SWITCHED GRAPHS OF SOME STRONGLY REGULAR GRAPHS RELATED TO THE SYMPLECTIC GRAPH

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Abstract. Applying a method of Godsil and McKay [6] to some graphs related to the symplectic graph, a series of new infinite families of strongly regular graphs with parameters $(2^n \pm 2^{(n-1)/2}, 2^{n-1} \pm 2^{(n-1)/2}, 2^{n-2} \pm 2^{(n-3)/2}, 2^{n-2} \pm 2^{(n-1)/2})$ are constructed for any odd $n \geq 5$. The construction is described in terms of geometry of quadric in projective space. The binary linear codes of the switched graphs are $[2^n \mp 2^{\frac{n-1}{2}}, n+3, 2^{t+1}]_2$-code or $[2^n \mp 2^{\frac{n-1}{2}}, n+3, 2^{t+2}]_2$-code.

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1. Introduction

Consider the n-dimensional projective space $\mathrm{PG}(n, 2)$ over the finite field $\mathbb{F}_2$. That is, $\mathrm{PG}(n, 2) = \mathbb{F}_2^{n+1} \setminus \{0\}$. When n is odd, there are two non-equivalent non-singular quadrics in $\mathrm{PG}(n, 2)$, namely elliptic and hyperbolic. For general references, see [9, Ch. 5] and [10, Ch. 22]. Both quadrics define a symplectic polarity (null polarity) in $\mathrm{PG}(n, 2)$ [9, Thoerem 5.28].

Let $n \geq 5$ be an odd number. Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, 2)$. Define the graph $\Gamma_{\mathcal{Q}} = (V_{\mathcal{Q}}, E_{\mathcal{Q}})$ as follows. The vertex set $V_{\mathcal{Q}}$ is the set of points of $\mathrm{PG}(n, 2)$ not in $\mathcal{Q}$. Two vertices x and y are adjacent in $\Gamma_{\mathcal{Q}}$ if and only if the line xy joining them is an external line of $\mathcal{Q}$. $\Gamma_{\mathcal{Q}}$ is the complement of a subgraph of the symplectic graph $Sp(n+1, 2)$, which is the graph of the perpendicular relation induced by a non-degenerate symplectic form of $\mathbb{F}_2^{n+1}$ on the non-zero vectors of $\mathbb{F}_2^{n+1}$. In [8, 7, 11], $\Gamma_{\mathcal{Q}}$ is denoted by $\overline{\mathcal{N}_{n+1}^{\epsilon}}$, where ϵ is $+$ (plus) if $\mathcal{Q}$ is hyperbolic, and $-$ (minus) if $\mathcal{Q}$ is elliptic.

A *strongly regular graph* with parameters (v, k, λ, μ) is a graph with v vertices such that each vertex lies on exactly k edges; any two adjacent vertices have exactly λ neighbours in common; and any two non-adjacent vertices have exactly μ neighbours in common. The adjacency matrix of a strongly regular graph has exactly three eigenvalues. One is k with multiplicity 1, and the remaining two are usually denoted by r and s, $r > s$ with multiplicities f and g respectively. For general references, see

[4, Ch.9] and [5, Ch.2]. It is well-known that $\Gamma_{\mathcal{Q}}$ defined above is a strongly regular graph. Table 1 shows the parameters of $\Gamma_{\mathcal{Q}}$ for the different quadrics in $\mathrm{PG}(n,2)$ (see [7]).

$\mathcal{Q}$	graph	v	k	λ	μ
elliptic	$\Gamma_{\mathcal{Q}} = \overline{\mathcal{N}_{n+1}^{-}}$	$2^n + 2^{\frac{n-1}{2}}$	$2^{n-1} + 2^{\frac{n-1}{2}}$	$2^{n-2} + 2^{\frac{n-3}{2}}$	$2^{n-2} + 2^{\frac{n-1}{2}}$
hyperbolic	$\Gamma_{\mathcal{Q}} = \overline{\mathcal{N}_{n+1}^{+}}$	$2^n - 2^{\frac{n-1}{2}}$	$2^{n-1} - 2^{\frac{n-1}{2}}$	$2^{n-2} - 2^{\frac{n-3}{2}}$	$2^{n-2} - 2^{\frac{n-1}{2}}$
$\mathcal{Q}$	graph	r	s	f	g
elliptic	$\Gamma_{\mathcal{Q}} = \overline{\mathcal{N}_{n+1}^{-}}$	$2^{\frac{n-3}{2}}$	$-2^{\frac{n-1}{2}}$	$\frac{1}{3}(2^{n+1}-4)$	$\frac{2^n+1}{3} + 2^{\frac{n-1}{2}}$
hyperbolic	$\Gamma_{\mathcal{Q}} = \overline{\mathcal{N}_{n+1}^{+}}$	$2^{\frac{n-1}{2}}$	$-2^{\frac{n-3}{2}}$	$\frac{2^n+1}{3} - 2^{\frac{n-1}{2}}$	$\frac{1}{3}(2^{n+1}-4)$

TABLE 1. Parameters of $\Gamma_{\mathcal{Q}}$

Godsil and McKay (1982) introduced a method to generate graphs with the same adjacency spectrum [6] i.e. the adjacency matrices of the graphs have equal multisets of eigenvalues. The method is described as follows. Let Γ be a graph. Let S be a subset of the vertex set such that the subgraph of Γ with vertex set S is regular. Suppose any vertex outside S has 0, $|S|$ or $\frac{1}{2}|S|$ neighbours in S. Consider the graph Γ' obtained by switching Γ as follows: for any vertex x of Γ outside S, if x has $\frac{1}{2}|S|$ neighbours in S, then delete those $\frac{1}{2}|S|$ edges and join x to the other $\frac{1}{2}|S|$ vertices. We call S a *Godsil and McKay switching set* of Γ. By Godsil and McKay [6], Γ' has the same adjacency spectrum as Γ. In the case where Γ is a strongly regular graph, Γ' has the same adjacency spectrum as Γ and thus is also a strongly regular graph with the same parameters (see [4]). Recently, there has been interest in constructing new strongly regular graphs from known ones using the method of Godsil-McKay described above, see for example [1] and [3].

In this article, we apply the method of Godsil-McKay to $\Gamma_{\mathcal{Q}}$ as described above. The paper is organized as follows: After a brief description of our terminology in Section 2, we give two constructions of Godsil-McKay switching sets for $\Gamma_{\mathcal{Q}}$ in Section 3. In Sections 4 and 5, we study the binary code spanned by the rows of the adjacency matrix $\Gamma_{\mathcal{Q}}$ and that of its switched graphs. In Section 6, we give a number of switched graphs found and find the parameters of the codes of the switched graphs.

2. Terminology and notation

For any $m = 0, 1, 2, \cdots, n-1$, a *subspace of dimension* m, or m-*space*, of $\mathrm{PG}(n,2)$ is a set of points all of whose representing vectors form, together with the zero, a subspace of dimension $m+1$ of $\mathbb{F}_2^{n+1}$. The number of points of an m-space in $\mathrm{PG}(n,2)$ is $2^{m+1}-1$ [9, Theorem 3.1].

A *quadric* Q_n in $\mathrm{PG}(n,2)$ is the set of points $[X_0, X_1, \cdots, X_n]$ satisfying a non-zero homogeneous equation of degree two, i.e. $\sum_{i\leq j,i,j=0}^{n} a_{ij}X_iX_j = 0$ for some $a_{ij} \in \mathbb{F}_q$, not all zero. If the equation can be reduced to fewer than $n+1$ variables by a change of basis, Q_n is called *singular*. Otherwise, it is *non-singular*.

Depending on the parity of n, there is one or there are two quadrics under the action of the automorphism group of $\mathrm{PG}(n, 2)$. For n odd, there are two distinct non-singular quadrics, respectively the elliptic quadric with canonical equation $f(X_0, X_1)+X_2X_3+\cdots+X_{n-1}X_n = 0$ where f is an irreducible binary quadratic form, and the hyperbolic quadric with canonical equation $X_0X_1+X_2X_3+\cdots+X_{n-1}X_n = 0$. For n even, there is the parabolic quadric with canonical equation $X_0^2+X_1X_2+\cdots+X_{n-1}X_n = 0$. For a parabolic quadric Q_n, there is an unique point in $\mathrm{PG}(n, 2) \setminus Q_n$, called *the nucleus* of Q_n, such that all line through the nucleus is tangent to Q_n (see [10, page10]). Table 2 shows the number of points of different non-singular quadrics.

quadric Q_n	Elliptic	Hyperbolic	Parabolic
number of points	$2^n - 2^{\frac{n-1}{2}} - 1$	$2^n + 2^{\frac{n-1}{2}} - 1$	$2^n - 1$

TABLE 2. Number of points in non-singular quadrics

A singular quadric in $\mathrm{PG}(n, 2)$ is either an m-space, $m < n$, or a *cone* $\Pi_{n-t-1}Q_t$ which is the set of points on the lines joining an $(n-t-1)$-space Π_{n-t-1} to a non-singular quadric Q_t in a t-space Π_t with $\Pi_{n-t-1} \cap \Pi_t = \emptyset$. The number of points of such a cone is

$$|\Pi_{n-t-1}Q_t| = (2^{n-t} - 1) + 2^{n-t}|Q_t|. \tag{2.1}$$

A *polarity* ρ of $\mathrm{PG}(n, 2)$ is an order-two bijection on its subspaces that reverses containment. That is, for an m-space Π_m and m'-space $\Pi_{m'}$ of $\mathrm{PG}(n, 2)$, if $\Pi_m \subset \Pi_{m'}$, then $\Pi_{m'}^{\rho} \subset \Pi_m^{\rho}$. In particular, a polarity interchanges m-spaces and $(n-1-m)$-spaces. For a general reference on polarities, see [9, Section 2.1].

The *(binary linear) code* $C(\Gamma)$ of a graph $\Gamma = (V, E)$ is the subspace in the vector space $\mathbb{F}_2^{|V|}$ generated by the rows of the adjacency matrix of Γ modulo 2. The *length* n of $C(\Gamma)$ is $|V|$, and the *dimension* k of $C(\Gamma)$ is the dimension of $C(\Gamma)$ as a subspace in $\mathbb{F}_2^{|V|}$. For any vector $w = (w_x)_{x\in V} \in \mathbb{F}_2^{|V|}$, the *weight* $\mathrm{wt}(w)$ of w is

$$\mathrm{wt}(w) = |\{x \in V | w_x \neq 0\}|.$$

The *minimum weight* d of a code is the minimum of the weight of its non-zero codewords. A binary linear code of length n, dimension k and minimum weight d will be referred to as an $[n, k, d]_2$. For any subset $U \subset V$, the *characteristic vector of* U, denoted by v^U, is the vector $(w_x)_{x\in V}$ where $w_x = 1$ if $x \in U$, and $w_x = 0$ if $x \notin U$. For a general reference on codes, see [2].

For the graph $\Gamma_Q = (V_{\mathcal{Q}}, E_{\mathcal{Q}})$ defined in Section 1, $C(\Gamma_{\mathcal{Q}})$ is a $[2^n + 2^{\frac{n-1}{2}}, n + 1, 2^{n-1}]_2$ code if $\mathcal{Q}$ is elliptic, and is a $[2^n + 2^{\frac{n-1}{2}}, n + 1, 2^{n-1} - 2^{\frac{n-1}{2}}]_2$ code if $\mathcal{Q}$ is hyperbolic. A vector $w \in \mathbb{F}_2^{|V_{\mathcal{Q}}|}$ is a codeword of $C(\Gamma_{\mathcal{Q}})$ if and only if it is the characteristic vector of $(\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \Sigma$ for some $(n-1)$-space Σ in $\mathrm{PG}(n,2)$. The weight distribution of $C(\Gamma_{\mathcal{Q}})$ is shown in Tables 3 and 4 (see for example [7]).

weight	0	2^{n-1}	$2^{n-1}+2^{\frac{n-1}{2}}$
number of codewords	1	$2^n-2^{\frac{n-1}{2}}-1$	$2^n+2^{\frac{n-1}{2}}$

Table 3. Weight distribution of $C(\Gamma_{\mathcal{Q}})$ if $\mathcal{Q}$ is elliptic

weight	0	$2^{n-1}-2^{\frac{n-1}{2}}$	2^{n-1}
number of codewords	1	$2^n-2^{\frac{n-1}{2}}$	$2^n+2^{\frac{n-1}{2}}-1$

Table 4. Weight distribution of $C(\Gamma_{\mathcal{Q}})$ if $\mathcal{Q}$ is hyperbolic

3. Two constructions of Godsil-McKay switching sets of $\Gamma_{\mathcal{Q}}$

In this section, we will prove Theorems 3.A and 3.B, which give constructions of Godsil-McKay switching sets of the graph $\Gamma_{\mathcal{Q}}$ defined in Section 1 for quadrics $\mathcal{Q}$ in $\mathrm{PG}(n,2)$.

Theorems 3.A and 3.B are as follows.

Theorem 3.A. Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n,2)$ where $n \geq 5$ is odd. Let t be an integer such that $0 < t \leq \frac{n-3}{2}$, α be a t-space in $\mathcal{Q}$, and Π be a $(t+1)$-space meeting $\mathcal{Q}$ in exactly α. Let $\Gamma_{\mathcal{Q}}$ be as defined in Section 1. Then

$$S_t := \Pi \setminus \alpha \tag{3.1}$$

is a Godsil-McKay switching set of $\Gamma_{\mathcal{Q}}$ of size 2^{t+1}. Let $\Gamma_{\mathcal{Q},t}$ be the graph obtained by Godsil-McKay switching with switching set S_t. Then $\Gamma_{\mathcal{Q},t}$ is a strongly regular graph with the same parameters as $\Gamma_{\mathcal{Q}}$ (which are listed as in Table 1). Furthermore, if $\perp$ is the polarity of $\mathrm{PG}(n,2)$ induced by $\mathcal{Q}$, then

$$T_t := (\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \alpha^{\perp} \tag{3.2}$$

is the set of vertices in $\Gamma_{\mathcal{Q}}$ outside S_t which have exactly $\frac{1}{2}|S_t|$ neighbours in S_t.

Theorem 3.B. Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n,2)$ where $n \geq 5$ is odd. If $\mathcal{Q}$ is elliptic, then let t be an integer such that $0 < t \leq \frac{n-3}{2}$. If $\mathcal{Q}$ is hyperbolic, then let t be an integer such that $0 < t \leq \frac{n-5}{2}$. In $\mathrm{PG}(n,2)$ where $n \geq 5$ is odd, let $\mathcal{Q}$ be a non-singular quadric. Let α be a t-space in $\mathcal{Q}$. Let Π, Π' be distinct $(t+1)$-spaces meeting $\mathcal{Q}$ in exactly α such that the space spanned by Π and Π' meet $\mathcal{Q}$ in exactly α. Let $\Gamma_{\mathcal{Q}}$ be as defined in Section 1. Then

$$S_{t,t} := (\Pi \cup \Pi') \setminus \alpha \tag{3.3}$$

is a Godsil-McKay switching set of $\Gamma_{\mathcal{Q}}$. Let $\Gamma_{\mathcal{Q},t,t}$ be the graph obtained by Godsil-McKay switching with switching set $S_{t,t}$. Then $\Gamma_{\mathcal{Q},t,t}$ is a strongly regular graph with the same parameters as $\Gamma_{\mathcal{Q}}$ (these are listed as in Table 1). Furthermore, if $\perp$ is the polarity of $\mathrm{PG}(n, 2)$ induced by $\mathcal{Q}$, then

$$T_{t,t} = T_t \cup [((\Pi^{\perp} \triangle \Pi'^{\perp}) \setminus S_{t,t}) \setminus \mathcal{Q}] \tag{3.4}$$

is the set of vertices in $\Gamma_{\mathcal{Q}}$ outside $S_{t,t}$ which have exactly $|\frac{1}{2}S_{t,t}|$ neighbours in $S_{t,t}$, where $\triangle$ is the symmetric difference.

Remark. In both Theorems 3.A and 3.B, $t \leq \frac{n-3}{2}$ or $t \leq \frac{n-5}{2}$. This is a necessary and sufficient condition for the existence of α, Π and Π' by [10, Theorem 22.8.3].

Remark. In Theorem 3.B, by the dimension theorem for subspaces, the space $\langle \Pi, \Pi' \rangle$ spanned by Π and Π' is an $(t + 2)$-subspace. By [9, Theorem 3.1], there are exactly three planes through α in $\langle \Pi, \Pi' \rangle$. Let Π'' be the plane through α other than Π and Π'. Since $\mathcal{Q}$ is a quadric, either $\langle \Pi, \Pi' \rangle \cap \mathcal{Q} = \Pi''$ or $\langle \Pi, \Pi' \rangle \cap \mathcal{Q} = \alpha$ holds by [10, Theorem 22.8.3]. In the former case, since $\langle \Pi, \Pi' \rangle$ is one dimension higher than that of Π'', $(\Pi \cup \Pi') \setminus \alpha = \langle \Pi, \Pi' \rangle \setminus \Pi''$ is a Godsil-McKay switching set of $\Gamma_{\mathcal{Q}}$ by Theorem 3.A. The latter case is treated in Theorem 3.B.

Throughout this article, we will work under the assumptions of Theorems 3.A or 3.B. In particular, the symbols n,$\mathcal{Q}$, $\perp$, t, α, Π, Π', $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$, S_t, $S_{t,t}$, T_t and $T_{t,t}$ are preserved as defined in Theorems 3.A or 3.B.

We first check the size of switching sets described in Theorems 3.A and 3.B.

Lemma 3.1. *The size $|S_t|$ of S_t and the size $|S_{t,t}|$ of $S_{t,t}$ are respectively 2^{t+1} and 2^{t+2}.*

Proof. Since Π is a $(t + 1)$-space and α is a t-space,

$$|S_t| = |\Pi \setminus \alpha| = (2^{t+2} - 1) - (2^{t+1} - 1) = 2^{t+1}.$$

For $S_{t,t}$, because of $\Pi \cap \Pi' = \alpha$, we have

$$|S_{t,t}| = |(\Pi \cup \Pi') \setminus \alpha| = 2(2^{t+2} - 1) - (2^{t+1} - 1) - (2^{t+1} - 1) = 2^{t+2}.$$

□

We now determine the structure of the subgraphs of $\Gamma_{\mathcal{Q}}$ with vertex sets S_t and $S_{t,t}$ respectively.

Lemma 3.2. *The subgraph of $\Gamma_{\mathcal{Q}}$ with vertex set S_t is null.*

Proof. Since α is one dimension less than that of Π, a line in Π either lies in α or is tangent to α, and thus to $\mathcal{Q}$. In other words, no two vertices are joined in $\Gamma_{\mathcal{Q}}$. □

Lemma 3.3. *The subgraph of $\Gamma_{\mathcal{Q}}$ with vertex set $S_{t,t}$ is a regular subgraph of degree 2^{t+1}.*

Proof. Let x be a point in $S_{t,t}$. Without loss of generality, assume $x \in \Pi$. We count the number of neighbours of x. By the same argument used in Lemma 3.2, x is not adjacent to any vertex in $\Pi \setminus \alpha$.

Since the span of Π and Π' meets $\mathcal{Q}$ in exactly α by assumption, any line through x and a point in $\Pi' \setminus \alpha$ is an external line of $\mathcal{Q}$. In other words, the vertex is adjacent to any vertex in $\Pi' \setminus \alpha$. Since the size of $\Pi' \setminus \alpha$ is 2^{t+1}, the result follows. □

Lemma 3.4. *S_t is a Godsil-McKay switching set of $\Gamma_{\mathcal{Q}}$.*

Proof. By Lemma 3.2, the subgraph of $\Gamma_{\mathcal{Q}}$ with vertex set S_t is null.

By Lemma 3.1, $|S_t| = 2^{t+1}$. Let x be a point in $(\mathrm{PG}(n, 2) \setminus \mathcal{Q}) \setminus S_t$. It suffices to show that either any line joining x and a point of S_t meets $\mathcal{Q}$, or there are exactly 2^t or 2^{t+1} points y in S_t such that the line xy is an external line to $\mathcal{Q}$. Since any line in $\mathrm{PG}(n, 2)$ has exactly three points, any line through two external points of $\mathcal{Q}$ is either a tangent or an external line. Thus, it also suffices to show that either any line joining x and a point of S_t is an external line, or there are exactly 2^{t+1}, 2^t points y in S_t such that the line xy is tangent to $\mathcal{Q}$.

Suppose the line xy joining x and a point y of S_t is tangent to $\mathcal{Q}$. Since every line has only three points, the unique point z on xy other than x and y is a point of $\mathcal{Q}$. Let Σ be the $(t + 1)$-space spanned by z and α. Then Σ meets $\mathcal{Q}$ in the t-space α and at least one point not in α, namely z. By [10, Theorem 22.8.3], Σ either lies in $\mathcal{Q}$ or meets $\mathcal{Q}$ in exactly two t-spaces.

If Σ lies in $\mathcal{Q}$, then every line through x and a point of S_t is a tangent to $\mathcal{Q}$, and we are done.

If Σ meets $\mathcal{Q}$ in exactly two t-spaces, say α and α', then α and α' meet in a $(t - 1)$-space. Then a line xy' through x and a point y' of S_t is tangent to $\mathcal{Q}$ if and only if y' is in $S_t \cap \langle x, \alpha' \rangle$. Since

$$\begin{aligned} S_t \cap \langle x, \alpha' \rangle &= (\Pi \setminus \alpha) \cap \langle x, \alpha' \rangle \\ &= (\Pi \cap \langle x, \alpha' \rangle) \setminus (\alpha \cap \langle x, \alpha' \rangle) \\ &= \langle y, \alpha \cap \alpha' \rangle \setminus (\alpha \cap \alpha') \end{aligned}$$

has $(2^{t+1} - 1) - (2^t - 1) = 2^t$ points, the result follows. □

We need to make use of a property of $\perp$ to prove the following two lemmas. Recall from [10, Lemma 22.3.3] that, for any point $y \in \mathcal{Q}$, $y^\perp$ comprises the points on the tangents to $\mathcal{Q}$ at y and the lines in $\mathcal{Q}$ through y; for any point $y \notin \mathcal{Q}$, $y^\perp$ consists of the points on the tangents to $\mathcal{Q}$ through y.

Lemma 3.5. *The following inclusions hold:*

(1) $S_t \subset \Pi^\perp \subset \alpha^\perp$.
(2) $S_{t,t} \subset \Pi^\perp \triangle \Pi'^\perp \subset \alpha^\perp$.

Proof. Since α is a subset of Π and Π', by the definition of a polarity, $\Pi^\perp$ and $\Pi'^\perp$ are subsets of $\alpha^\perp$.

Since the line through any two points of Π is either a tangent of $\mathcal{Q}$ or a line of $\mathcal{Q}$. By [10, Lemma 22.3.3], Π is a subset of $\Pi^\perp$. Similarly, Π' is a subset $\Pi'^\perp$. Hence $S_t \subset \Pi^\perp$ and $S_{t,t} \subset \Pi^\perp \cup \Pi'^\perp$.

As stated in Theorem 3.B, the space spanned by Π and Π' meets $\mathcal{Q}$ in exactly α. Thus, any line through joining a point of $\Pi \setminus \alpha$ and a point of $\Pi' \setminus \alpha$ is an external line of $\mathcal{Q}$. By [10, Lemma 22.3.3], $\Pi \cap \Pi'^\perp = \emptyset$ and $\Pi^\perp \cap \Pi' = \emptyset$, and so $S_{t,t} \cap \Pi^\perp \cap \Pi'^\perp = \emptyset$. The result follows. $\square$

To determine T_t, we prepare a lemma about polarities in $\mathrm{PG}(n, 2)$.

Lemma 3.6. *Let ρ be a polarity of $\mathrm{PG}(n, 2)$. Let Σ be an $(m+1)$-space of $\mathrm{PG}(n, 2)$ where $0 \le m < n-1$. Let x be a point in $\mathrm{PG}(n, 2)$. Then exactly one of the following cases occurs.*

(1) x is in Σ^ρ.
(2) x is in $\pi^\rho \setminus \Sigma^\rho$ for exactly one m-space π in Σ.

Proof. By [9, Theorem 3.1], there are exactly $N = 2^{m+2} - 1$ m-spaces in Σ. Let $\pi_1, \pi_2, \cdots, \pi_N$ be the m-spaces contained in Σ. Since ρ is a polarity, π_i^ρ, $i = 1, 2, \cdots, N$, are $(n-1-m)$-spaces containing the $(n-2-m)$-space Σ^ρ. For distinct $i, j \in \{1, 2, \cdots, N\}$, $\pi_i^\rho \cap \pi_j^\rho = \langle \pi_i, \pi_j \rangle^\rho = \Sigma^\rho$. Thus, the number of points in $\bigcup_{i=1}^N \pi_i^\rho$ is $|\Sigma^\rho| + \sum_{i=1}^N |\pi_i^\rho \setminus \Sigma^\rho| = (2^{n-1-m} - 1) + N[(2^{n-m} - 1) - (2^{n-1-m} - 1)] = 2^{n+1} - 1$, which is the number of points in $\mathrm{PG}(n, 2)$. Now, the result follows. $\square$

Lemma 3.7. *Let x be a point not in $\mathcal{Q}$. Then exactly one of the following cases occurs.*

(1) x is in $\Pi^\perp$; any line joining x and a point in S_t is not an external line of $\mathcal{Q}$.
(2) x is in $\pi^\perp \setminus \Pi^\perp$ for exactly one t-space $\pi \neq \alpha$ in Π; the line xy through x and a point $y \in S_t$ is an external line of $\mathcal{Q}$ if and only if $y \notin \pi$. Furthermore, there are 2^t such points y.
(3) x is in $\alpha^\perp \setminus \Pi^\perp$; any line through a point of S_t and x is an external line of $\mathcal{Q}$.

Proof. By [9, Theorem 3.1], there are exactly $N = 2^{t+2} - 1$ t-spaces in Π. Let $\pi_0, \pi_1, \cdots, \pi_{N-1}$ be the t-spaces contained in Π. Without loss of generality, assume $\pi_0 = \alpha$.

Let x be a point not in $\mathcal{Q}$. By Lemma 3.6, x is either in $\Pi^\perp$ or in $\pi_i^\perp \setminus \Pi^\perp$ for exactly one $i \in \{0, 1, \cdots, N-1\}$.

(1) Suppose $x \in \Pi^\perp$. Then $x \in y^\perp$ for all $y \in \Pi$. Thus the line through x and a point $y \in \Pi \setminus \mathcal{Q}$ is a tangent to $\mathcal{Q}$. In order words, no point y in $\Pi \setminus \mathcal{Q} = \Pi \setminus \alpha = S_t$ satisfies the condition that the line xy is an external line of $\mathcal{Q}$.

(2) Suppose $x \in \pi_i^\perp \setminus \Pi^\perp$ for exactly one $i \neq 0$. By a similar argument, it follows that no point y in $\pi_i \setminus \mathcal{Q} = \pi_i \setminus \alpha$ satisfies the condition that the line xy is an external line of $\mathcal{Q}$. Suppose there exists $z \in S_t \setminus \pi_i$ such that the line through xz is not an external line of $\mathcal{Q}$. Since every line contains exactly three points, that line is tangent to $\mathcal{Q}$ and thus x is in $z^\perp$. Then $x \in z^\perp \cap \pi_i^\perp = \langle z, \pi_i \rangle^\perp = \Pi^\perp$. This gives a contradiction, and thus the line through x and a point $y \in S_t$ is an external line of $\mathcal{Q}$ if and only if $y \notin \pi_i$. Since $\alpha \cap \pi_i$ is a $(t-1)$-space, there are exactly

$$|S_t \setminus \pi_i| = |(\Pi \setminus \pi_i) \setminus (\alpha \setminus \pi_i)| = [(2^{t+2} - 1) - (2^{t+1} - 1)] - [(2^{t+1} - 1) - (2^t - 1)] = 2^t$$

points y in S_t such that the line xy is an external line of $\mathcal{Q}$.

(3) Suppose $x \in \alpha^\perp \setminus \Pi^\perp$. Suppose there exists $y \in S_t$ such that the line xy is not an external line of $\mathcal{Q}$. Then that line is a tangent to $\mathcal{Q}$ and thus $x \in y^\perp$. Then $x \in y^\perp \cap \alpha^\perp = \langle y, \alpha \rangle^\perp = \Pi^\perp$. This gives a contradiction and the result follows.

□

We are ready to give a proof of Theorem 3.A.

Proof of Theorem 3.A. By Lemma 3.4, S_t is a Godsil-McKay switching set for $\Gamma_{\mathcal{Q}}$.

By Godsil and McKay [6], $\Gamma_{\mathcal{Q},t}$ has a same adjacency spectrum as $\Gamma_{\mathcal{Q}}$. Since $\Gamma_{\mathcal{Q}}$ is a strongly regular graph, $\Gamma_{\mathcal{Q},t}$ is also a strongly regular graph with the same parameters (see the first three paragraphs on [4, Subsection 14.5.1]), where the parameters are listed as in Table 1 on 2.

By the definition of T_t and Lemma 3.7,

$$T_t = (\mathrm{PG}(n,2) \setminus \mathcal{Q}) \cap [(\bigcup_{\pi \neq \alpha} \pi^\perp \setminus \Pi^\perp) \setminus S_t]$$

where π runs over all t-space of Π except α. By Lemma 3.6, $\left(\bigcup_{\pi \neq \alpha} \pi^\perp \setminus \Pi^\perp\right) = \mathrm{PG}(n,2) \setminus \alpha^\perp$. Since S_t is in $\alpha^\perp$, the result follows. □

With Lemma 3.7, we prove Theorem 3.B.

Proof of Theorem 3.B. By Lemma 3.3, the subgraph of $\Gamma_{\mathcal{Q}}$ with vertex set $S_{t,t}$ is a regular subgraph of degree 2^{t+1}.

Let x be a point in $(\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus S_{t,t}$. By Lemma 3.6, one of the following cases occurs.

(1) $x \in \Pi^\perp$ and $x \in \Pi'^\perp$.
(2) $x \in \Pi^\perp$ and $x \in \alpha^\perp \setminus \Pi'^\perp$.
(3) $x \in \Pi^\perp$ and $x \in \pi' \setminus \Pi'^\perp$ for some t-space $\pi' \neq \alpha$ of Π'.
(4) $x \in \alpha^\perp \setminus \Pi^\perp$ and $x \in \Pi'^\perp$.

(5) $x \in \alpha^\perp \setminus \Pi^\perp$ and $x \in \alpha^\perp \setminus \Pi'^\perp$.
(6) $x \in \pi^\perp \setminus \Pi^\perp$ for some t-space $\pi \neq \alpha$ of Π, and $x \in \Pi'^\perp$.
(7) $x \in \pi^\perp \setminus \Pi^\perp$ for some t-space $\pi \neq \alpha$ of Π, and $x \in \pi'^\perp \setminus \Pi'^\perp$ for some t-space $\pi' \neq \alpha$ of Π'.

Note that case (3) never occurs. Indeed, since α is a subset of Π, we have $\Pi^\perp \subset \alpha^\perp$. Indeed, if x is in $\Pi^\perp$, then x is in $\alpha^\perp$ by Lemma 3.5. By Lemma 3.6, $\alpha^\perp = (\alpha^\perp \setminus \Pi'^\perp) \cup \Pi'^\perp$ is disjoint from $\pi'^\perp \setminus \Pi'^\perp$. Similarly, case (6) never occurs.

For the remaining cases, by Lemma 3.7, there are respectively $0+0=0$, $0+2^{t+1} = 2^{t+1}$, $2^{t+1}+0 = 2^{t+1}$, $2^{t+1}+2^{t+1} = 2^{t+2}$, $2^t + 2^t = 2^{t+1}$ points y in $(\Pi \cup \Pi') \setminus \alpha$ such that the line xy is an external line of $\mathcal{Q}$. Therefore, $S_{t,t}$ is a Godsil-McKay switching set of $\Gamma_\mathcal{Q}$ because we have $|S_{t,t}| = 2^{t+2}$ by Lemma 3.1.

Similarly, by Godsil and McKay [6], $\Gamma_{\mathcal{Q},t,t}$ has a same adjacency spectrum as $\Gamma_\mathcal{Q}$. By [4], $\Gamma_{\mathcal{Q},t,t}$ is also a strongly regular graph with the same parameters, where the parameters are listed as in Table 1.

The vertex x is adjacent to none or all vertices in $S_{t,t}$, if and only if case (1) or (5) holds, if and only if $x \in \alpha^\perp \setminus (\Pi^\perp \triangle \Pi'^\perp)$. The result for $T_{t,t}$ now follows. $\square$

4. Some codewords of the switched graphs

We shall use the same notation n, $\mathcal{Q}$, $\perp$, t, α, Π, Π', $\Gamma_\mathcal{Q}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$, S_t, $S_{t,t}$, T_t, $T_{t,t}$, as described in Theorems 3.A or 3.B. Recall from Section 2 that $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$ are respectively the code of $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$. In this section, we aim to prove $v^{S_t}, v^{T_t} \in C(\Gamma_{\mathcal{Q},t})$ and $v^{S_{t,t}}, v^{T_{t,t}} \in C(\Gamma_{\mathcal{Q},t,t})$.

Since we will need frequently the number of external lines of a non-singular quadric through a point, we give these numbers in the following lemma for ease of reference.

Lemma 4.1. *Let Q_m be a non-singular quadric in $\mathrm{PG}(m,2)$. Let x be a point not in Q_m. If m is odd, there are $2^{m-2} \pm 2^{\frac{m-3}{2}}$ external lines through x, where the upper sign of $\pm$ is taken when if Q_m is elliptic, and otherwise if Q_m is hyperbolic. If m is even, there are 0 or $2^{m-2}-1$ external lines through x, depending on whether x is the nucleus of Q_m or not.*

Proof. When m is odd, Q_m has $2^m \mp 2^{\frac{m-1}{2}} - 1$ points (see Table 1). Thus, there are $|\mathrm{PG}(m,2)| - |Q_m| = 2^m \pm 2^{(m-1)/2}$ non-quadric points. By [10, Theorem 22.6.6], these non-quadric points are in the same orbit under the subgroup $\mathrm{Aut}(Q_m)$ of the automorphism group of $\mathrm{PG}(m,2)$ which fixes Q_m. Thus, through each point, there are a same number of external lines. The result follows because there are $\frac{1}{3}(2^{m-2})(2^{\frac{m+1}{2}} \pm 1)(2^{\frac{m-1}{2}} \pm 1)$ external lines in $\mathrm{PG}(m,2)$ [10, Lemma 22.8.1].

Similarly, when m is even, there are 2^m non-quadric points. Recall from Section 2 that all line through the nucleus of Q_n is tangent to Q_n. By [10, Theorem 22.6.6], any non-quadric points, other than the nucleus, are in the same orbit under $\mathrm{Aut}(Q_m)$. The result follows similarly because there are $\frac{1}{3}(2^{m-2})(2^m - 1)$ external lines in $\mathrm{PG}(m, 2)$ [10, Lemma 22.8.1]. $\square$

In the following lemma, whenever we use the signs $\pm$ or $\mp$, the upper sign is always taken when $\mathcal{Q}$ is elliptic, and lower sign is always taken when $\mathcal{Q}$ is hyperbolic.

Lemma 4.2. *There is an external line l of $\mathcal{Q}$ such that l and $\alpha^\perp$ are disjoint.*

Proof. Let x be a non-quadric point not in $\alpha^\perp$. Let Σ be the $(n-t)$-space spanned by $\{x, \alpha^\perp\}$. If there is an external line of $\mathcal{Q}$ through x but not in Σ, then such a line will be disjoint from $\alpha^\perp$ and we are done.

We first consider the case for $t = 1$. By Lemma 3.6, $x \in \pi^\perp$ for a unique 1-space π of Π. Since x is not in $\alpha^\perp$, we have $\pi \neq \alpha$ and so $\pi \cap \alpha$ is a point of $\mathcal{Q}$. By Theorem [10, Theorem 22.7.2], $\Sigma \cap \mathcal{Q}$ is a parabolic quadric. If x is the nucleus of $\Sigma \cap \mathcal{Q}$, then there is no external line (of both $\mathcal{Q}$ and $\Sigma \cap \mathcal{Q}$) in Σ and through x, as desired. If x is not the nucleus of $\Sigma \cap \mathcal{Q}$, then there are

$$2^{n-3} - 1$$

external lines in Σ and through x by Lemma 4.1. Since n is not less than 5, this number is less than the number of external lines in $\mathrm{PG}(n, 2)$ through x found in Lemma 4.1, and thus there is an external line of $\mathcal{Q}$ through x but not in Σ, as desired.

Similarly, in case $t = 2$, Σ is an $(n-2)$-space meeting $\mathcal{Q}$ in a line cone $\Pi_1 \mathcal{Q}^-_{n-4}$ over an elliptic quadric $\mathcal{Q}^-_{n-4}$ if $\mathcal{Q}$ is elliptic, and a line cone $\Pi_1 \mathcal{Q}^+_{n-4}$ over a hyperbolic quadric $\mathcal{Q}^+_{n-4}$ if $\mathcal{Q}$ is hyperbolic. Since $\Pi_1 \mathcal{Q}^-_{n-4}$ has

$$|\Pi_1 \mathcal{Q}^-_{n-4}| = 3 + 4(2^{n-4} - 2^{\frac{n-5}{2}} - 1) = 2^{n-2} - 2^{(n-1)/2} - 1 \tag{4.1}$$

points and $\Pi_1 \mathcal{Q}^+_{n-4}$ has

$$|\Pi_1 \mathcal{Q}^+_{n-4}| = 3 + 4(2^{n-4} + 2^{\frac{n-5}{2}} - 1) = 2^{n-2} + 2^{(n-1)/2} - 1 \tag{4.2}$$

points, there are

$$|\Sigma| - |\Pi_1 \mathcal{Q}^\epsilon_{n-4}| = 2^{n-2} \pm 2^{\frac{n-1}{2}}, \epsilon \in \{-, +\}$$

non-quadric points in the $(n-2)$-space Σ. Thus, there are at most

$$\frac{2^{n-2} \pm 2^{\frac{n-1}{2}}}{2} = 2^{n-3} \pm 2^{\frac{n-3}{2}}$$

external lines in Σ through x. Since this number is less than the number of external lines in $\mathrm{PG}(n, 2)$ through x found in Lemma 4.1, there is an external line of $\mathcal{Q}$ through x but not in Σ, as desired.

We now consider the case for $t > 2$. By [9, Theorem 3.1], through x, there are

$$2^{n-t} - 1$$

lines in the $(n-t)$-space Σ. Since this number is less than the number of external lines through x found in Lemma 4.1, there is an external line of $\mathcal{Q}$ through x but not in Σ, as desired. $\square$

Lemma 4.3. *The vector v^{S_t} is in $C(\Gamma_{\mathcal{Q},t})$. The vector $v^{S_{t,t}}$ is in $C(\Gamma_{\mathcal{Q},t,t})$.*

Proof. Let $l = \{x_1, x_2, x_3\}$ be an external line of $\mathcal{Q}$ such that l and $\alpha^\perp$ are disjoint. This exists by Lemma 4.2.

For each $i = 1, 2, 3$, let r_i, $\dot{r}_i$ and $\ddot{r}_i$ respectively be the row of the adjacency matrices of $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$ corresponding to x_i. Then r_i is the characteristic vector of $(\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus x_i^\perp$. By Lemma 3.6, $\mathrm{PG}(n,2) \setminus \mathcal{Q}$ is the disjoint union of $l^\perp \setminus \mathcal{Q}$, $(x_1^\perp \setminus l^\perp) \setminus \mathcal{Q}$, $(x_2^\perp \setminus l^\perp) \setminus \mathcal{Q}$ and $(x_3^\perp \setminus l^\perp) \setminus \mathcal{Q}$. Since $l^\perp$ is a subset of $x_i^\perp$ for $i = 1, 2, 3$, we have

$$r_1 + r_2 + r_3 = 0 \tag{4.3}$$

in $\mathbb{F}_2^{|V_{\mathcal{Q}}|}$.

Since l is disjoint from $\alpha^\perp$, we have $l \subset T_t$ and $l \subset T_{t,t}$. By the definitions of $\Gamma_{\mathcal{Q},t}$ and $\Gamma_{\mathcal{Q},t,t}$, for each $i = 1, 2, 3$, we have

$$\dot{r}_i = r_i + v^{S_t} \tag{4.4}$$

and

$$\ddot{r}_i = r_i + v^{S_{t,t}}. \tag{4.5}$$

By (4.3) and (4.4), $\dot{r}_1 + \dot{r}_2 + \dot{r}_3 = v^{S_t}$ and so v^{S_t} is a codeword of $C(\Gamma_{\mathcal{Q},t})$. Similarly, $v^{S_{t,t}}$ is a codeword of $C(\Gamma_{\mathcal{Q},t,t})$ because $\ddot{r}_1 + \ddot{r}_2 + \ddot{r}_3 = v^{S_{t,t}}$ by (4.3) and (4.5). $\square$

The purpose and proof of following lemma are similar to those of Lemma 4.2, and we apply this lemma to prove $v^{T_t} \in C(\Gamma_{\mathcal{Q},t})$ and $v^{T_{t,t}} \in C(\Gamma_{\mathcal{Q},t,t})$.

Lemma 4.4. *Let x be a non-quadric point in $\alpha^\perp$. Then there is an external line l of $\mathcal{Q}$ through x such that l is tangent to $\alpha^\perp$ at x.*

Proof. To prove the lemma, it suffices to show some of external line through x does not lie in $\alpha^\perp$.

We first consider the case for $t = 1$. Then $\alpha^\perp$ is an $(n-2)$-space. By [10, Theorem 22.7.2], $\alpha^\perp \cap \mathcal{Q}$ is a line cone $\Pi_1 \mathcal{Q}^-_{n-4}$ over an elliptic quadric $\mathcal{Q}^-_{n-4}$ if $\mathcal{Q}$ is elliptic, and a line cone $\Pi_1 \mathcal{Q}^+_{n-4}$ over a hyperbolic quadric $\mathcal{Q}^+_{n-4}$ if $\mathcal{Q}$ is hyperbolic. For either $\mathcal{Q}$ elliptic or hyperbolic, the set of points y's in $\alpha^\perp \cap \mathcal{Q}$ such that the line xy is tangent to $\alpha^\perp \cap \mathcal{Q}$ forms a line cone $\Pi_1 \mathcal{Q}_{n-5}$ over a parabolic quadric $\mathcal{Q}_{n-5}$. Since $\mathcal{Q}_{n-5}$ has $2^{n-5} - 1$ points [9, Theorem 5.21], there are

$$|\Pi_1 \mathcal{Q}_{n-5}| = [3 + 4(2^{n-5} - 1)] = 2^{n-3} - 1$$

tangents in $\alpha^\perp$ through x. Using (4.1) and (4.2), there are

$$\frac{|\alpha^\perp \cap \mathcal{Q}| - |\Pi_1 \mathcal{Q}_{n-5}|}{2} = 2^{n-4} \mp 2^{(n-3)/2}$$

secants in $\alpha^\perp$ through x. Since there are $2^{n-2} - 1$ lines in $\alpha^\perp$ through x [9, Theorem 3.1], there are

$$(2^{n-2} - 1) - (2^{n-4} \mp 2^{(n-3)/2}) - (2^{n-3} - 1) = 2^{n-4} \pm 2^{(n-3)/2} \tag{4.6}$$

external lines of $\mathcal{Q}$ in $\alpha^\perp$ through x, where the upper signs of $\pm$ and $\mp$ are taken if $\mathcal{Q}$ is elliptic and the lower sign if $\mathcal{Q}$ is hyperbolic. Since the number in (4.6) is less than the number of external lines through x found in Lemma 4.1, there is an external line of $\mathcal{Q}$ through x but not in $\alpha^\perp$, as desired.

We now consider the case for $t > 1$. By [9, Theorem 3.1], through x, there are only

$$2^{n-1-t} - 1$$

lines of the $(n-1-t)$-space $\alpha^\perp$. Since this number is less than the number of external lines through x found in Lemma 4.1, there is an external line of $\mathcal{Q}$ through x but not in $\alpha^\perp$, as desired. $\square$

Lemma 4.5. *The vector v^{T_t} is in $C(\Gamma_{\mathcal{Q},t})$. The vector $v^{T_{t,t}}$ is in $C(\Gamma_{\mathcal{Q},t,t})$.*

Proof. Let $x_1 \in S_t$. Note that $x_1 \in \alpha^\perp$. Take an external line $l = \{x_1, x_2, x_3\}$ of $\mathcal{Q}$ through x such that l is tangent to $\alpha^\perp$ at x_1. It exists by Lemma 4.4.

For each $i = 1, 2, 3$, let r_i, $\dot{r}_i$ and $\ddot{r}_i$ respectively be the row of the adjacency matrices of $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$ corresponding to x_i. By the same argument used in the proof of Lemma 4.3, we have

$$r_1 + r_2 + r_3 = 0. \tag{4.7}$$

Because of $x_1 \in S_t \subset S_{t,t}$, by the definitions of $\Gamma_{\mathcal{Q},t}$ and $\Gamma_{\mathcal{Q},t,t}$, we have

$$\dot{r}_1 = r_1 + v^{T_t} \tag{4.8}$$

and

$$\ddot{r}_1 = r_1 + v^{T_{t,t}}. \tag{4.9}$$

Since x_2, x_3 are not in $\alpha^\perp$, they are in T_t and $T_{t,t}$ by (3.2) and (3.4). So, for $i = 2, 3$, we have

$$\dot{r}_i = r_i + v^{S_t} \tag{4.10}$$

and

$$\ddot{r}_i = r_i + v^{S_{t,t}}. \tag{4.11}$$

By (4.7), (4.8) and (4.10), $\dot{r_1} + \dot{r_2} + \dot{r_3} = v^{T_t}$ and so v^{T_t} is a codeword of $C(\Gamma_{\mathcal{Q},t})$. Similarly, $v^{T_{t,t}}$ is a codeword of $C(\Gamma_{\mathcal{Q},t,t})$ because $\ddot{r_1} + \ddot{r_2} + \ddot{r_3} = v^{T_{t,t}}$ by (4.7), (4.9) and (4.11). □

5. The minimum word of $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$

In this section, we use the same notation n, $\mathcal{Q}$, $\perp$, t, α, Π, Π', $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$, S_t, $S_{t,t}$, T_t and $T_{t,t}$ as in Section 4, except requiring $n \geq 7$.

Let

$$C_t = \left\langle C(\Gamma_{\mathcal{Q},t}), v^{S_t}, v^{T_t} \right\rangle \tag{5.1}$$

and

$$C_{t,t} = \left\langle C(\Gamma_{\mathcal{Q},t,t}), v^{S_{t,t}}, v^{T_{t,t}} \right\rangle \tag{5.2}$$

In this section, we aim to prove the minimum word of C_t and $C_{t,t}$ are respectively v^{S_t} and $v^{S_{t,t}}$. This will give the minimum word of $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$ once we prove that $C_t = C(\Gamma_{\mathcal{Q},t})$ and $C_{t,t} = C(\Gamma_{\mathcal{Q},t,t})$ in the next section.

Lemma 5.1. *Let $w \in C(\Gamma_{\mathcal{Q}})$. Then $\mathrm{wt}(w + v^{S_t}) > 2^{t+1}$ and $\mathrm{wt}(w + v^{S_{t,t}}) > 2^{t+2}$.*

Proof. From Table 3, if $\mathcal{Q}$ is elliptic, the weight $\mathrm{wt}(w)$ of w satisfies $\mathrm{wt}(w) \geq 2^{n-1}$. By Lemma 3.1, $\mathrm{wt}(v^{S_t}) = 2^{t+1}$ and $\mathrm{wt}(v^{S_{t,t}}) = 2^{t+2}$. So,

$$\mathrm{wt}(w + v^{S_t}) \geq \mathrm{wt}(w) - \mathrm{wt}(v^{S_t}) = 2^{n-1} - 2^{t+1},$$

$$\mathrm{wt}(w + v^{S_{t,t}}) \geq \mathrm{wt}(w) - \mathrm{wt}(v^{S_{t,t}}) = 2^{n-1} - 2^{t+2}.$$

Since we have assumed $n \geq 7$ in this section and we have $t \leq \frac{n-3}{2}$ under the assumption in Theorems 3.A and 3.B, it is straightforward to verify that $\mathrm{wt}(w + v^{S_t}) > 2^{t+1}$ and $\mathrm{wt}(w + v^{S_{t,t}}) > 2^{t+2}$.

From Table 4, if $\mathcal{Q}$ is hyperbolic, then $\mathrm{wt}(w) \geq 2^{n-1} - 2^{\frac{n-1}{2}}$. Similarly, since $n \geq 7$, it is straightforward to verify that $\mathrm{wt}(w + v^{S_t}) > 2^{t+1}$ and $\mathrm{wt}(w + v^{S_{t,t}}) > 2^{t+2}$ with t in the range stated in Theorems 3.A and 3.B. □

For any subset U of points of $\mathrm{PG}(n, 2)$, denoted by $\widehat{U}$ the set $U \setminus \mathcal{Q}$. Recall that whenever we use the signs $\pm$ or $\mp$, the upper sign is always taken when $\mathcal{Q}$ is elliptic, and lower sign is always taken when $\mathcal{Q}$ is hyperbolic.

Lemma 5.2. *(1) $|\widehat{\alpha^\perp}| = 2^{n-t-1} \pm 2^{\frac{n-1}{2}}$.*

(2) Let $A = (\Pi^\perp \triangle \Pi'^\perp) \setminus S_{t,t}$. Then $|\widehat{A}| = 2^{n-t-2} \pm 2^{\frac{n-1}{2}} - 2^{t+2}$.

(3) Let Σ be an $(n-1)$-space. Then exactly one of the following holds:

(a) $\Sigma \cap \mathcal{Q} = \mathcal{Q}_{n-1}$; $|\widehat{\Sigma}| = 2^{n-1}$.

(b) $\Sigma \cap \mathcal{Q} = \Pi_0 \mathcal{Q}_{n-2}$ where $\mathcal{Q}_{n-2}$ and $\mathcal{Q}$ are both elliptic or both hyperbolic; $|\widehat{\Sigma}| = 2^{n-1} \pm 2^{\frac{n-1}{2}}$.

Proof. (1) Since α is in $\mathcal{Q}$, by [10, Theorem 22.8.3], $\alpha^\perp \cap \mathcal{Q}$ is a cone $\Pi_t Q_{n-2t-2}$ where Q_{n-2t-2} is elliptic if $\mathcal{Q}$ is elliptic, and is hyperbolic otherwise. By (2.1) and Table 1, we have

$$|\alpha^\perp \cap \mathcal{Q}| = (2^{t+1}-1) + 2^{t+1}(2^{n-2t-2} \mp 2^{\frac{n-2t-3}{2}} - 1) = 2^{n-t-1} \mp 2^{\frac{n-1}{2}} - 1.$$

Since $\alpha^\perp$ is an $(n-t-1)$-space, it follows that

$$\begin{aligned}|\widehat{\alpha^\perp}| =& |\alpha^\perp| - |\alpha^\perp \cap \mathcal{Q}| \\ =& (2^{n-t}-1) - [2^{n-t-1} \mp 2^{\frac{n-1}{2}} - 1] = 2^{n-t-1} \pm 2^{\frac{n-1}{2}}.\end{aligned}$$

(2) Similar to (1), we have

$$\begin{aligned}|\widehat{\Pi^\perp}| =& |\Pi^\perp| - |\Pi^\perp \cap \mathcal{Q}| = |\Pi^\perp| - |\Pi_t Q_{n-2t-3}| \\ =& (2^{n-t-1}-1) - [(2^{t+1}-1) + 2^{t+1}(2^{n-2t-3}-1)] = 2^{n-t-2}\end{aligned}$$

and

$$\begin{aligned}|\widehat{\langle \Pi, \Pi' \rangle^\perp}| =& |\, \langle \Pi, \Pi' \rangle^\perp \,| - |(\langle \Pi, \Pi' \rangle^\perp) \cap \mathcal{Q}| = |\, \langle \Pi, \Pi' \rangle^\perp \,| - |\Pi_t \mathcal{Q}_{n-2t-4}| \\ =& (2^{n-t-2}-1) - [(2^{t+1}-1) + 2^{t+1}(2^{n-2t-4} \pm 2^{\frac{n-2t-5}{2}} - 1)] \\ =& 2^{n-t-3} \mp 2^{\frac{n-3}{2}}.\end{aligned}$$

where $\mathcal{Q}_{n-2t-4}$ is hyperbolic if $\mathcal{Q}$ is elliptic; $\mathcal{Q}_{n-2t-4}$ is elliptic if $\mathcal{Q}$ is hyperbolic. Recall from Lemma 3.5, $S_{t,t} \subset \Pi^\perp \triangle \Pi'^\perp$. Now using Lemma 3.1, we deduce

$$|\widehat{A}| = |\widehat{\Pi^\perp}| + |\widehat{\Pi'^\perp}| - 2|\widehat{\langle \Pi, \Pi' \rangle^\perp}| - |S_{t,t}| = 2^{n-t-2} \pm 2^{\frac{n-1}{2}} - 2^{t+2}.$$

(3) By [10, Theorem 22.8.5], $\Sigma \cap \mathcal{Q}$ is either (a) $\mathcal{Q}_{n-1}$ or (b) $\Pi_0 \mathcal{Q}_{n-2}$ where $\mathcal{Q}_{n-2}$ and $\mathcal{Q}$ are both elliptic or both hyperbolic. The result follows by (2.1) and Table 1.

$\square$

Lemma 5.3. *The size of T_t and $T_{t,t}$ are respectively $|T_t| = 2^n - 2^{n-t-1}$ and $|T_{t,t}| = 2^n \pm 2^{\frac{n-1}{2}} - 2^{n-t-2} - 2^{t+2}$. Furthermore, the following holds:*

(1) $|T_t| > 2^{t+1}$.
(2) $|T_t \triangle S_t| > 2^{t+1}$.
(3) $|T_{t,t}| > 2^{t+2}$.
(4) $|T_{t,t} \triangle S_{t,t}| > 2^{t+2}$.

Proof. Using (3.2) and Lemma 5.2(1), we obtain

$$|T_t| = |\mathrm{PG}(n,2)| - |\mathcal{Q}| - |\widehat{\alpha^\perp}| = (2^{n+1}-1) - (2^n \mp 2^{\frac{n-1}{2}} - 1) - (2^{n-t-1} \pm 2^{\frac{n-1}{2}}) = 2^n - 2^{n-t-1}.$$

Since $0 < t \leq \frac{n-3}{2}$, we have

$$|T_t| - 2^{t+1} = 2^{n-t-1}(2^{t+1} - 1) - 2^{t+1} > 3 \cdot 2^{n-t-1} - 2^{t+1} > 0.$$

So, $|T_t| > 2^{t+1}$.

Using (3.4) and Lemma 5.2(2), we have

$$|T_{t,t}| = |T_t| + |\widehat{A}| = 2^n - 2^{n-t-2} \pm 2^{\frac{n-1}{2}} - 2^{t+2}$$

where $A = (\Pi^{\perp} \triangle \Pi'^{\perp}) \setminus S_{t,t}$. Because of $t > 0$, we have

$$|T_{t,t}| - 2^{t+2} = 2^{n-t-2}(2^{t+2} - 1) \pm 2^{\frac{n-1}{2}} - 2^{t+3} \geq 7 \cdot 2^{n-t-2} \pm 2^{\frac{n-1}{2}} - 2^{t+3}.$$

When $\mathcal{Q}$ is elliptic, $t \leq \frac{n-3}{2}$ and so

$$7 \cdot 2^{n-t-2} + 2^{\frac{n-1}{2}} - 2^{t+3} > 0.$$

When $\mathcal{Q}$ is hyperbolic, $t \leq \frac{n-5}{2}$ and so

$$7 \cdot 2^{n-t-2} - 2^{\frac{n-1}{2}} - 2^{t+3} \geq 7 \cdot 2^{\frac{n-1}{2}} - 2^{\frac{n-1}{2}} - 2^{t+3} > 0.$$

In both cases, $|T_{t,t}| > 2^{t+2}$. The results of $T_t \triangle S_t$ and $T_{t,t} \triangle S_{t,t}$ follow because of $T_t \cap S_t = \emptyset$ and $T_{t,t} \cap S_{t,t} = \emptyset$. $\square$

Lemma 5.4. *Let $R = (\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \Sigma$ for some $(n-1)$-space Σ of $\mathrm{PG}(n,2)$. Then the following holds:*

(1) $|R \triangle T_t| > 2^{t+1}$.
(2) $|R \triangle T_t \triangle S_t| > 2^{t+1}$.
(3) $|R \triangle T_{t,t}| > 2^{t+2}$.
(4) $|R \triangle T_{t,t} \triangle S_{t,t}| > 2^{t+2}$.

Proof. The complement R^c of R in $\mathrm{PG}(n,2) \setminus \mathcal{Q}$ is

$$R^c = \widehat{\Sigma}.$$

Let $A := ((\Pi^{\perp} \triangle \Pi'^{\perp}) \setminus S_{t,t}) \setminus \mathcal{Q}$. By (3.2) and (3.4), we have

$$\begin{aligned} T_t{}^c &= \widehat{\alpha^{\perp}}, \\ T_{t,t} &= T_t \cup A. \end{aligned} \tag{5.3}$$

Recall for any subsets U_1, U_2, U_3 of $\mathrm{PG}(n,2) \setminus \mathcal{Q}$, we have $U_1 \triangle U_2 = U_1^c \triangle U_2^c$; $(U_1 \cup U_2)^c = U_1^c \cap U_2^c$; $(U_1 \triangle U_2) \triangle U_3 = U_1 \triangle (U_2 \triangle U_3)$; $U_1 \triangle U_2 \supset U_1 \setminus U_2$, and equality holds if and only if $U_1 \subset U_2$. Further because of $S_t, S_{t,t} \subset \alpha^{\perp}$ by Lemma 3.5 and

$S_{t,t} \cap A = \emptyset$, we have

$$
\begin{aligned}
R\triangle T_t &= \widehat{\Sigma}\triangle\widehat{\alpha^\perp} \supset \widehat{\Sigma} \setminus \widehat{\alpha^\perp};\\
R\triangle T_t\triangle S_t &= (\widehat{\Sigma}\triangle\widehat{\alpha^\perp})\triangle S_t = \widehat{\Sigma}\triangle(\widehat{\alpha^\perp} \setminus S_t) \supset \widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus S_t) \supset \widehat{\Sigma} \setminus \widehat{\alpha^\perp};\\
R\triangle T_{t,t} &= \widehat{\Sigma}\triangle(\widehat{\alpha^\perp} \cap \widehat{A^c}) = \widehat{\Sigma}\triangle(\widehat{\alpha^\perp} \setminus \widehat{A}) \supset \widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A});\\
R\triangle T_{t,t}\triangle S_{t,t} &= \widehat{\Sigma}\triangle[(\widehat{\alpha^\perp} \setminus \widehat{A}) \setminus S_{t,t}] \supset \widehat{\Sigma} \setminus [(\widehat{\alpha^\perp} \setminus \widehat{A}) \setminus S_{t,t}] \supset \widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A}).
\end{aligned}
\tag{5.4}
$$

Thus, it suffices to show (i) $|\widehat{\Sigma} \setminus \widehat{\alpha^\perp}| > 2^{t+1}$ and (ii) $|\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A})| > 2^{t+2}$ for t within the range mentioned in Theorems 3.A and 3.B. By [10, Theorem 22.8.3], $\Sigma \cap \mathcal{Q}$ is either (a) a parabolic quadric $\mathcal{Q}_{n-1}$, or (b) a point cone $\Pi_0\mathcal{Q}_{n-2}$ where $\mathcal{Q}_{n-2}$ and $\mathcal{Q}$ are both elliptic or both hyperbolic.

(a) If $\Sigma \cap \mathcal{Q} = Q_{n-1}$, then by [10, Theorem 22.7.2], we have $\Sigma^\perp \notin \mathcal{Q}$ and so $\Sigma^\perp \notin \alpha$. By the definition of a polarity, we have $\alpha^\perp \not\subset \Sigma$. Since Σ is a hyperplane and $\alpha^\perp$ is an $(n-1-t)$-space, $\Sigma \cap \alpha^\perp$ is a $(n-2-t)$-space.
(i) By Lemma 5.2(3a) and $0 < t \le \frac{n-3}{2}$, we have

$$
\begin{aligned}
|\widehat{\Sigma} \setminus \widehat{\alpha^\perp}| - 2^{t+1} \ge& |\widehat{\Sigma}| - |\Sigma \cap \alpha^\perp| - 2^{t+1}\\
=& 2^{n-1} - (2^{n-t-1} - 1) - 2^{t+1} = 2^{n-t-1}(2^t - 1) + 1 - 2^{t+1}\\
\ge& 2^{n-t-1} - 2^{t+1} + 1 > 0.
\end{aligned}
$$

(ii) Similarly, since $\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A}) \subset \widehat{\Sigma} \setminus \widehat{\alpha^\perp}$, we have

$$
\begin{aligned}
|\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A})| - 2^{t+2} \ge& |\widehat{\Sigma}| - |\Sigma \cap \alpha^\perp| - 2^{t+2}\\
\ge& 2^{n-t-1} - 2^{t+2} + 1 > 0.
\end{aligned}
$$

(b) (i) If $\Sigma \cap \mathcal{Q} = \Pi_0\mathcal{Q}_{n-2}$, then by Lemma 5.2(1,3b) and because of $t > 0$, we have

$$
\begin{aligned}
|\widehat{\Sigma} \setminus \widehat{\alpha^\perp}| - 2^{t+1} \ge& |\widehat{\Sigma}| - |\widehat{\alpha^\perp}| - 2^{t+1}\\
=& (2^{n-1} \pm 2^{\frac{n-1}{2}}) - (2^{n-t-1} \pm 2^{\frac{n-1}{2}}) - 2^{t+1}\\
=& 2^{n-1-t}(2^t - 1) - 2^{t+1} \ge 2^{n-1-t} - 2^{t+1} > 0.
\end{aligned}
$$

(ii) If $\Sigma \cap \mathcal{Q} = \Pi_0\mathcal{Q}_{n-2}$, then by Lemma 5.2(1,2,3b) and because of $t > 0$, we have

$$\begin{aligned}
&|\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A})| - 2^{t+2} \\
\geq &|\widehat{\Sigma}| - |\widehat{\alpha^\perp}| + |\widehat{A}| - 2^{t+2} \\
= &(2^{n-1} \pm 2^{\frac{n-1}{2}}) - (2^{n-t-1} \pm 2^{\frac{n-1}{2}}) + (2^{n-t-2} \pm 2^{\frac{n-1}{2}} - 2^{t+2}) - 2^{t+2} \\
= &2^{n-2-t}(2^{t+1} - 1) \pm 2^{\frac{n-1}{2}} - 2^{t+3} \\
\geq &3 \cdot 2^{n-2-t} \pm 2^{\frac{n-1}{2}} - 2^{t+3}
\end{aligned} \tag{5.5}$$

where the last equality holds if and only if $t = 1$.

If $\mathcal{Q}$ is elliptic, then because of $t \leq \frac{n-3}{2}$, we have

$$3 \cdot 2^{n-2-t} + 2^{\frac{n-1}{2}} - 2^{t+3} \geq 0 \tag{5.6}$$

where the equality holds if and only if $t = \frac{n-3}{2}$. Because of $n \geq 7$, it is impossible to have $1 = t = \frac{n-3}{2}$. Combining (5.5) and (5.6), we have $|\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A})| > 2^{t+2}$.

If $\mathcal{Q}$ is hyperbolic, then because of $t \leq \frac{n-5}{2}$, we have

$$3 \cdot 2^{n-2-t} - 2^{\frac{n-1}{2}} - 2^{t+3} > 0. \tag{5.7}$$

Combining (5.5) and (5.7), we have $|\widehat{\Sigma} \setminus (\widehat{\alpha^\perp} \setminus \widehat{A})| > 2^{t+2}$.

□

Proposition 5.5. *Let u be a non-zero vector in C_t. Then $\mathrm{wt}(u) \geq 2^{t+1}$, and equality holds if and only if $u = v^{S_t}$.*

Proof. Let u be a non-zero vector in C_t. Then u is one of the following: w, $w + v^{S_t}$, $w + v^{T_t}$, $w + v^{T_t} + v^{S_t}$, v^{T_t}, $w^{T_t} + v^{S_t}$ or v^{S_t} for some $w \in C(\Gamma_{\mathcal{Q}})$. By Tables 3 and 4, $\mathrm{wt}(w) > 2^{t+1}$, and by Lemma 5.1, $\mathrm{wt}(w + v^{S_t}) > 2^{t+1}$. Note that for any subsets U_1, U_2 of $\mathrm{PG}(n, 2) \setminus \mathcal{Q}$, $v^{U_1} + v^{U_2} = v^{U_1 \triangle U_2}$. The result follows from Lemmas 3.1, 5.3 and 5.4 because $w = v^R$ where $R = (\mathrm{PG}(n, 2) \setminus \mathcal{Q}) \setminus \Sigma$ for some $(n-1)$-space Σ. □

Proposition 5.6. *Let u be a non-zero vector in $C_{t,t}$. Then $\mathrm{wt}(u) \geq 2^{t+2}$, and equality holds if and only if $u = v^{S_{t,t}}$.*

Proof. It follows using arguments that are similar to those in the proof of Proposition 5.5. □

6. Numbers of switched graphs found

With the notation as given in Section 5 for n, $\mathcal{Q}$, $\perp$, t, α, Π, Π', $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$, $\Gamma_{\mathcal{Q},t,t}$, S_t, $S_{t,t}$, T_t and $T_{t,t}$, we assume $n \geq 7$. In this section, we will prove $C(\Gamma_{\mathcal{Q},t}) = C_t$ and

$C(\Gamma_{\mathcal{Q},t,t}) = C_{t,t}$ as claimed in Section 5, and then count the number of non-isomorphic graphs constructed through Theorems 3.A and 3.B.

Let $A, A_t, A_{t,t}$ be the adjacency matrices of $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},t}$ and $\Gamma_{\mathcal{Q},t,t}$.

Since $S_t \subset S_{t,t}$ and $T_t \subset T_{t,t}$, we may assume that the first $|S_t|$ rows and columns of $A, A_t, A_{t,t}$ correspond to points of $\mathrm{PG}(n,2) \setminus \mathcal{Q}$ in S_t; the next $|S_{t,t} \setminus S_t|$ rows and columns correspond to those in $S_{t,t} \setminus S_t$; the last $|T_{t,t}|$ rows and columns correspond to points in $T_{t,t}$ such that the last $|T_t|$ rows and columns correspond to points in T_t. By the definition of $\Gamma_{\mathcal{Q},t}$,

$$A_t = A + M_t, \text{where } M_t = \begin{pmatrix} O & O & J_t \\ O & O & O \\ J_t' & O & O \end{pmatrix} \tag{6.1}$$

where J_t is the $|S_t|$-by-$|T_t|$ all-ones matrix. Similarly, by the definition of $\Gamma_{\mathcal{Q},t,t}$,

$$A_{t,t} = A + M_{t,t}, \text{where } M_{t,t} = \begin{pmatrix} O & O & J_{t,t} \\ O & O & O \\ J_{t,t}' & O & O \end{pmatrix} \tag{6.2}$$

where $J_{t,t}$ is the $|S_{t,t}|$-by-$|T_{t,t}|$ all-ones matrix.

Lemma 6.1. *None of v^{T_t} or $v^{T_{t,t}}$ is in $C(\Gamma_{\mathcal{Q}})$.*

Proof. Suppose v^{T_t} is in $C(\Gamma_{\mathcal{Q}})$. Recall any codeword in $C(\Gamma_{\mathcal{Q}})$ is v^R where $R = (\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \Sigma$ for some $(n-1)$-space Σ. By (3.2),

$$(\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \alpha^{\perp} = (\mathrm{PG}(n,2) \setminus \mathcal{Q}) \setminus \Sigma.$$

This implies $\Sigma \setminus \mathcal{Q} = \alpha^{\perp} \setminus \mathcal{Q}$. Considering the size of $\Sigma \setminus \mathcal{Q}$ and $\alpha^{\perp} \setminus \mathcal{Q}$ given in Lemma 5.2, we have $n = 3$ or $t = 0$, which contradicts the range of n and t stated in Theorem 3.A or Theorem 3.B. $\square$

We now prove $C(\Gamma_{\mathcal{Q},t}) = C_t$ and $C(\Gamma_{\mathcal{Q},t,t}) = C_{t,t}$ as announced in Section 5.

Lemma 6.2. *$C(\Gamma_{\mathcal{Q},t}) = \langle C(\Gamma_{\mathcal{Q},t}), v^{S_t}, v^{T_t} \rangle$ and the 2-rank of $C(\Gamma_{\mathcal{Q},t})$ is $n+3$.*

Proof. By Lemmas 4.3 and 4.5, v^{S_t} and v^{T_t} are codewords of $C(\Gamma_{\mathcal{Q},t})$. By (6.1), a row of the adjacency matrix of $\Gamma_{\mathcal{Q},t}$ either is a row of the adjacency matrix of $\Gamma_{\mathcal{Q}}$ or differs from such a row by v^{S_t} or v^{T_t}. Thus, any row of the adjacency matrix of $\Gamma_{\mathcal{Q}}$ is a codeword of $C(\Gamma_{\mathcal{Q},t})$.

By Lemma 6.1, $v^{T_t} \notin C(\Gamma_{\mathcal{Q}})$ and by Proposition 5.5, for any $w \in C(\Gamma_{\mathcal{Q}})$, we have that none of w and $w + v^{T_t}$ is the vector v^{S_t}. Thus, v^{S_t}, v^{T_t} and a basis of $C(\Gamma_{\mathcal{Q}})$ form a linearly independent set of size $2 + (n+1) = n+3$.

In (6.1), since the 2-rank of M_t is 2, the 2-rank of $C(\Gamma_{\mathcal{Q}})$ differs from that of $C(\Gamma_{\mathcal{Q},t})$ by at most 2. Since v^{S_t} and v^{T_t} and a basis of $C(\Gamma_{\mathcal{Q}})$ form a linearly independent set in $C(\Gamma_{\mathcal{Q},t})$ with size two more than the 2-rank of $C(\Gamma_{\mathcal{Q}})$, they form a basis of $C(\Gamma_{\mathcal{Q},t})$. $\square$

Lemma 6.3. *$C(\Gamma_{\mathcal{Q},t,t}) = \left\langle C(\Gamma_{\mathcal{Q},t,t}), v^{S_{t,t}}, v^{T_{t,t}} \right\rangle$ and the 2-rank of $C(\Gamma_{\mathcal{Q},t,t})$ is $n+3$.*

Proof. The proof is similar to that of Lemma 6.2. $\square$

We now give the parameters of $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$. Recall the upper sign of $\mp$ is taken when if $\mathcal{Q}$ is elliptic, and otherwise if $\mathcal{Q}$ is hyperbolic.

Theorem 6.4. *$C(\Gamma_{\mathcal{Q},t})$ is a $[2^n \mp 2^{\frac{n-1}{2}}, n+3, 2^{t+1}]_2$-code. $C(\Gamma_{\mathcal{Q},t,t})$ is a $[2^n \mp 2^{\frac{n-1}{2}}, n+3, 2^{t+2}]_2$-code*

Proof. The length of $C(\Gamma_{\mathcal{Q},t})$ and $C(\Gamma_{\mathcal{Q},t,t})$ are the number of vertices of their respective graphs, which is $2^n \mp 2^{\frac{n-1}{2}}$. Other parameters of the codes follow from Lemmas 6.2, 6.3, and Proportions 5.5, 5.6. $\square$

Theorem 6.5. *The graphs $\Gamma_{\mathcal{Q}}, \Gamma_{\mathcal{Q},1}, \Gamma_{\mathcal{Q},2}, \cdots, \Gamma_{\mathcal{Q},\frac{n-3}{2}}, \Gamma_{\mathcal{Q},1,1}, \Gamma_{\mathcal{Q},2,2}, \cdots, \Gamma_{\mathcal{Q},m,m}$ are distinct up to isomorphism, where $m = \frac{n-3}{2}$ if $\mathcal{Q}$ is elliptic and $m = \frac{n-5}{2}$ if $\mathcal{Q}$ is hyperbolic.*

Proof. $\Gamma_{\mathcal{Q}}$ is distinct from other graphs in the list because it has a 2-rank $n+1$ [11, Theorem 5.3] but others do not by Lemmas 6.2 and 6.3. Let Γ, Γ' be two graphs listed above other than $\Gamma_{\mathcal{Q}}$. Let S and S' be switching sets of $\Gamma_{\mathcal{Q}}$ such that Γ, Γ' are obtained from $\Gamma_{\mathcal{Q}}$ with switching sets respectively S and S'.

Suppose there is an isomorphism ϕ between Γ and Γ'. Then ϕ induces a code isomorphism Φ between $C(\Gamma)$ and $C(\Gamma')$. Since Φ maps minimum word(s) of $C(\Gamma)$ to those of $C(\Gamma')$, we have $\Phi(S) = S'$ by Propositions 5.5 and 5.6. Considering the size of the switching sets given in Lemma 3.1, we may assume without loss of generality that $\Gamma = \Gamma_{\mathcal{Q},t+1}$ and $\Gamma' = \Gamma_{\mathcal{Q},t,t}$ for some t. By Lemma 3.2, the subgraph of $\Gamma_{\mathcal{Q}}$, and hence of Γ, with vertex set $S = S_{t+1}$ is null. But by Lemma 3.3, the subgraph of $\Gamma_{\mathcal{Q}}$, and hence of Γ', with vertex set $S' = S_{t,t}$ is not null. This contradicts $\Phi(S) = S'$, and so Γ and Γ' are non-isomorphic. $\square$

Since we work under the assumption that $n \geq 7$ in Sections 5 and 6, Theorem 6.5 is valid under the same assumption. However, it can be checked directly that in case $\mathcal{Q}$ is elliptic and $n = 5$, $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q},1}$ and $\Gamma_{\mathcal{Q},1,1}$ are non-isomorphic; in case $\mathcal{Q}$ is hyperbolic and $n = 5$, $\Gamma_{\mathcal{Q}}$ and $\Gamma_{\mathcal{Q},1}$ are non-isomorphic. In conclusion, for $n \geq 5$, if $\mathcal{Q}$ is an elliptic quadric in $\mathrm{PG}(n,2)$, then Theorems 3.A and 3.B give $n-3$ non-isomorphic graphs, other than $\Gamma_{\mathcal{Q}}$, with the same parameters as $\Gamma_{\mathcal{Q}}$, where the parameters are shown in Table 1. For $n \geq 5$, if $\mathcal{Q}$ is a hyperbolic quadric in $\mathrm{PG}(n,2)$, then Theorems 3.A and 3.B give $n-2$ non-isomorphic graphs, other than $\Gamma_{\mathcal{Q}}$, with the same parameters as $\Gamma_{\mathcal{Q}}$.

References

[1] Abiad, A. & Haemers, W.H. Switched symplectic graphs and their 2-ranks. Appeared online in *Des. Codes Cryptogr.*

[2] Assmus, E.F. & Key, J.D. (1992). *Designs and their codes*. Cambridge University Press, Cambridge.

[3] Barwick, S.G., Jackson, W-A. & Penttila, T. New families of strongly regular graphs. Manuscript.

[4] Brouwer, A.E. & Haemers, W.H. (2011). *Spectra of graphs*. Springer, New York.

[5] Cameron, P.J. & van Lint, J.H. (1991). *Designs, graphs, codes and their links*, Cambridge University Press, Cambridge.

[6] Godsil, C.D. & McKay, B.D. (1982). Constructing cospectral graphs. *Aequationes Math*, **25**, 257–268.

[7] Haemers, W.H., Peeters, M.J.P. & van Rijckevorsel, J.M. (1999). Binary codes of strongly regular graphs. *Des. Codes Cryptogr.*, **17**, 187–209.

[8] Hall, J.I. & Shult, E.E. (1985). Locally cotriangular graphs. *Geometriae Dedicata*, **18**, 113–159.

[9] Hirschfeld, J.W.P. (1979). *Projective geometries over finite fields*. Oxford Univ. Press.

[10] Hirschfeld, J.W.P. & Thas, J.A. (1991). *General Galois geometries*. Oxford Sci. Publ., Clarendon Press.

[11] Peeters, M.J.P. (1995). Uniqueness of strongly regular graphs having minimal p-rank. *Linear Algebra Appl.* **226–228**, 9–31.