# Combinatorial Constructions of Optimal ( $m, n, 4,2$ ) Optical Orthogonal Signature Pattern Codes* 

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#### Abstract

Optical orthogonal signature pattern codes (OOSPCs) play an important role in a novel type of optical code division multiple access (OCDMA) network for 2-dimensional image transmission. There is a one-to-one correspondence between an $(m, n, w, \lambda)$-OOSPC and a $(\lambda+1)-(m n, w, 1)$ packing design admitting a point-regular automorphism group isomorphic to $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. In 2010, Sawa gave the first infinite class of ( $m, n, 4,2$ )-OOSPCs by using $S$-cyclic Steiner quadruple systems. In this paper, we use various combinatorial designs such as strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $s$-fan designs, strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G$-designs and rotational Steiner quadruple systems to present some constructions for ( $m, n, 4,2$ )-OOSPCs. As a consequence, our new constructions yield more infinite families of optimal ( $m, n, 4,2$ )-OOSPCs. Especially, we see that in some cases an optimal ( $m, n, 4,2$ )-OOSPC can not achieve the Johnson bound. We also use Witt's inversive planes to obtain optimal ( $p, p, p+1,2$ )-OOSPCs for all primes $p \geq 3$.


Keywords: Automorphism group, packing design, optical orthogonal code, optical orthogonal signature pattern code, spatial optical CDMA.

## 1 Introduction

Optical code division multiple access (OCDMA) allows many nodes, which technically collide with one another, to transmit and be accessed simultaneously and dynamically with no waiting time at the same wavelength [16], [53], [42], [43]. Spatial OCDMA, an extension of OCDMA to a twodimensional (2-D) space coding for image transmission and multiple access, can exploit the inherent parallelism of optics. The beauty of parallelism is that a light beam can carry the information of a 2-D array of pixels of a binary-digitized image, and hence the 2-D array of pixels of an image can be transmitted and processed simultaneously through multicore fiber without parallel-to-serial conversion. For details on fundamentals and applications of OCDMA and spatial OCDMA, the reader is referred to books [29], [41], [58]. The approach, which combines the features of optical image processing with multiple access inherent in OCDMA, has been proposed and demonstrated for encoding two dimensional pixel arrays by Kitayama and colleagues [28, 30]. This 2-D image multiplexing has many applications such as transmission of medical images, parallel optical interconnections between processors and memory in high performance computing, etc [41], [58]. And the spatial OCDMA provides higher throughput comparing with the traditional OCDMA [29].

In spatial OCDMA, each magnified 2-D bit plane, comprising individual pixels of an image, is encoded using a $(0,1) 2$-D matrix called a $2-\mathrm{D}$ optical orthogonal signature pattern (OOSP). The encoded image is constructed by taking the Hadamard product of the overlapping matrix elements of the magnified bit plane and OOSP. Each node on the network uses a unique OOSP to encode the

[^0]planar data from its input image. When a receiver which possesses the signature pattern receives the encoded signal, it extracts the input image from the received encoded signal by correlating the received encoded signal and its own OOSP with a specific signature function. For more details, the interested readers may refer to [24], [28].

As pointed out in [28], one of the keys to spatial OCDMA is the methodology in the construction of the 2-D OOSPs. The construction of the OOSPs for 2-D data encoding follows many of the same principles typical of optical CDMA codes including code orthogonality and large cardinality. In addition, each OOSP is distinguishable from space-shifted versions of themselves (auto-correlation) on a 2-D plane and any two different OOSPs in a set are distinguishable from each other (crosscorrelation), even with the existence of vertical or horizontal space shifts in the plane [24], [28]. The constraints require that the correlation is much lower than the weight (the number of " 1 ") of the OOSP. Now we introduce the formal definition of an optical orthogonal signature pattern code.

Let $m, n, w, \lambda$ be positive integers with $m n>w \geq \lambda$. An optical orthogonal signature pattern code with $m$ wavelengths, time-spreading length $n$, constant weight $w$ and the maximum collision parameter $\lambda$, or briefly $(m, n, w, \lambda)$-OOSPC, is a family, $\mathcal{C}$, of $m \times n(0,1)$-matrices (codewords) with constant Hamming weight $w$ (i.e., the number of ones) such that the following correlation properties hold:
(1) (Auto-Correlation Property)

$$
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i, j} a_{i \oplus_{m} \delta, j \oplus_{n} \tau} \leq \lambda
$$

for any matrix $A=\left(a_{i, j}\right) \in \mathcal{C}(0 \leq i \leq m-1,0 \leq j \leq n-1)$ and any integers $\delta$ and $\tau$ with $0 \leq \delta<m, 0 \leq \tau<n$ and $(\delta, \tau) \neq(0,0) ;$
(2) (Cross-Correlation Property)

$$
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i, j} b_{i \oplus_{m} \delta, j \oplus_{n} \tau} \leq \lambda
$$

for any two distinct matrices $A=\left(a_{i, j}\right), B=\left(b_{i, j}\right) \in \mathcal{C}(0 \leq i \leq m-1,0 \leq j \leq n-1)$, and any integers $\delta$ and $\tau$ with $0 \leq \delta<m$ and $0 \leq \tau<n$, where $\oplus_{m}$ (resp. $\oplus_{n}$ ) denotes addition modulo $m$ (resp. modulo $n$ ) and equality holds in each of the two inequalities for at least one instance.

When the auto-correlation property is replaced by $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i, j} a_{i, j \oplus_{n} \tau} \leq \lambda(0<\tau<n)$ and the cross-correlation property is replaced by $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{i, j} b_{i, j \oplus_{n} \tau} \leq \lambda(0 \leq \tau<n)$, this defines a two-dimensional optical orthogonal code (2-D $(m \times n, w, \lambda)$-OOC). Clearly, an OOSPC is a special 2-D OOC, for example, see [56] for other variations of 2-D OOCs. Many constructions for 2-D $(m \times n, w, \lambda)$-OOCs have been given, see [2], [3], [11], [17], [25], [37], [52], [56]. Note that in the particular case where $m=1$ and $n=v$, a $2-\mathrm{D}(m, n, w, \lambda)$-OOSPC is nothing else than a one-dimensional $(v, w, \lambda)$ optical orthogonal code (briefly, 1-D $(v, w, \lambda)$-OOC). For details on 1-D OOCs, the reader is referred to [12], [15], [34], [44]. So far, many constructions of 1-D OOCs with maximum size and many results have been made, for example, [1], [6], [8], [9], [10], [13], [20], [44], [57].

Throughout this paper we always denote by $\mathbb{Z}_{n}$ the additive group of integers modulo $n$. For each $(0,1)$-matrix $A=\left(a_{i j}\right) \in \mathcal{C}$, whose rows are indexed by $\mathbb{Z}_{m}$ and columns are indexed by $\mathbb{Z}_{n}$, we define $X_{A}=\left\{(i, j) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}: a_{i j}=1\right\}$. Then, $\mathcal{F}=\left\{X_{A}: A \in \mathcal{C}\right\}$ is a set-theoretic representation of an $(m, n, w, \lambda)$-OOSPC. Thus, an $(m, n, w, \lambda)$-OOSPC is a set $\mathcal{F}$ of $w$-subsets of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ in which each $w$-subset $X$ corresponds to a signature pattern $\left(a_{i j}\right)$ such that $a_{i j}=1$ if and only if $(i, j) \in X$, where the two correlation properties are given as follows:
(1') (Auto-Correlation Property)

$$
|X \cap(X+(\delta, \tau))| \leq \lambda
$$

for each $X \in \mathcal{F}$ and every $(\delta, \tau) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n} \backslash\{(0,0)\}$;

$$
|X \cap(Y+(\delta, \tau))| \leq \lambda
$$

for any distinct $X, Y \in \mathcal{F}$ and every $(\delta, \tau) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$.
The number of codewords in an OOSPC is called the size of the OOSPC. For given integers $m, n, w$ and $\lambda$, let $\Theta(m, n, w, \lambda)$ be the largest possible size among all ( $m, n, w, \lambda$ )-OOSPCs. An $(m, n, w, \lambda)$-OOSPC with size $\Theta(m, n, w, \lambda)$ is said to be optimal. Based on the Johnson bound [27] for constant weight codes, an upper bound on the largest possible size $\Theta(m, n, w, \lambda)$ of an ( $m, n, w, \lambda$ )-OOSPC was given below:

$$
\begin{equation*}
\Theta(m, n, w, \lambda) \leq J(m, n, w, \lambda)=\left\lfloor\frac{1}{w}\left\lfloor\frac{m n-1}{w-1}\left\lfloor\frac{m n-2}{w-2}\left\lfloor\cdots\left\lfloor\frac{m n-\lambda}{w-\lambda}\right\rfloor \cdots\right\rfloor\right\rfloor\right\rfloor\right\rfloor . \tag{1.1}
\end{equation*}
$$

When $m$ and $n$ are coprime, it has been shown in [55] that an ( $m, n, w, \lambda$ )-OOSPC is actually a 1-D $(m n, w, \lambda)$-OOC. However, when $m$ and $n$ are not coprime, the problem of constructing optimal ( $m, n, w, \lambda$ )-OOSPCs becomes difficult. Some infinite classes of optimal ( $m, n, w, 1$ )-OOSPCs have been given for specific values of $m, n, w$, see [7], [38], [39], [46], [55]. To our knowledge, the only known optimal OOSPCs with $\lambda \geq 2$ were obtained by Sawa [45]. He showed that there is an optimal $\left(2^{\epsilon} x, n, 4,2\right)$-OOSPC where $\epsilon \in\{1,2\}$, and each prime factor of $x, n$ is less than 500000 and congruent to 53 or 77 modulo 120 or belongs to $S=\{5,13,17,25,29,37,41,53$, $61,85,89,97,101,113,137,149,157,169,173,193,197,229,233,289,293,317\}$. In this paper, We use various combinatorial structures to present more infinite families of optimal ( $m, n, 4,2$ )-OOSPCs.

This paper is organized as follows. In Section II, a correspondence between an ( $m, n, w, \lambda$ )OOSPC and a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $(\lambda+1)-(m n, w, 1)$ packing design is described. Based on this correspondence we give an improved upper bound on $\Theta(m, n, 4,2)$ by analyzing the leave of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 3-( $m n, 4,1$ ) packing design. We also construct an optimal ( $p, p, p+1,2$ )OOSPC from an inversive plane of prime order $p$. Section III introduces a concept of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e n, 4,3\right)$ design, from which we can obtain a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 3 -( $m n, 4,1$ ) packing design. We also use a cyclic $\operatorname{SQS}(m)$ to construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ design. In Section IV, we give a recursive construction for strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}, e n, 4,3\right)$ design. Section V uses 1-fan designs to present a recursive construction for strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, w, 3)$ design. Based on known $S$-cyclic SQSs and rotational SQSs, many new optimal ( $m, n, 4,2$ )-OOSPCs are established in Section VI. Finally, Section VII gives a brief conclusion. Our main results are summarized in Table I.

## 2 Combinatorial characterization

In this section, we describe a correspondence between an ( $m, n, k, \lambda$ ) -OOSPC and a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ invariant $(\lambda+1)-(m n, k, 1)$ packing design. Based on this correspondence we give an improved upper bound on $\Theta(m, n, 4,2)$ and construct an optimal ( $p, p, p+1,2$ )-OOSPC for any prime $p$.

Let $t, w, n$ be positive integers. A $t-(n, w, 1)$ packing design consists of an $n$-element set $X$ and a collection $\mathcal{B}$ of $w$-element subsets of $X$, called blocks, such that every $t$-element subset of $X$ is contained in at most one block. A 3-(n,4,1) packing design is called a packing quadruple system and denoted by $\operatorname{PQS}(n)$. When "at most" is replaced by "exactly", this defines a Steiner system, denoted by $S(t, w, n)$. An $S(2,3, n)$ is called a Steiner triple system and denoted by $\operatorname{STS}(n)$. An $S(3,4, n)$ is called a Steiner quadruple system and denoted by $\operatorname{SQS}(n)$. It is well known that there is an $\operatorname{SQS}(n)$ if and only if $n \equiv 2,4(\bmod 6)$ [21].

A $t$ - $(n, w, 1)$ packing design is optimal if it has the largest possible number $D(n, w, t)$ of blocks. It is well known [27] that

$$
D(n, w, t) \leq\left\lfloor\frac{n}{w}\left\lfloor\frac{n-1}{w-1}\left\lfloor\cdots\left\lfloor\frac{n-t+1}{w-t+1}\right\rfloor \cdots\right\rfloor\right\rfloor\right\rfloor .
$$

For $t=2$ and $w \in\{3,4\}$, the numbers $D(n, w, 2)$ have been completely determined, there is also much work on $D(n, 5,2)$, see [51]. For $t \geq 3$, only numbers $D(n, 4,3)$ have been completely determined [4].

An automorphism $\sigma$ of a packing design $(X, \mathcal{B})$ is a permutation on $X$ leaving $\mathcal{B}$ invariant. All automorphisms of a packing design form a group, called the full automorphism group of the packing design. Any subgroup of the full automorphism group is called an automorphism group of the packing design. Let $G$ be an automorphism group of a packing design. For any block $B$ of the packing design, the subgroup

$$
\left\{\sigma \in G: B^{\sigma}=B\right\}
$$

is called the stabilizer of $B$ in $G$, where $B^{\sigma}$ stands for $\sigma$ acting on $B$. The orbit of $B$ under $G$ is the collection $\operatorname{Orb}_{G}(B)$ of all distinct images of $B$ under $G$, i.e.,

$$
\operatorname{Orb}_{G}(B)=\left\{B^{\sigma}: \sigma \in G\right\}
$$

It is clear that $\mathcal{B}$ can be partitioned into some orbits under $G$. An arbitrary set of representatives for each orbit of $\mathcal{B}$ is called the set of base blocks of the packing design. A packing design $(X, \mathcal{B})$ is said to be $G$-invariant if it admits $G$ as a point-regular automorphism group, that is, $G$ is an automorphism group such that for any $x, y \in X$, there exists exactly one element $\sigma \in G$ such that $x^{\sigma}=y$. In particular, a $\mathbb{Z}_{n}$-invariant packing design is cyclic. Moreover, a packing design $(X, B)$ is said to be strictly $G$-invariant if it is $G$-invariant and the stabilizer of each $B \in \mathcal{B}$ under $G$ equals the identity of $G$. A strictly $G$-invariant $t-(n, w, 1)$ packing design is called optimal if it contains the largest possible number of base blocks.

For a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $t$ - $(m n, w, 1)$ packing design $(X, \mathcal{B})$, without loss of generality we can identify $X$ with $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ and the automorphisms can be taken as translations $\sigma_{a}$ defined by $x^{\sigma_{a}}=$ $x+a$ for $x \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, where $a \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. Thus, given an arbitrary family of all base blocks of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $t$ - $(m n, w, 1)$ packing design, we can obtain the packing design by successively adding $(i, j)$ to each base block, where $(i, j) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$.

Based on the set-theoretic representation of an $(m, n, w, \lambda)$-OOSPC, the following connection is then obtained.

Theorem 2.1 [45] $A n(m, n, w, \lambda)$-OOSPC of size $u$ is equivalent to a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $(\lambda+1)-(m n, w, 1)$ packing design having $u$ base blocks.

Based on this connection, Sawa established a tighter upper bound on $\Theta(m, n, 4,2)$ with $m n \equiv 0$ $(\bmod 24)$ than the Johnson bound.

Lemma 2.2 [45] Let $m$ and $n$ be positive integers. If $m n \equiv 0(\bmod 24)$ then $\Theta(m, n, 4,2) \leq$ $J(m, n, 4,2)-1$.

Sawa [45] also posed an open problem: Does there exist an optimal ( $6, n, 4,2$ )-OOSPC attaining the Johnson bound (1.1) for a positive integer $n$, not being a multiple of 4 in general? By analyzing the leave of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $\operatorname{PQS}(m n)$, we show that there does not exist an ( $m, n, 4,2$ )OOSPC attaining the upper bound (1.1) for $m, n \equiv 0(\bmod 3)$ with $m n \equiv 18,36(\bmod 72)$.

The triples of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ are partitioned into equivalence classes called orbits of triples under the action of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. The number of triples (resp. quadruples) contained in an orbit is called the length of the orbit. If the length of an orbit is $m n$ then it is called full. The set of triples not contained in any quadruple of a $\operatorname{PQS}(v)$ is called the leave of this packing design.

For $(a, b) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n} \backslash\{(0,0)\}$, denote $T_{(a, b)}=\left\{\{(0,0),(a, b),(x, y)\}:(x, y) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n} \backslash\right.$ $\{(0,0),(a, b)\}\}$. Clearly, $\left|T_{(a, b)}\right|=m n-2$. Each quadruple of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ either contains two triples in $T_{(a, b)}$ or does not contain any triple in $T_{(a, b)}$. So, the number of triples which are from $T_{(a, b)}$ and in the leave of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $\operatorname{PQS}(m n)$ has the same parity as $m n$.

Lemma 2.3 If $m \equiv 0(\bmod 3), n \equiv 0(\bmod 3)$ and $m n \equiv 0,18$ or $36(\bmod 72)$, then $\Theta(m, n, 4,2) \leq$ $J(m, n, 4,2)-1$.

Proof It is easy to see that the orbits generated by $\left\{(0,0),\left(0, \frac{n}{3}\right),\left(0, \frac{2 n}{3}\right)\right\},\left\{(0,0),\left(\frac{m}{3}, \frac{n}{3}\right),\left(\frac{2 m}{3}, \frac{2 n}{3}\right)\right\}$, $\left\{(0,0),\left(\frac{m}{3}, \frac{2 n}{3}\right),\left(\frac{2 m}{3}, \frac{n}{3}\right)\right\}$ and $\left\{(0,0),\left(\frac{m}{3}, 0\right),\left(\frac{2 m}{3}, 0\right)\right\}$ under $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ are short. They have length $\frac{m n}{3}$ and they must be in the leave of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant PQS $(m n)$. Also, the other orbits of triples under $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ are all full. Consequently, there are $\left(\frac{m n(m n-1)(m n-2)}{6}-\frac{4 m n}{3}\right) / m n=$ $\frac{m^{2} n^{2}-3 m n-6}{6}$ full orbits of triples. On the other hand, since the number of triples containing any given two points in the leave is even, there must be at least one triple of the form $\left\{(0,0),(x, y),\left(x^{\prime}, y^{\prime}\right)\right\}$ with $\left(x^{\prime}, y^{\prime}\right) \neq(2 x, 2 y)$ in the leave for $(x, y) \in\left\{\left(0, \frac{n}{3}\right),\left(0, \frac{2 n}{3}\right),\left(\frac{m}{3}, \frac{n}{3}\right),\left(\frac{2 m}{3}, \frac{2 n}{3}\right),\left(\frac{m}{3}, \frac{2 n}{3}\right),\left(\frac{2 m}{3}, \frac{n}{3}\right),\left(\frac{m}{3}, 0\right)\right.$, $\left.\left(\frac{2 m}{3}, 0\right)\right\}$. Clearly, one full orbit of triples can not cover all such eight triples. It follows that there are at most $\frac{m^{2} n^{2}-3 m n-6}{6}-2$ full orbits of triples occurring in a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant PQS $(m n)$. Therefore, $\Theta(m, n, 4,2) \leq\left\lfloor\frac{m^{2} n^{2}-3 m n-18}{24}\right\rfloor=J(m, n, 4,2)-1$.

We use an example to show that the improved upper bound in Lemma 2.3 is tight. The corresponding optimal $(6,6,4,2)$-OOSPC gives an answer to the problem listed in Table I in [45]

Example 2.4 There exists a strictly $\mathbb{Z}_{6} \times \mathbb{Z}_{6}$-invariant $P Q S(36)$ with 48 base blocks, whose size meets the upper bound in Lemma 2.3.

Proof The following 48 base blocks generate the block set of a strictly $\mathbb{Z}_{6} \times \mathbb{Z}_{6}$-invariant PQS(36) over $\mathbb{Z}_{6} \times \mathbb{Z}_{6}$.

$$
\begin{aligned}
& \{(0,0), a,-a,(3,3)\} \text {, where } a \in\{(0,1),(1,0),(1,2),(1,4)\} \text {; } \\
& \{(0,0), b,(3,0),(0,3)+b\} \text {, where } b \in\{(0,1),(0,2),(1,0) \text {, } \\
& (1,1),(1,2),(2,0),(2,1)(2,2)\} \text {; } \\
& \{(0,0),(0,1),(1,0),(1,1)\}, \quad\{(0,0),(0,1),(1,2),(1,3)\}, \\
& \{(0,0),(0,1),(1,4),(1,5)\}, \quad\{(0,0),(0,1),(2,0),(2,1)\} \text {, } \\
& \{(0,0),(0,1),(2,2),(2,3)\}, \quad\{(0,0),(0,1),(2,4),(3,2)\} \text {, } \\
& \{(0,0),(0,1),(2,5),(4,2)\}, \quad\{(0,0),(0,1),(3,5),(4,3)\}, \\
& \{(0,0),(0,2),(1,0),(1,2)\}, \quad\{(0,0),(0,2),(1,1),(1,3)\} \text {, } \\
& \{(0,0),(0,2),(1,4),(2,1)\}, \quad\{(0,0),(0,2),(1,5),(2,0)\} \text {, } \\
& \{(0,0),(0,2),(2,2),(3,3)\}, \quad\{(0,0),(0,2),(2,3),(2,5)\} \text {, } \\
& \{(0,0),(0,2),(2,4),(4,4)\}, \quad\{(0,0),(0,2),(3,5),(4,0)\} \text {, } \\
& \{(0,0),(0,2),(4,1),(5,4)\}, \quad\{(0,0),(0,2),(4,2),(5,3)\} \text {, } \\
& \{(0,0),(1,0),(2,1),(3,1)\}, \quad\{(0,0),(1,0),(2,2),(5,4)\} \text {, } \\
& \{(0,0),(1,0),(2,3),(5,1)\}, \quad\{(0,0),(1,0),(2,4),(3,5)\} \text {, } \\
& \{(0,0),(1,0),(2,5),(5,3)\}, \quad\{(0,0),(1,0),(3,2),(4,2)\}, \\
& \{(0,0),(1,0),(4,1),(5,2)\}, \quad\{(0,0),(1,1),(2,3),(3,4)\} \text {, } \\
& \{(0,0),(1,1),(3,2),(4,5)\}, \quad\{(0,0),(1,2),(2,0),(5,2)\} \text {, } \\
& \{(0,0),(1,2),(2,1),(4,0)\}, \quad\{(0,0),(1,2),(3,1),(4,3)\} \text {, } \\
& \{(0,0),(1,2),(3,2),(5,1)\}, \quad\{(0,0),(1,3),(2,2),(3,5)\} \text {, } \\
& \{(0,0),(1,3),(3,1),(4,0)\}, \quad\{(0,0),(1,3),(3,3),(4,2)\} \text {, } \\
& \{(0,0),(1,4),(2,3),(4,5)\}, \quad\{(0,0),(1,4),(3,5),(5,1)\} \text {. }
\end{aligned}
$$

We finish this section by giving an optimal ( $p, p, p+1,2$ )-OOSPC from an inversive plane.
Let $q$ be a prime power and $G F(q)$ the finite field of order $q$. Suppose that $a, b, c, d \in G F(q)$ and $a d-b c \neq 0$. Define a linear fractional mapping $\pi\left(\begin{array}{cc}a & b \\ c & d\end{array}\right): G F(q) \cup\{\infty\} \rightarrow G F(q) \cup\{\infty\}$ as follows:

$$
\pi\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)(x)= \begin{cases}\frac{a x+b}{c x+d}, & \text { if } x \in G F(q), c x+d \neq 0 \\
\infty, & \text { if } x \in G F(q), a x+b \neq 0, c x+d=0 \\
\frac{a}{c}, & \text { if } x=\infty, c \neq 0 \\
\infty, & \text { if } x=\infty, c=0\end{cases}
$$

Then $\left.\pi_{\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \text { is a permutation of } G F(q) \cup\{\infty\} \text {, and the permutations } \pi\binom{a}{c} \text { and } \pi\binom{r a r b}{c c} .} \begin{array}{l}d\end{array}\right)$ are identical if $r \neq 0$. Define $\operatorname{PGL}(2, q)$ to consist of all the distinct permutations $\pi\left(\begin{array}{l}a b \\ c \\ c\end{array}\right)$, where $a, b, c, d \in G F(q)$, and $a d-b c \neq 0$. It is well known that $\operatorname{PGL}(2, q)$ is a sharply 3 -transitive permutation group acting on the set $G F(q) \cup\{\infty\}$, i.e., for all choices of six elements $x_{1}, x_{2}, x_{3}$, $y_{1}, y_{2}, y_{3} \in G F(q) \cup\{\infty\}$ such that $x_{1}, x_{2}, x_{3}$ are distinct and $y_{1}, y_{2}, y_{3}$ are distinct, there is exactly one permutation $\pi \in \mathrm{PGL}(2, q)$ such that $\pi\left(x_{i}\right)=y_{i}$ for all $i, 1 \leq i \leq 3$. Witt proved that the $\operatorname{PGL}\left(2, q^{2}\right)$ orbit of the set $G F(q) \cup\{\infty\}$ is an $S\left(3, q+1, q^{2}+1\right)$ (called an inversive plane) with the point set $G F\left(q^{2}\right) \cup\{\infty\}[54]$.

Theorem 2.5 [54] For any prime power $q$, there is an $S\left(3, q+1, q^{2}+1\right)$.

Theorem 2.6 For any prime $p$, there is an optimal ( $p, p, p+1,2$ )-OOSPC with the size attaining the upper bound (1.1).

Proof Start with an inversive plane $S\left(3, p+1, p^{2}+1\right)$ whose point set is $G F\left(p^{2}\right) \cup\{\infty\}$ and whose block set is the $\operatorname{PGL}\left(2, p^{2}\right)$ orbit of the set $G F(p) \cup\{\infty\}$. Since the $\operatorname{PGL}\left(2, p^{2}\right)$ contains the permutation group $G=\left\{\pi\left(\begin{array}{ll}1 & b \\ 0 & 1\end{array}\right): b \in G F\left(p^{2}\right)\right\}$, the inversive plane $S\left(3, p+1, p^{2}+1\right)$ admits an automorphism group $G$. Since each automorphism $\pi_{(1 b}$ ) fixes the point $\infty$, the set of all $\left(\begin{array}{ll}1 & b \\ 0 & 1\end{array}\right)$
blocks containing $\infty$ admits the automorphism group $G$. On the other hand, all blocks containing $\infty$ with $\infty$ deleted form the set of an $S\left(2, p, p^{2}\right)$ (called an affine plane). There are total $p^{2}+p$ blocks containing $\infty$. Thus, deleting $\infty$ and all blocks containing $\infty$ from the inversive plane yields a $3-\left(p^{2}, p+1,1\right)$ packing design which admits $G$ as a point-regular automorphism group and has $p^{2}(p-1)$ blocks. Since $\operatorname{gcd}\left(p+1, p^{2}\right)=1$, the stabilizer of each block in the $3-\left(p^{2}, p+1,1\right)$ packing design under $G$ equals the identity of $G$, thus the $3-\left(p^{2}, p+1,1\right)$ packing design with point set $G F\left(p^{2}\right)$ is strictly $G$-invariant. This packing design is in fact a strictly $\left(G F\left(p^{2}\right),+\right)$-invariant $3-\left(p^{2}, p+1,1\right)$ packing design. Since $\left(G F\left(p^{2}\right),+\right)$ is isomorphic to $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, there is a strictly $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$-invariant $3-\left(p^{2}, p+1,1\right)$ packing design with $p-1$ full orbits of blocks. By Theorem 2.1 and the upper bound (1.1), there is an optimal ( $p, p, p+1,2$ )-OOSPC.

## 3 Construction of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ designs via cyclic $\operatorname{SQS}(m)$ s

In this section, we introduce a concept of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ design and present a construction of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $\mathrm{PQS}(m n)$ from a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ design. We also use a strictly semi-cyclic $G(2, n, 4,3)$ design and a cyclic $\operatorname{SQS}(m)$ to construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ design.

Let $m, n, t$ be positive integers and $K$ a set of some positive integers. A $G(m, n, K, t)$ design is a triple $(X, \Gamma, \mathcal{B})$, where $X$ is a set of $m n$ points, $\Gamma$ is a set of subsets of $X$ which is partition of $X$ into $m$ sets of size $n$ (called groups) and $\mathcal{B}$ is a set of subsets of $X$ with cardinalities from $K$, called blocks, such that each $t$-set of points not contained in any group occurs in exactly one block and each $t$-subset of each group does not occur in any block. When $K=\{k\}$, we simply write $k$ for $K$.
$G$-designs were introduced by Mills [36] who determined the existence of $G(m, 6,4,3)$ design. Recently, Zhuralev et al. [59] showed that there exists a $G(m, n, 4,3)$ design if and only if $n=1$ and $m \equiv 2,4(\bmod 6)$, or $n$ is even and $n(m-1)(m-2) \equiv 0(\bmod 3)$.

Let $m, n, t$ be positive integers and $K$ a set of some positive integers. An $H(m, n, K, t)$ design is a triple $(X, \Gamma, \mathcal{B})$, where $X$ is a set of $m n$ points, $\Gamma$ is a partition of $X$ into $m$ sets of size $n$ (called
groups) and $\mathcal{B}$ is a set of subsets of $X$ with cardinalities from $K$, called blocks, such that each block intersect each group in at most one point and each $t$-set of points from $t$ distinct groups occurs in exactly one block.

The early idea of an $H$-design can be found in Hanani [22], who used different terminology. Mills used the terminology $H$-design in [36] and determined the existence of an $H(m, n, 4,3)$ design except for $m=5$ [35]. Recently, the second author proved that an $H(5, n, 4,3)$ design exists if $n$ is even, $n \neq 2$ and $n \not \equiv 10,26(\bmod 48)[26]$.

An automorphism $\alpha$ of a $G$-design (resp. $H$-design) $(X, \Gamma, \mathcal{B})$ is a permutation on $X$ leaving $\Gamma$ and $\mathcal{B}$ invariant. All automorphisms of an $G$-design (resp. $H$-design) form a group, called the full automorphism group of the $G$-design (resp. $H$-design). Any subgroup of the full automorphism group is called an automorphism group of the $G$-design (resp. $H$-design). A $G$-design (resp. $H$-design) $(X, \Gamma, \mathcal{B})$ is said to be $Q$-invariant if it admits $Q$ as a point-regular automorphism group. Moreover, it is said to be strictly $Q$-invariant if it is $Q$-invariant and the stabilizer of each $B \in \mathcal{B}$ under $Q$ equals the identity of $Q$. For a $Q$-invariant $G$-design (resp. $H$-design), we always identify the point set $X$ with $Q$ and the automorphisms are regarded as translations $\sigma_{a}$ defined by $\sigma_{a}(x)=x+a$ for $x \in Q$, where $a \in Q$. Then all blocks of this $G$-design (resp. $H$-design) can be partitioned into some orbits under the permutation group $\left\{\sigma_{a}: a \in Q\right\}$. Let $L$ be a subgroup of $Q$. If the group set of a $Q$-invariant $G$-design (resp. $H$-design) is a set of cosets of $L$ in $Q$, then it is a $Q$-invariant $G$-design (resp. $H$-design) relative to $L$. A $G(m, n, K, t)$ (resp. $H(m, n, K, t)$ ) design is said to be semi-cyclic if the $G$-design (resp. $H$-design) admits an automorphism $\sigma$ consisting of $m$ cycles of length $n$ and leaving each group invariant. Note that the stabilizer of each block $B$ of a semi-cyclic $H$-design under $\left\{\sigma^{i}: 0 \leq i<n\right\}$ equals the identity, i.e., a semi-cyclic $H$-design is always strictly.

Example 3.1 There is a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $G(5,4,4,3)$ design relative to $5 \mathbb{Z}_{10} \times \mathbb{Z}_{2}$.

Proof The following base blocks under $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$ generate the set of blocks of a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2^{-}}$ invariant $G(5,4,4,3)$ design over $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$ with groups $\{i, i+5\} \times \mathbb{Z}_{2}, 0 \leq i<5$.

$$
\begin{array}{ll}
\{(0,0),(1,0),(9,0),(0,1)\}, & \{(0,0),(2,0),(8,0),(0,1)\}, \\
\{(0,0),(3,0),(7,0),(0,1)\}, & \{(0,0),(4,0),(6,0),(0,1)\}, \\
\{(0,0),(1,0),(3,0),(4,0)\}, & \{(0,0),(1,0),(5,0),(6,1)\}, \\
\{(0,0),(1,0),(6,0),(5,1)\}, & \{(0,0),(1,0),(2,1),(3,1)\}, \\
\{(0,0),(1,0),(4,1),(7,1)\}, & \{(0,0),(2,0),(5,0),(7,1)\}, \\
\{(0,0),(2,0),(7,0),(5,1)\}, & \{(0,0),(2,0),(3,1),(9,1)\}, \\
\{(0,0),(2,0),(4,1),(8,1)\}, & \{(0,0),(3,0),(1,1),(4,1)\} .
\end{array}
$$

For a semi-cyclic $G(m, n, K, t)$ (resp. $H(m, n, K, t)$ ) design, without loss of generality we can identify the point set $X$ with $I_{m} \times \mathbb{Z}_{n}$, and the automorphism $\sigma$ can be taken as $(i, j) \mapsto(i, j+1)$ $(-, \bmod n),(i, j) \in I_{m} \times \mathbb{Z}_{n}$, where $I_{m}=\{1, \ldots, m\}$. Then all blocks of this $G$-design (resp. $H$ design) can be partitioned into some orbits under the action of $\sigma$. A set of base blocks is a set of representatives for the orbits and each element is called a base block.

Let $n$ be a positive integer. It is not hard to see that

$$
\left\{\{(1, x),(2, x+y),(3, x+z),(4, x+y+z)\}: x, y, z \in \mathbb{Z}_{n}\right\}
$$

is the set of a semi-cyclic $H(4, n, 4,3)$ design on $I_{4} \times \mathbb{Z}_{n}$ with groups $\{i\} \times \mathbb{Z}_{n}, i \in I_{4}$. Such a result has been stated in [17].

Lemma 3.2 [17] For any positive integers $n$, there exists a semi-cyclic $H(4, n, 4,3)$ design.

The following construction is simple but very useful.

Construction 3.3 Let $e, m, n$ be positive integers such that $m$ is divisible by $e$. Suppose there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}\right.$, en, $\left.k, 3\right)$ design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. If there exists a strictly $\mathbb{Z}_{e} \times \mathbb{Z}_{n}$-invariant 3-(en, $\left.k, 1\right)$ packing design having $b$ base blocks, then there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ invariant $3-(m n, k, 1)$ packing design having $b+\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}$ base blocks. Further, if $b=J(e, n, k, 2)$ then $b+\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}=J(m, n, k, 2)$.

Proof Let $\mathcal{F}$ be the family of base blocks of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e n, k, 3\right)$ design with group set $\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}$. Then $|\mathcal{F}|=\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}$. Let $\mathcal{A}$ be the family of $b$ base blocks of a strictly $\left(\frac{m}{e} \mathbb{Z}_{m}\right) \times \mathbb{Z}_{n}$-invariant 3 - $(e n, k, 1)$ packing design, where $\frac{m}{e} \mathbb{Z}_{m}=\left\{0, \frac{m}{e}, \frac{2 m}{e}, \ldots, m-\frac{m}{e}\right\}$. Such a design exists by assumption. Then $\mathcal{F} \cup \mathcal{A}$ is the set of base blocks of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 3 - $(m n, k, 1)$ packing design having $b+$ $\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}$ base blocks.

When $b=J(e, n, k, 2)$, the same discussion as the proof of [18, Theorem 6.6] shows that

$$
\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}+\left\lfloor\frac{1}{k}\left\lfloor\frac{e n-1}{k-1}\left\lfloor\frac{e n-2}{k-2}\right\rfloor\right\rfloor\right\rfloor=\left\lfloor\frac{1}{k}\left\lfloor\frac{m n-1}{k-1}\left\lfloor\frac{m n-2}{k-2}\right\rfloor\right\rfloor\right\rfloor .
$$

For completeness, we give its proof again. First we will prove that if $a, b, c$ are positive integers, then $\left\lfloor\frac{1}{a}\left\lfloor\frac{c}{b}\right\rfloor\right\rfloor=\left\lfloor\frac{c}{a b}\right\rfloor$. Let $c=x b+y, 0 \leq y \leq b-1$. Let $x=x_{1} a+y_{1}, 0 \leq y_{1} \leq a-1$. It follows that $\left\lfloor\frac{c}{a b}\right\rfloor=\left\lfloor\frac{x}{a}+\frac{y}{a b}\right\rfloor=\left\lfloor x_{1}+\frac{y_{1}}{a}+\frac{y}{a b}\right\rfloor=\left\lfloor x_{1}+\frac{y_{1} b+y}{a b}\right\rfloor=x_{1}=\left\lfloor\frac{1}{a}\left\lfloor\frac{c}{b}\right\rfloor\right\rfloor$.

For any given pair $P$ of points from a group, consider all triples containing $P$. By the definition of a $G$-design, there are total $m n-e n$ such triples and each block containing $P$ contains $k-2$ such triples. Therefore, $m n-e n$ is divisible by $k-2$. Similarly, for any given pair $P$ of points from distinct groups, consider all triples containing $P$, we then obtain that $m n-2$ is divisible by $k-2$, thereby, $e n-2$ is also divisible by $k-2$. Then $\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}+\left\lfloor\frac{1}{k}\left\lfloor\frac{e n-1}{k-1}\left\lfloor\frac{e n-2}{k-2}\right\rfloor\right\rfloor\right\rfloor$ is equal to

$$
\frac{(m n-1)(m n-2)-(e n-1)(e n-2)}{k(k-1)(k-2)}+\left\lfloor\frac{(e n-1)(e n-2)}{k(k-1)(k-2)}\right\rfloor=\left\lfloor\frac{(m n-1)(m n-2)}{k(k-1)(k-2)}\right\rfloor=\left\lfloor\frac{1}{k}\left\lfloor\frac{m n-1}{k-1}\left\lfloor\frac{m n-2}{k-2}\right\rfloor\right\rfloor\right\rfloor
$$

as desired.
Construction 3.3 shows that it is useful to find some strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e n, 4,3\right)$ designs.

A block-orbit of a cyclic $\operatorname{SQS}(v)$ is said to be quarter if the block-orbit contains the block $\{0, v / 4, v / 2,3 v / 4\}$, while a block-orbit of a cyclic $\operatorname{SQS}(v)$ is said to be half if the block-orbit contains a block of the form $\{0, i, v / 2, v / 2+i\}, 0<i<v / 4$. It is easy to see that in a cyclic $\operatorname{SQS}(v)$, each block-orbit is full, half, or quarter.

Construction 3.4 If there is a cyclic $S Q S(m)$ and a strictly semi-cyclic $G(2, n, 4,3)$ design with $n>1$, then there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{2}, 2 n, 4,3\right)$ design relative to $\frac{m}{2} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$.

Proof By the necessary condition of a $G(2, n, 4,3)$ design, we have that $n$ is even. First, we construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-invariant $H(m, 2,4,3)$ design over $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$ with the group set $\left\{\{i\} \times \mathbb{Z}_{2}: i \in \mathbb{Z}_{m}\right\}$.

Let $\left(\mathbb{Z}_{m}, \mathcal{B}\right)$ be a cyclic $\operatorname{SQS}(m)$. Let $\widetilde{\mathcal{B}}_{1}$ be the set of base blocks generating all full block-orbits, $\widetilde{\mathcal{B}}_{2}$ the set of base blocks generating all half block-orbits, $\widetilde{\mathcal{B}}_{3}$ the set of the base block generating the unique quarter block-orbit. Note that $\widetilde{\mathcal{B}}_{2}, \widetilde{\mathcal{B}}_{3}$ may be empty sets.

Take any base block $B=\{x, y, z, w\} \in \widetilde{\mathcal{B}}_{1} \cup \widetilde{\mathcal{B}}_{2} \cup \widetilde{\mathcal{B}}_{3}$, construct a semi-cyclic $H(4,2,4,3)$ design on $B \times \mathbb{Z}_{2}$ with the group set $\left\{\{x\} \times \mathbb{Z}_{2}: x \in B\right\}$ and the following eight blocks:

$$
\begin{aligned}
& \{(x, 0),(y, 0),(z, 0),(w, 1)\},\{(x, 1),(y, 1),(z, 1),(w, 0)\}, \\
& \{(x, 0),(y, 0),(z, 1),(w, 0)\},\{(x, 1),(y, 1),(z, 0),(w, 1)\}, \\
& \{(x, 0),(y, 1),(z, 0),(w, 0)\},\{(x, 1),(y, 0),(z, 1),(w, 1)\}, \\
& \{(x, 1),(y, 0),(z, 0),(w, 0)\},\{(x, 0),(y, 1),(z, 1),(w, 1)\} .
\end{aligned}
$$

Clearly, the four blocks on the right are obtained by adding $(0,1)$ to the four blocks on the left. When $B \in \widetilde{\mathcal{B}}_{2}$, it must be of the form $\{0, i, m / 2, i+m / 2\}+j$. Let $x=j, y=j+i, z=j+m / 2$ and $w=j+i+m / 2$. Then it is easy to see that the later four blocks are obtained by adding $(m / 2,0)$ to the first four blocks, respectively. When $B \in \widetilde{\mathcal{B}}_{3}$, it must be of the form $\{0, m / 4, m / 2,3 m / 4\}+j$. Let $x=j, y=j+m / 4, z=j+m / 2$ and $w=j+3 m / 4$. Then it is easy to see that the later six blocks are obtained by adding $(m / 4,0),(m / 2,0),(3 m / 4,0)$ to the first two blocks, respectively. Let $\mathcal{A}_{B}^{1}$ consist of the four blocks on the left if $B \in \widetilde{\mathcal{B}}_{1}$, let $\mathcal{A}_{B}^{2}$ consist of the first two blocks on the left if $B \in \widetilde{\mathcal{B}}_{2}$, and let $\mathcal{A}_{B}^{3}$ consist of the first block if $B \in \widetilde{\mathcal{B}}_{3}$.

Denote

$$
\mathcal{A}=\left(\bigcup_{B \in \widetilde{\mathcal{B}}_{1}} \mathcal{A}_{B}^{1}\right) \bigcup\left(\bigcup_{B \in \widetilde{\mathcal{B}}_{2}} \mathcal{A}_{B}^{2}\right) \bigcup\left(\bigcup_{B \in \widetilde{\mathcal{B}}_{3}} \mathcal{A}_{B}^{3}\right)
$$

It is routine to check that $\mathcal{A}$ is the set of base blocks of the required strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-invariant $H(m, 2,4,3)$ design.

Secondly, we construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $H(m, n, 4,3)$ design over $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ with the group set $\left\{\{i\} \times \mathbb{Z}_{n}: i \in \mathbb{Z}_{m}\right\}$.

For $n>2$, write $n=2 n^{\prime}$. For each base block $A \in \mathcal{A}$, construct a semi-cyclic $H\left(4, n^{\prime}, 4,3\right)$ design on $A \times \mathbb{Z}_{n^{\prime}}$ with groups $\left\{\{x\} \times \mathbb{Z}_{n^{\prime}}: x \in A\right\}$. Such a design exists by Lemma 3.2. Denote the family of base blocks of this design by $\mathcal{C}_{A}$. Define a mapping $\varphi: \mathbb{Z}_{m} \times \mathbb{Z}_{2} \times \mathbb{Z}_{n^{\prime}} \rightarrow \mathbb{Z}_{m} \times \mathbb{Z}_{n}$ by $\varphi(i, \ell, k)=(i, \ell+2 k)$ for $(i, \ell, k) \in \mathbb{Z}_{m} \times \mathbb{Z}_{2} \times \mathbb{Z}_{n^{\prime}}$. Denote $\mathcal{D}=\left\{\{\varphi(z): z \in C\}: C \in \mathcal{C}_{A}, A \in \mathcal{A}\right\}$ and let $\mathcal{D}^{\prime}=\left\{D+\delta: D \in \mathcal{D}, \delta \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}\right\}$.

Simple computation shows that $|\mathcal{D}|=|\mathcal{A}| \cdot\left|\mathcal{C}_{A}\right|=\frac{(m-1)(m-2)}{6}\left|\mathcal{C}_{A}\right|=\left(n^{\prime}\right)^{2} \cdot \frac{(m-1)(m-2)}{6}=$ $\frac{(m-1)(m-2) n^{2}}{24}$, which is the right number of base blocks of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $H(m, n, 4,3)$ design. We need only to show that each triple from three distinct groups appears in at least one block of $\mathcal{D}^{\prime}$. Let $T=\left\{\left(i_{1}, \ell_{1}+2 k_{1}\right),\left(i_{2}, \ell_{2}+2 k_{2}\right),\left(i_{3}, \ell_{3}+2 k_{3}\right)\right\}$ be such a triple, where $i_{1}, i_{2}, i_{3}$ are distinct, $\ell_{j} \in\{0,1\}$ and $k_{j} \in\left\{0,1, \ldots, n^{\prime}-1\right\}$ for $j \in\{1,2,3\}$.

Since $\left\{A+\tau: A \in \mathcal{A}, \tau \in \mathbb{Z}_{m} \times \mathbb{Z}_{2}\right\}$ is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-invariant $H(m, 2,4,3)$ design, there is a base block $A=\left\{\left(a_{1}, c_{1}\right),\left(a_{2}, c_{2}\right),\left(a_{3}, c_{3}\right),\left(a_{4}, c_{4}\right)\right\} \in \mathcal{A}$ and an element $\left(\delta_{1}, \delta_{2}\right), 0 \leq \delta_{1}<m$ and $\delta_{2} \in$ $\{0,1\}$, such that $\left\{\left(i_{1}, \ell_{1}\right),\left(i_{2}, \ell_{2}\right),\left(i_{3}, \ell_{3}\right)\right\} \subset\left\{\left(a_{1}, c_{1}\right),\left(a_{2}, c_{2}\right),\left(a_{3}, c_{3}\right),\left(a_{4}, c_{4}\right)\right\}+\left(\delta_{1}, \delta_{2}\right)$. Without loss of generality, let $a_{j}+\delta_{1} \equiv i_{j}(\bmod m)$ and $\ell_{j} \equiv c_{j}+\delta_{2}(\bmod 2)$. Denote $c_{j}+\delta_{2}=\ell_{j}+2 \sigma_{j}$, $\sigma_{j} \in\{0,1\}$. Since $\mathcal{C}_{A}$ is the set of base blocks of a semi-cyclic $H\left(4, n^{\prime}, 4,3\right)$ design over $A \times \mathbb{Z}_{n^{\prime}}$, there is a base block $C=\left\{\left(a_{1}, c_{1}, d_{1}\right),\left(a_{2}, c_{2}, d_{2}\right),\left(a_{3}, c_{3}, d_{3}\right),\left(a_{4}, c_{4}, d_{4}\right)\right\} \in \mathcal{C}_{A}$ and an element $\delta_{3} \in$ $\left\{0,1, \ldots, n^{\prime}-1\right\}$ such that $k_{j}-\sigma_{j} \equiv d_{j}+\delta_{3}\left(\bmod n^{\prime}\right)$. It follows that $T \subset \varphi(C)+\left(\delta_{1}, \delta_{2}+2 \delta_{3}\right) \in \mathcal{D}^{\prime}$.

Finally, we construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{2}, 2 n, 4,3\right)$ design relative to $\frac{m}{2} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$.
For $1 \leq i<\frac{m}{2}$, construct a strictly semi-cyclic $G(2, n, 4,3)$ design on $\{0, i\} \times \mathbb{Z}_{n}$ with groups $\{0\} \times \mathbb{Z}_{n}$ and $\{i\} \times \mathbb{Z}_{n}$. Such a design exists by assumption. Denote the set of base blocks by $\mathcal{F}_{i}$ and let $\mathcal{F}=\cup_{1 \leq i<m / 2} \mathcal{F}_{i}$. It is easy to see that $\mathcal{D} \cup \mathcal{F}$ is the set of base blocks of the required strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{2}, 2 n, 4,3\right)$ design.

We illustrate the idea of Construction 3.4 with $m=4$ and $n=8$.

Example 3.5 There is a strictly $\mathbb{Z}_{4} \times \mathbb{Z}_{8}$-invariant $G(2,16,4,3)$ design relative to $2 \mathbb{Z}_{4} \times \mathbb{Z}_{8}$ and an optimal $(4,8,4,2)$-OOSPC with the size meeting the upper bound (1.1).

- Step 1: Since the trivial cyclic SQS(4) has a unique block $\{0,1,2,3\}$, we have $A=\{(0,0),(1,0)$, $(2,0),(3,1)\}$ is the unique base block of a strictly $\mathbb{Z}_{4} \times \mathbb{Z}_{2}$-invariant $H(4,2,4,3)$ design with groups $\{i\} \times \mathbb{Z}_{2}, i \in \mathbb{Z}_{4}$, i.e., $\mathcal{A}=\{A\}$.
- Step 2: Since $\{\{(0,0,0),(1,0, x),(2,0, y),(3,1, x+y)\}: 0 \leq x, y<4\}$ is the set of base blocks of a semi-cyclic $H(4,4,4,3)$ design on $A \times \mathbb{Z}_{4}$ with groups $\{z\} \times \mathbb{Z}_{4}(z \in A)$, by the mapping $\varphi: \mathbb{Z}_{4} \times \mathbb{Z}_{2} \times \mathbb{Z}_{4} \rightarrow \mathbb{Z}_{4} \times \mathbb{Z}_{8}$ defined by $\varphi(i, \ell, k)=(i, \ell+2 k)$ for $(i, \ell, k) \in \mathbb{Z}_{4} \times \mathbb{Z}_{2} \times \mathbb{Z}_{4}$ we
have

$$
\mathcal{D}=\{\{(0,0),(1,2 x),(2,2 y),(3,1+2 x+2 y)\}: 0 \leq x, y<4\}
$$

is the set of base blocks of a strictly $\mathbb{Z}_{4} \times \mathbb{Z}_{8}$-invariant $H(4,8,4,3)$ design with groups $\{i\} \times \mathbb{Z}_{8}$, $i \in \mathbb{Z}_{4}$.

- Step 3: Construct a strictly semi-cyclic $G(2,8,4,3)$ design on $\{0,1\} \times \mathbb{Z}_{8}$ with groups $\{i\} \times \mathbb{Z}_{8}$, $i \in\{0,1\}$, whose set $\mathcal{F}$ of base blocks consists of the following:

$$
\begin{array}{ll}
\{(0,0),(0,1),(1,0),(1,1)\}, & \{(0,0),(0,1),(1,2),(1,4)\}, \\
\{(0,0),(0,1),(1,3),(1,7)\}, & \{(0,0),(0,1),(1,5),(1,6)\}, \\
\{(0,0),(0,2),(1,0),(1,2)\}, & \{(0,0),(0,2),(1,1),(1,6)\}, \\
\{(0,0),(0,2),(1,3),(1,4)\}, & \{(0,0),(0,2),(1,5),(1,7)\}, \\
\{(0,0),(0,3),(1,0),(1,4)\}, & \{(0,0),(0,3),(1,1),(1,7)\}, \\
\{(0,0),(0,3),(1,2),(1,5)\}, & \{(0,0),(0,3),(1,3),(1,6)\}, \\
\{(0,0),(0,4),(1,0),(1,5)\}, & \{(0,0),(0,4),(1,2),(1,3)\} .
\end{array}
$$

Then $\mathcal{D} \cup \mathcal{F}$ is the set of base blocks of a strictly $\mathbb{Z}_{4} \times \mathbb{Z}_{8}$-invariant $G(2,16,4,3)$ design with groups $\{i, i+2\} \times \mathbb{Z}_{8}, 0 \leq i<2$.

Since there is a $(2,8,4,2)$-OOSPC with $J(2,8,4,2)$ codewords from [45], there is a strictly $\mathbb{Z}_{2} \times \mathbb{Z}_{8^{-}}$ invariant $P Q S(16)$ with $J(2,8,4,2)$ base blocks by Theorem 2.1. By Construction 3.3, there is a strictly $\mathbb{Z}_{4} \times \mathbb{Z}_{8}$-invariant $P Q S(32)$ with $J(4,8,4,2)$ base blocks, which leads to an optimal $(4,8,4,2)$ OOSPC with the size meeting the upper bound (1.1) by Theorem 2.1.

## 4 Constructions of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}(m, n, 4,3)$ designs

In this section, we give product constructions of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}(m, n, 4,3)$ designs.
Let $e, m, n$ be positive integers such that $m$ is divisible by $e$. Let $m-e \equiv n \equiv 0(\bmod 2)$. In a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e n, 4,3\right)$ design $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n},\left\{\left\{i, i+\frac{m}{e}, \ldots, m-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}, \mathcal{B}\right)$, there exist $n(m-e) / 2$ triples of the form $\{(0,0),(i, j),(-i,-j)\},(i, j) \in \mathbf{I} \times \mathbb{Z}_{n}$, and $(m-e) n$ triples of the form $\{(0,0),(i, j),(0, n / 2)\},(i, j) \in\left(\mathbb{Z}_{m} \backslash \frac{m}{e} \mathbb{Z}_{m}\right) \times \mathbb{Z}_{n}$, respectively, where $\mathbf{I}=\{k: 1 \leq k \leq$ $\left.\left\lfloor\frac{m}{2}\right\rfloor, k \not \equiv 0\left(\bmod \frac{m}{e}\right)\right\}$ and $\frac{m}{e} \mathbb{Z}_{m}=\left\{0, \frac{m}{e}, \ldots, m-\frac{m}{e}\right\}$. If any triple of the form $\{y, y+x, y-x\}$ or $\{y, y+z, y+(0, n / 2)\}$, where $x \in \mathbf{I} \times \mathbb{Z}_{n}, z \in\left(\mathbb{Z}_{m} \backslash \frac{m}{e} \mathbb{Z}_{m}\right) \times \mathbb{Z}_{n}$ and $y \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, is contained in the block $\{y, y+a, y-a, y+(0, n / 2)\}$ for some $a \in \mathbf{I} \times\left\{0,1, \ldots, \frac{n}{2}-1\right\}$, then such a $G\left(\frac{m}{e}, e n, 4,3\right)$ design is denoted by $G^{*}\left(\frac{m}{e}, e n, 4,3\right)$.

In Example 3.1, the first four base blocks generate 80 blocks which contain all triples of the form $\{y, y+x, y-x\},\{y, y+z, y+(0,1)\}$, where $x \in\{1,2,3,4\} \times \mathbb{Z}_{2}, z \in\left(\mathbb{Z}_{10} \backslash\{0,5\}\right) \times \mathbb{Z}_{2}$ and $y \in \mathbb{Z}_{10} \times \mathbb{Z}_{2}$, thereby, this $G$-design is also a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $G^{*}(5,4,4,3)$.

Two constructions for $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}$-designs are presented in Constructions 4.1 and 4.2. The proofs of constructions are of design theory. Here, we only describe how to construct them. The detailed proof of Construction 4.1 is moved to Appendix A. The detailed proof of Construction 4.2 is omitted, which is similar to that of Construction 4.1.

Construction 4.1 Let $m, n, e, g$ be positive integers such that $m$ is divisible by $e$, both $n$ and $m-e$ are even, $g$ is odd and $g \geq 3$. If there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}\right.$, en, 4,3$)$ design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, then there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $G^{*}\left(\frac{m}{e}\right.$, eng, 4, 3) design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n g}$.

Proof Let $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n},\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}, \mathcal{B}\right)$ be a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n^{-}}$ invariant $G^{*}\left(\frac{m}{e}, e n, 4,3\right)$ design. Let $\mathbf{I}=\left\{i: 1 \leq i \leq\left\lfloor\frac{m}{2}\right\rfloor, i \not \equiv 0\left(\bmod \frac{m}{e}\right)\right\}$. Denote the family
of base blocks of this design by $\mathcal{F}=\mathcal{F}_{1} \bigcup \mathcal{F}_{2}$, where $\mathcal{F}_{1}$ consists of all base blocks in the form of $\left\{(0,0),\left(0, \frac{n}{2}\right),(i, j),(-i,-j)\right\}$ for $i \in \mathbf{I}$ and $0 \leq j<\frac{n}{2}$, and $\mathcal{F}_{2}$ consists of all the other base blocks. It is easy to see that $\left|\mathcal{F}_{1}\right|=n(m-e) / 4$ and $\left|\mathcal{F}_{2}\right|=n(m-e)(m n+e n-9) / 24$. We construct the required $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $G^{*}\left(\frac{m}{e}, e n g, 4,3\right)$ design on $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ with group set $\mathcal{G}=\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n g}: 0 \leq i<\frac{m}{e}\right\}$.

Define

$$
\begin{aligned}
& \mathcal{C}_{1}=\left\{\left\{\left(i_{0}, j_{0}\right),\left(i_{1}, j_{1}+k_{1} n\right),\left(i_{2}, j_{2}+k_{2} n\right),\left(i_{3}, j_{3}+k_{1} n+k_{2} n\right)\right\}:\left\{\left(i_{0}, j_{0}\right), \ldots,\left(i_{3}, j_{3}\right)\right\} \in \mathcal{F}_{2},\right. \\
& \mathcal{C}_{2}=\left\{\left\{(0,0),\left(i, j^{\prime}\right),\left(-i,-j^{\prime}\right),\left(0, \frac{n g}{2}\right)\right\}: i \in \mathbf{I}, 0 \leq j^{\prime}<\frac{n g}{2}\right\},\left.0 \leq k_{1}, k_{2}<g\right\}, \\
& \mathcal{C}_{3}=\left\{\left\{(0,0),(i, j+\ell n),\left(i, j+\ell^{\prime} n\right),\left(2 i, 2 j+\ell n+\ell^{\prime} n\right)\right\}: i \in \mathbf{I}, 0 \leq j<n, 0 \leq \ell<\ell^{\prime}<g\right\}, \\
& \mathcal{C}_{4}=\left\{\left\{(0,0),\left(0, \frac{n}{2}+\ell n\right),\left(i, j+\ell^{\prime} n\right),\left(i, j+\frac{n}{2}+\ell n+\ell^{\prime} n\right)\right\}: i \in \mathbf{I}, 0 \leq j<\frac{n}{2}, 0 \leq \ell<\frac{g-1}{2},\right. \\
&\left.0 \leq \ell^{\prime}<g\right\} .
\end{aligned}
$$

Note that for each base block $B=\left\{\left(i_{0}, j_{0}\right),\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right),\left(i_{3}, j_{3}\right)\right\} \in \mathcal{F}_{2}, \mathcal{A}_{B}=\left\{\left\{\left(i_{0}, j_{0}\right),\left(i_{1}, j_{1}+\right.\right.\right.$ $\left.\left.\left.k_{1} n\right),\left(i_{2}, j_{2}+k_{2} n\right),\left(i_{3}, j_{3}+k_{1} n+k_{2} n\right)\right\}: 0 \leq k_{1}, k_{2}<g\right\}$ is the set of base blocks of a semi-cyclic $H(4, g, 4,3)$ design on $\{(x, y+k n):(x, y) \in B, 0 \leq k<g\}$ with group set $\{\{(x, y+k n): 0 \leq k<$ $g\}:(x, y) \in B\}$ through $+(0, n) \bmod (m, n g)$.

Let $\mathcal{C}_{i}^{\prime}=\left\{C+\delta: C \in \mathcal{C}_{i}, \delta \in \mathbb{Z}_{m} \times \mathbb{Z}_{n g}\right\}$ for $1 \leq i \leq 4$. Denote $\mathcal{C}^{\prime}=\mathcal{C}_{1}^{\prime} \cup \mathcal{C}_{2}^{\prime} \cup \mathcal{C}_{3}^{\prime} \cup \mathcal{C}_{4}^{\prime}$. We claim that $\mathcal{C}^{\prime}$ is the set of blocks of the required strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $G^{*}\left(\frac{m}{e}, e n g, 4,3\right)$ design.

Construction 4.2 Let $m, n, e, g$ be positive integers such that $m$ is divisible by $e$, both $n$ and $m-e$ are even, $g$ is odd and $g \geq 3$. If there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}\right.$, en, 4, 3) design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, then there exists a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}\right.$, egn, 4,3$)$ design relative to $\frac{m}{e} \mathbb{Z}_{m g} \times \mathbb{Z}_{n}$.

Proof We keep the notations of Construction 4.1 and we adapt the proof to the present situation. Define

$$
\begin{aligned}
& \mathcal{D}_{1}=\left\{\left\{\left(i_{0}, j_{0}\right),\left(i_{1}+k_{1} m, j_{1}\right),\left(i_{2}+k_{2} m, j_{2}\right),\left(i_{3}+k_{1} m+k_{2} m, j_{3}\right)\right\}:\left\{\left(i_{0}, j_{0}\right), \ldots,\left(i_{3}, j_{3}\right)\right\} \in \mathcal{F}_{2},\right. \\
&\left.0 \leq k_{1}, k_{2}<g\right\}, \\
& \mathcal{D}_{2}=\left\{\left\{(0,0),(i+\ell m, j),(-i-\ell m,-j),\left(0, \frac{n}{2}\right)\right\}: i \in \mathbf{I}, 0 \leq j<\frac{n}{2}, 0 \leq \ell<g\right\} \\
& \mathcal{D}_{3}=\left\{\left\{(0,0),(i+\ell m, j),\left(i+\ell^{\prime} m, j\right),\left(2 i+\ell m+\ell^{\prime} m, 2 j\right)\right\}: i \in \mathbf{I}, 0 \leq j<n, 0 \leq \ell<\ell^{\prime}<g\right\}, \\
& \mathcal{D}_{4}=\left\{\left\{(0,0),\left(\ell m, \frac{n}{2}\right),\left(i+\ell^{\prime} n, j\right),\left(i+\ell m+\ell^{\prime} m, \frac{n}{2}+j\right)\right\}: i \in \mathbf{I}, 0 \leq j<\frac{n}{2}, 1 \leq \ell \leq \frac{g-1}{2}, 0 \leq \ell^{\prime}<g\right\} .
\end{aligned}
$$

Let $\mathcal{D}_{i}^{\prime}=\left\{D+\delta: D \in \mathcal{D}_{i}, \delta \in \mathbb{Z}_{m g} \times \mathbb{Z}_{n}\right\}$ for $1 \leq i \leq 4$. Denote $\mathcal{D}^{\prime}=\mathcal{D}_{1}^{\prime} \cup \mathcal{D}_{2}^{\prime} \cup \mathcal{D}_{3}^{\prime} \cup \mathcal{D}_{4}^{\prime}$. Similar to the proof of Theorem 4.1, it is readily checked that $\mathcal{D}^{\prime}$ is the set of blocks of the required strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}\right.$, egn, 4,3$)$ design relative to $\frac{m}{e} \mathbb{Z}_{m g} \times \mathbb{Z}_{n}$.

Example 4.3 There is a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$-invariant $G^{*}(5,20,4,3)$ design relative to $5 \mathbb{Z}_{10} \times \mathbb{Z}_{10}$ and an optimal (10, 10, 4, 2)-OOSPC with the size meeting the upper bound (1.1).

Proof As it has been pointed out before Construction 4.1, the strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $G(5,4,4,3)$ design relative to $5 \mathbb{Z}_{10} \times \mathbb{Z}_{2}$ in Example 3.1 is also a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $G^{*}(5,4,4,3)$ design. Applying Construction 4.1 with $g=5$ yields a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$-invariant $G^{*}(5,20,4,3)$ design relative to $5 \mathbb{Z}_{10} \times \mathbb{Z}_{10}$. Since a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $G^{*}(5,4,4,3)$ design is also a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{2}$-invariant $P Q S(20)$ with $J(10,2,4,3)$ base blocks, by Construction 3.3 there is a strictly $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$-invariant $P Q S(100)$ with $J(10,10,4,2)$ base blocks, which leads to an optimal $(10,10,4,2)$ OOSPC with the size meeting the upper bound (1.1) by Theorem 2.1.

## 5 Constructions of strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G(m, n, 4,3)$ designs via 1-fan designs

In this section, use $s$-fan designs admitting an automorphism group to construct strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n^{-}}$ invariant $G(m, n, 4,3)$ designs.

Let $s$ be a non-negative integer and $K_{0}, K_{1}, \ldots, K_{s}$ be sets of positive integers. An $s$-fan design is an $(s+3)$-tuple $\left(X, \mathcal{G}, \mathcal{B}_{0}, \ldots, \mathcal{B}_{s}\right)$ where $X$ is a set of $m n$ points, $\mathcal{G}$ is a partition of $X$ into $m$ sets of size $n$ (called groups) and $\mathcal{B}_{0}, \ldots, \mathcal{B}_{s}$ are sets of subsets of $X$ satisfying that each $\left(X, \mathcal{G}, \mathcal{B}_{i}\right)$ is an $H\left(m, n, K_{i}, 2\right)$ design for $0 \leq i<s$ and $\left(X, \mathcal{G}, \mathcal{B}_{0} \cup \cdots \cup \mathcal{B}_{s}\right)$ is a $G\left(m, n, K_{1} \cup \cdots \cup K_{s}, 3\right)$ design. For simplicity, it is denoted by $s$ - $\mathrm{FG}\left(3,\left(K_{0}, \ldots, K_{s}\right), m n\right)$ of type $n^{m}$. Note that a $0-\mathrm{FG}$ is nothing but a $G$-design. For general $s$-fan designs, the reader is referred to [23].

An automorphism group of an $s$-fan design $\left(X, \mathcal{G}, \mathcal{B}_{0}, \ldots, \mathcal{B}_{s}\right)$ is a permutation group on $X$ leaving $\mathcal{G}, \mathcal{B}_{0}, \ldots, \mathcal{B}_{s}$ invariant, respectively. All automorphisms of an $s$-fan design form a group, called the full automorphism group of the $s$-fan design. Any subgroup of the full automorphism group is called an automorphism group of the $s$-fan design. An $s$-fan design $\left(X, \Gamma, \mathcal{B}_{0}, \ldots, \mathcal{B}_{s}\right)$ is said to be $G$ invariant if it admits $G$ as a point-regular automorphism group. Moreover, it is said to be strictly $G$-invariant if it is $G$-invariant and the stabilizer of each $B \in \mathcal{B}$ under $G$ equals the identity of $G$. For a $G$-invariant $s$-FG, we always identify the point set $X$ with $G$ and the automorphisms are regarded as translations $\sigma_{a}$ defined by $\sigma_{a}(x)=x+a$ for $x \in G$, where $a \in G$. Let $L$ be a subgroup of $G$. If the group set of a $G$-invariant $s$-FG is a set of cosets of $L$ in $G$, then it is a $G$-invariant $s$-FG relative to $L$.

Example $5.1\left(\mathbb{Z}_{3} \times \mathbb{Z}_{3},\left\{\{i\} \times \mathbb{Z}_{3}: i \in \mathbb{Z}_{3}\right\}, \mathcal{B}_{0}, \mathcal{B}_{1}\right)$ is a $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$-invariant 1- $F G(3,(3,4), 9)$ of type $3^{3}$, where $\mathcal{B}_{0}$ and $\mathcal{B}_{1}$ are as follows:

$$
\begin{aligned}
& \mathcal{B}_{0}=\bigcup_{0 \leq i \leq 2}\{\{(0, i),(1, i),(2, i)\},\{(i, 0),(i+1,1)(i+2,2)\},\{(i, 0),(i+2,1),(i+1,2)\}\}, \\
& \mathcal{B}_{1}=\bigcup_{0 \leq i, j \leq 2}\{\{(i, j+1),(i, j+2),(i+1, j),(i+2, j)\},\{(i, j),(i, j+1),(i+1, j),(i+1, j+1)\}\} .
\end{aligned}
$$

An $s$-fan design of type $n^{m}\left(X, \Gamma, \mathcal{B}_{0}, \ldots, \mathcal{B}_{s}\right)$ is said to be semi-cyclic if the $s$-fan design admits an automorphism $\sigma$ consisting of $m$ cycles of length $n$ and leaving each group, $\mathcal{B}_{0}, \ldots, \mathcal{B}_{s}$ invariant. For a semi-cyclic $s$-fan design of type $n^{m}$, without loss of generality we can identify the point set $X$ with $I_{m} \times \mathbb{Z}_{n}$, and the automorphism $\sigma$ can be taken as $(i, j) \mapsto(i, j+1)(-, \bmod n),(i, j) \in I_{m} \times \mathbb{Z}_{n}$.

A rotational $\operatorname{SQS}(m+1)$ is an $\operatorname{SQS}(m+1)$ with an automorphism consisting of a cycle of length $m$ and one fixed point. Such a design is denoted by RoSQS $(m+1)$. As pointed out in [19], there is an equivalence between 1-FGs and RoSQSs as follows.

Lemma 5.2 [19] An $\operatorname{RoSQS}(m+1)$ with $m \equiv 1(\bmod 6)$ is equivalent to a strictly cyclic 1$F G(3,(3,4), m)$ of type $1^{m}$. An $\operatorname{RoSQS}(m+1)$ with $m \equiv 3(\bmod 6)$ is equivalent to a strictly cyclic $1-F G(3,(3,4), m)$ of type $3^{m / 3}$.

Bitan and Etzion have pointed out in [5] that the existence of an $\operatorname{RoSQS}(v+1)$ implies the existence of an optimal 1-D $(v, 4,2)$-OOC. Similarly, we can give the following relationship.

Lemma 5.3 Let $m n \equiv 1,3(\bmod 6)$. Then there is an optimal $(m, n, 4,2)$-OOSPC with the size attaining the upper bound (1.1) if and only if there is a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, (3, 4), mn) of type $1^{m n}$.

Proof Suppose that $\mathcal{C}$ is an $(m, n, 4,2)$-OOSPC with the size attaining the upper bound (1.1). By Theorem 2.1, there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant PQS $(m n)$ with $\frac{(m n-1)(m n-3)}{24}$ base blocks, whose set of base blocks is denoted by $\mathcal{B}$. Then, there are $\frac{m n(m n-1)}{6}$ triples in the leave $\mathcal{L}$, and the leave is $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant. Clearly, for any pair $\left\{\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right\}$ of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ there is at least one triple in
the leave containing $\left\{\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right\}$ since $m n-2$ is odd. It follows that there are at least $\frac{m n(m n-1)}{6}$ triples in the leave. Consequently, each pair occurs in exactly one triple in the leave, i.e., the leave is the block set of an $\operatorname{STS}(m n)$ over $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ admitting $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ as a point-regular automorphism group. So, $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n},\left\{\{x\}: x \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}\right\}, \mathcal{L}, \mathcal{B}\right)$ is a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, (3, 4),mn) of type $1^{m n}$.

Conversely, there are $\frac{m n(m n-1)(m n-3)}{24}$ quadruples in a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, (3, 4), mn) of type $1^{m n}$, and all orbits of quadruples are full under $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. Therefore, all quadruples form a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant PQS $(m n)$, which leads to an ( $m, n, 4,2$ )-OOSPC by Theorem 2.1 with the size attaining the upper bound (1.1).

In [19], Feng et al. showed that there exists a semi-cyclic 1-FG $(3,(3,4), 3 h)$ of type $h^{3}$ if there exists an RoSQS $(h+1)$. By the necessary condition, $h$ must be odd. It follows that all orbits of quadruples of a semi-cyclic 1-FG $(3,(3,4), 3 h)$ of type $h^{3}$ are full. Since the blocks of size three are from three distinct groups, all block-orbits of size three in a semi-cyclic 1-FG(3, $(3,4), 3 h)$ are also full. So, a semi-cyclic 1-FG $(3,(3,4), 3 h)$ of type $h^{3}$ must be strictly semi-cyclic.

Lemma 5.4 If there exists an RoSQS(h+1), then there exists a strictly semi-cyclic 1-FG(3, (3, 4), 3h) of type $h^{3}$.

Hartman established a fundamental construction for 3-designs [23]. By using it, Hartman gave a new existence proof of Steiner quadruple systems. The following is a special case.

Theorem 5.5 [23] Suppose there is a $1-F G\left(3,\left(K_{0}, K_{1}\right)\right.$, mn) of type $n^{m}$ (called a master design). If there exists an $s-F G\left(3,\left(L_{0}, L_{1}, \ldots, L_{s}\right), g k\right)$ of type $g^{k}$ for any $k \in K_{0}$ and an $H\left(k, g, L_{s}, 3\right)$ design for any $k \in K_{s}$, then there exists an $s-F G\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(n g)^{m}$.

By using Theorem 5.5, Feng et al. established a recursive construction for strictly cyclic s-fan designs [18]. We generalize it as follows. The detailed proof Construction 5.6 is moved to Appendix B.

Construction 5.6 Suppose there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, $\left(K_{0}, K_{1}\right)$, mn) of type $(e n)^{m / e}$ relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$ (called a master design). If there exists a strictly semi-cyclic s$F G\left(3,\left(L_{0}, L_{1}, \ldots, L_{s}\right), g k\right)$ of type $g^{k}$ for any $k \in K_{0}$, and a semi-cyclic $H\left(k, g, L_{s}, 3\right)$ design for any $k \in K_{1}$, then there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant s-FG(3, $\left.L_{0}, \ldots, L_{s}\right)$, mng) of type (eng $)^{m / e}$ relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ and a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant s-FG(3,(L, $\left., \ldots, L_{s}\right)$, mng) of type $(\text { eng })^{m / e}$ relative to $\frac{m}{e} \mathbb{Z}_{m g} \times \mathbb{Z}_{n}$.

Proof Let $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n}, \mathcal{G}, \mathcal{B}_{0}, \mathcal{B}_{1}\right)$ be a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG $\left(3,\left(K_{0}, K_{1}\right)\right.$, mn $)$ of type $(e n)^{m / e}$ where $\mathcal{G}=\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}$. Denote the family of base blocks of this design by $\mathcal{F}=\mathcal{F}_{0} \cup \mathcal{F}_{1}$, where $\mathcal{F}_{0}$ and $\mathcal{F}_{1}$ generate all blocks of $\mathcal{B}_{0}$ and $\mathcal{B}_{1}$, respectively.

For each base block $B \in \mathcal{F}_{0}$, construct a strictly semi-cyclic $s$ - $\mathrm{FG}\left(3,\left(L_{0}, \ldots, L_{s}\right),|B| g\right)$ of type $g^{|B|}$ on $B \times \mathbb{Z}_{g}$ with group set $\left\{\{x\} \times \mathbb{Z}_{g}: x \in B\right\}$. Denote the family of base blocks of the $j$-th subdesign $H\left(|B|, g, L_{j}, 2\right)$ design by $\mathcal{A}_{B}^{j}$ for $0 \leq j<s$, and denote the family of all the other base blocks by $\mathcal{A}_{B}^{s}$. Let $\mathcal{A}_{B}=\bigcup_{j=0}^{s} \mathcal{A}_{B}^{j}$.

For each base block $B \in \mathcal{F}_{1}$, construct a semi-cyclic $H\left(|B|, g, L_{s}, 3\right)$ design on $B \times \mathbb{Z}_{g}$ with groups $\left\{\{x\} \times \mathbb{Z}_{g}: x \in B\right\}$. Denote the family of base blocks of this design by $\mathcal{D}_{B}$.

Let $\mathcal{A}_{j}=\bigcup_{B \in \mathcal{F}_{0}} \mathcal{A}_{B}^{j}$ for $0 \leq j<s$ and $\mathcal{A}_{s}=\left(\bigcup_{B \in \mathcal{F}_{0}} \mathcal{A}_{B}^{s}\right) \bigcup\left(\bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}\right)$, and $\mathcal{G}^{\prime}=\{\{i, i+$ $\left.\left.\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n g}: 0 \leq i<\frac{m}{e}\right\}$. Define a mapping $\tau$ from $\mathbb{Z}_{m} \times \mathbb{Z}_{n} \times \mathbb{Z}_{g}$ to $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ by $\tau(x, y, z)=(x, y+z n)$. Now we construct a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $s$-FG $\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(e n g)^{m / e}$ as follows: For each $C \in\left(\bigcup_{0 \leq j \leq s} \mathcal{A}_{j}\right)$, define $\tau(C)=\{\tau(c): c \in C\}$. For $0 \leq j \leq s$, let

$$
\mathcal{A}_{j}^{*}=\bigcup_{C \in \mathcal{A}_{j}} \tau(C)
$$

$$
\mathcal{A}_{j}^{\prime}=\left\{A+\delta: A \in \mathcal{A}_{j}^{*}, \delta \in \mathbb{Z}_{m} \times \mathbb{Z}_{n g}\right\}
$$

where $A+\delta=\{u+\delta: u \in A\}$. We claim that $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n g}, \mathcal{G}^{\prime}, \mathcal{A}_{0}^{\prime}, \ldots, \mathcal{A}_{s}^{\prime}\right)$ is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g^{-}}$ invariant $s$-FG $\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(e n g)^{m / e}$.

Let $\mathcal{G}^{\prime \prime}=\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m g-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}$. Define a mapping $\varphi$ from $\mathbb{Z}_{m} \times$ $\mathbb{Z}_{n} \times \mathbb{Z}_{g}$ to $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$ by $\tau(x, y, z)=(x+z m, y)$. Now we construct a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $s$-FG $\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(e n g)^{m / e}$ as follows: For each $C \in\left(\bigcup_{0 \leq j \leq s} \mathcal{A}_{j}\right)$, define $\varphi(C)=$ $\{\varphi(c): c \in C\}$. For $0 \leq j \leq s$, let

$$
\begin{gathered}
\mathcal{A}_{j}^{* *}=\bigcup_{C \in \mathcal{A}_{j}} \varphi(C) \\
\mathcal{A}_{j}^{\prime \prime}=\left\{A+\delta: A \in \mathcal{A}_{j}^{* *}, \delta \in \mathbb{Z}_{m g} \times \mathbb{Z}_{n}\right\},
\end{gathered}
$$

where $A+\delta=\{u+\delta: u \in A\}$. Similarly, it is readily checked that $\left(\mathbb{Z}_{m g} \times \mathbb{Z}_{n}, \mathcal{G}^{\prime \prime}, \mathcal{A}_{0}^{\prime \prime}, \ldots, \mathcal{A}_{s}^{\prime \prime}\right)$ is a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $s$-FG $\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(e n g)^{m / e}$.

Corollary 5.7 Suppose that there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}\right.$, en, 4,3$)$ design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$ such that all elements of order 2 are contained in $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. If there is a strictly semicyclic $G(2, g, 4,3)$ design, then there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $G\left(\frac{m}{e}\right.$, emg, 4,3$)$ design relative to $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ and a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}\right.$, emg, 4,3) design relative to $\frac{m}{e} \mathbb{Z}_{m g} \times$ $\mathbb{Z}_{n}$.

Proof Let $\mathcal{B}_{1}$ be the set of blocks of a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e n, 4,3\right)$ design on $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ with group set $\mathcal{G}=\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times \mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}$. Let $\mathcal{F}_{0}=\{\{(0,0),(i, j)\}$ : $\left.1 \leq i \leq\left\lfloor\frac{m}{2}\right\rfloor, i \not \equiv 0\left(\bmod \frac{m}{e}\right), j \in \mathbb{Z}_{n}\right\}$ and $\mathcal{B}_{0}=\left\{P+\delta: P \in \mathcal{F}_{0}, \delta \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}\right\}$. Since all elements of order 2 are contained in $\frac{m}{e} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, the quadruple $\left(X, \mathcal{G}, \mathcal{B}_{0}, \mathcal{B}_{1}\right)$ is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n^{-}}$ invariant 1-FG $(3,(2,4), m n)$ of type $(e n)^{m / e}$. Since there is a strictly semi-cyclic $G(2, g, 4,3)$ design by assumption and a semi-cyclic $H(4, g, 4,3)$ design by Lemma 3.2, applying Construction 5.6 yields a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $G\left(\frac{m}{e}, e m g, 4,3\right)$ design and a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{n}$-invariant $G\left(\frac{m}{e}, e m g, 4,3\right)$ design.

Corollary 5.8 Suppose there is an $\operatorname{RoSQS}(m+1)$ and an $\operatorname{RoSQS}(n+1)$. If $m \equiv 1(\bmod 6)$ then there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $1-F G(3,(3,4), m n)$ of type $n^{m}$ relative to $\{0\} \times \mathbb{Z}_{n}$. If $m \equiv 3$ $(\bmod 6)$ then there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $1-F G(3,(3,4), m n)$ of type $(3 n)^{m / 3}$ relative to $\frac{m}{3} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$.

Proof Since there is an $\operatorname{RoSQS}(m+1)$ by assumption, there is a strictly cyclic 1-FG( $3,(3,4), m)$ of type $1^{m}$ if $m \equiv 1(\bmod 6)$, a strictly cyclic $1-\mathrm{FG}(3,(3,4), m)$ of type $3^{m / 3}$ if $m \equiv 3(\bmod 6)$ by Lemma 5.2. Since there is an $\operatorname{RoSQS}(n+1)$, there is a strictly semi-cyclic $1-F G(3,(3,4), 3 n)$ of type $n^{3}$ by Lemma 5.4. Also, there is a semi-cyclic $H(4, n, 4,3)$ design by Lemma 3.2. Therefore, applying Construction 5.6 gives a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, (3, 4), mn) of type $n^{m}$ if $m \equiv 1$ $(\bmod 6)$, a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $1-\mathrm{FG}(3,(3,4), m n)$ of type $(3 n)^{m / 3}$ if $m \equiv 3(\bmod 6)$.

Corollary 5.9 If there is an $\operatorname{RoSQS}(m+1)$ and a strictly semi-cyclic $G(3, g, 4,3)$ design, then there exists a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{g}$-invariant $G(m, g, 4,3)$ design relative to $\{0\} \times \mathbb{Z}_{g}$ if $m \equiv 1(\bmod 6)$, and a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{g}$-invariant $G\left(\frac{m}{3}, 3 g, 4,3\right)$ design relative to $\frac{m}{3} \mathbb{Z}_{m} \times \mathbb{Z}_{g}$ if $m \equiv 3(\bmod 6)$.

Proof Since there is an RoSQS $(m+1)$ by assumption, there is a strictly cyclic 1-FG $(3,(3,4), m)$ of type $1^{m}$ if $m \equiv 1(\bmod 6)$, a strictly cyclic $1-\mathrm{FG}(3,(3,4), m)$ of type $3^{m / 3}$ if $m \equiv 3(\bmod 6)$ by Lemma 5.2. Since there is a strictly semi-cyclic $G(3, g, 4,3)$ design by assumption and a semi-cyclic $H(4, g, 4,3)$ design by Lemma 3.2, applying Construction 5.6 gives the conclusion.

## 6 New ( $m, n, 4,2$ )-OOSPCs

In this section, we use constructions in Sections 3, 4 and 5 to establish new optimal ( $m, n, 4,2$ )OOSPCs.

Since the survey of Lindner and Rosa [32], many recursive constructions for cyclic SQSs have been given, including the doubling construction, product constructions. Recently, Feng et al. established some recursive constructions for strictly cyclic 3-designs, as corollaries, many known constructions for strictly cyclic Steiner quadruple systems were unified [18]. The work of Köhler on $S$-cyclic SQS has been extended by Bitan and Etzion [5], Siemon [47], [48], [49], [50]. Although a great deal has been done on cyclic SQSs, the spectrum remains wide open.

A cyclic $\operatorname{SQS}(v)\left(\mathbb{Z}_{v}, \mathcal{B}\right)$ is said to be $S$-cyclic if each block satisfies $-B=B+a$ for some $a \in \mathbb{Z}_{v}$. Piotrowski gave necessary and sufficient conditions for the existence of an $S$-cyclic SQS $(v)$ [40].

Theorem $6.1[40]$ An $S$-cyclic $S Q S(v)$ exists if and only if $v \equiv 0(\bmod 2), v \not \equiv 0(\bmod 3), v \not \equiv 0$ $(\bmod 8), v \geq 4$ and if for any prime divisor $p$ of $v$ there exists an $S$-cyclic $S Q S(2 p)$.

Theorem $6.2[5]$ For any prime $p \equiv 5(\bmod 12)$ with $p<1500000$, there is an $S$-cyclic $S Q S(4 p)$.

Theorem 6.3 [33] For each prime $p \equiv 1(\bmod 4)$ with $p \leq 10^{5}$, there is an $S$-cyclic $S Q S(2 p)$.

Let $n$ be even and $m n \equiv n(\bmod 4)$. In a cyclic $G(m, n, 4,3)$ design, if any triple of form $\{j, j+i, j+2 i\}$ or $\{j, j+i, j+m n / 2\}$, where $1 \leq i \leq m n / 2, i \not \equiv 0(\bmod m)$ and $0 \leq j \leq m n-1$, is contained in the block $\left\{j, j+a, j-a, j+\frac{m n}{2}\right\}$ for some $1 \leq a \leq\left\lfloor\frac{m n}{4}\right\rfloor$ and $a \not \equiv 0(\bmod m)$, then such a cyclic $G$-design is denoted by cyclic $G^{*}(m, n, 4,3)$ design. As pointed out in [18], a cyclic $G^{*}(m, n, 4,3)$ is always strictly cyclic. The following recursive construction for cyclic $G^{*}$-designs was given in [18]. For the completeness, we describe how to construct a cyclic $G^{*}(m, n g, 4,3)$ design from a cyclic $G^{*}(m, n, 4,3)$ design here.

Theorem 6.4 [18] If there exists a cyclic $G^{*}(m, n, 4,3)$ design, then there exists a cyclic $G^{*}(m, n g, 4,3)$ design for any odd integer $g$.

Proof Let $\left(\mathbb{Z}_{m n},\{\{i, i+m, \ldots, i+m m-m\}: 0 \leq i<m\}, \mathcal{B}\right)$ be a cyclic $G^{*}(m, n, 4,3)$ design. Denote the family of base blocks of this design by $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2}$, where $\mathcal{F}_{1}$ consists of all base blocks in the form of $\left\{0, \frac{m n}{2}, i,-i,\right\}, 1 \leq i \leq\left\lfloor\frac{m n}{4}\right\rfloor$ and $i \not \equiv 0(\bmod m)$, and $\mathcal{F}_{2}$ consists of all the other base blocks. It is easy to see that $\left|\mathcal{F}_{1}\right|=n(m-1) / 4$ and $\left|\mathcal{F}_{2}\right|=n(m-1)(m n+n-9) / 24$.

Define

$$
\begin{aligned}
& \mathcal{D}_{1}=\left\{\left\{i_{0}, i_{1}+k_{1} m n, i_{2}+k_{2} m n, i_{3}+k_{1} m n+k_{2} m n\right\}:\left\{i_{0}, i_{1}, i_{2}, i_{3}\right\} \in \mathcal{F}_{2}, 0 \leq k_{1}, k_{2}<g\right\}, \\
& \mathcal{D}_{2}=\left\{\left\{0, i,-i, \frac{m n g}{2}\right\}: 1 \leq i \leq \frac{m n g}{4}, i \not \equiv 0(\bmod m)\right\}, \\
& \mathcal{D}_{3}=\left\{\left\{0, i+\ell m n, i+\ell^{\prime} m n, 2 i+\ell m n+\ell^{\prime} m n\right\}: 1 \leq i \leq \frac{m n}{2}, i \neq 0(\bmod m), 0 \leq \ell<\ell^{\prime}<g\right\}, \\
& \mathcal{D}_{4}=\left\{\left\{0, \frac{m n}{2}+\ell m n, i+\ell^{\prime} m n, i+\frac{m n}{2}+\ell m n+\ell^{\prime} m n\right\}: 1 \leq i \leq \frac{m n}{4}, i \neq 0(\bmod m),\right. \\
& \left.0 \leq \ell<\frac{g-1}{2}, 0 \leq \ell^{\prime}<g\right\} .
\end{aligned}
$$

Let $\mathcal{D}_{i}^{\prime}=\left\{D+\delta: D \in \mathcal{D}_{i}, \delta \in \mathbb{Z}_{m n g}\right\}$ for $1 \leq i \leq 4$ and $\mathcal{D}^{\prime}=\mathcal{D}_{1}^{\prime} \cup \mathcal{D}_{2}^{\prime} \cup \mathcal{D}_{3}^{\prime} \cup \mathcal{D}_{4}^{\prime}$. Then $\mathcal{D}^{\prime}$ is the set of blocks of the required cyclic $G^{*}(m, n g, 4,3)$ design on $\mathbb{Z}_{m n g}$ with the group set $\{\{i, i+m, \ldots, i+m n g-m\}: 0 \leq i<m\}$.

Theorem 6.5 Let $m, n, g$ be odd integers such that there is an $S$-cyclic $S Q S(2 p)$ for each prime divisor $p$ of $m$ and $n$. If there is an optimal $1-D\left(2^{\epsilon} g, 4,2\right)-O O C$ with $J\left(1,2^{\epsilon} g, 4,2\right)$ codewords, then there is an optimal $\left(m, 2^{\epsilon} n g, 4,2\right)$-OOSPC and an optimal $\left(m g, 2^{\epsilon} n, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1), where $\epsilon \in\{1,2\}$.

Proof Since there is an $S$-cyclic $\operatorname{SQS}(2 p)$ for each prime divisor $p$ of $m$ and $n$, there is an $S$ cyclic $\operatorname{SQS}\left(2^{\varepsilon} m\right)$ and an $S$-cyclic $\operatorname{SQS}\left(2^{\varepsilon} n\right)$ by Theorem 6.1 , where $\epsilon \in\{1,2\}$. Since $\mathbb{Z}_{2^{\varepsilon} m}$ is isomorphic to $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\epsilon}}$, the existence of an $S$-cyclic $\operatorname{SQS}\left(2^{\varepsilon} m\right)$ implies that there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\text {- }} \text { invariant }} G^{*}\left(m, 2^{\varepsilon}, 4,3\right)$ design. Applying Construction 4.1 gives a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\epsilon} n g^{-}}$ invariant $G^{*}\left(m, 2^{\varepsilon} n g, 4,3\right)$ design relative to $\{0\} \times \mathbb{Z}_{2^{\epsilon} n g}$ and applying Construction 4.2 gives a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{2^{\epsilon} n^{-} \text {-invariant }} G^{*}\left(m, 2^{\varepsilon} n g, 4,3\right)$ design relative to $m \mathbb{Z}_{m g} \times \mathbb{Z}_{2^{\epsilon} n}$.

Since an $S$-cyclic $\operatorname{SQS}\left(2^{\varepsilon} n\right)$ implies the existence of a cyclic $G^{*}\left(n, 2^{\epsilon}, 4,3\right)$ design, applying Theorem 6.4 gives a cyclic $G^{*}\left(n, 2^{\varepsilon} g, 4,3\right)$ design. Since there is an optimal 1-D $\left(2^{\epsilon} g, 4,2\right)$-OOC with $J\left(1,2^{\epsilon} g, 4,2\right)$ codewords by assumption which corresponds to a strictly cyclic PQS $\left(2^{\epsilon} g\right)$ with $J\left(1,2^{\epsilon} g, 4,2\right)$ base blocks, there is a strictly cyclic $\operatorname{PQS}\left(2^{\epsilon} n g\right)$ with $J\left(1,2^{\varepsilon} n g, 4,2\right)$ base blocks by Construction 3.3. When we input this cyclic $\operatorname{PQS}\left(2^{\epsilon} n g\right)$ into the strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\epsilon} n g \text {-invariant }}$ $G^{*}\left(m, 2^{\varepsilon} n g, 4,3\right)$ design, applying Construction 3.3 gives a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\epsilon} n g \text {-invariant } \operatorname{PQS}\left(2^{\epsilon} m n g\right) ~}^{\text {- }}$ with $J\left(m, 2^{\epsilon} n g, 4,2\right)$ base blocks, which leads to an optimal $\left(m, 2^{\varepsilon} n g, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1).

Since an $S$-cyclic $\operatorname{SQS}\left(2^{\varepsilon} n\right)$ implies the existence of a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{2^{\epsilon} \text {-invariant }} G^{*}\left(n, 2^{\varepsilon}, 4,3\right)$ design, applying Construction 4.1 gives a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{2^{\epsilon} g^{\prime} \text {-invariant } G^{*}\left(n, 2^{\varepsilon} g, 4,3\right) \text { design. Since }}$ there is a strictly cyclic $\operatorname{PQS}\left(2^{\epsilon} g\right)$ with $J\left(1,2^{\epsilon} g, 4,2\right)$ base blocks, there is a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{2^{\epsilon} g}$-invariant $\operatorname{PQS}\left(2^{\epsilon} n g\right)$ with $J\left(n, 2^{\varepsilon} g, 4,2\right)$ base blocks by Construction 3.3. Since $\mathbb{Z}_{n} \times \mathbb{Z}_{2^{\epsilon} g}$ is isomorphic to $\mathbb{Z}_{2^{\epsilon} n} \times \mathbb{Z}_{g}$, there is a strictly $\mathbb{Z}_{2^{\epsilon} n} \times \mathbb{Z}_{g}$-invariant $\operatorname{PQS}\left(2^{\epsilon} n g\right)$ with $J\left(2^{\varepsilon} n, g, 4,2\right)$ base blocks. Further, we put this PQS into the strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{2^{\epsilon} n}$-invariant $G^{*}\left(m, 2^{\varepsilon} n g, 4,3\right)$ design relative to $m \mathbb{Z}_{m g} \times \mathbb{Z}_{2^{\epsilon} n}$. By applying Construction 3.3 we obtain a strictly $\mathbb{Z}_{m g} \times \mathbb{Z}_{2^{\epsilon} n \text {-invariant PQS }\left(2^{\epsilon} m n g\right) ~}^{n}$ ) with $J\left(m g, 2^{\epsilon} n, 4,2\right)$ base blocks, which leads to an optimal ( $m g, 2^{\varepsilon} n, 4,2$ )-OOSPC with the size attaining the upper bound (1.1).

Lemma $6.6[14,18]$ There is an optimal 1-D ( $n, 4,2$ )-OOC with $J(1, n, 4,2)$ codewords for all $7 \leq n \leq 100$ with the definite exceptions of $n \in\{9,12,13,24,48,72,96\}$ and possible exceptions of $n \in\{45,47,53,55,59,60,65,66,69,71,76,77,81,83,84,85,89,91,92,95,97,99\}$. There is an optimal 1-D (n, 4, 2)-OOC with $J(1, n, 4,2)-1$ codewords for each $n \in\{9,12,13,24,48,72$, $96\}$.

Corollary 6.7 Let $m$ and $n$ be composite numbers whose prime divisors each belong to $\{p \equiv 1$ $(\bmod 12): p$ is a prime, $\left.p<10^{5}\right\} \cup\{p \equiv 5(\bmod 12): p$ is a prime, $p<1500000\}$. Then, there is an optimal $(m, 2 n g, 4,2)$-OOSPC (resp. $(m g, 2 n, 4,2)$-OOSPC) with the size attaining the upper bound (1.1) for $g \in\{1,3,5, \ldots, 49\} \backslash\{33\}$, and an optimal ( $m, 4 n g, 4,2$ )-OOSPC (resp. $(m g, 4 n, 4,2)$ $O O S P C$ ) with the size attaining the upper bound (1.1) for $g \in\{1,3,5, \ldots, 13\} \backslash\{3\}$.

Proof Since there is an $S$-cyclic $\operatorname{SQS}(2 p)$ for any prime divisor $p$ of $m$ and $n$ by Theorems 6.26.3 and an optimal 1-D $(2 g, 4,2)$-OOC with $J(1,2 g, 4,2)$ codewords for $g \in\{1,3,5, \ldots, 49\} \backslash\{33\}$ by Lemma 6.6, applying Theorem 6.5 gives an optimal ( $m, 2 n g, 4,2$ )-OOSPC (resp. ( $m g, 2 n, 4,2$ )OOSPC) with the size attaining the upper bound (1.1). Similarly, since there is an optimal 1-D $(4 g, 4,2)$-OOC with $J(1,4 g, 4,2)$ codewords for $g \in\{1,3,5, \ldots, 13\} \backslash\{3\}$ by Lemma 6.6 , applying Theorem 6.5 gives an optimal ( $m, 4 n g, 4,2$ )-OOSPC (resp. ( $m g, 4 n, 4,2$ )-OOSPC) with the size attaining the upper bound (1.1).

Remark: The optimal ( $m, 2^{\epsilon} n, 4,2$ )-OOSPC in [45] is obtained again in Corollary 6.7. Comparing with Sawa's method, our construction seems easier.

Lemma 6.8 There exists an optimal $(3,12,4,2)$-OOSPC with the size attaining the upper bound in Lemma 2.3.

Proof The following 48 base blocks generate the block set of a strictly $\mathbb{Z}_{3} \times \mathbb{Z}_{12}$-invariant PQS(36), which corresponds to an optimal $(3,12,4,2)$-OOSPC with the size attaining the upper bound in Lemma 2.3.

$$
\begin{array}{ll}
\{(0,0),(0,1),(0,11),(0,6)\}, & \{(0,0),(1,1),(2,11),(0,6)\}, \\
\{(0,0),(1,3),(2,9),(0,6)\}, & \{(0,0),(1,5),(2,7),(0,6)\}, \\
\{(0,0),(0,1),(0,3),(0,4)\}, & \{(0,0),(0,1),(0,5),(0,8)\}, \\
\{(0,0),(0,1),(1,0),(1,1)\}, & \{(0,0),(0,1),(1,2),(1,3)\}, \\
\{(0,0),(0,1),(1,4),(1,5)\}, & \{(0,0),(0,1),(1,6),(1,7)\}, \\
\{(0,0),(0,1),(1,8),(1,9)\}, & \{(0,0),(0,1),(1,10),(1,11)\}, \\
\{(0,0),(0,2),(0,5),(1,0)\}, & \{(0,0),(0,10),(0,7),(2,0)\}, \\
\{(0,0),(0,2),(0,6),(1,2)\}, & \{(0,0),(0,10),(0,6),(2,10)\}, \\
\{(0,0),(0,2),(1,1),(1,3)\}, & \{(0,0),(0,2),(1,4),(1,6)\}, \\
\{(0,0),(0,2),(1,5),(1,7)\}, & \{(0,0),(0,2),(1,8),(1,10)\}, \\
\{(0,0),(0,2),(1,9),(1,11)\}, & \{(0,0),(0,3),(1,0),(1,3)\}, \\
\{(0,0),(0,3),(1,1),(1,4)\}, & \{(0,0),(0,3),(1,2),(1,5)\}, \\
\{(0,0),(0,3),(1,6),(1,9)\}, & \{(0,0),(0,3),(1,7),(2,4)\}, \\
\{(0,0),(0,9),(2,5),(1,8)\}, & \{(0,0),(0,3),(1,8),(2,7)\}, \\
\{(0,0),(0,4),(1,1),(1,5)\}, & \{(0,0),(0,4),(1,2),(1,10)\}, \\
\{(0,0),(0,4),(1,3),(1,8)\}, & \{(0,0),(0,8),(2,9),(2,4)\}, \\
\{(0,0),(0,4),(1,4),(2,5)\}, & \{(0,0),(0,8),(2,8),(1,7)\}, \\
\{(0,0),(0,4),(1,6),(2,10)\}, & \{(0,0),(0,4),(1,7),(2,9)\}, \\
\{(0,0),(0,4),(1,9),(2,7)\}, & \{(0,0),(0,5),(1,1),(2,4)\}, \\
\{(0,0),(0,5),(1,2),(2,3)\}, & \{(0,0),(0,5),(1,3),(2,2)\}, \\
\{(0,0),(0,5),(1,5),(2,8)\}, & \{(0,0),(0,7),(2,7),(1,4)\}, \\
\{(0,0),(0,5),(1,6),(2,11)\}, & \{(0,0),(0,5),(1,7),(2,7)\}, \\
\{(0,0),(0,7),(2,5),(1,5)\}, & \{(0,0),(0,5),(1,11),(2,6)\}, \\
\{(0,0),(0,6),(1,0),(2,8)\}, & \{(0,0),(0,6),(2,0),(1,4)\},
\end{array}
$$

Theorem 6.9 Let $m, n$ be equal to 1 or the composite numbers of primes as in Corollary 6.7. Then there is an optimal $(3 m, b n, 4,2)$-OOSPC with $J(3 m, b n, 4,2)-1$ codewords attaining the upper bound in Lemma 2.3 for $b \in\{6,12\}$.

Proof Start with a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{\epsilon \cdot 3 n}}$-invariant $G^{*}\left(m, 2^{\varepsilon} \cdot 3 n, 4,3\right)$ design relative to $\{0\} \times \mathbb{Z}_{2^{\epsilon \cdot 3 n}}$, $\epsilon \in\{1,2\}$, which exists from the proof of Theorem 6.5. Applying Construction 4.2 gives a strictly $\mathbb{Z}_{3 m} \times \mathbb{Z}_{2^{\epsilon \cdot 3 n} \text {-invariant }} G^{*}\left(m, 2^{\varepsilon} \cdot 9 n, 4,3\right)$ design relative to $m \mathbb{Z}_{3 m} \times \mathbb{Z}_{2^{\epsilon \cdot 3 n}}$. Similarly, there is a strictly $\mathbb{Z}_{3 n} \times \mathbb{Z}_{2^{\epsilon \cdot 3}}$-invariant $G^{*}\left(n, 2^{\epsilon} \cdot 9,4,3\right)$ design relative to $n \mathbb{Z}_{3 n} \times \mathbb{Z}_{2^{\epsilon .3}}$. Since $\mathbb{Z}_{3 n} \times \mathbb{Z}_{2^{\epsilon \cdot 3}}$ is isomorphism to $\mathbb{Z}_{2^{\varepsilon \cdot 3 n}} \times \mathbb{Z}_{3}$, there is a strictly $\mathbb{Z}_{2^{\epsilon} \cdot 3 n} \times \mathbb{Z}_{3}$-invariant $G^{*}\left(n, 2^{\epsilon} \cdot 9,4,3\right)$ design relative to $n \mathbb{Z}_{2^{\epsilon \cdot 3 n}} \times \mathbb{Z}_{3}$. Since there is an optimal (3, $\left.6,4,2\right)$-OOSPC from [45] and an optimal (3, 12, 4, 2)OOSPC by Lemma 6.8 with the size attaining the upper bound in Lemma 2.3 which is equivalent to a strictly $\mathbb{Z}_{3} \times \mathbb{Z}_{b}$-invariant $\operatorname{PQS}(3 b)$ for $b \in\{6,12\}$, applying Construction 3.3 yields a strictly $\mathbb{Z}_{3} \times \mathbb{Z}_{b n} \mathrm{PQS}(3 b n)$ with the size attaining the upper bound in Lemma 2.3. Further, input this PQS into the strictly $\mathbb{Z}_{3 m} \times \mathbb{Z}_{2^{\epsilon \cdot 3 n}}$-invariant $G^{*}\left(m, 2^{\epsilon} \cdot 9 n, 4,3\right)$ design and apply Construction 3.3. We then obtain an optimal strictly $\mathbb{Z}_{3 m} \times \mathbb{Z}_{b n}$-invariant $\operatorname{PQS}(3 b m n)$ with the size attaining the upper bound in Lemma 2.3, which leads to an optimal ( $3 m, 3 \cdot 2^{\epsilon} n, 4,2$ )-OOSPC.

Lemma $6.10[18]$ There exists a strictly cyclic $G(2, g, 4,3)$ design for each integer $g \equiv 0(\bmod 8)$.

Lemma 6.11 [18, Corollary 6.21] Suppose there is a cyclic $S Q S(n)$ with $n \equiv 2$ or $10(\bmod 12)$ and a strictly cyclic $P Q S(g)$ of the size $J(1, g, 4,2)-1$ attaining the upper bound in Lemma 2.2 with $g \equiv 0(\bmod 24)$. Then there is a strictly cyclic $P Q S\left(2^{a} 3^{b} 5^{c} n^{d} g\right)$ of the size $J\left(1,2^{a} 3^{b} 5^{c} n^{d} g, 4,2\right)-1$ attaining the upper bound in Lemma 2.2 for any nonnegative integers $a, b, c, d$.

Theorem 6.12 Let $m, n$ be equal to 1 or the composite numbers of primes as in Corollary 6.7. Then there is an optimal $\left(m, 2^{a} 3^{b} n, 4,2\right)$-OOSPC with $J\left(m, 2^{a} 3^{b} n, 4,2\right)-1$ codewords attaining the upper bound in Lemma 2.2 for $a \geq 4, b \geq 1$.

Proof Start with a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2 n}$-invariant $G^{*}(m, 2 n, 4,3)$ design relative to $\{0\} \times \mathbb{Z}_{2 n}$, which exists from the proof of Theorem 6.5. Since there is a strictly semi-cyclic $G\left(2,2^{a-1} 3^{b}, 4,3\right)$ design by Lemma 6.10 , there is strictly a $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{a} 3^{b} n}$-invariant $G\left(m, 2^{a} 3^{b} n, 4,3\right)$ design relative to $\{0\} \times \mathbb{Z}_{2^{a} 3^{b} n}$. Since there is an $S$-cyclic $\operatorname{SQS}(2 n)$ and an optimal 1-D ( $24,4,2$ )-OOC with $J(1,24,4,2)-1$ codewords by Lemma 6.6 which is equivalent to a strictly cyclic PQS(24), by Lemma 6.11 there is a strictly cyclic PQS $\left(2^{a} 3^{b} n\right)$ with $J\left(1,2^{a} 3^{b} n, 4,2\right)-1$ base blocks. Further, input this PQS into the strictly
 strictly cyclic $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{a} 3^{b} n}$-invariant $\operatorname{PQS}\left(2^{a} 3^{b} m n\right)$ with $J\left(m, 2^{a} 3^{b} n, 4,2\right)-1$ base blocks attaining the upper bound in Lemma 2.2, which leads to an optimal ( $m, 2^{a} 3^{b} n, 4,2$ )-OOSPC.

Theorem 6.13 Let $m, n$ be equal to 1 or odd integers such that there is an $S$-cyclic $S Q S(2 p)$ for each prime divisor $p$ of $m$ and $n$. Let $g \equiv 0(\bmod 8)$. If there is an optimal $\left(2^{\epsilon}, g, 4,2\right)-O O S P C$ with $J\left(2^{\varepsilon}, g, 4,2\right)$ codewords, then there is an optimal $\left(2^{\epsilon} m, n g, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1), where $\epsilon \in\{1,2\}$. If $2^{\epsilon} g \equiv 0(\bmod 24)$ and there is an optimal $\left(2^{\epsilon}, g, 4,2\right)$ OOSPC with $J\left(2^{\varepsilon}, g, 4,2\right)-1$ codewords, then there is an optimal $\left(2^{\epsilon} m, n g, 4,2\right)$-OOSPC with the size attaining the upper bound in Lemma 2.2, where $\epsilon \in\{1,2\}$.

Proof For $m=1$ and $n>1$, from the proof of Theorem 6.5 , there is a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{2^{\epsilon} \text {-invariant }}$ $G\left(n, 2^{\epsilon}, 4,3\right)$ design. Since there is a strictly semi-cyclic $G(2, g, 4,3)$ design by Lemma 6.10, Corollary 5.7 shows that there is a strictly $\mathbb{Z}_{n g} \times \mathbb{Z}_{2^{\epsilon}}$-invariant $G\left(n, 2^{\epsilon} g, 4,3\right)$ design relative to $n \mathbb{Z}_{n g} \times \mathbb{Z}_{2^{\epsilon}}$. Since there is an $\left(2^{\epsilon}, g, 4,2\right)$-OOSPC with $J\left(2^{\epsilon}, g, 4,2\right)$ codewords by assumption, which is equivalent
 gives a strictly $\mathbb{Z}_{n g} \times \mathbb{Z}_{2^{\epsilon}-\text { invariant }} \operatorname{PQS}\left(2^{\epsilon} n g\right)$ with $J\left(n g, 2^{\epsilon}, 4,2\right)$ base blocks. Therefore, there is an optimal $\left(2^{\epsilon}, n g, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1) by Theorem 2.1.

For $m>1$, from the proof of Theorem 6.5 , there is an $S$-cyclic $\operatorname{SQS}\left(2^{\epsilon} m\right)$, which implies the existence of strictly cyclic $G\left(m, 2^{\epsilon}, 4,3\right)$ design. Since there is a strictly semi-cyclic $G(2, n g, 4,3)$ design by Lemma 6.10 , Corollary 5.7 shows that there is a strictly $\mathbb{Z}_{2^{\epsilon} m} \times \mathbb{Z}_{n g}$-invariant $G\left(m, 2^{\epsilon} n g, 4,3\right)$ design relative to $m \mathbb{Z}_{2^{\epsilon} m} \times \mathbb{Z}_{n g}$. Applying Construction 3.3 with the known strictly $\mathbb{Z}_{2^{\epsilon}} \times \mathbb{Z}_{n g}$-invariant PQS $\left(2^{\epsilon} n g\right)$ with $J\left(2^{\epsilon}, n g, 4,2\right)$ base blocks gives a strictly $\mathbb{Z}_{2^{\epsilon} m} \times \mathbb{Z}_{n g}$-invariant PQS( $\left.2^{\epsilon} m n g\right)$ with $J\left(2^{\epsilon} m, n g, 4,2\right)$ base blocks. Therefore, there is an optimal $\left(2^{\epsilon} m, n g, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1).

When $2^{\epsilon} g \equiv 0(\bmod 24)$, similar discussion as above gives the conclusion.

Theorem 6.14 Let $m$, $n$ be equal to 1 or the composite numbers of primes as in Corollary 6.7 and $\epsilon \in\{1,2\}$. Then there is an optimal $\left(2^{\epsilon} m, 8 n, 4,2\right)$-OOSPC with the size attaining the upper bound (1.1).

Proof For $m=n=1$, there is a $(2,8,4,2)$-OOSPC with $J(2,8,4,2)$ codewords [45]. By Example 3.5 , there is an optimal $(4,8,4,2)$-OOSPC with $J(4,8,4,2)$ codewords. For other values $m$ and $n$, applying Theorem 6.13 gives the conclusion.

Denote $U=\left\{4^{r}-1: r\right.$ is a positive integer $\} \cup\{1,27,33,39,51,87,123,183\}$, and $P=\{p \equiv 7$ $(\bmod 12): p$ is a prime $\} \cup\left\{2^{n}-1:\right.$ odd integer $\left.n \geq 1\right\} \cup\{25,37,61,73,109,157,181,229,277\}$, $V=\{v: v \in P$ or $v$ is a product of integers from the set $P\}$ and $M=\{u v: u \in U, v \in V\} \cup\left\{21^{r} u\right.$ : $r \geq 0, u \in\{3,15,21,27,33,39,51,57,63,75,87,93\}\}$.

Lemma 6.15 [19] There exists an $\operatorname{RoSQS}(m+1)$ for $m \in M$.
Theorem 6.16 For $m, n \in M$, there is an optimal ( $m, n, 4,2$ )-OOSPC with the size attaining the upper bound (1.1).

Proof If $m \equiv 1(\bmod 6)$, by Corollary 5.8 , there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG $(3,(3,4)$, $m n)$ of type $n^{m}$ relative to $\{0\} \times \mathbb{Z}_{n}$. Construct a cyclic 1-FG(3, $\left.(3,4), n\right)$ on $\{0\} \times \mathbb{Z}_{n}$ which is obtained
by deleting the fixed point. Then we obtain a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG(3, $\left.(3,4), m n\right)$ type $1^{m n}$. By Lemma 5.3, there is an optimal ( $m, n, 4,2$ )-OOSPC.

If $m \equiv 3(\bmod 6)$, by Corollary 5.8 , there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1 -FG $(3,(3,4), m n)$ of type $(3 n)^{m / 3}$ relative to $\frac{m}{3} \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. Also by Corollary 5.8 , there is a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{3}$-invariant 1-FG $(3,(3,4), m n)$ of type $3^{n}$ if $n \equiv 1(\bmod 6)$, and a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{3}$-invariant 1-FG $(3,(3,4), 3 n)$ of type $9^{n / 3}$ relative to $\frac{n}{3} \mathbb{Z}_{n} \times \mathbb{Z}_{3}$ if $n \equiv 3(\bmod 6)$. Since there is a $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$-invariant 1-FG $(3,(3,4), 9)$ of type $3^{3}$ by Example 5.1 or from the proof of Theorem 2.6, there is a $\mathbb{Z}_{n} \times \mathbb{Z}_{3}$-invariant 1 $\mathrm{FG}(3,(3,4), 3 n)$ of type $1^{3 n}$ whenever $n \equiv 1(\bmod 6)$ or $n \equiv 3(\bmod 6)$. Therefore, there is a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant 1-FG $(3,(3,4), m n)$ type $1^{m n}$. By Lemma 5.3, there is an optimal $(m, n, 4,2)$ OOSPC.

Lemma 6.17 [18, Lemma 4.8] There exists a strictly cyclic $G(3, g, 4,3)$ design for each integer $g \equiv 0(\bmod 12)$.

Lemma 6.18 [18, Corollary 6.14] If there is an $\operatorname{RoSQS}(n+1)$, then for any integers $a, b \geq 0$, there is an optimal 1-D $\left(3^{a} 5^{b} n \cdot 36,4,2\right)$-OOC with the size attaining the upper bound (1.1).

Theorem 6.19 Let $m, n \in M$ and $a, b$ nonnegative integers. If $m \equiv 1(\bmod 6)$ then there is an optimal ( $m, 2^{2} 3^{a+2} 5^{b} n, 4,2$ )-OOSPC with the size attaining the upper bound (1.1).

Proof Since there is an RoSQS $(m+1)$ by Lemma 6.15 and a strictly semi-cyclic $G\left(3,2^{2} 3^{a+2} 5^{b} n, 4,3\right)$ design by Lemma 6.17 , there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{2} 3^{a+2} 5^{b} n}$-invariant $G\left(m, 2^{2} 3^{a+2} 5^{b} n, 4,3\right)$ design relative to $\{0\} \times \mathbb{Z}_{2^{2} 3^{a+2} 5^{b} n}$ by Corollary 5.9. By Lemma 6.18, there is an optimal 1-D $\left(3^{a} 5^{b} n \cdot 36,4,2\right)$ OOC with $J\left(1,3^{a} 5^{b} n \cdot 36,4,2\right)$ codewords, which is equivalent to a strictly $\operatorname{PQS}\left(3^{a} 5^{b} n \cdot 36\right)$. Applying Construction 3.3 gives an optimal $\left(m, 3^{a} 5^{b} n \cdot 36,4,2\right)$-OOSPC with the size attaining the upper bound (1.1).

Theorem 6.20 Let $m, n \in M$. If $m \equiv n \equiv 3(\bmod 6)$, then there is an optimal $(m, 4 n, 4,2)$ OOSPC with the size attaining the upper bound in Lemma 2.3.

Proof Since there is an RoSQS $(m+1)$ by Lemma 6.15 and a strictly semi-cyclic $G(3,4 n, 4,3)$ design by Lemma 6.17 , there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{4 n}$-invariant $G\left(\frac{m}{3}, 12 n, 4,3\right)$ design relative to $\frac{m}{3} \mathbb{Z}_{m} \times \mathbb{Z}_{4 n}$ by Corollary 5.9. Since there is an $\operatorname{RoSQS}(n+1)$ by assumption and a strictly semicyclic $G(3,12,4,3)$ design by Lemma 6.17 , there is a strictly $\mathbb{Z}_{n} \times \mathbb{Z}_{12}$-invariant $G(n / 3,36,4,3)$ design relative to $\frac{n}{3} \mathbb{Z}_{n} \times \mathbb{Z}_{12}$ by Corollary 5.9, thereby, there is a strictly $\mathbb{Z}_{4 n} \times \mathbb{Z}_{3}$-invariant $G(n / 3,36,4,3)$ design $\frac{n}{3} \mathbb{Z}_{4 n} \times \mathbb{Z}_{3}$. Since there is an optimal ( $3,12,4,2$ )-OOSPC with the size attaining the upper bound in Lemma 2.3 , there is a strictly $\mathbb{Z}_{4 n} \times \mathbb{Z}_{3}$-invariant $\operatorname{PQS}(12 n)$ with $J(4 n, 3,4,2)-1$ base blocks. Applying Construction 3.3 gives optimal ( $m, 4 n, 4,2$ )-OOSPC with the size attaining the upper bound in Lemma 2.3.

Theorem 6.21 Let $m \in M$ and $n$ be composite numbers of primes as in Corollary 6.7 and let $a, b$ be two nonnegative integers. If $m \equiv 1(\bmod 6)$, then there is an optimal $\left(m, 2^{a+2} 3^{b+1} n, 4,2\right)$-OOSPC with the size attaining the upper bound in Lemma 2.2.

Proof Since there is an RoSQS $(m+1)$ by Lemma 6.15 and a strictly semi-cyclic $G\left(3,2^{a+2} 3^{b+1} n, 4,3\right)$ design by Lemma 6.17 , there is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{a+2} 3^{b+1}}$-invariant $G\left(m, 2^{a+2} 3^{b+1} n, 4,3\right)$ design relative to $\{0\} \times \mathbb{Z}_{2^{a+2} 3^{b+1} n}$ by Corollary 5.9. Since $n$ is a composite number of primes as in Corollary 6.7, there is a cyclic $\operatorname{SQS}(2 n)$ by Theorem 6.1. By Lemma 6.11, there is an optimal 1-D $\left(2^{a+2} 3^{b+1} n, 4,2\right)$-OOC with the size attaining the upper bound in Lemma 2.2, which is equivalent to a strictly cyclic PQS $\left(2^{a+2} 3^{b+1} n\right)$. Applying Construction 3.3 gives an optimal $\left(m, 2^{a+2} 3^{b+1} n, 4,2\right)$ OOSPC with the size attaining the upper bound in Lemma 2.2.

Table I
New Infinite families of optimal ( $m, n, k, 2$ )-OOSPCs

| Parameters | Conditions | Size | Source |
| :---: | :---: | :---: | :---: |
| $(p, p, p+1,2)$ | $p$ is a prime | $J(p, p, p+1,2)$ | Theorem 2.6 |
| $(m, 2 n g, 4,2)$ | $m, n \in W$ | $J(m, 2 n g, 4,2)$ | Corollary 6.7 |
| $(m g, 2 n, 4,2)$ | $g \in\{1,3,5, \ldots, 49\} \backslash\{33\}$ | $J(m g, 2 n, 4,2)$ |  |
| $(m, 4 n g, 4,2)$ | $m, n \in W$ | $J(m, 4 n g, 4,2)$ | Corollary 6.7 |
| $(m g, 4 n, 4,2)$ | $g \in\{1,3,5, \ldots, 13\} \backslash\{3\}$ | $J(m g, 4 n, 4,2)$ |  |
| $(3 m, b n, 4,2)$ | $m, n \in W, b \in\{6,12\}$ | $J(3 m, b n, 4,2)-1$ | Theorem 6.9 |
| $\left(m, 2^{a} 3^{b} n, 4,2\right)$ | $m, n \in W, a \geq 4, b>1$ | $J\left(m, 2^{a} 3^{b} n, 4,2\right)-1$ | Theorem 6.12 |
| $\left(2^{\epsilon} m, 8 n, 4,2\right)$ | $m, n \in W, \epsilon \in\{1,2\}$ | $J\left(2^{\epsilon} m, 8 n, 4,2\right)$ | Theorem 6.14 |
| $(m, n, 4,2)$ | $m, n \in M$ | $J(m, n, 4,2)$ | Theorem 6.16 |
| $\left(m, 2^{2} 3^{a+2} 5^{b} n, 4,2\right)$ | $m, n \in M, m \equiv 1 \bmod 6)$, | $J\left(m, 2^{2} 3^{a+2} 5^{b} n, 4,2\right)$ | Theorem 6.19 |
| $(m, 4 n, 4,2)$ | $m, n \in M, m, n \equiv 3 \bmod 6)$ | $J(m, 4 n, 4,2)-1$ | Theorem 6.20 |
| $\left(m, 2^{a+2} 3^{b+1} n, 4,2\right)$ | $m \in M, m \equiv 1 \bmod 6)$, | $J\left(m, 2^{a+2} 3^{b+1} n, 4,2\right)-1$ | Theorem 6.21 |
|  | $n \in W, a, b \geq 0$ |  |  |

$W=\left\{p_{1}^{a_{1}} p_{2}^{a_{2}} \cdots p_{r}^{a_{r}}:\right.$ each prime $p_{i} \equiv 1(\bmod 12)$ and $p_{i}<10^{5}$, or $p_{i} \equiv 5(\bmod 12)$ and $\left.p<1500000\right\} ;$
$M=\{u v: u \in U, v \in V\} \cup\left\{21^{r} u: r \geq 0, u \in\{3,15,21,27,33,39,51,57,63,75,87,93\}\right\}$, where $U=\left\{4^{r}-1:\right.$ $r$ is a positive integer $\} \cup\{1,27,33,39,51,87,123,183\}$, and $P=\{p \equiv 7(\bmod 12): p$ is a prime $\} \cup\left\{2^{n}-1\right.$ : odd integer $n \geq 1\} \cup\{25,37,61,73,109,157,181,229,277\}, V=\{v: v \in P$ or $v$ is a product of integers from the set $P\}$.

## 7 Concluding Remark

In this paper, we gave some combinatorial constructions for optimal ( $m, n, 4,2$ )-OOSPCs. As applications, many infinite families of optimal ( $m, n, 4,2$ )-OOSPCs were obtained. We summarized all infinite families obtained in this table I. As pointed out in the remark of Lemma 6.7, Sawa's result in [45] was obtained again and our construction seemed easier. We also obtained some infinite classes of optimal $(m, n, 4,2)$-OOSPCs with $\operatorname{gcd}(m, n)$ being divisible by 2 or 3 . Our constructions strength the importance of $S$-cyclic SQSs and RoSQSs. They are worth studying.

By Lemma 2.2 and Lemma 2.3, we see that the Johnson bound can not be achieved in some cases. The problem of constructing optimal $(m, n, w, \lambda)$-OOSPC is apparently a difficult and challenging task in general weight. Although the case of $w=4$ is too small for practical application, we hope that it may help us to study the other larger cases.

## APPENDIX A

Proof of Construction 4.1: Firstly, we compute the number of blocks in $\mathcal{C}^{\prime}$. Since $m-e$ is even, the cardinality of $\mathbf{I}$ is $\frac{m-e}{2}$. It is easy to see that $\left|\mathcal{C}_{1}\right|=g^{2}\left|\mathcal{F}_{2}\right|=g^{2} n(m-e)(m n+e n-9) / 24$, $\left|\mathcal{C}_{2}\right|=n g(m-e) / 4,\left|\mathcal{C}_{3}\right|=n g(g-1)(m-e) / 4$ and $\left|\mathcal{C}_{4}\right|=n g(m-e)(g-1) / 8$. Thus,

$$
\left|\mathcal{C}^{\prime}\right|=\operatorname{mng}\left(\left|\mathcal{C}_{1}\right|+\left|\mathcal{C}_{2}\right|+\left|\mathcal{C}_{3}\right|+\left|\mathcal{C}_{4}\right|\right)=\frac{n^{2} g^{2} m(m-e)(m n g+e n g-3)}{24}
$$

which is the expected number of quadruples. Also, it is $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant, thereby, it suffices to show that each triple containing $(0,0)$ and not contained in any group appears in at least one quadruple of $\mathcal{C}^{\prime}$. Let $T=\left\{(0,0),\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right)\right\}$ be such a triple, where $x_{k} \in \mathbb{Z}_{m}, 0 \leq y_{k}<n$, $0 \leq z_{k}<g$ and $1 \leq k \leq 2$. Clearly, at most one of $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ belongs to $\left(\frac{m}{e} \mathbb{Z}_{m}\right) \times \mathbb{Z}_{n}$. The proof is divided into two cases.

Case 1: Two of $(0,0),\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ are equal. When $\left(x_{1}, y_{1}\right)=\left(x_{2}, y_{2}\right)$, we have $x_{1} \notin$ $\frac{m}{e} \mathbb{Z}_{m}$. If $x_{1} \in \mathbf{I}$ then there is a block $B=\left\{(0,0),\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{1}, y_{1}+z_{2} n\right),\left(2 x_{1}, 2 y_{1}+z_{1} n+\right.\right.$ $\left.\left.z_{2} n\right)\right\} \in \mathcal{C}_{3} \subset \mathcal{C}_{3}^{\prime}$ such that $T \subset B$. If $x_{1} \notin \mathbf{I}$, then $-x_{1} \in \mathbf{I}$, thereby there is a base block $B=\left\{(0,0),\left(-x_{1},-y_{1}-z_{1} n\right),\left(-x_{1},-y_{1}-z_{2} n\right),\left(-2 x_{1},-2 y_{1}-z_{1} n-z_{2} n\right)\right\} \in \mathcal{C}_{3}$ such that $T \subset$
$B+\left(2 x_{1}, 2 y_{1}+z_{1} n+z_{2} n\right) \in \mathcal{C}_{3}^{\prime}$. When one of $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is equal to $(0,0)$, without loss of generality let $\left(x_{1}, y_{1}\right)=(0,0)$, consider the triple $T-\left(x_{2}, y_{2}+z_{2} n\right)$. Similarly, there is a block $C \in \mathcal{C}^{\prime}$ such that $T-\left(x_{2}, y_{2}+z_{2} n\right) \subset C$, thereby, there is a block $C^{\prime} \in \mathcal{C}^{\prime}$ containing $T$.

Case 2: $(0,0),\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ are distinct. Since $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n},\left\{\left\{i, i+\frac{m}{e}, \ldots, i+m-\frac{m}{e}\right\} \times\right.\right.$ $\left.\left.\mathbb{Z}_{n}: 0 \leq i<\frac{m}{e}\right\}, \mathcal{B}\right)$ is a $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant $G^{*}\left(\frac{m}{e}, e n, 4,3\right)$ design, there is a base block $B=$ $\left\{\left(x_{1}^{\prime}, y_{1}^{\prime}\right),\left(x_{2}^{\prime}, y_{2}^{\prime}\right),\left(x_{3}^{\prime}, y_{3}^{\prime}\right),\left(x_{4}^{\prime}, y_{4}^{\prime}\right)\right\} \in \mathcal{F}$ and $(\tau, \mu) \in \mathbb{Z}_{m} \times \mathbb{Z}_{n}$ such that $\left\{(0,0),\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right\} \subset$ $B+(\tau, \mu)$. Without loss of generality, let $\left(x_{k}, y_{k}\right)=\left(x_{k}^{\prime}, y_{k}^{\prime}\right)+(\tau, \mu)$ for $1 \leq k \leq 2$ and $(0,0)=$ $\left(x_{3}^{\prime}, y_{3}^{\prime}\right)+(\tau, \mu)$. If $B \in \mathcal{F}_{2}$ then let $y_{k}^{\prime}+\mu=a_{k} n+y_{k}, a_{k} \in\{0,1\}$ for $1 \leq k \leq 3$ where $y_{3}=0$. Since $\mathcal{A}_{B}$ is the set of base blocks of a semi-cyclic $H(4, g, 4,3)$ design on $\left\{\left(x_{l}^{\prime}, y_{k}^{\prime}+u n\right): 1 \leq k \leq\right.$ $4,0 \leq u<g\}$ with groups $\left\{\left(x_{k}^{\prime}, y_{k}^{\prime}+u n\right): 0 \leq u<g\right\}(1 \leq k \leq 4)$, there is a unique base block $A=\left\{\left(x_{1}^{\prime}, y_{1}^{\prime}+z_{1}^{\prime} n\right),\left(x_{2}^{\prime}, y_{2}^{\prime}+z_{2}^{\prime} n\right),\left(x_{3}^{\prime}, y_{3}^{\prime}+z_{3}^{\prime} n\right),\left(x_{4}^{\prime}, y_{4}^{\prime}+z_{4}^{\prime} n\right)\right\} \in \mathcal{A}_{B}$ and an element $\rho \in$ $\{0,1, \ldots, g-1\}$ such that $\left\{\left(x_{1}^{\prime}, y_{1}^{\prime}+z_{1} n-a_{1} n\right),\left(x_{2}^{\prime}, y_{2}^{\prime}+z_{2} n-a_{2} n\right),\left(x_{3}^{\prime}, y_{3}^{\prime}+z_{3} n-a_{3} n\right)\right\} \subset A+(0, \rho n)$. It follows that $T \subset A+(\tau, \mu+\rho n) \in \mathcal{C}_{1}^{\prime}$. If $B \in \mathcal{F}_{1}$, then $\left\{(0,0),\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right\}$ is of the form $\{(-x,-y),(0,0),(x, y)\}+(\tau, \mu)$ where $(x, y) \in \mathbf{I} \times \mathbb{Z}_{n}$, or of the form $\left\{(0,0),(x, y),\left(0, \frac{n}{2}\right)\right\}+(\tau, \mu)$ where $(x, y) \in\left(\mathbb{Z}_{m} \backslash \frac{m}{e} \mathbb{Z}_{m}\right) \times\left\{0,1, \ldots, \frac{n-2}{2}\right\}$.

Suppose that $\left\{(0,0),\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right\}$ is of the form $(\tau, \mu)+\{(0,0),(x, y),(-x,-y)\}, x \in \mathbf{I}, 0 \leq$ $y<n$. Then there is a triple of the form $\left\{(0,0),(x, y+z n),\left(-x,-y+z^{\prime} n\right)\right\}\left(0 \leq z, z^{\prime}<g\right)$ in the orbit generated by $\left\{(0,0),\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right)\right\}$ under $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. If $z^{\prime}=-z$ then $\left\{(0,0),(x, y+z n),\left(-x,-y+z^{\prime} n\right)\right\} \subset\left\{(0,0),(x, y+z n),\left(-x,-y+z^{\prime} n\right),\left(0, \frac{g n}{2}\right)\right\} \in \mathcal{C}_{2}$. Otherwise, $\left\{(0,0),(x, y+z n),\left(-x,-y+z^{\prime} n\right)\right\} \subset\left\{(0,0),(x, y+z n),\left(x, y+\left(g-z^{\prime}\right) n\right),\left(2 x, 2 y+z n-z^{\prime} n\right)\right\}-$ $\left(x, y-z^{\prime} n\right) \in \mathcal{C}_{3}^{\prime}$. It follows that $T$ occurs in a block of $\mathcal{C}_{2}^{\prime} \cup \mathcal{C}_{3}^{\prime}$.

Suppose that $\left\{(0,0),\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right\}$ is of the form $(\tau, \mu)+\left\{(0,0),(x, y),\left(0, \frac{n}{2}\right)\right\}, x \in \mathbb{Z}_{m} \backslash$ $\frac{m}{e} \mathbb{Z}_{m}, 0 \leq y<\frac{n}{2}$. Then there is a triple of the form $\left\{(0,0),(x, y+z n),\left(0, \ell n+\frac{n}{2}\right)\right\}(0 \leq z, \ell<g)$ in the orbit generated by $\left\{(0,0),\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right)\right\}$ under $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. For $x \in \mathbf{I}$, if $0 \leq \ell<\frac{g-1}{2}$ then $\left\{(0,0),(x, y+z n),\left(0, \ell n+\frac{n}{2}\right)\right\} \subset\left\{(0,0),\left(0, \ell n+\frac{n}{2}\right),(x, y+z n),\left(x, y+\frac{n}{2}+\ell n+z n\right)\right\} \in \mathcal{C}_{4}$, if $\ell=\frac{g-1}{2}$ then $\left\{(0,0),(x, y+z n),\left(0, \ell n+\frac{n}{2}\right)\right\} \subset\left\{(0,0),(x, y+z n),(-x,-y-z n),\left(0, \ell n+\frac{n}{2}\right)\right\} \in \mathcal{C}_{2}$ or $\left\{(0,0),(x, y+z n),\left(0, \ell n+\frac{n}{2}\right)\right\} \subset\left\{(0,0),(x, y+z n),(-x,-y-z n),\left(0, \ell n+\frac{n}{2}\right)\right\}+\left(0, \frac{n g}{2}\right) \in \mathcal{C}_{2}^{\prime}$ according to $0 \leq y+z n<\frac{n g}{2}$ or not, otherwise, $\left\{(0,0),(x, y+z n),\left(0, \ell n+\frac{n}{2}\right)\right\} \subset\{(0,0),(0,(g-$ $\left.\left.\ell-1) n+\frac{n}{2}\right),\left(x, y-\frac{n}{2}+z n-\ell n\right),(x, y+z n)\right\}+\left(0, \frac{n}{2}+l n\right) \in \mathcal{C}_{4}^{\prime}$. It follows that $T$ occurs in a block of $\mathcal{C}_{4}^{\prime}$. For $-x \in \mathbf{I}$, similar discussion shows that $T$ is contained a block of $\mathcal{C}^{\prime}$.

## APPENDIX B

Proof of Construction 5.6: For checking the required design, it suffices to show that: (1) the resulting design is strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant; (2) any triple $T, T \subset \mathbb{Z}_{m} \times \mathbb{Z}_{n g},\left|T \cap G^{\prime}\right|<3$ for all $G^{\prime} \in \mathcal{G}^{\prime}$, is contained in a unique block of the resulting design; (3) any pair of points $P$, $P \subset \mathbb{Z}_{m} \times \mathbb{Z}_{n g},\left|P \cap G^{\prime}\right|<2$ for all $G^{\prime} \in \mathcal{G}^{\prime}$, is contained in a unique block of $\mathcal{A}_{j}^{\prime}$ for each $0 \leq j<s$.
(1) Suppose that $A=\left\{\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right), \ldots,\left(x_{r}, y_{r}+z_{r} n\right)\right\}$ is a base block of the resulting design, where $x_{l} \in \mathbb{Z}_{m}, 0 \leq y_{l} \leq n-1,0 \leq z_{l} \leq g-1,1 \leq l \leq r$. We need to show that the stabilizer of $A$ is trivial, i.e. $A+\delta=A$ if and only if $\delta=(0,0)$. The sufficiency follows immediately, so we consider the necessity.

Assume that $\delta=\left(\delta_{1}, \delta_{2}+\delta_{3} n\right), \delta_{1} \in \mathbb{Z}_{m}, 0 \leq \delta_{2} \leq n, 0 \leq \delta_{3}<g$. If $A+\delta=A$ then

$$
\left\{\left(x_{l}, y_{l}+z_{l} n\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}+\delta_{1}, y_{l}+z_{l} n+\delta_{2}+\delta_{3} n\right): 1 \leq l \leq r\right\}
$$

where the arithmetic is in the ring $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$. It follows that

$$
\left\{\left(x_{l}, y_{l}\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}+\delta_{1}, y_{l}+\delta_{2}\right): 1 \leq l \leq r\right\},
$$

where the arithmetic is in the ring $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$. Let $U=\left\{\left(x_{l}, y_{l}\right): 1 \leq l \leq r\right\}$.
If $A \in \mathcal{A}_{j}^{\prime}, 0 \leq j<s$, then $|U|=r \geq 2$. Since the $\operatorname{subdesign}\left(X, \mathcal{G}, \mathcal{B}_{0}\right)$ of the master design 1-FG $\left(3,\left(K_{0}, K_{1}\right), m n\right)$ of type $(e n)^{m / e}\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n}, \mathcal{G}, \mathcal{B}_{0}, \mathcal{B}_{1}\right)$ is strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant and it requires that any 2 -subset of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ which intersects any group of $\mathcal{G}$ in at most one point occurs in exactly one block, we have $\left(\delta_{1}, \delta_{2}\right)=(0,0)$.

If $A \in \mathcal{A}_{s}^{\prime}$, without loss of generality we can always assume that $A \in \mathcal{A}_{s}^{*}$. If $A=\tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_{1}} \mathcal{D}_{B}$, then $|U|=r \geq 3$. Since the master design 1-FG $\left(3,\left(K_{0}, K_{1}\right) \text {, mn ) of type (en }\right)^{m / e}$ is strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$-invariant and it requires that any 3 -subset of $\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ which intersects any group of $\mathcal{G}$ in at most two points occurs in exactly one block, we have $\left(\delta_{1}, \delta_{2}\right)=(0,0)$. If $A=\tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_{0}} \mathcal{A}_{B}^{s}$, then $|U| \geq 2$. Note that in this case $U$ may be a multiset, i.e. $|U|$ may be not equal to $r$. By similar arguments as the case of $A \in \mathcal{A}_{j}^{\prime}$, we have $\left(\delta_{1}, \delta_{2}\right)=(0,0)$.

Hence,

$$
\left\{\left(x_{l}, y_{l}+z_{l} n\right): 1 \leq l \leq r\right\}=\left\{\left(x_{l}, y_{l}+z_{l} n+\delta_{3} n\right): 1 \leq l \leq r\right\}
$$

where the arithmetic is in the ring $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$. Since the input designs are all strictly semi-cyclic, we have $\delta_{3}=0$. Thus the resulting design is strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant.
(2) Take any triple $T=\left\{\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right),\left(x_{3}, y_{3}+z_{3} n\right)\right\} \subset \mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ which is not contained in any group of $\mathcal{G}^{\prime}$, where $x_{l} \in \mathbb{Z}_{m}, 0 \leq y_{l} \leq n-1,0 \leq z_{l} \leq g-1,1 \leq l \leq 3$ and $x_{1}, x_{2}, x_{3}$ are not congruent to the same number modulo $\frac{m}{e}$. We consider the following cases.

Case 1. Suppose that $x_{1}, x_{2}, x_{3}$ are pairwise distinct modulo $\frac{m}{e}$. Then there exists a unique base block $B$ in $\mathcal{F}$ and unique elements $\delta_{1}, \delta_{2}$ with $0 \leq \delta_{1}<m$ and $0 \leq \delta_{2}<n$, such that $\left\{\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right),\left(x_{3}, y_{3}\right)\right\} \subseteq B+\left(\delta_{1}, \delta_{2}\right)$. Let $\left(x_{l}^{*}, y_{l}^{*}\right) \in B$ satisfy $x_{l} \equiv x_{l}^{*}+\delta_{1}(\bmod m)$ and $y_{l}^{*}+\delta_{2}=y_{l}+\sigma_{l} n$ for some $\sigma_{l} \in\{0,1\}, 1 \leq l \leq 3$. Note that $x_{1}^{*}, x_{2}^{*}, x_{3}^{*}$ are also pairwise distinct modulo $\frac{m}{e}$.

If $B \in \mathcal{F}_{0}$, then there exists a unique base block $C \in \mathcal{A}_{B}$ and a unique element $\delta_{3}$ with $0 \leq \delta_{3}<g$, such that $\left\{\left(x_{1}^{*}, y_{1}^{*}, z_{1}^{*}\right),\left(x_{2}^{*}, y_{2}^{*}, z_{2}^{*}\right),\left(x_{3}^{*}, y_{3}^{*}, z_{3}^{*}\right)\right\} \subseteq C$ and $\left(x_{l}^{*}, y_{l}^{*}, z_{l}^{*}+\delta_{3}\right)=\left(x_{l}^{*}, y_{l}^{*}, z_{l}-\sigma_{l}+\sigma_{l}^{\prime} g\right)$ for some $\sigma_{l}^{\prime} \in\{0,1\}, 1 \leq l \leq 3$. By the mapping $\tau$, we have that $\left(x_{l}^{*}, y_{l}^{*}+z_{l}^{*} n\right)+\left(\delta_{1}, \delta_{2}+\delta_{3} n\right)=$ $\left(x_{l}^{*}+\delta_{1}, y_{l}^{*}+\delta_{2}+z_{l}^{*} n+\delta_{3} n\right)=\left(x_{l}, y_{l}+\sigma_{l} n+z_{l}^{*} n+\delta_{3} n\right)=\left(x_{l}, y_{l}+z_{l} n\right)$. Let $\delta=\left(\delta_{1}, \delta_{2}+\delta_{3} n\right)$. By (1) the resulting design is strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant, so $T$ is contained in the unique block $\tau(C)+\delta$, which is generated by $\tau(C)$. Similar arguments show that if $B \in \mathcal{F}_{1}$ then there is a unique base block $C \in \mathcal{D}_{B}$ and a unique element $\delta \in \mathbb{Z}_{m} \times \mathbb{Z}_{n g}$ such that $T \subset \tau(C)+\delta$.

Case 2. Suppose that $x_{1} \equiv x_{2} \not \equiv x_{3}\left(\bmod \frac{m}{e}\right), x_{1} \neq x_{2}$. By similar arguments as in Case 1 , there exists a unique base block $B \in \mathcal{F}_{1}$, a unique base block $C \in \mathcal{D}_{B}$ and unique elements $\delta_{1}, \delta_{2}, \delta_{3}$ with $0 \leq \delta_{1}<m, 0 \leq \delta_{2}<n$ and $0 \leq \delta_{3}<g$, such that $T$ is contained in the unique block $\tau(C)+\delta$, where $\delta=\left(\delta_{1}, \delta_{2}+\delta_{3} n\right)$, which is generated by $\tau(C)$.

Case 3. Suppose that $x_{1}=x_{2}, y_{1} \neq y_{2}$ and $x_{1} \not \equiv x_{3}\left(\bmod \frac{m}{e}\right)$. By similar arguments as in Case 1 , there exists a unique base block $B \in \mathcal{F}_{0}$, a unique base block $C \in \mathcal{A}_{B}$ and unique elements $\delta_{1}, \delta_{2}, \delta_{3}$ with $0 \leq \delta_{1}<m, 0 \leq \delta_{2}<n$ and $0 \leq \delta_{3}<g$, such that $T$ is contained in the unique block $\tau(C)+\delta$, where $\delta=\left(\delta_{1}, \delta_{2}+\delta_{3} n\right)$, which is generated by $\tau(C)$.
(3) Take any 2 -subset $P=\left\{\left(x_{1}, y_{1}+z_{1} n\right),\left(x_{2}, y_{2}+z_{2} n\right)\right\}$ which is not contained in any group of $\mathcal{G}^{\prime}$, where $x_{l} \in \mathbb{Z}_{m}, 0 \leq y_{l} \leq n-1,0 \leq z_{l} \leq g-1,1 \leq l \leq 2$. Then $x_{1} \not \equiv x_{2}\left(\bmod \frac{m}{e}\right)$ and there exists a unique base block $B$ in $\mathcal{F}_{0}$ and unique elements $\delta_{1}, \delta_{2}$ with $0 \leq \delta_{1}<m$ and $0 \leq \delta_{2}<n$, such that $\left\{\left(x_{1}^{*}, y_{1}^{*}\right),\left(x_{2}^{*}, y_{2}^{*}\right)\right\} \subseteq B$ and $x_{l}^{*}+\delta_{1} \equiv x_{l}(\bmod m)$ and $y_{l}^{*}+\delta_{2}=y_{1}+\sigma_{l} n$ for some $\sigma_{l} \in\{0,1\}$, $1 \leq l \leq 2$. Note that $x_{1}^{*}, x_{2}^{*}$ are also distinct modulo $\frac{m}{e}$.

Then, given any $0 \leq j<s$, there exists a unique base block $C_{j}$ in $\mathcal{A}_{j}^{*}$ and a unique element $\delta_{3} \in \mathbb{Z}_{g}$, such that $\left\{\left(x_{1}^{*}, y_{1}^{*}, z_{1}^{*}\right),\left(x_{2}^{*}, y_{2}^{*}, z_{2}^{*}\right)\right\} \subseteq C_{j}$ and $\left(x_{l}^{*}, y_{l}^{*}, z_{l}^{*}+\delta_{3}\right)=\left(x_{l}^{*}, y_{l}^{*}, z_{l}-\sigma_{l}+\sigma_{l}^{\prime} g\right)$ for some $\sigma_{l}^{\prime} \in\{0,1\}, 1 \leq l \leq 2$. By the mapping $\tau$, we have that $\left(x_{l}^{*}+\delta_{1}, y_{l}^{*}+\delta_{2}+z_{l}^{*} n+\delta_{3} n\right)=\left(x_{l}, y_{l}+z_{l} n\right)$. Let $\delta=\left(\delta_{1}, \delta_{2}+\delta_{3} n\right)$. By (1) the resulting design is strictly cyclic, so $P$ is contained in the unique block $\tau\left(C_{j}\right)+\delta$, which is generated by $\tau\left(C_{j}\right)$.

So, $\left(\mathbb{Z}_{m} \times \mathbb{Z}_{n g}, \mathcal{G}^{\prime}, \mathcal{A}_{0}^{\prime}, \ldots, \mathcal{A}_{s}^{\prime}\right)$ is a strictly $\mathbb{Z}_{m} \times \mathbb{Z}_{n g}$-invariant $s$ - $\mathrm{FG}\left(3,\left(L_{0}, \ldots, L_{s}\right), m n g\right)$ of type $(\text { eng })^{m / e}$.

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[^0]:    *Research supported by NSFC grants 11222113,11431003 , and a project funded by the priority academic program development of Jiangsu higher education institutions.
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