \text{Mat}$ is a Delsarte code such that $\dim(\mathcal{A}) = m \cdot \text{maxrk}(\mathcal{A})$, where $\text{maxrk}(\mathcal{A}) := \max\{\text{rk}(M) : M \in \mathcal{A}\}$.

Given a Delsarte code \mathcal{C} of dimension t and an integer $1 \leq r \leq t$, the **r -th generalized weight** of \mathcal{C} is

$$a_r(\mathcal{C}) := \frac{1}{m} \min\{\dim(\mathcal{A}) : \mathcal{A} \subseteq \text{Mat} \text{ is an anticode with } \dim(\mathcal{C} \cap \mathcal{A}) \geq r\}.$$

The next result relates the generalized weights of a code to the minimum distance and the rank defect of the dual code.

Proposition 22. Let \mathcal{C} be a Delsarte code with minimum distance d and dimension t . If $t < m$ then $d^\perp = 1$ and $\text{Rdef}(\mathcal{C}^\perp) = 0$. If $t \geq m$, write $t = \alpha m + \rho$ with $0 \leq \rho < m$. The minimum distance of the dual code \mathcal{C}^\perp is

$$d^\perp = \begin{cases} \alpha + 1 & \text{if } n + 1 - a_{t+1-\rho m}(\mathcal{C}) = \alpha, \\ \min\{1 \leq r \leq \alpha : n + 1 - a_{t+1-rm}(\mathcal{C}) > r\} & \text{otherwise,} \end{cases}$$

and the rank defect of \mathcal{C}^\perp is

$$\text{Rdef}(\mathcal{C}^\perp) = \alpha + 1 - d^\perp.$$

Proof. If $t < m$, then $d^\perp = 1$ by Corollary 10. Assume therefore that $t \geq m$. By Theorem 66 of [15] we have

$$d^\perp = a_1(\mathcal{C}^\perp) < a_{1+m}(\mathcal{C}^\perp) < \dots < a_{1+(n-\alpha-1)m}(\mathcal{C}^\perp) \leq n.$$

Define

$$W_1(\mathcal{C}^\perp) := \left\{ a_1(\mathcal{C}^\perp), a_{1+m}(\mathcal{C}^\perp), \dots, a_{1+(n-\alpha-1)m}(\mathcal{C}^\perp) \right\}$$

and

$$\overline{W}_{1+t}(\mathcal{C}) := \{n + 1 - a_{t+1-m}(\mathcal{C}), n + 1 - a_{t+1-2m}(\mathcal{C}), \dots, n + 1 - a_{t+1-\alpha m}(\mathcal{C})\}.$$

By [15, Corollary 78] we have that $W_1(\mathcal{C}^\perp) = [n] \setminus \overline{W}_{1+t}(\mathcal{C})$. Hence the result on the minimum distance follows from the fact that

$$n + 1 - a_{t+1-m}(\mathcal{C}) < n + 1 - a_{t+1-2m}(\mathcal{C}) < \dots < n + 1 - a_{t+1-\alpha m}(\mathcal{C})$$

by Theorem 66 of [15]. The formula for the rank defect now follows from the definition. \square

3 Rank distribution of Delsarte codes

In this section we prove that the rank distribution of a code \mathcal{C} is determined by its parameters, together with the number of codewords of small weight: $A_d(\mathcal{C}), \dots, A_{n-d^\perp}(\mathcal{C})$. In particular, we show that the rank distribution of MRD and dually QMRD codes only depends on their parameters. We start with a series of preliminary definitions and results.

Lemma 23. Let $a \geq 1$ be an integer and let M, N be the real $a \times a$ matrices defined by $M_{ij} := \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}$ and $N_{ij} := (-1)^{j-i} q^{\binom{j-i}{2}} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}$ for $i, j \in [a]$. Then $MN = NM = I_a$.

Proof. Since M and N are square matrices, it suffices to prove that $MN = I_a$. One can easily check that

$$(MN)_{ij} = \sum_{r=0}^{a-1} (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j \\ r \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix}.$$

By Lemma 8.3 we have

$$(MN)_{ij} = \sum_{r=0}^{a-1} (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j \\ r \end{bmatrix} \begin{bmatrix} r \\ i \end{bmatrix} = \begin{bmatrix} j \\ i \end{bmatrix} \sum_{r=0}^j (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j-i \\ j-r \end{bmatrix}.$$

Hence $(MN)_{ii} = 1$ for $i = 1, \dots, a$. If $i > j$, then $(MN)_{ij} = 0$. If $i < j$, then

$$\begin{aligned} (MN)_{ij} &= \begin{bmatrix} j \\ i \end{bmatrix} \sum_{r=0}^j (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j-i \\ j-r \end{bmatrix} \\ &= \begin{bmatrix} j \\ i \end{bmatrix} \sum_{r=0}^j (-1)^r q^{\binom{r}{2}} \begin{bmatrix} j-i \\ r \end{bmatrix} \\ &= \begin{bmatrix} j \\ i \end{bmatrix} \sum_{r=0}^{j-i} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} j-i \\ r \end{bmatrix} \\ &= 0, \end{aligned}$$

where the last equality follows from Lemma 8.5. □

Lemma 24. Let $a \in \mathbb{N}_{\geq 1}$. For $j = 0, \dots, a$ we have

$$\sum_{i=0}^j (-1)^i q^{\binom{i}{2}} \begin{bmatrix} a \\ i \end{bmatrix} = (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} a-1 \\ j \end{bmatrix}.$$

Proof. By induction on j . If $j = 0$ then the result is immediate. Assume $j > 0$. By induction hypothesis and Lemma 8.4 we have

$$\begin{aligned} \sum_{i=0}^j (-1)^i q^{\binom{i}{2}} \begin{bmatrix} a \\ i \end{bmatrix} &= (-1)^{j-1} q^{\binom{j}{2}} \begin{bmatrix} a-1 \\ j-1 \end{bmatrix} + (-1)^j q^{\binom{j}{2}} \begin{bmatrix} a \\ j \end{bmatrix} \\ &= (-1)^j q^{\binom{j+1}{2}} q^{-j} \left(\begin{bmatrix} a \\ j \end{bmatrix} - \begin{bmatrix} a-1 \\ j-1 \end{bmatrix} \right) \\ &= (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} a-1 \\ j \end{bmatrix}, \end{aligned}$$

as claimed. □

Now we state our main result.

Theorem 25. Let \mathcal{C} be a t -dimensional code, with minimum distance d and dual minimum distance d^\perp . Let $\delta = 1$ if $d + d^\perp = n + 2$, and $\delta = 0$ otherwise. For all $1 \leq r \leq d^\perp$ we have

$$\begin{aligned} A_{n-d^\perp+r}(\mathcal{C}) &= (-1)^r q^{\binom{r}{2}} \sum_{j=d^\perp}^{n-d} \begin{bmatrix} j \\ d^\perp - r \end{bmatrix} \begin{bmatrix} j - d^\perp + r - 1 \\ r - 1 \end{bmatrix} A_{n-j}(\mathcal{C}) \\ &\quad + \begin{bmatrix} n \\ d^\perp - r \end{bmatrix} \sum_{i=0}^{r-1-\delta} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n - d^\perp + r \\ i \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{m(d^\perp-r+i)}} - 1 \right). \end{aligned}$$

In particular, n, m, t, d, d^\perp and $A_d(\mathcal{C}), \dots, A_{n-d^\perp}(\mathcal{C})$ determine the rank distribution of \mathcal{C} .

Proof. We only prove the theorem in the case $d + d^\perp \leq n + 1$. The proof in the case $d + d^\perp = n + 2$ is analogous and easier. Let A, B be the real matrices of size $d^\perp \times d^\perp$ and $d^\perp \times (n - d - d^\perp + 1)$, defined by $A_{r,j} = \begin{bmatrix} j-1 \\ r-1 \end{bmatrix}$ and $B_{r,i} = \begin{bmatrix} i+d^\perp-1 \\ r-1 \end{bmatrix}$ for $r, j \in [d^\perp]$ and $i \in [n - d - d^\perp + 1]$. When $d + d^\perp = n + 1$ we only have the matrix A . Throughout the proof we write A_i for $A_i(\mathcal{C})$. The second part of the statement of Theorem 9 may be written in the form

$$(A|B) (A_n, \dots, A_d)^t = \left((|\mathcal{C}| - 1) \begin{bmatrix} n \\ 0 \end{bmatrix}, \dots, \left(\frac{|\mathcal{C}|}{q^{m(n-d)}} - 1 \right) \begin{bmatrix} n \\ d^\perp - 1 \end{bmatrix} \right)^t.$$

Multiplying by A^{-1} we get

$$(I_{d^\perp} | A^{-1}B) (A_n, \dots, A_d)^t = A^{-1} \left((|\mathcal{C}| - 1) \begin{bmatrix} n \\ 0 \end{bmatrix}, \dots, \left(\frac{|\mathcal{C}|}{q^{m(n-d)}} - 1 \right) \begin{bmatrix} n \\ d^\perp - 1 \end{bmatrix} \right)^t.$$

By Lemma 23, $(A^{-1})_{r,j} = (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j-1 \\ r-1 \end{bmatrix}$ for $r, j \in [d^\perp]$. Hence for $1 \leq r < d^\perp + 1 \leq j \leq n - d + 1$ the entry of the matrix $(I_{d^\perp} | A^{-1}B)$ in position (r, j) is

$$\begin{aligned} (I_{d^\perp} | A^{-1}B)_{r,j} &= \sum_{i=0}^{d^\perp-1} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \\ &= \sum_{i=0}^{n-d} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \\ &\quad - \sum_{i=d^\perp}^{n-d} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \\ &= \delta_{r-1, j-1} - \sum_{i=d^\perp}^{n-d} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \\ &= - \sum_{i=d^\perp}^{n-d} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \\ &= - \sum_{i=d^\perp}^{j-1} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix}. \end{aligned}$$

As a consequence, for $r = 1, \dots, d^\perp$ we obtain

$$\begin{aligned} A_{n-r+1} + \sum_{j=d^\perp+1}^{n-d+1} \left(- \sum_{i=d^\perp}^{j-1} (-1)^{i-r+1} q^{\binom{i-r+1}{2}} \begin{bmatrix} i \\ r-1 \end{bmatrix} \begin{bmatrix} j-1 \\ i \end{bmatrix} \right) A_{n-j+1} &= \\ &= \sum_{j=1}^{d^\perp} (-1)^{j-r} q^{\binom{j-r}{2}} \begin{bmatrix} j-1 \\ r-1 \end{bmatrix} \begin{bmatrix} n \\ j-1 \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{m(j-1)}} - 1 \right) \end{aligned}$$

or, equivalently,

$$\begin{aligned} A_{n-d^\perp+r} + \sum_{j=d^\perp}^{n-d} \left(- \sum_{i=d^\perp}^j (-1)^{i-d^\perp+r} q^{\binom{i-d^\perp+r}{2}} \begin{bmatrix} i \\ d^\perp-r \end{bmatrix} \begin{bmatrix} j \\ i \end{bmatrix} \right) A_{n-j} &= \\ &= \sum_{j=0}^{d^\perp-1} (-1)^{j-d^\perp+r} q^{\binom{j-d^\perp+r}{2}} \begin{bmatrix} j \\ d^\perp-r \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{mj}} - 1 \right). \end{aligned}$$

By Lemma 8 we have

$$\begin{aligned} A_{n-d^\perp+r} &= \sum_{j=d^\perp}^{n-d} \sum_{i=d^\perp}^j (-1)^{i-d^\perp+r} q^{\binom{i-d^\perp+r}{2}} \begin{bmatrix} i \\ d^\perp-r \end{bmatrix} \begin{bmatrix} j \\ i \end{bmatrix} A_{n-j} \\ &\quad + \sum_{j=0}^{d^\perp-1} (-1)^{j-d^\perp+r} q^{\binom{j-d^\perp+r}{2}} \begin{bmatrix} j \\ d^\perp-r \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{mj}} - 1 \right) \\ &= \sum_{j=d^\perp}^{n-d} \sum_{i=d^\perp}^j (-1)^{i-d^\perp+r} q^{\binom{i-d^\perp+r}{2}} \begin{bmatrix} i \\ d^\perp-r \end{bmatrix} \begin{bmatrix} j \\ i \end{bmatrix} A_{n-j} \\ &\quad + \sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} d^\perp-r+i \\ d^\perp-r \end{bmatrix} \begin{bmatrix} n \\ d^\perp-r+i \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{m(d^\perp-r+i)}} - 1 \right) \\ &= \sum_{j=d^\perp}^{n-d} \begin{bmatrix} j \\ d^\perp-r \end{bmatrix} \left(\sum_{i=d^\perp}^j (-1)^{i-d^\perp+r} q^{\binom{i-d^\perp+r}{2}} \begin{bmatrix} j-d^\perp+r \\ j-i \end{bmatrix} \right) A_{n-j} \\ &\quad + \begin{bmatrix} n \\ d^\perp-r \end{bmatrix} \sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n-d^\perp+r \\ i \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{m(d^\perp-r+i)}} - 1 \right). \end{aligned}$$

Using Lemma 8 and Lemma 24, the first term of the sum can be simplified as follows:

$$\begin{aligned}
\sum_{i=d^\perp}^j (-1)^{i-d^\perp+r} q^{\binom{i-d^\perp+r}{2}} \begin{bmatrix} j-d^\perp+r \\ j-i \end{bmatrix} &= \sum_{i=r}^{j-d^\perp+r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j-d^\perp+r \\ j-d^\perp+r-i \end{bmatrix} \\
&= \sum_{i=r}^{j-d^\perp+r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j-d^\perp+r \\ i \end{bmatrix} \\
&= -\sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j-d^\perp+r \\ i \end{bmatrix} \\
&= (-1)^r q^{\binom{r}{2}} \begin{bmatrix} j-d^\perp+r-1 \\ r-1 \end{bmatrix}.
\end{aligned}$$

The theorem follows by combining the equalities. \square

Theorem 25, from which the next corollary easily follows, extends Theorem 5.6 of [5] on the rank distribution of MRD codes.

Corollary 26. Assume that \mathcal{C} is MRD or dually QMRD. Then the rank distribution of \mathcal{C} is given by

$$A_r(\mathcal{C}) = \begin{bmatrix} n \\ r \end{bmatrix} \sum_{i=0}^{r-d} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} \left(\frac{|\mathcal{C}|}{q^{m(n+i-r)}} - 1 \right)$$

for $r = d, \dots, n$. In particular, it is completely determined by n, m , and d .

Finally, we show that Corollary 26 does not hold for codes which are QMRD, but not dually QMRD. In particular, for such codes the parameters m, n , and d do not determine the weight distribution.

Example 27. Let $2 \leq n \leq m$, and let \mathcal{C} be a code of dimension 1 and minimum distance $d < n$. In Example 16 we showed that \mathcal{C}^\perp is QMRD, but not dually QMRD. Theorem 9 for $r = 1$ yields the identity

$$\begin{bmatrix} n \\ 1 \end{bmatrix} + A_d(\mathcal{C}) \begin{bmatrix} n-d \\ 1 \end{bmatrix} = q^{1-m} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} + A_1(\mathcal{C}^\perp) \begin{bmatrix} n-1 \\ 0 \end{bmatrix} \right)$$

from which

$$A_1(\mathcal{C}^\perp) = \frac{q^{m+n-1} + q^{m+n-d} - q^{m+n-d-1} - q^m - q^n + 1}{q-1}.$$

In particular, the weight distribution of \mathcal{C}^\perp depends on d and not only on m, n, d^\perp .

4 Codes with small rank defect

In Corollary 26 we established a very special property of non-trivial Delsarte codes \mathcal{C} whose minimum distance and dual minimum distance satisfy $n+1 \leq d+d^\perp \leq n+2$. In this section we study codes \mathcal{C} such that $d+d^\perp = n$ and $m \mid t$, proving that the sets of minimum-rank codewords of \mathcal{C} and \mathcal{C}^\perp have the same cardinality.

Notation 28. Given a subspace $U \subseteq \mathbb{F}_q^n$, we set $\text{Mat}(U) := \{M \in \text{Mat} : \text{colsp}(M) \subseteq U\}$ and $\mathcal{C}(U) := \mathcal{C} \cap \text{Mat}(U)$. The dual of a subspace $U \subseteq \mathbb{F}_q^n$ with respect to the standard inner product of \mathbb{F}_q^n is denoted by U^\perp .

We need the following technical result.

Lemma 29 ([16], Remark 22 and Lemma 24). Let $U \subseteq \mathbb{F}_q^n$ be an \mathbb{F}_q -subspace. The following hold:

1. $\text{Mat}(U)$ is an \mathbb{F}_q -vector subspace of Mat with $\dim(\text{Mat}(U)) = m \cdot \dim(U)$.
2. $\text{Mat}(U)^\perp = \text{Mat}(U^\perp)$.

For a given subspace $U \subseteq \mathbb{F}_q^n$, the dimension of $\mathcal{C}(U)$ and the dimension of $\mathcal{C}^\perp(U^\perp)$ relate as follows.

Proposition 30. Let \mathcal{C} be a Delsarte code of dimension t , let $U \subseteq \mathbb{F}_q^n$ be a subspace. We have

$$\dim(\mathcal{C}(U)) = \dim(\mathcal{C}^\perp(U^\perp)) + t - m(n - \dim(U)).$$

Proof. By Remark 3, $\dim(\mathcal{C}(U)^\perp) = mn - \dim(\mathcal{C}(U))$. On the other hand, by Remark 3 and Lemma 29.2 we have $\mathcal{C}(U)^\perp = \mathcal{C}^\perp + \text{Mat}(U^\perp)$. By Lemma 29.1 we have

$$\begin{aligned} mn - \dim(\mathcal{C}(U)) &= \dim(\mathcal{C}(U)^\perp) \\ &= \dim(\mathcal{C}^\perp) + \dim(\text{Mat}(U^\perp)) - \dim(\mathcal{C}^\perp \cap \text{Mat}(U^\perp)) \\ &= mn - t + m(n - \dim(U)) - \dim(\mathcal{C}^\perp(U^\perp)). \end{aligned}$$

The proposition follows. □

Theorem 31. Let \mathcal{C} be a t -dimensional code, with minimum distance d and dual minimum distance d^\perp . Assume that $d + d^\perp = n$ and $m \mid t$. Then $\text{Rdef}(\mathcal{C}) = \text{Rdef}(\mathcal{C}^\perp) = 1$ and

$$A_d(\mathcal{C}) = A_{d^\perp}(\mathcal{C}^\perp).$$

Proof. Since \mathcal{C} has minimum distance d , for all subspaces $U, U' \subseteq \mathbb{F}_q^n$ with $\dim(U) = \dim(U') = d$ and $U \neq U'$ we have $\mathcal{C}(U) \cap \mathcal{C}(U') = \{0\}$. Similarly, for all subspaces $U, U' \subseteq \mathbb{F}_q^n$ with $\dim(U) = \dim(U') = d^\perp = n - d$ and $U \neq U'$ we have $\mathcal{C}^\perp(U) \cap \mathcal{C}^\perp(U') = \{0\}$. As a consequence, the number of minimum-rank codewords of \mathcal{C} and \mathcal{C}^\perp is, respectively,

$$A_d(\mathcal{C}) = \sum_{\substack{U \subseteq \mathbb{F}_q^n \\ \dim_{\mathbb{F}_q}(U)=d}} (|\mathcal{C}(U)| - 1), \quad A_{d^\perp}(\mathcal{C}^\perp) = \sum_{\substack{U \subseteq \mathbb{F}_q^n \\ \dim_{\mathbb{F}_q}(U)=n-d}} (|\mathcal{C}^\perp(U)| - 1).$$

Write $t = mk$ with $k \in \mathbb{N}$. Since $d + d^\perp = n$, then \mathcal{C} is not MRD by Corollary 19. Theorem 4 applied to \mathcal{C} and \mathcal{C}^\perp gives $k \leq n - d$ and $n - k \leq n - d^\perp$. Since $d + d^\perp = n$, then we have $k = n - d = d^\perp$, and $\text{Rdef}(\mathcal{C}) = \text{Rdef}(\mathcal{C}^\perp) = 1$. Hence for any subspace $U \subseteq \mathbb{F}_q^n$ with $\dim(U) = d$ we have

$$t - m(n - \dim(U)) = m(k - n + d) = 0.$$

Therefore, $\dim(\mathcal{C}(U)) = \dim(\mathcal{C}^\perp(U^\perp))$ by Proposition 30. It follows that

$$\begin{aligned}
A_d(\mathcal{C}) &= \sum_{\substack{U \subseteq \mathbb{F}_q^n \\ \dim_{\mathbb{F}_q}(U)=d}} (|\mathcal{C}(U)| - 1) \\
&= \sum_{\substack{U \subseteq \mathbb{F}_q^n \\ \dim_{\mathbb{F}_q}(U)=d}} (|\mathcal{C}^\perp(U^\perp)| - 1) \\
&= \sum_{\substack{U \subseteq \mathbb{F}_q^n \\ \dim_{\mathbb{F}_q}(U)=n-d}} (|\mathcal{C}^\perp(U)| - 1) \\
&= A_{d^\perp}(\mathcal{C}^\perp),
\end{aligned}$$

as claimed. \square

Remark 32. In Theorem 31 we prove that, if $d + d^\perp = n$ and $m \mid t$, then $\text{Rdef}(\mathcal{C}) = \text{Rdef}(\mathcal{C}^\perp) = 1$. If instead $d + d^\perp = n$ and $m \nmid t$, then it is easy to show that $\text{Rdef}(\mathcal{C}) + \text{Rdef}(\mathcal{C}^\perp) = 1$.

5 Gabidulin codes

In this section we discuss how the results from Section 3 specialize to Gabidulin codes.

Definition 33. A (**rank-metric Gabidulin**) **code** of length n and dimension $0 \leq k \leq n$ is a k -dimensional \mathbb{F}_{q^m} -subspace $C \subseteq \mathbb{F}_{q^m}^n$. The **rank** of a vector $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ is

$$\text{rk}(v) := \dim_{\mathbb{F}_q} \text{span}_{\mathbb{F}_q} \{v_1, \dots, v_n\}.$$

The **minimum distance** of a code $C \neq \{0\}$ is

$$d(C) := \min\{\text{rk}(v) : v \in C, v \neq 0\}.$$

The **rank distribution** of a code C is the collection $(A_i(C))_{i \in \mathbb{N}}$, where

$$A_i(C) := |\{v \in C : \text{rk}(v) = i\}|.$$

The **dual** of a Gabidulin code $C \subseteq \mathbb{F}_{q^m}^n$ is the Gabidulin code

$$C^\perp := \{v \in \mathbb{F}_{q^m}^k : \langle c, v \rangle = 0 \text{ for all } c \in C\} \subseteq \mathbb{F}_{q^m}^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{F}_{q^m}^n$. A code is **trivial** if $C = \{0\}$ or $C = \mathbb{F}_{q^m}^n$.

Notation 34. Throughout this section, C denotes a non-trivial Gabidulin code $C \subseteq \mathbb{F}_{q^m}^n$ with minimum distance d , dual minimum distance d^\perp , of dimension k over \mathbb{F}_{q^m} . We assume that $n \leq m$.

There is a natural way to associate a Delsarte code to a Gabidulin code.

Definition 35. Let $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$, and let $\mathcal{G} = \{\gamma_1, \dots, \gamma_m\}$ be a basis of \mathbb{F}_{q^m} over \mathbb{F}_q . The matrix **associated** to v with respect to the basis \mathcal{G} is the $n \times m$ matrix $M_{\mathcal{G}}(v)$ with entries in \mathbb{F}_q such that

$$v_i = \sum_{j=1}^m M_{\mathcal{G}}(v) \gamma_j.$$

The Delsarte code **associated** to the Gabidulin code $C \subseteq \mathbb{F}_{q^m}^n$ with respect to the basis \mathcal{G} is

$$\mathcal{C}_{\mathcal{G}}(C) := \{M_{\mathcal{G}}(v) : v \in C\}.$$

We will need the following properties of associated Delsarte codes.

Theorem 36. Let \mathcal{C} be a Delsarte code associated to the Gabidulin code $C \subseteq \mathbb{F}_{q^m}^n$. Then:

1. $\dim(\mathcal{C}) = mk$.
2. C has the same rank distribution as \mathcal{C} . In particular, if C is non-zero then they have the same minimum distance.
3. C^\perp has the same rank distribution as \mathcal{C}^\perp .

Proof. The first two parts of the statement easily follows from Definition 35. The third part is Theorem 18 of [16]. \square

Combining Theorem 4 and Theorem 36 we obtain the following result.

Proposition 37. Let $C \subseteq \mathbb{F}_{q^m}^n$ be a Gabidulin code with minimum distance d and dimension k . Then $k \leq n - d + 1$.

Definition 38. A Gabidulin code $C \subseteq \mathbb{F}_{q^m}^n$ with minimum distance d and dimension k is **MRD** if $k = n - d + 1$.

It is well known that a Gabidulin code C is MRD if and only if C^\perp is MRD. We obtain the next result by combining Corollary 19 and Theorem 36.

Proposition 39. Let $C \subseteq \mathbb{F}_{q^m}^n$ be a Gabidulin code with minimum distance d and dual minimum distance d^\perp . One of the following holds:

1. C is MRD, and $d + d^\perp = n + 2$.
2. $d + d^\perp \leq n$.

We notice in particular that the case $d + d^\perp = n + 1$ does not occur for Gabidulin codes. Hence Corollary 26 reads as follows for Gabidulin codes.

Corollary 40. Let $C \subseteq \mathbb{F}_{q^m}^n$ be a Gabidulin code with minimum distance d and dimension k . If C is MRD, then the rank distribution of C satisfies

$$A_r(C) = \binom{n}{r} \sum_{i=0}^{r-d} (-1)^i q^{\binom{i}{2}} \binom{r}{i} \left(q^{m(k-n+i-r)} - 1 \right)$$

for $r = d, \dots, n$. In particular, it is completely determined by n , m , and d .

Similarly, Theorem 25 and Theorem 31 read as follows for Gabidulin codes.

Corollary 41. Let $C \subseteq \mathbb{F}_q^n$ be a Gabidulin code with minimum distance d and dimension k . If $d + d^\perp \leq n$, then the rank distribution of C satisfies

$$\begin{aligned} A_{n-d^\perp+r}(C) &= (-1)^r q^{\binom{r}{2}} \sum_{j=d^\perp}^{n-d} \begin{bmatrix} j \\ d^\perp - r \end{bmatrix} \begin{bmatrix} j - d^\perp + r - 1 \\ r - 1 \end{bmatrix} A_{n-j}(C) \\ &\quad + \begin{bmatrix} n \\ d^\perp - r \end{bmatrix} \sum_{i=0}^{r-1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n - d^\perp + r \\ i \end{bmatrix} \left(q^{m(k-d^\perp+r-i)} - 1 \right). \end{aligned}$$

In particular, n , m , d , d^\perp , and $A_d(C), \dots, A_{n-d^\perp}(C)$ determine the rank distribution of C .

Corollary 42. Let $C \subseteq \mathbb{F}_q^n$ be a Gabidulin code with minimum distance d and dual minimum distance d^\perp . Assume that $d + d^\perp = n$. Then $A_d(C) = A_{d^\perp}(C^\perp)$.

Remark 43. The rank defect of C is $\text{Rdef}(C) = n + 1 - k - d$. The following are easy consequences of Proposition 37.

1. $\text{Rdef}(C) = 0$ if and only if C is MRD.
2. $\text{Rdef}(C) = \text{Rdef}(C^\perp) = 1$ if and only if $d + d^\perp = n$.

Following the standard terminology, we say that C is **AMRD** (Almost MRD) if $\text{Rdef}(C) = 1$. We say that C is **dually AMRD** if $\text{Rdef}(C) = \text{Rdef}(C^\perp) = 1$. Then Corollary 41 is the analogue of [6, Theorem 9], and Corollary 42 is the analogue of [6, Proposition 14]. Notice that the proof of Corollary 42 is substantially different from the proof of [6, Proposition 14].

References

- [1] G. E. Andrews, *The Theory of Partitions*. Encyclopedia of Mathematics and its Applications, vol. 2, G.C. Rota Editor. Addison-Wesley, 1976.
- [2] R. Ahlswede, N. Cai, S.-Y.R. Li, R.W. Yeung, *Network information flow*. IEEE Transactions on Information Theory 46 (2000), 4, pp. 1204 – 1216.
- [3] M. A. de Boer, *Almost MDS codes*. Designs, Codes and Cryptography, 9 (1996), pp. 143 – 155.
- [4] J. de la Cruz, M. Kiermaier, A. Wassermann, W. Willems, *Algebraic structures of MRD Codes*. Online preprint: <http://arxiv.org/abs/1502.02711>.
- [5] P. Delsarte, *Bilinear forms over a finite field, with applications to coding theory*. J. Combin. Theory Ser. A 25 (1978) pp. 226 – 241.
- [6] A. Faldum, W. Willems, *Codes of small defect*. Designs, Codes and Cryptography, 10 (1997), pp. 341 – 350.
- [7] E. Gabidulin *Theory of codes with maximum rank distance*. Problems of Information Transmission, 1 (1985), 2, pp. 1 – 12.

- [8] M. Gaduleau, Z. Yan, *MacWilliams identity for Codes with the Rank Metric*. EURASIP Journal on Wireless Communications and Networking, 2008.
- [9] R. Kötter, F. R. Kschischang, *Coding for Errors and Erasures in Random Network Coding*. IEEE Transactions on Information Theory, 54 (2008), 8, pp. 3579 – 3591.
- [10] S.-Y.R. Li, R.W. Yeung, N. Cai, *Linear network coding*. IEEE Transactions on Information Theory, 49 (2003), 2, pp. 371 – 381.
- [11] M. Médard, A. Sprintson (editors), *Network Coding: Fundamentals and Applications*. Elsevier, 2012.
- [12] J. Kurihara, R. Matsumoto, T. Uyematsu, *Relative Generalized Rank Weight of Linear Codes and Its Applications to Network Coding*. Online preprint: <http://arxiv.org/abs/1301.5482>.
- [13] M. Giorgetti, A. Previtoli, *Galois invariance, trace codes and subfield subcodes*. Finite Fields and Their Applications, 16 (2010), 2, pp. 96 – 99.
- [14] F. Oggier, A. Sboui, *On the Existence of Generalized Rank Weights*. IEEE International Symposium on Information Theory and its Applications (2012).
- [15] A. Ravagnani, *Generalized weights: an anticode approach*. Online preprint: <http://arxiv.org/abs/1410.7207v1>.
- [16] A. Ravagnani, *Rank-metric codes and their duality theory*. Designs, Codes and Cryptography (to appear). Online preprint: <http://arxiv.org/abs/1410.1333>.
- [17] J. Sheekey, *A new family of linear maximum rank distance codes*. Online preprint: <http://arxiv.org/abs/1504.01581>.
- [18] D. Silva, F. R. Kschischang, *On metrics for error correction in network coding*. IEEE Transactions on Information Theory, 55 (2009), 12, pp. 5479 – 5490.
- [19] D. Silva, F. R. Kschischang, *Universal Secure Network Coding via Rank-Metric Codes*. IEEE Transactions on Information Theory, 57 (2011), 2, pp. 1124 – 1135.
- [20] V. K. Wei, *Generalized Hamming Weights for Linear Codes*. IEEE Transactions on Information Theory, 37 (1991), 5, pp. 1412 – 1418.