## Derivation of Cameron-Liebler line classes

### Alexander L. Gavrilyuk

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, PR China and

> Krasovskii Institute of Mathematics and Mechanics, Kovalevskaya str., 16, Ekaterinburg 620990, Russia e-mail: alexander.gavriliouk@gmail.com

#### Ilia Matkin

Faculty of Mathematics, Chelyabinsk State University, Kashirinykh str., 129, Chelyabinsk 454001, Russia e-mail: ilya.matkin@gmail.com

#### Tim Penttila

Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA e-mail: penttila@math.colostate.edu

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#### Abstract

We construct a new infinite family of Cameron-Liebler line classes in PG(3, q) with parameter  $x = \frac{q^2+1}{2}$  for all odd q.

## 1 Introduction

Let PG(3,q) denote the 3-dimensional projective space over the finite field  $\mathbb{F}_q$ . For a set  $\mathcal{L}$  of lines in PG(3,q), let  $\overline{\mathcal{L}}$  denote the complementary set of lines. A *spread* of PG(3,q) is a set of  $q^2 + 1$  lines that partition the set of points.

We say that  $\mathcal{L}$  is a Cameron-Liebler line class with parameter x in PG(3,q), if there exists a non-negative integer x such that, for every spread S of PG(3,q), one has:

$$|S \cap \mathcal{L}| = x$$
.

It can be seen from the definition that  $\overline{\mathcal{L}}$  is then a Cameron-Liebler line class with parameter  $q^2+1-x$ , so that we may assume  $x \leq \frac{q^2+1}{2}$ . An empty set of lines (x=0), the set of all lines in a plane (x=1) or, dually, through a point (x=1) are trivial examples of Cameron-Liebler line classes. If the point is not in the plane, then the union of the previous two examples with x=1 gives a slightly less trivial Cameron-Liebler line class with parameter x=2.

Cameron-Liebler line classes first appeared in the study [3] of collineation groups of PG(n,q),  $n \ge 3$ , that have equally many orbits on lines and on points (and were given their name in [12]). Under the Klein correspondence, Cameron-Liebler line classes are translated to tight sets of the Klein quadric being, thus, a special case of a tight set of a polar space (see [5, 1]). For more comprehensive background on this topic, we refer to the recent papers [7], [9], [11], [8], [1].

It was conjectured in [3] that the only Cameron-Liebler line classes are the examples mentioned above, i.e.,  $x \le 2$ . The first counterexample was found by Drudge [6] in PG(3,3) with x = 5, which was generalised later by Bruen and Drudge [2] to an infinite family having parameter  $x = \frac{q^2+1}{2}$  for all odd q. The first counterexample in characteristic 2 was found in [10]. With the aid of computer and using some clever ideas about possible symmetries of Cameron-Liebler line classes, Rodgers [14] constructed many more new examples for certain x and prime powers q. Very recently, some of them have been shown in [1], [7] to be a part of a new infinite family of Cameron-Liebler line classes with parameter  $x = \frac{q^2+1}{2}$  for  $q \equiv 5$  or 9 (mod 12). (In fact, a line class of the family found in [1], [7] has parameter  $\frac{q^2-1}{2}$ , however, it is disjoint with a plane, which is a Cameron-Liebler line class with parameter 1, so that the union of their lines is a Cameron-Liebler line class with parameter  $\frac{q^2-1}{2} + 1$ .)

In this note, we first describe a switching-like operation in Cameron-Liebler line classes that satisfy some necessary conditions (see Lemma 2.1). We then show in Lemma 2.3 that these conditions may only hold for line classes with  $x = q^2$  or  $x = \frac{q^2+1}{2}$ . Applying this switching operation to the line classes found by Bruen and Drudge, we construct another infinite family of Cameron-Liebler line classes in PG(3, q) with parameter  $x = \frac{q^2+1}{2}$  for all odd q, and show that they are not equivalent to the line classes of Bruen and Drudge, unless q = 3 (see Theorem 3.3).

# 2 Switching in Cameron-Liebler line classes

For a point P and a plane  $\pi$  of PG(3, q), let  $\mathsf{Star}(P)$  and  $\mathsf{Line}(\pi)$  denote the set of all lines on P or in  $\pi$ , respectively.

**Lemma 2.1** Let  $\mathcal{L}$  be a Cameron-Liebler line class such that there exists an incident point-plane pair  $(P, \pi)$  satisfying the following conditions:

- (1)  $(\mathsf{Line}(\pi) \setminus \mathsf{Star}(P)) \cap \mathcal{L} = \emptyset$ ,
- (2)  $\mathsf{Star}(P) \setminus \mathsf{Line}(\pi) \subseteq \mathcal{L}$ .

Then

$$\mathcal{L}' := \mathcal{L} \cup (\mathsf{Line}(\pi) \setminus \mathsf{Star}(P)) \setminus (\mathsf{Star}(P) \setminus \mathsf{Line}(\pi))$$

is a Cameron-Liebler line class with the same parameter.

*Proof:* For any spread S of  $\operatorname{PG}(3,q)$  we have that S contains either a line of  $\operatorname{Star}(P) \cap \operatorname{Line}(\pi)$ , or a line  $\ell \in \operatorname{Line}(\pi) \setminus \operatorname{Star}(P)$  and a line  $m \in \operatorname{Star}(P) \setminus \operatorname{Line}(\pi)$ . In the former case,  $S \cap \mathcal{L} = S \cap \mathcal{L}'$ , while in the latter case  $S \cap \mathcal{L}' = (S \cap \mathcal{L}) \cup \{m\} \setminus \{\ell\}$ . Thus,  $|S \cap \mathcal{L}'| = |S \cap \mathcal{L}|$  holds in both cases, and so  $\mathcal{L}'$  is a Cameron-Liebler line class.

Let  $\mathcal{L}$  be a Cameron-Liebler line class, and  $\ell$  a line of PG(3,q). Then  $\ell$  lies in q+1 planes  $\pi_1, \ldots, \pi_{q+1}$  and contains q+1 points  $P_1, \ldots, P_{q+1}$ . Define the square matrix  $T(\ell) = (t_{ij})$  of size q+1 with integer entries given by

$$t_{ij} := |((\mathsf{Line}(\pi_i) \cap \mathsf{Star}(P_j)) \setminus \{\ell\}) \cap \mathcal{L}|, \quad 1 \leq i, j \leq q+1.$$

The set consisting of the matrix T, and every matrix obtained from this one by a permutation of the rows and a permutation of the columns is called the *pattern* of  $\ell$  with respect to  $\mathcal{L}$ . We represent each pattern by one of its matrices. This concept was introduced in [9], where the following result has been proved.

**Proposition 2.2** Let  $\mathcal{L}$  be a Cameron-Liebler line class with parameter x, let  $\ell$  be a line of PG(3,q), and  $T=(t_{ij})$  the pattern of  $\ell$ .

(a) For any  $i \in \{1, ..., q+1\}$ 

$$\sum_{j=1}^{q+1} t_{ij} = |\mathsf{Line}(\pi_i) \cap \mathcal{L} \setminus \{\ell\}| \quad and \quad \sum_{j=1}^{q+1} t_{ji} = |\mathsf{Star}(P_i) \cap \mathcal{L} \setminus \{\ell\}|.$$

(b) For all  $k, l \in \{1, \dots, q+1\}$ 

$$\sum_{i=1}^{q+1} t_{il} + \sum_{j=1}^{q+1} t_{kj} = \begin{cases} x + (q+1)t_{kl} & \text{if } \ell \notin \mathcal{L} \\ x + (q+1)(t_{kl}+1) - 2 & \text{if } \ell \in \mathcal{L}. \end{cases}$$

(c)  $t_{kl} + t_{rs} = t_{ks} + t_{rl}$  for all  $k, l, r, s \in \{1, \dots, q+1\}$ .

(d)  $\sum_{i,j=1}^{q+1} t_{ij}^2 = \begin{cases} x(q+x) & \text{if } \ell \notin \mathcal{L} \\ q^3 + q^2 + (x-1)^2 + q(x-1) & \text{if } \ell \in \mathcal{L}. \end{cases}$ 

**Lemma 2.3** Let  $\mathcal{L}$  be a Cameron-Liebler line class such that there exists an incident point-plane pair  $(P, \pi)$  satisfying the conditions of Lemma 2.1. Then the parameter x of  $\mathcal{L}$  is equal to  $q^2$  or  $\frac{q^2+1}{2}$ .

Proof: Up to taking the complement to a line set and the point-plane duality in PG(3, q), we may assume that there exists a line  $\ell$  of  $\mathsf{Star}(P) \cap \mathsf{Line}(\pi) \setminus \mathcal{L}$ . Let T be the pattern of  $\ell$  such that, without loss of generality, its first row corresponds to  $\pi$ , and its first column corresponds to P. Then the conditions of Lemma 2.1 imply that  $t := t_{11} = |\mathsf{Star}(P) \cap \mathsf{Line}(\pi) \cap \mathcal{L}|$ , and  $t_{1,j} = q$  and  $t_{j,1} = 0$  for all  $j \in \{2, \ldots, q+1\}$ . By Proposition 2.2 (c), we see that  $t_{ij} = q - t_{11}$  for all  $i, j \in \{2, \ldots, q+1\}$ .

Further, Proposition 2.2 (b) applied to the first row and column of T, and Proposition 2.2 (d) applied to the pattern T give the following equations:

$$\begin{cases} t + q^2 + t = x + t(q+1), \\ t^2 + q^3 + q^2(q-t)^2 = x(q+x), \end{cases}$$

which yield t=0 and  $x=q^2$  (and thus  $\mathcal{L}$  is the complement to a Cameron-Liebler line class with parameter 1), or  $t=\frac{q+1}{2}$  and  $x=\frac{q^2+1}{2}$ .

## 3 Application of switching

From Lemma 2.3 we see that the only non-trivial case, where the switching operation of Lemma 2.1 may be applied, is the case  $x = \frac{q^2+1}{2}$ . There exist at least two infinite families of Cameron-Liebler line classes with parameter  $x = \frac{q^2+1}{2}$ , namely, the first counterexamples to the Cameron-Liebler conjecture constructed by Bruen and Drudge in [2] and the line classes recently found in [1] and independently in [7]. Fortunately, the former satisfy the conditions of Lemma 2.1 (while the latter do not), and applying the switching operation indeed produces a new Cameron-Liebler line class, not equivalent to the original one, if q > 3. In this section we give the necessary details.

First of all, let us recall the construction by Bruen and Drudge. Let q be an odd prime power, and  $\mathcal{Q}$  an elliptic quadric of  $\mathrm{PG}(3,q)$  with the corresponding quadratic form  $\mathbb{Q}$ . The set of q+1 tangents  $\mathcal{T}_P$  to a point  $P \in \mathcal{Q}$  can be divided into two subsets, say  $\mathcal{T}_P^1$ ,  $\mathcal{T}_P^2$ , of size (q+1)/2 each, depending on whether a tangent line contains a point  $P' \neq P$  such that  $\mathbb{Q}(P')$  is a square in  $\mathbb{F}_q$ . Note if  $\mathbb{Q}(P')$  is a square in  $\mathbb{F}_q$ , then all the points on the tangent PP' satisfy this property, as  $\mathbb{Q}(P+cP')=c^2\mathbb{Q}(P')$ .

Denote by  $\mathcal{T}^i$  the set  $\cup_{P\in\mathcal{Q}}\mathcal{T}_P^i$ ,  $i\in\{1,2\}$ . Let  $\mathcal{S}$  and  $\mathcal{E}$  be the sets of secant and external lines to  $\mathcal{Q}$ , respectively. Then any of

$$\mathcal{S} \cup \mathcal{T}^i, \quad \mathcal{E} \cup \mathcal{T}^j, \quad i, j \in \{1, 2\},$$

is a Cameron-Liebler line class of parameter  $\frac{q^2+1}{2}$ .

Since all these line classes are equivalent under the action of  $P\Gamma L(4,q)$  and the polarity induced by  $\mathcal{Q}$  (see [5]), we may choose, without loss of generality,  $\mathcal{L}$  to be  $\mathcal{S} \cup \mathcal{T}^1$ . For a point  $P_1$  of  $\mathcal{Q}$  and its tangent plane  $\tau_{P_1}$ , one can see that

$$(\mathsf{Line}(\tau_{P_1}) \setminus \mathsf{Star}(P_1)) \subset \mathcal{E} \subset \overline{\mathcal{L}}, \ \ \mathsf{Star}(P_1) \setminus \mathsf{Line}(\tau_{P_1}) \subset \mathcal{S} \subset \mathcal{L},$$

so that  $(P_1, \tau_{P_1})$  satisfies the condition of Lemma 2.1, and the line class  $\mathcal{L}'$  defined by

$$\mathcal{L}' := \mathcal{L} \cup (\mathsf{Line}(\tau_{P_1}) \setminus \mathsf{Star}(P_1)) \setminus (\mathsf{Star}(P_1) \setminus \mathsf{Line}(\tau_{P_1}))$$

is a Cameron-Liebler line class with parameter  $\frac{q^2+1}{2}$ .

Our aim now is to show that  $\mathcal{L}'$  is not equivalent to  $\mathcal{L}$  unless q=3. For q=3, we can either apply Drudge's classification of Cameron-Liebler line classes in PG(3,3) [6] that determined that, up to equivalence, there is a unique Cameron-Liebler line class with parameter 5, or it can be checked with the aid of computer that  $\mathcal{L}'$  is projectively equivalent to  $\overline{\mathcal{L}}$  for this value of q. From now on, we assume that q>3.

**Lemma 3.1** A plane  $\pi$  of PG(3,q) contains  $\frac{q+1}{2}$ , or  $\frac{q(q+1)}{2}$ , or  $\frac{(q+1)(q+2)}{2}$  lines of  $\mathcal{L}$ .

*Proof:* If  $\pi$  is a tangent plane to  $\mathcal{Q}$ , then  $|\mathsf{Line}(\pi) \cap \mathcal{L}| = \frac{q+1}{2}$  by the construction of  $\mathcal{L}$ . Suppose that  $\pi$  is a secant plane so that  $\pi \cap \mathcal{Q}$  is a conic. Under the polarity, say  $\rho$ , induced by  $\mathcal{Q}$ , every tangent line to the conic in  $\pi$  is mapped to a tangent line to  $\mathcal{Q}$  on  $\rho(\pi)$ . Therefore, all tangent lines to the conic in  $\pi$  are either in  $\mathcal{T}^1$  or in  $\mathcal{T}^2$ . In the former case,  $\pi$  contains  $\binom{q+1}{2} + q + 1$  lines from  $\mathcal{L}$ , in the latter case  $|\mathsf{Line}(\pi) \cap \mathcal{L}| = \binom{q+1}{2}$ .

**Lemma 3.2** A point P of PG(3,q) is on  $q^2 + \frac{q+1}{2}$ , or  $\frac{q(q-1)}{2}$ , or  $\frac{q(q+1)}{2} + 1$  lines of  $\mathcal{L}$ .

Proof: If  $P \in \mathcal{Q}$ , then  $|\mathsf{Star}(P) \cap \mathcal{L}| = \frac{q+1}{2} + q^2$  by the construction of  $\mathcal{L}$ . Suppose that  $P \notin \mathcal{Q}$ . If P is on a tangent line from  $\mathcal{T}^i$  for  $i \in \{1,2\}$ , then all tangent lines to  $\mathcal{Q}$  through P are in  $\mathcal{T}^i$ . Let P' be a point of  $\mathcal{Q}$  such that PP' is a tangent line to  $\mathcal{Q}$ , and consider all secant planes  $\pi_1, \ldots, \pi_q$  containing the line PP'. Recall that every point not on a conic in a projective plane of odd order lies on 0 or 2 tangents, see [13, 15]. Since  $\pi_i \cap \mathcal{Q}$  is a conic, and PP' is a tangent line to the conic, we conclude that P lies on 2 tangents and  $\frac{q-1}{2}$  secants to  $\pi_i \cap \mathcal{Q}$ . Thus,  $|\mathsf{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2}$ , if  $PP' \in \mathcal{T}^2$ , or  $|\mathsf{Star}(P) \cap \mathcal{L}| = \frac{q(q-1)}{2} + q + 1$ , if  $PP' \in \mathcal{T}^1$ .

**Theorem 3.3** The line classes  $\mathcal{L}$  and  $\mathcal{L}'$  are not equivalent under the action of  $P\Gamma L(4,q)$  or a duality.

Proof: Following the notation from the above, one can see that the plane  $\tau_{P_1}$  contains  $\frac{q+1}{2} + q^2$  lines of  $\mathcal{L}'$ . Since, for a point  $P_2 \in \mathcal{Q}$ ,  $P_2 \neq P_1$ , one has  $\tau_{P_1} \cap \tau_{P_2} \in \mathcal{E}$ , the plane  $\tau_{P_2}$  contains  $\frac{q+1}{2} + 1$  lines of  $\mathcal{L}'$ . It now follows from Lemmas 3.1, 3.2 that the intersection numbers of  $\mathcal{L}'$  with respect to planes and points of PG(3,q) are different from those of  $\mathcal{L}$  or  $\overline{\mathcal{L}}$ .

We also note that  $\mathcal{L}'$  is not equivalent to a line class of the family found in [1], [7], since there is no plane (or, dually, a point with all lines on it) contained in or disjoint from  $\mathcal{L}'$ . In particular, in PG(3,5), there exist at least three pairwise non-equivalent Cameron-Liebler line classes with  $x = \frac{q^2+1}{2} = 13$  (namely, the example by Bruen and Drudge, its switched mate by Theorem 3.3, and the example found in [7] and [1]). In fact, up to equivalence, these are the only Cameron-Liebler line classes with given x in PG(3,5) (the details will be given elsewhere).

The line class  $\mathcal{L}'$  contains only the one incident point-plane pair, namely,  $(P_1, \tau_{P_1})$ , satisfying the conditions of Lemma 2.1, and, clearly, switching of  $\mathcal{L}'$  with respect to it gives the line class  $\mathcal{L}$ . Since, for q > 3, there is a unique switched mate for  $\mathcal{L}'$  (namely,  $\mathcal{L}$ ), it follows that its stabiliser  $G_{\mathcal{L}'}$  is a subgroup of the stabiliser  $G_{\mathcal{L}}$ . The stabiliser  $G_{\mathcal{L}}$  of a Bruen-Drudge line class is a subgroup of index two of Pro-(4, q), i.e., the subgroup that fixes  $\mathcal{T}^1$  and  $\mathcal{T}^2$ . Thus,  $G_{\mathcal{L}'}$  is the stabiliser of the point  $P_1$  in  $G_{\mathcal{L}}$ . Then, for  $q = p^h$ , where p is a prime,  $G_{\mathcal{L}'}$  has order  $q^2(q^2 - 1)h$ , and is isomorphic to AGL $(1, q^2) \times C_h$ .

We expect that the only non-trivial Cameron-Liebler line classes satisfying the conditions of Lemma 2.1 are the examples of Bruen and Drudge and their switched mates.

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### References

- [1] J. De Beule, J. Demeyer, K. Metsch, M. Rodgers. A new family of tight sets in  $Q^+(5,q)$ . Des. Codes Cryptogr. 78 (2016) 655–678.
- [2] A. A. Bruen, Keldon Drudge. The construction of Cameron-Liebler line classes in PG(3, q). Finite Fields Appl.  $\mathbf{5(1)}$  (1999) 35–45.
- [3] P. J. Cameron, R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.* **46** (1982) 91–102.
- [4] A. Cossidente, F. Pavese. Intriguing sets of quadrics in PG(5, q). Adv. Geom., in press.
- [5] Keldon Drudge. Extremal sets in projective and polar spaces. *Ph.D. Thesis*, University of Western Ontario, 1998.
- [6] Keldon Drudge. On a conjecture of Cameron and Liebler. European J. Combin. **20(4)** (1999) 263–269.
- [7] T. Feng, K. Momihara, Q. Xiang. Cameron-Liebler line classes with parameter  $x = \frac{q^2-1}{2}$ . J. Combin. Theory Ser. A 133 (2015) 307–338.

- [8] A. L. Gavrilyuk, K. Metsch. A modular equality for Cameron-Lieber line classes. *J. Combin. Theory Ser. A* **127** (2014) 224–242.
- [9] Alexander L. Gavrilyuk, Ivan Yu. Mogilnykh. Cameron-Liebler line classes in PG(n, 4). Des. Codes Cryptogr. **73(3)** (2014) 969–982.
- [10] Patrick Govaerts, Tim Penttila. Cameron-Liebler line classes in PG(3,4). Bull. Belg. Math. Soc. Simon Stevin 12(5) (2005) 793–804.
- [11] Klaus Metsch. An improved bound on the existence of Cameron–Liebler line classes. *J. Combin. Theory Ser. A* **121** (2014) 89–93.
- [12] Tim Penttila. Cameron-Liebler line classes in PG(3,q). Geom. Dedicata 37(3) (1991) 245–252.
- [13] Bertil Qvist. Some remarks concerning curves of the second degree in a finite plane. Suomalainen tiedeakatemia (Sci. Fennica) 134 (1952) 4–27.
- [14] Morgan Rodgers. Cameron-Liebler line classes. Des. Codes Cryptogr. 68(1-3) (2013) 33–37.
- [15] Beniamino Segre. Ovals in a finite projective plane. Canad. J. Math. 7 (1955) 414–416.