# AG codes and AG quantum codes from the GGS curve 

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#### Abstract

In this paper, algebraic-geometric (AG) codes associated with the GGS maximal curve are investigated. The Weierstrass semigroup at all $\mathbb{F}_{q^{2}}$-rational points of the curve is determined; the Feng-Rao designed minimum distance is computed for infinite families of such codes, as well as the automorphism group. As a result, some linear codes with better relative parameters with respect to one-point Hermitian codes are discovered. Classes of quantum and convolutional codes are provided relying on the constructed AG codes.


Keywords: GGS curve, AG code, quantum code, convolutional code, code automorphisms.

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## 1 Introduction

In [13, 14] Goppa used algebraic curves to construct linear error correcting codes, the so called algebraic geometric codes (AG codes). The construction of an AG code with alphabet a finite field $\mathbb{F}_{q}$ requires that the underlying curve is $\mathbb{F}_{q}$-rational and involves two $\mathbb{F}_{q}$-rational divisors $D$ and $G$ on the curve.

In general, to construct a "good" AG code over $\mathbb{F}_{q}$ we need a curve $\mathcal{X}$ with low genus $g$ with respect to its number of $\mathbb{F}_{q}$-rational points. In fact, from the Goppa bounds on the parameters of the code it follows that the relative Singleton defect is upper bounded by the ratio $g / N$, where $N$ can be as large as the number of $\mathbb{F}_{q}$-rational points of $\mathcal{X}$ not in the

[^0]support of $G$. Maximal curves over $\mathbb{F}_{q}$ attain the Hasse-Weil upper bound for the number of $\mathbb{F}_{q}$-rational points with respect to their genus and for this reason they have been used in a number of works. Examples of such curves are the Hermitian curve, the GK curve [12], the GGS curve [10], the Suzuki curve [7], the Klein quartic when $\sqrt{q} \equiv 6(\bmod 7)$ [33], together with their quotient curves. Maximal curves often have large automorphism groups which in many cases can be inherited by the code: this can bring good performances in encoding [25] and decoding [17].

Good bounds on the parameters of one-point codes, that is AG codes arising from divisors $G$ of type $n P$ for a point $P$ of the curve, have been obtained by investigating the Weierstrass semigroup at $P$. These results have been later generalized to codes and semigroups at two or more points; see e.g. [4, 5, 20, 21, 27, 30, 31].

AG codes from the Hermitian curve have been widely investigated; see [8, 22, 24, 37, 39,40 and the references therein. Other constructions based on the Suzuki curve and the curve with equation $y^{q}+y=x^{q^{r}+1}$ can be found in [32] and [36]. More recently, AG Codes from the GK curve have been constructed in [1, 3, 9].

In the present work we investigate one-point $A G$ codes from the $\mathbb{F}_{q^{2 n} \text {-maximal }}$ GGS curve, $n \geq 5$ odd. The GGS curve has more short orbits under its automorphism group than other maximal curves, see [15], and hence more possible structures for the Weierstrass semigroups at one point. On the one hand this makes the investigation more complicated; on the other hand it gives more chances of finding one-point AG codes with good parameters. One achievement of this work is the determination of the Weierstrass semigroup at any $\mathbb{F}_{q^{2}}$-rational point.

We show that the one-point codes at the infinite point $P_{\infty}$ inherit a large automorphism group from the GGS curve; for many of such codes, the full automorphism group is obtained. Moreover, for $q=2$, we compute explicitly the Feng-Rao designed minimum distance, which improves the Goppa designed minimum distance. As an application, we provide families of codes with $q=2$ whose relative Singleton defect goes to zero as $n$ goes to infinity. We were not able to produce analogous results for an $\mathbb{F}_{q^{2}}$-rational affine point $P_{0}$, because of the more complicated structure of the Weierstrass semigroup. In a comparison between onepoint codes from $P_{\infty}$ and one-point codes from $P_{0}$, it turns out that the best codes come sometimes from $P_{\infty}$, other times from $P_{0}$; we give evidence of this fact with tables for the case $q=2, n=5$.

Note that in general, many of our codes are better than the comparable one-point Hermitian codes on the same alphabet. In fact, let $C_{1}$ be a code from a one-point divisor $G_{1}$ on the $\mathbb{F}_{q^{2 n} \text {-maximal }}$ GGS curve with genus $g_{1}$, with alphabet $\mathbb{F}_{q^{2 n}}$, length $N_{2}$, designed dimension $k_{1}^{*}=\operatorname{deg} G_{1}-g_{1}+1$, and designed minimum distance $d_{1}^{*}=\operatorname{deg} G_{1}-\left(2 g_{1}-2\right)$. In the same way, let $C_{2}$ be a code from a one-point divisor $G_{2}$ on the $\mathbb{F}_{q^{2 n} \text {-maximal Hermi- }}$ tian curve with genus $g_{2}$, with the same alphabet $\mathbb{F}_{q^{2 n}}$ and length $N_{2}=N_{1}$ as $C_{1}$, designed
dimension $k_{2}^{*}=\operatorname{deg} G_{2}-g_{2}+1$, and designed minimum distance $d_{2}^{*}=\operatorname{deg} G_{2}-\left(2 g_{2}-2\right)$. In order to compare $C_{1}$ and $C_{2}$, we can choose $G_{1}$ and $G_{2}$ such that $k_{1}^{*}=k_{2}^{*}$. Then the difference $d_{1}^{*}-d_{2}^{*}$, like the difference $\delta_{2}^{*}-\delta_{1}^{*}$ between the designed Singleton defects, is equal to $g_{2}-g_{1}=\frac{1}{2}\left(q^{2 n}-q^{n+2}+q^{3}-q^{2}\right) \gg 0$.

Finally, we apply our results on AG codes to construct families of quantum codes and convolutional codes.

## 2 Preliminaries

### 2.1 Curves and codes

Let $\mathcal{X}$ be a projective, geometrically irreducible, nonsingular algebraic curve of genus $g$ defined over the finite field $\mathbb{F}_{q}$ of size $q$. The symbols $\mathcal{X}\left(\mathbb{F}_{q}\right)$ and $\mathbb{F}_{q}(\mathcal{X})$ denote the set of $\mathbb{F}_{q}$-rational points and the field of $\mathbb{F}_{q}$-rational functions, respectively. A divisor $D$ on $\mathcal{X}$ is a formal sum $n_{1} P_{1}+\cdots+n_{r} P_{r}$, where $P_{i} \in \mathcal{X}\left(\mathbb{F}_{q}\right), n_{i} \in \mathbb{Z}, P_{i} \neq P_{j}$ if $i \neq j$. The divisor $D$ is $\mathbb{F}_{q}$-rational if it coincides with its image $n_{1} P_{1}^{q}+\cdots+n_{r} P_{r}^{q}$ under the Frobenius map over $\mathbb{F}_{q}$. For a function $f \in \mathbb{F}_{q}(\mathcal{X}), \operatorname{div}(f)$ and $(f)_{\infty}$ indicate the divisor of $f$ and its pole divisor. Also, the Weierstrass semigroup at $P$ will be indicated by $H(P)$. The Riemann-Roch space associated with an $\mathbb{F}_{q}$-rational divisor $D$ is

$$
\mathcal{L}(D):=\left\{f \in \mathcal{X}\left(\mathbb{F}_{q}\right) \backslash\{0\}: \operatorname{div}(f)+D \geq 0\right\} \cup\{0\}
$$

and its dimension over $\mathbb{F}_{q}$ is denoted by $\ell(D)$.
Let $P_{1}, \ldots, P_{N} \in \mathcal{X}\left(\mathbb{F}_{q}\right)$ be pairwise distinct points and consider the divisor $D=P_{1}+$ $\cdots+P_{N}$ and another $\mathbb{F}_{q}$-rational divisor $G$ whose support is disjoint from the support of $D$. The AG code $C(D, G)$ is the image of the linear map $\eta: \mathcal{L}(G) \rightarrow \mathbb{F}_{q}^{N}$ given by $\eta(f)=\left(f\left(P_{1}\right), f\left(P_{2}\right), \ldots, f\left(P_{N}\right)\right)$. The code has length $N$ and if $N>\operatorname{deg}(G)$ then $\eta$ is an embedding and the dimension $k$ of $C(D, G)$ is equal to $\ell(G)$. The minimum distance $d$ satisfies $d \geq d^{*}=N-\operatorname{deg}(G)$, where $d^{*}$ is called the designed minimum distance of $C(D, G)$; if in addition $\operatorname{deg}(G)>2 g-2$, then by the Riemann-Roch Theorem $k=\operatorname{deg}(G)-g+1$; see [19, Th. 2.65]. The dual code $C^{\perp}(D, G)$ is an $A G$ code with dimension $k^{\perp}=N-k$ and minimum distance $d^{\perp} \geq \operatorname{deg} G-2 g+2$. If $G=\alpha P, \alpha \in \mathbb{N}, P \in \mathcal{X}\left(\mathbb{F}_{q}\right)$, the AG codes $C(D, G)$ and $C^{\perp}(D, G)$ are referred to as one-point AG codes. Let $H(P)$ be the Weierstrass semigroup associated with $P$, that is

$$
H(P):=\left\{n \in \mathbb{N}_{0} \mid \exists f \in \mathbb{F}_{q}(\mathcal{X}),(f)_{\infty}=n P\right\}=\left\{\rho_{1}=0<\rho_{2}<\rho_{3}<\cdots\right\}
$$

Denote by $f_{\ell} \in \mathbb{F}_{q}(\mathcal{X}), \ell \geq 1$, a rational function such that $\left(f_{\ell}\right)_{\infty}=\rho_{\ell} P$. For $\ell \geq 0$,
define the Feng-Rao function

$$
\nu_{\ell}:=\left|\left\{(i, j) \in \mathbb{N}_{0}^{2}: \rho_{i}+\rho_{j}=\rho_{\ell+1}\right\}\right| .
$$

Consider $C_{\ell}(P)=C^{\perp}\left(P_{1}+P_{2}+\cdots+P_{N}, \rho_{\ell} P\right), P, P_{1}, \ldots, P_{N}$ pairwise distint points in $\mathcal{X}\left(\mathbb{F}_{q}\right)$. The number

$$
d_{O R D}\left(C_{\ell}(P)\right):=\min \left\{\nu_{m}: m \geq \ell\right\}
$$

is a lower bound for the minimum distance $d\left(C_{\ell}(P)\right)$ of the code $C_{\ell}(P)$, called the order bound or the Feng-Rao designed minimum distance of $C_{\ell}(P)$; see [19, Theorem 4.13]. Also, by [19, Theorem 5.24], $d_{O R D}\left(C_{\ell}(P)\right) \geq \ell+1-g$ and equality holds if $\ell \geq 2 c-g-1$, where $c=\max \{m \in \mathbb{Z}: m-1 \notin H(P)\}$.

A numerical semigroup is called telescopic if it is generated by a sequence $\left(a_{1}, \ldots, a_{k}\right)$ such that

- $\operatorname{gcd}\left(a_{1}, \ldots, a_{k}\right)=1 ;$
- for each $i=2, \ldots, k, a_{i} / d_{i} \in\left\langle a_{1} / d_{i-1}, \ldots, a_{i-1} / d_{i-1}\right\rangle$, where $d_{i}=\operatorname{gcd}\left(a_{1}, \ldots, a_{i}\right)$;
see [28]. The semigroup $H(P)$ is called symmetric if $2 g-1 \notin H(P)$. The property of being symmetric for $H(P)$ gives rise to useful simplifications of the computation of $d_{O R D}\left(C_{\ell}(P)\right)$, when $\rho_{\ell}>2 g$. The following result is due to Campillo and Farrán; see [2, Theorem 4.6].

Proposition 2.1. Let $\mathcal{X}$ be an algebraic curve of genus $g$ and let $P \in \mathcal{X}\left(\mathbb{F}_{q}\right)$. If $H(P)$ is a symmetric Weierstrass semigroup then one has

$$
d_{O R D}\left(C_{\ell}(P)\right)=\nu_{\ell}
$$

for all $\rho_{\ell+1}=2 g-1+e$ with $e \in H(P) \backslash\{0\}$.

### 2.2 The automorphism group of an AG code $C(D, G)$

In the following we use the same notation as in [11,26]. Let $\mathcal{M}_{N, q} \leq \operatorname{GL}(N, q)$ be the subgroup of matrices having exactly one non-zero element in each row and column. For $\gamma \in \operatorname{Aut}\left(\mathbb{F}_{q}\right)$ and $M=\left(m_{i, j}\right)_{i, j} \in \operatorname{GL}(N, q)$, let $M^{\gamma}$ be the matrix $\left(\gamma\left(m_{i, j}\right)\right)_{i, j}$. Let $\mathcal{W}_{N, q}$ be the semidirect product $\mathcal{M}_{N, q} \rtimes \operatorname{Aut}\left(\mathbb{F}_{q}\right)$ with multiplication $M_{1} \gamma_{1} \cdot M_{2} \gamma_{2}:=M_{1} M_{2}^{\gamma} \cdot \gamma_{1} \gamma_{2}$. The automorphism group $\operatorname{Aut}(C(D, G))$ of $C(D, G)$ is the subgroup of $\mathcal{W}_{N, q}$ preserving $C(D, G)$, that is,

$$
M \gamma\left(x_{1}, \ldots, x_{N}\right):=\left(\left(x_{1}, \ldots, x_{N}\right) \cdot M\right)^{\gamma} \in C(D, G) \text { for any }\left(x_{1}, \ldots, x_{N}\right) \in C(D, G)
$$

Let $\operatorname{Aut}_{\mathbb{F}_{q}}(\mathcal{X})$ denote the $\mathbb{F}_{q}$-automorphism group of $\mathcal{X}$. Also, let

$$
\operatorname{Aut}_{\mathbb{F}_{q}, D, G}(\mathcal{X})=\left\{\sigma \in \operatorname{Aut}_{\mathbb{F}_{q}}(\mathcal{X}) \mid \sigma(D)=D, \sigma(G) \approx_{D} G\right\}
$$

where $G^{\prime} \approx_{D} G$ if and only if there exists $u \in \mathbb{F}_{q}(\mathcal{X})$ such that $G^{\prime}-G=(u)$ and $u\left(P_{i}\right)=1$ for $i=1, \ldots, N$, and

$$
\operatorname{Aut}_{\mathbb{F}_{q}, D, G}^{+}(\mathcal{X}):=\left\{\sigma \in \operatorname{Aut}_{\mathbb{F}_{q}}(\mathcal{X})|\sigma(D)=D, \sigma(|G|)=|G|\}\right.
$$

where $|G|=\left\{G+(f) \mid f \in \overline{\mathbb{F}}_{q}(\mathcal{X})\right\}$ is the linear series associated with $G$. Note that $\operatorname{Aut}_{\mathbb{F}_{q}, D, G}(\mathcal{X}) \subseteq \operatorname{Aut}_{\mathbb{F}_{q}, D, G}^{+}(\mathcal{X})$.

Remark 2.2. Suppose that $\operatorname{supp}(D) \cup \operatorname{supp}(G)=\mathcal{X}\left(\mathbb{F}_{q}\right)$ and each point in $\operatorname{supp}(G)$ has the same weight in $G$. Then

$$
\operatorname{Aut}_{\mathbb{F}_{q}, D, G}(\mathcal{X})=\operatorname{Aut}_{\mathbb{F}_{q}, D, G}^{+}(\mathcal{X})=\left\{\sigma \in \operatorname{Aut}_{\mathbb{F}_{q}}(\mathcal{X}) \mid \sigma(\operatorname{supp}(G))=\operatorname{supp}(G)\right\}
$$

In [11] the following result was proved.
Theorem 2.3. ([11, Theorem 3.4]) Suppose that the following conditions hold:

- $G$ is effective;
- $\ell(G-P)=\ell(G)-1$ and $\ell(G-P-Q)=\ell(G)-2$ for any $P, Q \in \mathcal{X}$;
- $\mathcal{X}$ has a plane model $\Pi(\mathcal{X})$ with coordinate functions $x, y \in \mathcal{L}(G)$;
- $\mathcal{X}$ is defined over $\mathbb{F}_{p}$;
- the support of $D$ is preserved by the Frobenius morphism $(x, y) \mapsto\left(x^{p}, y^{p}\right)$;
- $N>\operatorname{deg}(G) \cdot \operatorname{deg}(\Pi(\mathcal{X}))$.

Then

$$
\operatorname{Aut}(C(D, G)) \cong\left(\operatorname{Aut}_{\mathbb{F}_{q}, D, G}^{+}(\mathcal{X}) \rtimes \operatorname{Aut}\left(\mathbb{F}_{q}\right)\right) \rtimes \mathbb{F}_{q}^{*}
$$

If any non-trivial element of $\operatorname{Aut}_{\mathbb{F}_{q}}(\mathcal{X})$ fixes at most $N-1 \mathbb{F}_{q}$-rational points of $\mathcal{X}$ then $\operatorname{Aut}(C(D, G))$ contains a subgroup isomorphic to $\left(\operatorname{Aut}_{\mathbb{F}_{q}, D, G}(\mathcal{X}) \rtimes \operatorname{Aut}\left(\mathbb{F}_{q}\right)\right) \rtimes \mathbb{F}_{q}^{*}$; see $\mathbb{1}$, Proposition 2.3].

### 2.3 The GGS curve

Let $q$ be a prime power and consider an odd integer $n$. The GGS curve $G G S(q, n)$ is defined by the equations

$$
\left\{\begin{array}{l}
X^{q}+X=Y^{q+1}  \tag{1}\\
Y^{q^{2}}-Y=Z^{m}
\end{array}\right.
$$

where $m=\left(q^{n}+1\right) /(q+1)$; see [10]. The genus of $G G S(q, n)$ is $\frac{1}{2}(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)$, and $G G S(q, n)$ is $\mathbb{F}_{q^{2 n}}$-maximal.

Let $P_{0}=(0,0,0), P_{(a, b, c)}=(a, b, c)$, and let $P_{\infty}$ be the unique ideal point of $G G S(q, n)$. Note that $G G S(q, n)$ is singular, being $P_{\infty}$ its unique singular point. Yet, there is only one place of $G G S(q, n)$ centered at $P_{\infty}$; therefore, we can actually construct AG codes from $G G S(q, n)$ as described in Section 2.1 (see [38, Appendix B] and [18, Chapter 8] for an introduction to the concept of place of a curve). The divisors of the functions $x, y, z$ satisfying $x^{q}+x=y^{q+1}, y^{q^{2}}-y=z^{m}$ are

$$
\begin{aligned}
& (x)=m(q+1) P_{0}-m(q+1) P_{\infty}, \\
& (y)=m \sum_{\substack{\alpha^{q}+\alpha=0}} P_{(\alpha, 0,0)}-m q P_{\infty}, \\
& (z)=\sum_{\substack{\alpha^{q}+\alpha=\beta \\
\beta \in \mathbb{F}_{q^{2}}}} P_{(\alpha, \beta, 0)}-q^{3} P_{\infty} .
\end{aligned}
$$

Throughout the paper we indicate by $\bar{D}$ and $\tilde{D}$ the divisors

$$
\begin{equation*}
\bar{D}=\sum_{P \in G G S(q, n)\left(\mathbb{F}_{q^{2 n}}\right), P \neq P_{\infty}} P, \quad \tilde{D}=\sum_{P \in G G S(q, n)\left(\mathbb{F}_{q^{2 n}}\right), P \neq P_{0}} P . \tag{2}
\end{equation*}
$$

### 2.4 Structure of the paper

The paper is organized as follows. In Section 3 the value of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for $q=2$ and $n \geq 5$ is obtained, where $C_{\ell}\left(P_{\infty}\right)=C^{\perp}\left(\bar{D}, \ell P_{\infty}\right)$; this is applied in Section 3.5 to two families of codes with $q=2$ whose relative Singleton defect goes to zero as $n$ goes to infinity. In Section 4 we determine the Weierstrass semigroup at $P_{0}$, and hence at any $\mathbb{F}_{q^{2}}$-rational affine point of $G G S(q, n)$. The tables in Section 5 describe the parameters of $C_{\ell} P_{\infty}$ and $C_{\ell}\left(P_{0}\right)$ in the particular case $q^{2 n}=2^{10}$. Sections 6 and 7 provide families of quantum codes and convolutional codes constructed from $C_{\ell}\left(P_{\infty}\right)$ and $C_{\ell}\left(P_{0}\right)$. Finally, we compute in Section 8 the automorphism group of the AG code $C\left(\bar{D}, \ell P_{\infty}\right)$ for $q^{n}+1 \leq \ell \leq q^{n+2}-q^{3}$.

## 3 The computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for $q=2$

In this section we deal with the codes $C_{\ell}\left(P_{\infty}\right)=C^{\perp}\left(\bar{D}, \rho_{\ell} P_{\infty}\right)$, where $\bar{D}$ is as in (2). Our purpose is to exhibit the exact value of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for the case $q=2$. First of all we determine the values of $\nu_{\ell}$ (Subsection 3.1); in Subsections 3.2, 3.3, 3.4 we compute $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$.

### 3.1 The Feng-Rao function $\nu_{\ell}$ for $q=2$

Assume that $q=2$ and $n \geq 5$ is odd. Let $m=\frac{2^{n}+1}{3}$. Then, from [15, Corollary 3.5],

$$
H\left(P_{\infty}\right)=\left\{\left.i\left(2^{n}+1\right)+2 j \frac{2^{n}+1}{3}+8 k \right\rvert\, i \in\{0,1\}, j \in\{0,1,2,3\}, k \geq 0\right\}
$$

Remark 3.1. Let $\rho_{\ell}=i\left(2^{n}+1\right)+2 j \frac{2^{n}+1}{3}+8 k \in H\left(P_{\infty}\right)$. Then $\rho_{\ell}$ is uniquely determined by the triple $(i, j, k)$.

Proof. Assume that $i\left(2^{n}+1\right)+2 j \frac{2^{n}+1}{3}+8 k=i^{\prime}\left(2^{n}+1\right)+2 j^{\prime} \frac{2^{n}+1}{3}+8 k^{\prime}$. Then $i \equiv i^{\prime}(\bmod 2)$ and since $i, i^{\prime}<2$ we have that $i=i^{\prime}$. Thus, $2 j \frac{2^{n}+1}{3}+8 k=2 j^{\prime} \frac{n^{n}+1}{3}+8 k^{\prime}$. Since this implies that $j \equiv j^{\prime}(\bmod 4)$ and $j, j^{\prime}<4$, we have that $j=j^{\prime}$ and $k=k^{\prime}$ and the claim follows.

According to Remark [3.1, the notation $(i, j, k)$ is used to indicate the non-gap at $P_{\infty}$ associated with the choices of the parameters $i, j, k$. In order to compute $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ the following definition is required. Let $\rho_{\ell} \in H\left(P_{\infty}\right)$ be fixed. Assume that $\rho_{\ell+1}=(i, j, k)$. Then,

$$
\nu_{\ell}=\left|\left\{\left(i_{r}, j_{r}, k_{r}\right), r=1,2 \mid(i, j, k)=\left(i_{1}, j_{1}, k_{1}\right)+\left(i_{2}, j_{2}, k_{2}\right)\right\}\right| .
$$

In the following lemmas we determine the value of $\nu_{\ell}$.
Lemma 3.2. Let $\rho_{\ell} \in H\left(P_{\infty}\right)$ be fixed. Assume that $\rho_{\ell+1}=(1, j, k)$ for some $j=0,1,2,3$ and $k \geq 0$. Then,

$$
\nu_{\ell}=\left\{\begin{array}{l}
2(j+1)(k+1), \quad \text { if } k<m, \\
2(j+1)(k+1)+2(3-j)(k-m+1), \quad \text { otherwise } .
\end{array}\right.
$$

Proof. Let $i_{1}, i_{2}, j_{1}, j_{2}, k_{1}$, and $k_{2} \in \mathbb{N}$ be such that
$\left(2^{n}+1\right)+2 j m+8 k=\left(i_{1}+i_{2}\right)\left(2^{n}+1\right)+2\left(j_{1}+j_{2}\right) m+8\left(k_{1}+k_{2}\right)=3\left(i_{1}+i_{2}\right) m+2\left(j_{1}+j_{2}\right) m+8\left(k_{1}+k_{2}\right)$.
Then $i_{1}+i_{2} \equiv 1(\bmod 2)$ and since $i_{1}+i_{2} \leq 2$ we have that $i_{1}+i_{2}=1$. This implies that

$$
3 m+2 j m+8 k=3 m+2\left(j_{1}+j_{2}\right) m+8\left(k_{1}+k_{2}\right)
$$

and hence

$$
\begin{equation*}
j m+4 k=\left(j_{1}+j_{2}\right) m+4\left(k_{1}+k_{2}\right) . \tag{3}
\end{equation*}
$$

Assume that $j=0$. Then from (3),$\left(j_{1}+j_{2}\right) m \equiv 0(\bmod 4)$ and so, $j_{1}+j_{2}=4 h$ for some integer $h$. Since $0 \leq j_{1}+j_{2} \leq 6$ we have that $h=0$ or $h=1$. In the first case $k_{1}+k_{2}=k$, in the second case $k_{1}+k_{2}=k-m$. Since $k_{1}+k_{2} \geq 0$, if $k<m$ the second case cannot occur. Thus, if $k<m$, since we have 2 possible choices for $i_{1}$ and $(k+1)$ choices for $k_{1}$ (while $i_{2}$ and $k_{2}$ are determined according to the choices of $i_{1}$ and $k_{1}$, respectively), then $\nu_{\ell}=2(k+1)$. Also, if $k \geq m$ we have that
$\nu_{\ell}=2(k+1)+2 \cdot\left|\left\{\left(j_{1}, j_{2}\right): 0 \leq j_{1}, j_{2} \leq 3, j_{1}+j_{2}=4\right\}\right| \cdot(k-m+1)=2(k+1)+6(k-m+1)$
and the claim follows by direct checking.
Assume that $j=1$. Then from (3),$\left(j_{1}+j_{2}\right) m \equiv m(\bmod 4)$ and so $j_{1}+j_{2}=1+4 h$ for some integer $h$. Since $0 \leq j_{1}+j_{2} \leq 6$ we have that $h=0$ or $h=1$. In the first case $k_{1}+k_{2}=k$, in the second case $k_{1}+k_{2}=k-m$. Since $k_{1}+k_{2} \geq 0$ if $k<m$ the second case cannot occur. Thus, if $k<m$, since we have 2 possible choices for $i_{1}, 2$ possible choices for $j_{1}$ and $(k+1)$ choices for $k_{1}$, then $\nu_{\ell}=4(k+1)$. Also, if $k \geq m$ we have that,
$\nu_{\ell}=4(k+1)+2 \cdot\left|\left\{\left(j_{1}, j_{2}\right): 0 \leq j_{1}, j_{2} \leq 3, j_{1}+j_{2}=5\right\}\right| \cdot(k-m+1)=4(k+1)+4(k-m+1)$,
and the claim follows by direct checking.
Assume that $j=2$. Then from (3), $\left(j_{1}+j_{2}\right) m \equiv 2 m(\bmod 4)$ and so $j_{1}+j_{2}=2+4 h$, for some integer $h$. Since $0 \leq j_{1}+j_{2} \leq 6$ we have that $h=0$ or $h=1$. In the first case $k_{1}+k_{2}=k$, in the second case $k_{1}+k_{2}=k-m$. Since $k_{1}+k_{2} \geq 0$, if $k<m$ the second case cannot occur. Thus, if $k<m$, since we have 2 possible choices for $i_{1}, 3$ possible choices for $j_{1}$ and $(k+1)$ choices for $k_{1}$, then $\nu_{\ell}=6(k+1)$. Also, if $k \geq m$ we have that
$\nu_{\ell}=6(k+1)+2(k-m+1) \cdot\left|\left\{\left(j_{1}, j_{2}\right): 0 \leq j_{1}, j_{2} \leq 3, j_{1}+j_{2}=6\right\}\right|=6(k+1)+2(k-m+1)$, and the claim follows by direct checking.

Assume that $j=3$. Then from (3),$\left(j_{1}+j_{2}\right) m \equiv 3 m(\bmod 4)$ and so $j_{1}+j_{2}=3+4 h$, for some integer $h$. Since $0 \leq j_{1}+j_{2} \leq 6$ we have that $h=0$. Since this implies that $k_{1}+k_{2}=k$, we have that $\nu_{\ell}=8(k+1)$.

Using a similar approach we can prove the following.

Lemma 3.3. Let $\rho_{\ell} \in H\left(P_{\infty}\right)$ be fixed. Assume that $\rho_{\ell+1}=(0, j, k)$ for some $j=0,1,2,3$ and $k \geq 0$. Then,
$\nu_{\ell}=\left\{\begin{array}{l}(j+1)(k+1)+\left\lfloor\frac{j}{3}\right\rfloor(k+1), \quad \text { if } k<m, \\ (j+1)(k+1)+\left\lfloor\frac{j}{3}\right\rfloor(k+1)+(5-2 \max \{0, j-2\})(k-m+1), \quad \text { if } m \leq k<2 m, \\ (j+1)(k+1)+\left\lfloor\frac{j}{3}\right\rfloor(k+1)+ \\ +(5-2 \max \{0, j-2\})(k-m+1)+\max \{0,2-j\}(k-2 m+1), \quad \text { otherwise. }\end{array}\right.$

### 3.2 Computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for $\rho_{\ell+1}=(1, j, k)$ and $\rho_{\ell} \leq 2 g$

Let $\rho_{\ell} \in H\left(P_{\infty}\right)$. Assume that $\rho_{\ell+1}=(1, j, k)$ for some $j=0,1,2,3$ and $k \geq 0$. Recall that $C_{\ell}\left(P_{\infty}\right)$ is the dual code of the AG code $C\left(\bar{D}, \rho_{\ell} P_{\infty}\right)$, where $\bar{D}$ is as in (2).

Lemma 3.4. If $\rho_{\ell+1}=(1,0, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)= \begin{cases}2, & \text { if } k=0, \\ 3, & \text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\ 4, & \text { if } \frac{m}{8}<k \leq\left\lfloor\frac{m}{4}\right\rfloor, \\ 5, & \text { if } \frac{m}{4}<k \leq\left\lfloor\frac{3 m}{8}\right\rfloor, \\ 6, & \text { if } \frac{3 m}{8}<k \leq\left\lfloor\frac{m}{2}\right\rfloor, \\ 8, & \text { if } \frac{m}{2}<k \leq\left\lfloor\frac{3 m}{4}\right\rfloor, \\ 8\left(\left\lceil k-\frac{3 m}{4}\right\rceil+1\right), \quad \text { if } \frac{3 m}{4} \leq k \leq m-2, \\ \nu_{\ell}=2 m, \quad \text { if } k=m-1 .\end{cases}
$$

Proof. For $\rho_{s} \in H\left(P_{\infty}\right)$ the following system of inequalities is considered:

$$
\left\{\begin{array}{l}
\rho_{s+1} \geq \rho_{\ell+1}  \tag{4}\\
\nu_{s} \leq \nu_{\ell}
\end{array}\right.
$$

In order to compute $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ we take the minimum value of $\nu_{s}$ such that System (4) is satisfied. Also, a case-by-case analysis with respect to $a \in\{0,1\}$ is required. Assume that $\rho_{s+1}=(a, b, c)$ for some $a \in\{0,1\}, b \in\{0,1,2,3\}$ and $c \geq 0$. From Lemma 3.2, System (4) reads,

$$
\left\{\begin{array}{l}
3 a m+2 b m+8 c \geq 3 m+8 k  \tag{5}\\
\nu_{s} \leq 2(k+1)
\end{array}\right.
$$

Case 1: $a=1$ and $c<m$. From Lemma 3.2, System (5) reads

$$
\left\{\begin{array}{l}
2 b m+8 c \geq 8 k \\
2(b+1)(c+1) \leq 2(k+1)
\end{array}\right.
$$

- If $b=0$ then $c=k$ and so the unique solution is $\nu_{\ell}$ itself.
- If $b=1$ then $c \geq\left\lceil k-\frac{m}{4}\right\rceil$ and $c \leq\left\lfloor\frac{k-1}{2}\right\rfloor$. Such a $c$ exists if and only if $\left\lceil k-\frac{m}{4}\right\rceil \leq\left\lfloor\frac{k-1}{2}\right\rfloor$. Assume that $k$ is odd. Then $k-\left\lfloor\frac{m}{4}\right\rfloor=\left\lceil k-\frac{m}{4}\right\rceil \leq\left\lfloor\frac{k-1}{2}\right\rfloor=\frac{k-1}{2}$ if and only if $k \leq 2\left\lfloor\frac{m}{4}\right\rfloor-1$. Similarly if $k$ is even then $c$ exists if and only if $k-\left\lfloor\frac{m}{4}\right\rfloor \leq \frac{k-2}{2}$, that is $k \leq 2\left\lfloor\frac{m}{4}\right\rfloor-2$. For these cases the minimum is obtained taking $c=\max \left\{0,\left\lceil k-\frac{m}{4}\right\rceil\right\}$ and hence $\nu_{s}=4\left(\max \left\{0,\left\lceil k-\frac{m}{4}\right\rceil\right\}+1\right)$.
- If $b=2$ then $c \geq\left\lceil k-\frac{m}{2}\right\rceil$ and $c \leq\left\lfloor\frac{k-2}{3}\right\rfloor$. As before, such a $c$ exists if and only if $\left\lceil k-\frac{m}{2}\right\rceil \leq\left\lfloor\frac{k-2}{3}\right\rfloor$. This is equivalent to $k \leq \frac{3}{2}\left(\left\lfloor\frac{m}{2}\right\rfloor-1\right)$ if $k \equiv 0(\bmod 3)$, to $k \leq \frac{3}{2}\left\lfloor\frac{m}{2}\right\rfloor-2$ if $k \equiv 1(\bmod 3)$, to $k \leq \frac{3}{2}\left\lfloor\frac{m}{2}\right\rfloor-1$ if $k \equiv 2(\bmod 3)$. For these cases the minimum is obtained taking $c=\max \left\{0,\left\lceil k-\frac{m}{2}\right\rceil\right\}$ and hence $\nu_{s}=6\left(\max \left\{0,\left\lceil k-\frac{m}{2}\right\rceil\right\}+1\right)$.
- If $b=3$ then $c \geq\left\lceil k-\frac{3 m}{4}\right\rceil$ and $c \leq\left\lfloor\frac{k-3}{4}\right\rfloor$. As before, such a $c$ exists if and only if $\lceil k-$ $\left.\frac{3 m}{4}\right\rceil \leq\left\lfloor\frac{k-3}{4}\right\rfloor$. By direct checking, this is equivalent to $k \leq m-2$. Here the minimum is obtained taking $c=\max \left\{0,\left\lceil k-\frac{3 m}{4}\right\rceil\right\}$ and hence $\nu_{s}=8\left(\max \left\{0,\left\lceil k-\frac{3 m}{4}\right\rceil\right\}+1\right)$.

When $k>\frac{3 m}{4}$ and $k \leq m-2$ the minimum value above is obtained as $\nu_{s}=8\left(\left\lceil k-\frac{3 m}{4}\right\rceil+1\right)$. We observe that if $k=m-1$ then $\nu_{\ell}=2(k+1)=2 m$ and $8\left(\max \left\{0,\left\lceil k-\frac{3 m}{4}\right\rceil\right\}+1\right)=$ $\left.8\left(\left\lceil m-1-\frac{3 m}{4}\right\rceil\right\}+1\right)>2 m$. This implies that if $k=m-1$ then the minimum value is $\nu_{\ell}=2 m$ itself. Thus, combining the previous results we obtain

$$
\min \left\{\nu_{s} \mid a=1 \text { and } c<m\right\}=\left\{\begin{array}{l}
2, \text { if } k=0,  \tag{6}\\
4 \text { if } 1 \leq k \leq\left\lfloor\frac{m}{4}\right\rfloor \\
6, \text { if } \frac{m}{4}<k \leq\left\lfloor\frac{m}{2}\right\rfloor, \\
8, \text { if } \frac{m}{2}<k \leq\left\lfloor\frac{3 m}{4}\right\rfloor, \\
8\left(\left\lceil k-\frac{3 m}{4}\right\rceil+1\right), \text { if } \frac{3 m}{4}<k \leq m-2, \\
2 m=\nu_{\ell}, \text { if } k=m-1
\end{array}\right.
$$

Case 2: $a=1$ and $c \geq m$. From Lemma 3.2 System (5) reads,

$$
\left\{\begin{array}{l}
2 b m+8 c \geq 8 k \\
2(b+1)(c+1)+2(3-b)(c-m+1) \leq 2(k+1)
\end{array}\right.
$$

Since $2(b+1)(c+1)+2(3-b)(c-m+1) \geq 2(c+1)$ and $c>k$ this case cannot occur.
Case 3: $a=0$ and $c<m$. From Lemma 3.3 (5) reads,

$$
\left\{\begin{array}{l}
2 b m+8 c \geq 3 m+8 k, \\
(b+1)(c+1)+\left\lfloor\frac{b}{3}\right\rfloor(c+1) \leq 2(k+1)
\end{array}\right.
$$

- If $b=0$ then $c \geq\left\lceil k+\frac{3 m}{8}\right\rceil$ and $c \leq 2 k+1$. Such a $c$ exists if and only if $k \geq\left\lfloor\frac{3 m}{8}\right\rfloor$. For these cases, the minimum is obtained taking $c=\left\lceil k+\frac{3 m}{8}\right\rceil$ and hence $\nu_{s}=\left(\left\lceil k+\frac{3 m}{8}\right\rceil+1\right)$.
- The case $b=1$ cannot occur. In fact we have $c \geq\left\lceil k+\frac{m}{8}\right\rceil$ and $2(c+1) \leq 2(k+1)$, a contradiction.
- If $b=2$ then $c \geq\left\lceil k-\frac{m}{8}\right\rceil$ and $c \leq\left\lfloor\frac{2 k-1}{3}\right\rfloor$. Such a $c$ exists if and only if $k+\left\lceil-\frac{m}{8}\right\rceil \leq$ $\left\lfloor\frac{2 k+1}{3}\right\rfloor$. This is equivalent to $k \leq 3\left\lfloor\frac{m}{8}\right\rfloor-1$ if $2 k \equiv 1(\bmod 3)$, to $k \leq 3\left\lfloor\frac{m}{8}\right\rfloor+1$ if $2 k \equiv 2(\bmod 3)$, to $k \leq 3\left\lfloor\frac{m}{8}\right\rfloor-3$ if $2 k \equiv 0(\bmod 3)$. For these cases, the minimum is obtained taking $c=\max \left\{0,\left\lceil k-\frac{m}{8}\right\rceil\right\}$ and hence $\nu_{s}=3\left(\max \left\{0,\left\lceil k-\frac{m}{8}\right\rceil\right\}+1\right)$.
- If $b=3$ then $c \geq\left\lceil k-\frac{3 m}{8}\right\rceil$ and $c \leq\left\lfloor\frac{2 k-3}{5}\right\rfloor$. Such a $c$ exists if and only if $k+\left\lceil-\frac{3 m}{8}\right\rceil \leq$ $\left\lfloor\frac{2 k-3}{5}\right\rfloor$. This is equivalent to $k \leq \frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-\frac{5}{3}$ if $2 k \equiv 0(\bmod 5)$, to $k \leq \frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-2$ if $2 k \equiv 1(\bmod 5)$, to $k \leq \frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-\frac{7}{3}$ if $2 k \equiv 2(\bmod 5)$, to $k \leq \frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-1$ if $2 k \equiv 3$ $(\bmod 5)$, to $k \leq \frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-\frac{4}{3}$ if $2 k \equiv 4(\bmod 5)$. In these cases, the minimum is obtained taking $c=\max \left\{0,\left\lceil k-\frac{3 m}{8}\right\rceil\right\}$ and hence $\nu_{s}=5\left(\max \left\{0,\left\lceil k-\frac{3 m}{8}\right\rceil\right\}+1\right)$.
Thus, we obtain

$$
\min \left\{\nu_{s} \mid a=0 \text { and } c<m\right\}=\left\{\begin{array}{l}
3,  \tag{7}\\
\text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\
5, \\
\text { if } \frac{m}{8}<k \leq\left\lfloor\frac{3 m}{8}\right\rfloor, \\
5\left(\left\lceil k-\frac{3 m}{8}\right\rceil+1\right), \text { if }\left\lceil\frac{3 m}{8}\right\rceil \leq k \leq\left\lfloor\frac{5}{3}\left\lfloor\frac{3 m}{8}\right\rfloor-\frac{7}{3}\right\rfloor, \\
\left(\left\lceil k+\frac{3 m}{8}\right\rceil+1\right), \text { otherwise. }
\end{array}\right.
$$

Case 4: $a=0$ and $m \leq c<2 m$. From Lemma 3.3 System (5) reads,

$$
\left\{\begin{array}{l}
2 b m+8 c \geq 3 m+8 k \\
(b+1)(c+1)+\left\lfloor\frac{b}{3}\right\rfloor(c+1)+(5-2 \max \{0, b-2\})(c-m+1) \leq 2(k+1)
\end{array}\right.
$$

Since $(b+1)(c+1)+\left\lfloor\frac{b}{3}\right\rfloor(c+1)+(5-2 \max \{0, b-2\})(c-m+1) \geq(b+1)(c+1)$ and $c>k$, cases $b=1,2,3$ cannot occur. Thus $b=0$ and

$$
\left\{\begin{array}{l}
8 c \geq 3 m+8 k \\
(c+1)+5(c-m+1) \leq 2(k+1)
\end{array}\right.
$$

Hence $c \geq\left\lceil k+\frac{3 m}{8}\right\rceil$ and $c \leq\left\lfloor\frac{2 k+5 m-4}{6}\right\rfloor$. Since $c \geq m$ then $k \geq \frac{m+4}{2}$. The minimum value is obtained (when it is possible) for $c=\left\lceil k+\frac{3 m}{8}\right\rceil$. By direct checking the minimum $\nu_{\ell}$ is bigger than the one obtained in (6), and hence we can discard this case.

Case 5: $a=0$ and $c \geq 2 m$. From Lemma 3.3 System (15) reads,

$$
\left\{\begin{array}{l}
2 b m+8 c \geq 3 m+8 k, \\
(b+1)(c+1)+\left\lfloor\frac{b}{3}\right\rfloor(c+1)+(5-2 \max \{0, b-2\})(c-m+1)+\max \{0,2-b\}(c-2 m+1) \leq 2(k+1)
\end{array}\right.
$$

Since $(b+1)(c+1)+\left\lfloor\frac{b}{3}\right\rfloor(c+1)+(5-2 \max \{0, b-2\})(c-m+1)+\max \{0,2-b\}(c-2 m+1) \geq$ $(b+1)(c+1) \geq 2 m+1$ and $2(k+1) \leq 2 m$ this case cannot occur.

Taking the minimum of the values in (6) and (7) the claim follows.
Using the same arguments the following results are obtained.
Lemma 3.5. If $\rho_{\ell+1}=(1,3, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\nu_{\ell}
$$

Lemma 3.6. If $\rho_{\ell+1}=(1,1, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}
4, \quad \text { if } k=0, \\
5, \quad \text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\
6 \text { if } \frac{m}{8}<k \leq\left\lfloor\frac{m}{4}\right\rfloor, \\
8, \quad \text { if } \frac{m}{4}<k \leq\left\lfloor\frac{m}{2}\right\rfloor, \\
8\left(\left\lceil k-\frac{m}{2}\right\rceil+1\right), \quad \text { if }\left\lceil\frac{m}{2}\right\rceil \leq k \leq\left\lfloor\frac{3 m}{4}\right\rfloor-2, \\
2\left(\left\lceil\frac{m}{4}+k\right\rceil+1\right)+6\left(\left\lceil\frac{m}{4}+k\right\rceil-m+1\right), \quad \text { if }\left\lfloor\frac{3 m}{4}\right\rfloor-1 \leq k \leq m-2, \\
4 m, \quad \text { if } k=m-1 .
\end{array}\right.
$$

Lemma 3.7. If $\rho_{\ell+1}=(1,2, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}
6, \text { if } k=0, \\
8, \quad \text { if } k \leq\left\lfloor\frac{m}{4}\right\rfloor, \\
8\left(\left\lceil k-\frac{m}{4}\right\rceil+1\right), \quad \text { if }\left\lceil\frac{m}{4}\right\rceil \leq k \leq\left\lfloor\frac{m}{2}\right\rfloor-2, \\
2\left(\left\lceil k+\frac{m}{2}\right\rceil+1\right)+6\left(\left\lceil k+\frac{m}{2}\right\rceil-m+1\right), \quad \text { if }\left\lfloor\frac{m}{2}\right\rfloor-1 \leq k \leq\left\lfloor\frac{3 m}{4}\right\rfloor-2, \\
4\left(\left\lceil k+\frac{m}{4}\right\rceil+1\right)+4\left(\left\lceil k+\frac{m}{4}\right\rceil-m+1\right), \quad \text { if }\left\lfloor\frac{3 m}{4}\right\rfloor-1 \leq k \leq m-2, \\
\nu_{\ell}=6 m, \quad \text { if } k=m-1 .
\end{array}\right.
$$

### 3.3 Computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for $\rho_{\ell+1}=(0, j, k)$ and $\rho_{\ell} \leq 2 g$

Using the same arguments as above we obtain the following results in the case $\rho_{\ell} \leq 2 g$.
Lemma 3.8. If $\rho_{\ell+1}=(0,0, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)= \begin{cases}2, & \text { if } k \leq\left\lfloor\frac{3 m}{8}\right\rfloor \\ 3, & \text { if }\left\lceil\frac{3 m}{8}\right\rceil \leq k \leq\left\lfloor\frac{m}{2}\right\rfloor, \\ 4, & \text { if }\left\lceil\frac{m}{2}\right\rceil \leq k \leq\left\lfloor\frac{5 m}{8}\right\rfloor, \\ 5, & \text { if }\left\lceil\frac{5 m}{8}\right\rceil \leq k \leq\left\lfloor\frac{3 m}{4}\right\rfloor, \\ 6, & \text { if }\left\lceil\frac{3 m}{4}\right\rceil \leq k \leq\left\lfloor\frac{7 m}{8}\right\rfloor, \\ 8, & \text { if }\left\lceil\frac{7 m}{8}\right\rceil \leq k \leq m-1\end{cases}
$$

Lemma 3.9. If $\rho_{\ell+1}=(0,1, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)= \begin{cases}2, & \text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\ 3, & \text { if }\left\lceil\frac{m}{8}\right\rceil \leq k \leq\left\lfloor\frac{m}{4}\right\rfloor, \\ 4, & \text { if }\left\lceil\frac{m}{4}\right\rceil \leq k \leq\left\lfloor\frac{3 m}{8}\right\rfloor, \\ 5, & \text { if }\left\lceil\frac{3 m}{8}\right\rceil \leq k \leq\left\lfloor\frac{m}{2}\right\rfloor, \\ 6, & \text { if }\left\lceil\frac{m}{2}\right\rceil \leq k \leq\left\lfloor\frac{5 m}{8}\right\rfloor, \\ 8\left(\max \left\{0,\left\lceil k-\frac{7 m}{8}\right\rceil\right\}+1\right), \quad \text { if }\left\lceil\frac{5 m}{8}\right\rceil \leq k \leq m-1 .\end{cases}
$$

Lemma 3.10. If $\rho_{\ell+1}=(0,3, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}
6, \quad \text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\
8\left(\max \left\{0,\left\lceil\frac{3 m}{8}\right\rceil\right\}+1\right), \quad \text { if }\left\lceil\frac{m}{8}\right\rceil \leq k \leq m-2, \\
\nu_{\ell}=5(k+1), \quad \text { if } k=m-1
\end{array}\right.
$$

Lemma 3.11. If $\rho_{\ell+1}=(0,2, k)$ for some $k<m$ then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}
4, \text { if } k \leq\left\lfloor\frac{m}{8}\right\rfloor, \\
5, \text { if }\left\lceil\frac{m}{8}\right\rceil \leq k \leq\left\lfloor\frac{m}{4}\right\rfloor, \\
6, \quad \text { if }\left\lceil\frac{m}{4}\right\rceil \leq k \leq\left\lfloor\frac{3 m}{8}\right\rfloor, \\
8\left(\max \left\{0\left\lceil k-\frac{5 m}{8}\right\rceil\right\}+1\right), \quad \text { if }\left\lceil\frac{3 m}{8}\right\rceil \leq k \leq\left\lfloor\frac{7 m}{8}\right\rfloor-2, \\
2\left(\left\lceil k+\frac{m}{8}\right\rceil+1\right), \quad \text { if }\left\lfloor\frac{7 m}{8}\right\rfloor-1 \leq k \leq m-3, \\
3(k+1)=\nu_{\ell}, \quad \text { if } k \in\{m-2, m-1\} .
\end{array}\right.
$$

Lemma 3.12. If $\rho_{\ell+1}=(0,0, k)$ for some $m \leq k<2 m$ then
$d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}8\left(\left\lceil k-\frac{9 m}{8}\right\rceil+1\right), \quad \text { if } m \leq k<\left\lfloor\frac{11 m}{8}-1\right\rfloor, \\ 2\left(\left\lceil k-\frac{3 m}{8}\right\rceil+1\right)+\max \left\{0,6\left(\left\lceil k-\frac{3 m}{8}\right\rceil-m+1\right)\right\}, \quad \text { if }\left\lfloor\frac{11 m}{8}-1\right\rfloor \leq k<2 m .\end{array}\right.$

### 3.4 Computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for $\rho_{\ell}>2 g$

Proposition 3.13. The Weierstrass semigroup $H\left(P_{\infty}\right)=\left\langle q^{3}, m q, q^{n}+1\right\rangle$ is telescopic.
Proof. Let $a_{1}=q^{3}, a_{2}=m q, a_{3}=q^{n}+1, d_{0}=0, d_{1}=q^{3}, d_{2}=\operatorname{gcd}\left(q^{3}, m q\right)=q$, $d_{3}=\operatorname{gcd}\left(q^{3}, m q, q^{n}+1\right)=1$. Then $a_{i} / d_{i} \in\left\langle a_{1} / d_{i-1}, \ldots, a_{i-1} / d_{i-1}\right\rangle$ for $i=2,3$; that is, $H\left(P_{\infty}\right)$ is telescopic.

Proposition 3.13 implies that $H\left(P_{\infty}\right)$ is symmetric, from [28, Lemma 6.5]. This also follows from the fact that the divisor $(2 g-2) P_{\infty}$ is canonical; see [15, Lemma 3.8] and [28, Remark 4.4].

In the following, Proposition 2.1 is used to reduce the direct computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ with $\rho_{\ell}>2 g$, only to those cases for which $\rho_{\ell+1} \neq 2 g-1+e$ for any $e \in H\left(P_{\infty}\right) \backslash\{0\}$. Since the cases in which $\rho_{\ell+1}=(0,0, k)$ for $k<2 m$ or $\rho_{\ell+1}=(i, j, k)$ for $k<m$ have been already studied, they can be excluded.

Proposition 3.14. Let $\rho_{\ell} \in H\left(P_{\infty}\right)$ with $\rho_{\ell}>2 g$ and $\rho_{\ell+1}=(i, j, k)$ and $k \geq m$. If $\rho_{\ell+1} \neq(0,0, k)$ for any $k \in[m, 2 m)$, then $\rho_{\ell+1}-2 g+1 \notin H\left(P_{\infty}\right)$ if and only if $\rho_{\ell+1}=(0,1, k)$ for some $k \in[m, 2 m)$.

Proof. Write $k=m+s$ for some $s \geq 0$. We prove the claim using a case-by-case analysis with respect to the values of $i$ and $j$. We recall that $2 g-1=\left(2^{n+1}+2^{n}-4\right)-1=9 m-8$.

Case 1: $i=1$. Clearly, $\rho_{\ell+1}=3 m+2 j b+8 m+8 s$.

- If $j=0$, then $\rho_{\ell+1}=3 m+8 m+8 s=(9 m-8)+(2 m+8(s+1))=2 g-1+e$. Writing $e=(0,1, s+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.
- If $j=1$, then $\rho_{\ell+1}=3 m+2 m+8 m+8 s=(9 m-8)+(4 m+8(s+1))=2 g-1+e$. Writing $e=(0,2, s+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.
- If $j=2$, then $\rho_{\ell+1}=3 m+4 m+8 m+8 s=(9 m-8)+(6 m+8(s+1))=2 g-1+e$. Writing $e=(0,3, s+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.
- If $j=3$, then $\rho_{\ell+1}=3 m+6 m+8 k=(9 m-8)+(8(k+1))=2 g-1+e$. Writing $e=(0,0, k+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.

Case 2: $i=0$. Clearly, $\rho_{\ell+1}=2 j b+8 k=2 j b+8 m+8 s$.

- If $j=0$, then in particular we can write $k=2 m+t$ for $t \geq 0$, since $k \geq 2 m$. Thus, $\rho_{\ell+1}=16 m+8 t=(9 m-8)+(7 m+8(t+1))=2 g-1+e$. Writing $e=(1,2, t+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.
- If $j=1$, then $\rho_{\ell+1}=2 m+8 k$. We first assume that $k \geq 2 m$ and so that $k=2 m+t$ for some $t \geq 0$. In this case $\rho_{\ell+1}=2 m+16 m+8 t=(9 m-8)+(9 m+8(t+1))=2 g-1+e$. Writing $e=(1,3, t+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur. Thus, $k \in[m, 2 m)$. In this case, $\rho_{\ell+1}=2 m+8 m+8 s=(9 m-8)+(m+8(s+1))=2 g-1+e$. By direct computation $e \notin H\left(P_{\infty}\right)$ and the claim follows.
- If $j=2$, then $\rho_{\ell+1}=4 m+8 m+8 s=(9 m-8)+(3 m+8(s+1))=2 g-1+e$. Writing $e=(1,0, s+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.
- If $j=3$, then $\rho_{\ell+1}=6 m+8 m+8 s=(9 m-8)+(5 m+8(s+1))=2 g-1+e$. Writing $e=(1,1, s+1)$ we have that $e \in H\left(P_{\infty}\right)$, so this case cannot occur.

Since from Proposition 3.13 the Weierstrass semigroup $H\left(P_{\infty}\right)$ is symmetric, its conductor is $c=2 g$; equivalently, its largest gap is $2 g-1$. The following theorem shows that the exact value of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ is known for $\rho_{\ell+1} \geq 4 g$; see [2, Proposition 4.2 (iii)].

Theorem 3.15. Let $H(P)$ be a Weierstrass semigroup. Then $d_{O R D}\left(C_{\ell}(P)\right) \geq \ell+1-g$ and equality holds if $\rho_{\ell+1} \geq 4 g$.

According to the results obtained in the previous sections, Remark 3.14, and Theorem 3.15. to complete the computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ for every $\rho_{\ell} \in H\left(P_{\infty}\right)$, only the case $\rho_{\ell} \in[2 g, 4 g-1)$ with $\rho_{\ell+1}=(0,1, k)$ and $k \in[m, 2 m)$ has to be considered.

Proposition 3.16. Let $\rho_{\ell} \in H\left(P_{\infty}\right)$ be such that $\rho_{\ell}>2 g$ and $\rho_{\ell+1}=(0,1, k)<4 g$ for $k \in[m, 2 m)$. Then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\left\{\begin{array}{l}
\nu_{\ell+5}=8 k-7 m+13, \quad \text { if } k<\frac{9 m-11}{8} \\
\nu_{\ell+3}=8 k-7 m+11, \quad \text { if } \frac{9 m-11}{8} \leq k<\frac{11 m-9}{8} \\
\nu_{\ell+1}=8 k-7 m+9, \quad \text { if } k \geq \frac{11 m-9}{8}
\end{array}\right.
$$

Proof. Arguing as in the previous propositions one can prove that the value of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ is obtained by $\nu_{\ell+5}, \nu_{\ell+3}$, and $\nu_{\ell+1}$, if $k<\frac{9 m-11}{8}, \frac{9 m-11}{8} \leq k<\frac{11 m-9}{8}$, and $k \leq \frac{11 m-9}{8}$ respectively. Since $\rho_{\ell+1} \geq 2 g$, we have that $\rho_{\ell+t}=\rho_{\ell+1}+(t-1)$ for every $t \geq 1$.

Assume that $k<\frac{9 m-11}{8}$. By direct checking $\rho_{\ell+6}=\rho_{\ell+1}+5=(1,3, \tilde{k})$, where $\tilde{k}=$ $k-\frac{7 m-5}{8}$. Hence from Lemma 3.2, $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=8\left(k-\frac{7 m-5}{8}+1\right)=8 k-7 m+13$, as $\tilde{k}<\frac{9 m-11}{8}-\frac{7 m-5}{8}<m$.

Assume that $\frac{9 m-11}{8} \leq k<\frac{11 m-9}{8}$. By direct checking $\rho_{\ell+4}=\rho_{\ell+1}+3=(1,0, \tilde{k})$ where $\tilde{k}=k-\frac{m-3}{8}$. Hence $\tilde{k} \geq m-1$ and from Lemma 3.2, $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=2 m=8 k-7 m+11$ if $\tilde{k}=m-1$, while $d_{\text {ORD }}\left(C_{\ell}\left(P_{\infty}\right)\right)=2(\tilde{k}+1)+6(\tilde{k}-m+1)=8 k-7 m+11$ if $\tilde{k} \geq m$.

Assume that $k \geq \frac{11 m-9}{8}$. By direct checking $\rho_{\ell+2}=\rho_{\ell+1}+1=(1,1, \tilde{k})$ where $\tilde{k}=k-\frac{3 m-1}{8}$. Hence $\tilde{k} \geq m-1$ and from Lemma 3.2, $d_{\text {ORD }}\left(C_{\ell}\left(P_{\infty}\right)\right)=4 m=8 k-7 m+9$ if $\tilde{k}=m-1$, while $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=4(\tilde{k}+1)+4(\tilde{k}-m+1)=8 k-7 m+9$ if $\tilde{k} \geq m$.

For $q \neq 2$, we cannot determine $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right.$ for all $\ell$. Yet, this is possible for certain $\ell$, as shown in the following propositions.

Proposition 3.17. If $\rho_{\ell+1} \leq(q-1)\left(q^{n}+1\right)$, then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=j+1
$$

where $j \leq q-1$ satisfies $(j-1)\left(q^{n}+1\right)<\rho_{\ell+1} \leq j\left(q^{n}+1\right)$.
Proof. Since $H\left(P_{\infty}\right)$ is telescopic from Proposition 3.13, we can apply [28, Theorem 6.11]. The claim then follows because $q^{n}+1=\max \left\{\frac{q^{3}}{1}, \frac{m q}{1}, \frac{q^{n}+1}{1}\right\}$.
Proposition 3.18. If $\frac{3}{2}(q-1)\left(q^{n+1}+\frac{1}{3} q^{n}-q^{2}-\frac{2}{3}\right)-2<\ell \leq \frac{3}{2}(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)-2$, then

$$
d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\min \left\{\rho_{t} \mid \rho_{t} \geq \ell+1-g\right\}
$$

Proof. This is the claim of [28, Theorem 6.10].

### 3.5 Application for $q=2$ : families of AG codes with relative Singleton defect going to zero

In this section, we assume that $q=2$ and provide two families of codes of type $C_{\ell}\left(P_{\infty}\right)$ in the cases $\rho_{\ell}=9 m$ and $\rho_{\ell}=9 m+8$, with relative Singleton defect going to zero as $n$ goes to infinity. We denote by $\delta$ and $\Delta$ the Singleton defect and the relative Singleton defect of $C_{\ell}\left(P_{\infty}\right)$, respectively.

Lemma 3.19. Fix $n \geq 5$ odd. Then $9 m-1,9 m, 9 m+1 \in H\left(P_{\infty}\right)$.

Proof. A direct computation shows that $9 m-1=\left(0,3, \frac{2^{n}}{8}\right), 9 m=(1,3,0)$, and $9 m+1=$ ( $0,2, \frac{5 \cdot 2^{n-3}+1}{3}$ ), thus the claim follows.

We now assume that $\rho_{\ell}=9 m$. Since $\rho_{\ell+1}=9 m+1=\left(0,2, \frac{5 \cdot 2^{n-3}+1}{3}\right)$ the following result follows from Lemma 1.3.

Corollary 3.20. Assume that $\rho_{\ell}=(1,3,0)$. Then $\nu_{\ell}=3 \cdot\left(\frac{5 \cdot 2^{n-3}+1}{3}+1\right) \geq 24$.
Proposition 3.21. The code $C_{\ell}\left(P_{\infty}\right)$ is an $[N, k, d]_{2^{2 n}}$-linear code with

- $N=(3 m-1)^{2}+(3 m-1)(9 m-7)$,
- $k=N-\frac{9 m+9}{2}$,
- $d \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=16$,
- $\delta \leq N-k+1-d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\frac{9 m-21}{2}$,
- $\Delta=\frac{\delta}{N} \leq \frac{9 m-21}{2(3 m-1)(12 m-8)}$; hence, $\Delta$ goes to zero as $n$ goes to infinity.

Proof. Since $G G S(q, n)$ is $\mathbb{F}_{2^{2 n} \text {-maximal, we have }}$
$N=\left(2^{2 n}+1+2 g(G G S(q, n)) 2^{n}\right)-1=2^{2 n}+2^{n}(9 m-7)=(3 m-1)^{2}+(3 m-1)(9 m-7) ;$
the last equality follows from $m=\left(2^{n}+1\right) / 3$. Since $C_{\ell}\left(P_{\infty}\right)=C^{\perp}\left(\bar{D}, \rho_{\ell} P_{\infty}\right), k=N-\tilde{k}$ where $\tilde{k}$ is the dimension of $C\left(\bar{D}, \rho_{\ell} P_{\infty}\right)$. As $\operatorname{deg}\left(\rho_{\ell} P_{\infty}\right)>2 g(G G S(q, n))-2$, from the Riemann-Roch Theorem follows
$k=N-\tilde{k}=N-\left(\operatorname{deg}\left(\rho_{\ell} P_{\infty}\right)+1-g(G G S(q, n))\right)=N-\left(9 m+1-\frac{9 m-7}{2}\right)=r-\frac{9 m+9}{2}$.
By Lemma 3.11, $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right) \geq 16$. To prove the claim is sufficient to show that there exists $\rho_{s} \geq \rho_{\ell}$ such that $\nu_{s}=16$. To this end we take $\rho_{s+1}=(1,3,1)=9 m+8>9 m+1$. From Lemma 1.1, $\nu_{s}=2(b+1)(c+1)=2(3+1)(1+1)=16$ and the claim follows. Now the claim on $\delta$ and $\Delta$ follows by direct computation.

We now assume that $\rho_{\ell}=9 m+8$, so that $\rho_{\ell+1}=9 m+9=\left(0,2, \frac{5 \cdot 2^{n-3}+1}{3}+1\right)=\left(0,2, \frac{5 m+9}{8}\right)$. Arguing as in the proof of Proposition 3.21 and using Lemma 3.11, the following result is obtained.

Proposition 3.22. The code $C_{\ell}\left(P_{\infty}\right)$ is an $[r, k, d]_{2^{2 n}}$-linear code with

- $r=(3 m-1)^{2}+(3 m-1)(9 m-7)$,
- $k=r-\frac{9 m+25}{2}$,
- $d \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=2\left(\left\lceil\frac{6 m+9}{8}\right\rceil+1\right)$,
- $\delta \leq r-k+1-d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)=\frac{9 m+25}{2}-2\left(\left\lceil\frac{6 m+9}{8}\right\rceil+1\right)<\frac{6 m+21}{2}$,
- $\Delta=\frac{\delta}{r}<\frac{6 m+21}{2(3 m-1)(12 m-8)}$; hence, $\Delta$ goes to zero as $n$ goes to infinity.


## 4 Weierstrass semigroup at $P_{0}$

In this section we describe the Weierstrass semigroup at $P_{0}$, and hence at any $\mathbb{F}_{q^{2}}$-rational affine point by Lemma 8.1. Consider the functions

$$
\begin{equation*}
\frac{y^{r} z^{t}}{x^{s}}, \quad s \in\left[0, q^{2}-1\right], r \in[0, s], t \in\left[0,\left\lfloor\frac{s m(q+1)-r q m}{q^{3}}\right]\right] . \tag{8}
\end{equation*}
$$

All these functions belong to $H\left(P_{0}\right)$. In fact,

$$
\left(\frac{y^{r} z^{t}}{x^{s}}\right)=(m r+t-m(q+1) s) P_{0}+\left(m(q+1) s-m q r-t q^{3}\right) P_{\infty}
$$

and by assumption

$$
m(q+1) s-m q r-t q^{3} \geq 0
$$

Proposition 4.1. Let $t \in\left[0, \min \left(\left\lfloor\frac{s m(q+1)-r q m}{q^{3}}\right\rfloor, m-1\right)\right]$ and

$$
s \in[0, q], r \in[0, s]
$$

or

$$
s \in\left[q+1, q^{2}-1\right], r \in[0, q] .
$$

Then all the integers $m r+t-m(q+1) s$ are distinct.
Proof. Suppose $m r+t-m(q+1) s=m \bar{r}+\bar{t}-m(q+1) \bar{s}$. Then $t \equiv \bar{t}(\bmod m)$, which implies $t=\bar{t}$. Now, from $m r-m(q+1) s=m \bar{r}-m(q+1) \bar{s}, r \equiv \bar{r}(\bmod q+1)$, which yields $r=\bar{r}$ and $s=\bar{s}$.

Proposition 4.2. Consider the following sets

$$
\left.\begin{array}{l}
\mathcal{L}_{1}:=\{-t-r m+m(q+1) s \mid s \in[0, q], r \in[0, s], \\
\left.t \in\left[0,((s-r) q+s) \frac{m-q^{2}+q-1}{q^{3}}+s-r\right]\right\} ; \\
\mathcal{L}_{2}:=\left\{-t-r m+m(q+1) s \mid s \in\left[q+1, q^{2}-q\right], r \in[0, q],\right. \\
\left.t \in\left[0,((s-r) q+s) \frac{m-q^{2}+q-1}{q^{3}}+s-r\right]\right\} ; \\
\\
\mathcal{L}_{3}:=\left\{-t-r m+m(q+1) s \mid s \in\left[q^{2}-q+1, q^{2}-2\right],\right. \\
\left.r \in\left[0, q+s-q^{2}-1\right], t \in[0, m-1]\right\} ;
\end{array}\right\} \begin{array}{r}
\mathcal{L}_{4}:=\left\{-t-r m+m(q+1) s \mid s \in\left[q^{2}-q+1, q^{2}-2\right], r \in\left[q+s-q^{2}, q\right],\right. \\
\left.\quad t \in\left[0,((s-r) q+s) \frac{m-q^{2}+q-1}{q^{3}}+s-r\right]\right\} ; \\
\mathcal{L}_{5}:=\left\{-t+m(q+1)\left(q^{2}-1\right) \mid t \in\left[q^{3}, m-1\right]\right\} ; \\
\mathcal{L}_{6}:=\left\{-t-r m+m(q+1)\left(q^{2}-1\right) \mid r \in[1, q-2], t \in[0, m-1]\right\} ; \\
\mathcal{L}_{7}:=\left\{-t-r m+m(q+1)\left(q^{2}-1\right) \mid r \in[q-1, q],\right. \\
\left.t \in\left[0,\left(\left(q^{2}-1-r\right) q+q^{2}-1\right) \frac{m-q^{2}+q-1}{q^{3}}+q^{2}-1-r\right]\right\} .
\end{array}
$$

Then each $\mathcal{L}_{i}$ is contained in $\mathcal{L}\left((2 g-1) P_{0}\right)$.
Proof. By direct computations.
Finally, we can give the description of the Weierstrass semigroup $H\left(P_{0}\right)$.

## Proposition 4.3.

$$
\bigcup_{i=1}^{7} \mathcal{L}_{i}=H\left(P_{0}\right) \cap\{0, \ldots, 2 g-1\} .
$$

Proof. By direct computations, since

$$
\left|\mathcal{L}_{1}\right|=\left(\frac{q^{4}+5 q^{3}+8 q^{2}+4 q}{6}\right)\left(\frac{m-q^{2}+q-1}{q^{3}}\right)+\frac{(q+1)(q+2)(q+3)}{6}
$$

$$
\begin{gathered}
\left|\mathcal{L}_{2}\right|=\left(\frac{q^{6}-q^{5}-q^{4}-3 q^{2}-2 q}{2}\right)\left(\frac{m-q^{2}+q-1}{q^{3}}\right)+\frac{q^{5}-2 q^{4}+2 q^{3}-q^{2}-6 q}{2}, \\
\left|\mathcal{L}_{4}\right|=\left(\frac{3 q^{5}+2 q^{4}-20 q^{3}+q^{2}+8 q+12}{6}\right)\left(\frac{m-q^{2}+q-1}{q^{3}}\right)+\frac{3 q^{4}-q^{3}-18 q^{2}+22 q-12}{6}, \\
\left|\mathcal{L}_{3}\right|=\frac{m(q-2)(q-1)}{2}, \quad\left|\mathcal{L}_{5}\right|=m-q^{3}, \quad\left|\mathcal{L}_{6}\right|=\sum_{r=1}^{q-2} m=(q-2) m, \\
\left|\mathcal{L}_{7}\right|=\frac{m-q^{2}+q-1}{q^{3}}\left(2 q^{3}-q-2\right)+2 q^{2}-2 q+1
\end{gathered}
$$

and $\mathcal{L}_{i} \cap \mathcal{L}_{j}=\emptyset$ if $i \neq j$.
Let $C_{\ell}\left(P_{0}\right)=C^{\perp}\left(\tilde{D}, \tilde{\rho}_{\ell} P_{0}\right)$, where $\tilde{D}$ is as in (2) and $\tilde{\rho}_{\ell}$ is the $\ell$-th positive non-gap at $P_{0}$. In this case it has not been possible to determine $d_{O R D}\left(C_{\ell}\left(P_{0}\right)\right)$ for any $n$ since we do not have a basis of the Weierstrass semigroup at $P_{0}$. Nevertheless, Tables 1, 2, and ?? give evidence that for some specific values of $\ell$ the AG codes and AG quantum codes from $C_{\ell}\left(P_{0}\right)$ are better than $C_{\ell}\left(P_{\infty}\right)$, since the designed relative Singleton defect of $C_{\ell}\left(P_{0}\right)$ is smaller than the one of $C_{\ell}\left(P_{\infty}\right)$.

## 5 AG codes on the GGS curve for $q=2$ and $n=5$

In this section a more detailed description of the results obtained in the previous sections is given for the particular case $q=2, n=5$. Recall that in this case

$$
\begin{gathered}
H\left(P_{\infty}\right)=\{0,8,16,22,24,30,32,33,38,40,41,44,46,48,49,52\} \cup\{54, \ldots, 57\} \\
\cup\{60\} \cup\{62, \ldots, 66\} \cup\{68\} \cup\{70, \ldots, 74\} \cup\{76, \ldots, 82\} \cup\{84, \ldots, 90\} \cup\{92, \ldots\} .
\end{gathered}
$$

For the point $P_{0}$ (and hence for any $\mathbb{F}_{q^{2}}$-rational point), we have from Proposition 4.3
$H\left(P_{0}\right)=\{0,21,22\} \cup\{29, \ldots, 33\} \cup\{42,43,44\} \cup\{50, \ldots, 55\} \cup\{58, \ldots, 66\} \cup\{71, \ldots, 77\} \cup\{79, \ldots\}$.
Table 1 contains the parameters of the codes $C_{\ell_{\infty}}\left(P_{\infty}\right)$ and $C_{\ell_{0}}\left(P_{0}\right)$; in particular, their common length $N=3968$ and dimension $k$, their Feng-Rao designed minimum distance $d_{O R D}^{\infty}$ and $d_{O R D}^{0}$, their designed Singleton defects $\delta_{\infty}=N+1-k-d_{O R D}^{\infty}$ and $\delta_{0}=N+1-k-d_{O R D}^{0}$, and their designed relative Singleton defects $\Delta_{\infty}=\frac{\delta_{\infty}}{N}$ and $\Delta_{0}=\frac{\delta_{0}}{N}$.

Table 1: Codes $C_{\ell_{\infty}}\left(P_{\infty}\right)$ and $C_{\ell_{0}}\left(P_{0}\right), q^{n}=2^{5}$

| $N$ | $k$ | $\rho_{\ell \infty}$ | $d_{O R D}^{\infty}$ | $\delta_{\infty} \leq$ | $\Delta_{\infty} \leq$ | $\rho_{\ell_{0}}$ | $d_{O R D}^{0}$ | $\delta_{0} \leq$ | $\Delta_{0} \leq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3968 | 3966 | 8 | 2 | 1 | 0,0003 | 21 | 2 | 1 | 0,0003 |
| 3968 | 3965 | 16 | 2 | 2 | 0,0006 | 22 | 2 | 2 | 0,0006 |
| 3968 | 3964 | 22 | 2 | 3 | 0,0008 | 29 | 2 | 3 | 0,0008 |
| 3968 | 3963 | 24 | 2 | 4 | 0,0011 | 30 | 2 | 4 | 0,0011 |
| 3968 | 3962 | 30 | 2 | 5 | 0,0013 | 31 | 2 | 5 | 0,0013 |
| 3968 | 3961 | 32 | 2 | 6 | 0,0016 | 32 | 2 | 6 | 0,016 |
| 3968 | 3960 | 33 | 3 | 6 | 0,0016 | 33 | 3 | 6 | 0,016 |
| 3968 | 3959 | 38 | 3 | 7 | 0,0018 | 42 | 3 | 7 | 0,018 |
| 3968 | 3958 | 40 | 3 | 8 | 0,0021 | 43 | 3 | 8 | 0,021 |
| 3968 | 3957 | 41 | 3 | 9 | 0,0023 | 44 | 3 | 9 | 0,023 |
| 3968 | 3956 | 44 | 4 | 9 | 0,0023 | 50 | 3 | 10 | 0,026 |
| 3968 | 3955 | 46 | 4 | 10 | 0,0026 | 51 | 3 | 11 | 0,028 |
| 3968 | 3954 | 48 | 4 | 11 | 0,0028 | 52 | 3 | 12 | 0,031 |
| 3968 | 3953 | 49 | 4 | 12 | 0,0031 | 53 | 3 | 13 | 0,033 |
| 3968 | 3952 | 52 | 4 | 13 | 0,0033 | 54 | 3 | 14 | 0,036 |
| 3968 | 3951 | 54 | 4 | 14 | 0,0036 | 55 | 3 | 15 | 0,038 |
| 3968 | 3950 | 55 | 5 | 14 | 0,0036 | 58 | 4 | 15 | 0,038 |
| 3968 | 3949 | 56 | 5 | 15 | 0,0038 | 59 | 5 | 15 | 0,038 |
| 3968 | 3948 | 57 | 5 | 16 | 0,0041 | 60 | 5 | 16 | 0,041 |
| 3968 | 3947 | 60 | 5 | 17 | 0,0043 | 61 | 5 | 17 | 0,043 |
| 3968 | 3946 | 62 | 5 | 18 | 0,0046 | 62 | 5 | 18 | 0,046 |
| 3968 | 3945 | 63 | 5 | 19 | 0,0048 | 63 | 5 | 19 | 0,048 |
| 3968 | 3944 | 64 | 5 | 20 | 0,0051 | 64 | 5 | 20 | 0,051 |
| 3968 | 3943 | 65 | 5 | 21 | 0,0053 | 65 | 5 | 21 | 0,053 |
| 3968 | 3942 | 66 | 6 | 21 | 0,0053 | 66 | 6 | 21 | 0,053 |
| 3968 | 3941 | 68 | 6 | 22 | 0,0056 | 71 | 6 | 22 | 0,056 |
| 3968 | 3940 | 70 | 6 | 23 | 0,0058 | 72 | 6 | 23 | 0,058 |
| 3968 | 3939 | 71 | 6 | 24 | 0,0061 | 73 | 6 | 24 | 0,061 |
| 3968 | 3938 | 72 | 6 | 25 | 0,0064 | 74 | 6 | 25 | 0,064 |
| 3968 | 3937 | 73 | 6 | 26 | 0,0066 | 75 | 6 | 26 | 0,066 |
| 3968 | 3936 | 74 | 6 | 27 | 0,0069 | 76 | 6 | 27 | 0,069 |
| 3968 | 3935 | 76 | 6 | 28 | 0,0071 | 77 | 6 | 28 | 0,0071 |
| 3968 | 3934 | 77 | 8 | 27 | 0,0069 | 79 | 8 | 27 | 0,069 |
| 3968 | 3933 | 78 | 8 | 28 | 0,0071 | 80 | 8 | 28 | 0,0071 |
| 3968 | 3932 | 79 | 8 | 29 | 0,0074 | 81 | 8 | 29 | 0,0074 |
| 3968 | 3931 | 80 | 8 | 30 | 0,0076 | 82 | 8 | 30 | 0,0076 |
| 3968 | 3930 | 81 | 8 | 31 | 0,0079 | 83 | 8 | 31 | 0,0079 |
| 3968 | 3929 | 82 | 8 | 32 | 0,0081 | 84 | 8 | 32 | 0,0081 |
| 3968 | 3928 | 84 | 8 | 33 | 0,0084 | 85 | 8 | 33 | 0,0084 |
| 3968 | 3927 | 85 | 8 | 34 | 0,0086 | 86 | 8 | 34 | 0,0086 |
| 3968 | 3926 | 86 | 8 | 35 | 0,0089 | 87 | 8 | 35 | 0,0089 |
| 3968 | 3925 | 87 | 8 | 36 | 0,0091 | 88 | 8 | 36 | 0,0091 |
| 3968 | 3924 | 88 | 8 | 37 | 0,0094 | 89 | 8 | 37 | 0,0094 |
| 3968 | 3923 | 89 | 8 | 38 | 0,0096 | 90 | 8 | 38 | 0,0096 |
| 3968 | 3922 | 90 | 8 | 39 | 0,0099 | 91 | 8 | 39 | 0,0099 |
| 3968 | 3921 | 92 | 8 | 40 | 0,0101 | 92 | 8 | 40 | 0,0101 |
| 3968 | 3920 | 93 | 8 | 41 | 0,0104 | 93 | 8 | 41 | 0,0104 |
| 3968 | 3919 | 94 | 8 | 42 | 0,0106 | 94 | 8 | 42 | 0,0106 |
| 3968 | 3918 | 95 | 8 | 43 | 0,0109 | 95 | 8 | 43 | 0,0109 |
| 3968 | 3917 | 96 | 8 | 44 | 0,0111 | 96 | 8 | 44 | 0,0111 |
| 3968 | 3916 | 97 | 8 | 45 | 0,0114 | 97 | 8 | 45 | 0,0114 |

Table 1: continued from previous page

| $n$ | $k$ | $\rho_{\ell_{\infty}}$ | $d_{O R D}^{\infty}$ | $\delta_{\infty} \leq$ | $\Delta_{\infty} \leq$ | $\rho_{\ell_{0}}$ | $d_{O R D}^{0}$ | $\delta_{o} \leq$ | $\Delta_{0} \leq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3968 | 3915 | 98 | 8 | 46 | 0,0116 | 98 | 8 | 46 | 0,0116 |
| 3968 | 3914 | 99 | 16 | 39 | 0,0099 | 99 | 9 | 46 | 0,0116 |
| 3968 | 3913 | 100 | 16 | 40 | 0,0101 | 100 | 16 | 40 | 0,0101 |
| 3968 | 3912 | 101 | 16 | 41 | 0,0104 | 101 | 21 | 36 | 0,0091 |
| 3968 | 3911 | 102 | 16 | 42 | 0,0106 | 102 | 22 | 36 | 0,0091 |
| 3968 | 3910 | 103 | 16 | 43 | 0,0109 | 103 | 22 | 37 | 0,0094 |
| 3968 | 3909 | 104 | 16 | 44 | 0,0111 | 104 | 22 | 38 | 0,0096 |
| 3968 | 3908 | 105 | 16 | 45 | 0,0114 | 105 | 22 | 39 | 0,0099 |
| 3968 | 3907 | 106 | 16 | 46 | 0,0116 | 106 | 22 | 40 | 0,0101 |
| 3968 | 3906 | 107 | 22 | 41 | 0,0104 | 107 | 22 | 41 | 0,0104 |
| 3968 | 3905 | 108 | 22 | 42 | 0,0106 | 108 | 22 | 42 | 0,0106 |
| 3968 | 3904 | 109 | 22 | 43 | 0,0109 | 109 | 22 | 43 | 0,0109 |
| 3968 | 3903 | 110 | 22 | 44 | 0,0111 | 110 | 22 | 44 | 0,0111 |
| 3968 | 3902 | 111 | 22 | 45 | 0,0114 | 111 | 26 | 41 | 0,0104 |
| 3968 | 3901 | 112 | 22 | 46 | 0,0116 | 112 | 29 | 39 | 0,0099 |
| 3968 | 3900 | 113 | 24 | 45 | 0,0114 | 113 | 29 | 40 | 0,0101 |
| 3968 | 3899 | 114 | 24 | 46 | 0,0116 | 114 | 29 | 41 | 0,0104 |
| 3968 | 3898 | 115 | 30 | 41 | 0,0104 | 115 | 29 | 42 | 0,0106 |
| 3968 | 3897 | 116 | 30 | 42 | 0,0106 | 116 | 29 | 43 | 0,0109 |
| 3968 | 3896 | 117 | 30 | 43 | 0,0109 | 117 | 29 | 44 | 0,0111 |
| 3968 | 3895 | 118 | 30 | 44 | 0,0111 | 118 | 29 | 45 | 0,0114 |
| 3968 | 3894 | 119 | 30 | 45 | 0,0114 | 119 | 29 | 46 | 0,0116 |
| 3968 | 3893 | 120 | 30 | 46 | 0,0116 | 120 | 30 | 46 | 0,0116 |
| 3968 | 3892 | 121 | 32 | 45 | 0,0114 | 121 | 31 | 46 | 0,0116 |
| 3968 | 3891 | 122 | 32 | 46 | 0,0116 | 122 | 36 | 42 | 0,0106 |
| 3968 | 3890 | 123 | 33 | 46 | 0,0116 | 123 | 37 | 42 | 0,0106 |
| 3968 | 3889 | 124 | 38 | 42 | 0,0106 | 124 | 37 | 43 | 0,0109 |
| 3968 | 3888 | 125 | 38 | 43 | 0,0109 | 125 | 37 | 44 | 0,0111 |
| 3968 | 3887 | 126 | 38 | 44 | 0,0111 | 126 | 37 | 45 | 0,0114 |
| 3968 | 3886 | 127 | 38 | 45 | 0,0114 | 127 | 37 | 46 | 0,0116 |
| 3968 | 3885 | 128 | 38 | 46 | 0,0116 | 128 | 38 | 46 | 0,0116 |
| 3968 | 3884 | 129 | 40 | 45 | 0,0114 | 129 | 39 | 46 | 0,0116 |
| 3968 | 3883 | 130 | 40 | 46 | 0,0116 | 130 | 40 | 46 | 0,0116 |
| 3968 | 3882 | 131 | 41 | 46 | 0,0116 | 131 | 41 | 46 | 0,0116 |
| 3968 | 3881 | 132 | 44 | 44 | 0,0111 | 132 | 42 | 46 | 0,0116 |
| 3968 | 3880 | 133 | 44 | 45 | 0,0114 | 133 | 46 | 43 | 0,0109 |
| 3968 | 3879 | 134 | 44 | 46 | 0,0116 | 134 | 48 | 42 | 0,0106 |
| 3968 | 3878 | 135 | 46 | 45 | 0,0114 | 135 | 48 | 43 | 0,0109 |
| 3968 | 3877 | 136 | 46 | 46 | 0,0116 | 136 | 50 | 42 | 0,0106 |
| 3968 | 3876 | 137 | 48 | 45 | 0,0114 | 137 | 50 | 43 | 0,0109 |
| 3968 | 3875 | 138 | 48 | 46 | 0,0116 | 138 | 50 | 44 | 0,0111 |
| 3968 | 3874 | 139 | 49 | 46 | 0,0116 | 139 | 50 | 45 | 0,0114 |
| 3968 | 3873 | 140 | 52 | 44 | 0,0111 | 140 | 50 | 46 | 0,0116 |
| 3968 | 3872 | 141 | 52 | 45 | 0,0114 | 141 | 51 | 46 | 0,0116 |
| 3968 | 3871 | 142 | 52 | 46 | 0,0116 | 142 | 52 | 46 | 0,0116 |
| 3968 | 3870 | 143 | 54 | 45 | 0,0114 | 143 | 53 | 46 | 0,0116 |
| 3968 | 3869 | 144 | 54 | 46 | 0,0116 | 144 | 56 | 44 | 0,0111 |
| 3968 | 3868 | 145 | 55 | 46 | 0,0116 | 145 | 57 | 44 | 0,0111 |
| 3968 | 3867 | 146 | 56 | 46 | 0,0116 | 146 | 58 | 44 | 0,0111 |
| 3968 | 3866 | 147 | 57 | 46 | 0,0116 | 147 | 58 | 45 | 0,0114 |
| 3968 | 3865 | 148 | 60 | 44 | 0,0111 | 148 | 58 | 46 | 0,0116 |
| 3968 | 3864 | 149 | 60 | 45 | 0,0114 | 149 | 59 | 46 | 0,0116 |

Table 1: continued from previous page

| $n$ | $k$ | $\rho_{\ell_{\infty}}$ | $d_{O R D}^{\infty}$ | $\delta_{\infty} \leq$ | $\Delta_{\infty} \leq$ | $\rho_{\ell_{0}}$ | $d_{O R D}^{0}$ | $\delta_{o} \leq$ | $\Delta_{0} \leq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3968 | 3863 | 150 | 60 | 46 | 0,0116 | 150 | 60 | 46 | 0,0116 |
| 3968 | 3862 | 151 | 62 | 45 | 0,0114 | 151 | 61 | 46 | 0,0116 |
| 3968 | 3861 | 152 | 62 | 46 | 0,0116 | 152 | 62 | 46 | 0,0116 |
| 3968 | 3860 | 153 | 63 | 46 | 0,0116 | 153 | 63 | 46 | 0,0116 |
| 3968 | 3859 | 154 | 64 | 46 | 0,0116 | 154 | 64 | 46 | 0,0116 |
| 3968 | 3858 | 155 | 65 | 46 | 0,0116 | 155 | 66 | 45 | 0,0114 |
| 3968 | 3857 | 156 | 66 | 46 | 0,0116 | 156 | 66 | 46 | 0,0116 |
| 3968 | 3856 | 157 | 68 | 45 | 0,0114 | 157 | 67 | 46 | 0,0116 |
| 3968 | 3855 | 158 | 68 | 46 | 0,0116 | 158 | 68 | 46 | 0,0116 |
| 3968 | 3854 | 159 | 70 | 45 | 0,0114 | 159 | 69 | 46 | 0,0116 |
| 3968 | 3853 | 160 | 70 | 46 | 0,0116 | 160 | 70 | 46 | 0,0116 |
| 3968 | 3852 | 161 | 71 | 46 | 0,0116 | 161 | 71 | 46 | 0,0116 |
| 3968 | 3851 | 162 | 72 | 46 | 0,0116 | 162 | 72 | 46 | 0,0116 |
| 3968 | 3850 | 163 | 73 | 46 | 0,0116 | 163 | 73 | 46 | 0,0116 |
| 3968 | 3849 | 164 | 74 | 46 | 0,0116 | 164 | 74 | 46 | 0,0116 |
| 3968 | 3848 | 165 | 76 | 45 | 0,0114 | 165 | 75 | 46 | 0,0116 |
| 3968 | 3847 | 166 | 76 | 46 | 0,0116 | 166 | 76 | 46 | 0,0116 |
| 3968 | 3846 | 167 | 77 | 46 | 0,0116 | 167 | 77 | 46 | 0,0116 |
| 3968 | 3845 | 168 | 78 | 46 | 0,0116 | 168 | 78 | 46 | 0,0116 |
| 3968 | 3844 | 169 | 79 | 46 | 0,0116 | 169 | 79 | 46 | 0,0116 |
| 3968 | 3843 | 170 | 80 | 46 | 0,0116 | 170 | 80 | 46 | 0,0116 |
| 3968 | 3842 | 171 | 81 | 46 | 0,0116 | 171 | 81 | 46 | 0,0116 |
| 3968 | 3841 | 172 | 82 | 46 | 0,0116 | 172 | 82 | 46 | 0,0116 |
| 3968 | 3840 | 173 | 84 | 45 | 0,0114 | 173 | 83 | 46 | 0,0116 |
| 3968 | 3839 | 174 | 84 | 46 | 0,0116 | 174 | 84 | 46 | 0,0116 |
| 3968 | 3838 | 175 | 85 | 46 | 0,0116 | 175 | 85 | 46 | 0,0116 |
| 3968 | 3837 | 176 | 86 | 46 | 0,0116 | 176 | 86 | 46 | 0,0116 |
| 3968 | 3836 | 177 | 87 | 46 | 0,0116 | 177 | 87 | 46 | 0,0116 |
| 3968 | 3835 | 178 | 88 | 46 | 0,0116 | 178 | 88 | 46 | 0,0116 |
| 3968 | 3834 | 179 | 89 | 46 | 0,0116 | 179 | 89 | 46 | 0,0116 |
| 3968 | 3833 | 180 | 90 | 46 | 0,0116 | 180 | 90 | 46 | 0,0116 |
| 3968 | 3832 | 181 | 92 | 45 | 0,0114 | 181 | 91 | 46 | 0,0116 |
| 3968 | 3831 | 182 | 92 | 46 | 0,0116 | 182 | 92 | 46 | 0,0116 |
| 3968 | $3968-\ell \infty$ | $\rho_{\ell_{\infty}}$ | $\geq 183$ | $\ell_{\infty}-45$ | 46 | 0,0116 | $\rho_{\ell_{0}} \geq 183$ | $\ell_{0}-45$ | 46 |
| 0 | 0,0116 |  |  |  |  |  |  |  |  |

Table 2 provides some examples in which codes of type $C_{\ell_{0}}\left(P_{0}\right)$ have better parameters than codes of type $C_{\ell_{\infty}}\left(P_{\infty}\right)$. In particular, the length $n$ of the two codes is 3968 , the dimension $k_{0}$ and the Feng-Rao designed minimum distance $d_{O R D}^{0}$ of $C_{\ell_{0}}\left(P_{0}\right)$ are greater than or equal to the corresponding parameters $k_{\infty}$ and $d_{O R D}^{\infty}$ of $C_{\ell_{\infty}}\left(P_{\infty}\right)$, and the designed Singleton defect $\delta_{0}=n+1-k_{0}-d_{O R D}^{0}$ of $C_{\ell_{0}}\left(P_{0}\right)$ is strictly smaller than the designed Singleton defect $\delta_{\infty}=n+1-k_{\infty}-d_{O R D}^{\infty}$ of $C_{\ell_{\infty}}\left(P_{\infty}\right)$.

Table 2: Designed Singleton defect of $C_{\ell_{0}}\left(P_{0}\right)$ and $C_{\ell_{\infty}}\left(P_{\infty}\right), q^{n}=2^{5}$

| $\ell_{0}$ | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 19 | 20 | 21 | 22 | 23 | 24 | 26 | 27 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\ell_{\infty}$ | 4 | 5 | 6 | 7 | 9 | 10 | 11 | 20 | 21 | 22 | 23 | 24 | 25 | 27 | 28 |
| $\delta_{\infty}-\delta_{0}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\ell_{0}$ | 28 | 29 | 30 | 31 | 32 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 |
| $\ell_{\infty}$ | 29 | 30 | 31 | 32 | 33 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| $\delta_{\infty}-\delta_{0}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\ell_{0}$ | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 55 | 56 | 56 | 57 | 57 | 58 |
| $\ell_{\infty}$ | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 56 | 56 | 57 | 57 | 58 | 58 |
| $\delta_{\infty}-\delta_{0}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 6 | 6 | 7 | 6 |
| $\ell_{0}$ | 58 | 59 | 59 | 60 | 60 | 61 | 61 | 62 | 63 | 64 | 65 | 66 | 66 | 67 | 67 |
| $\ell_{\infty}$ | 59 | 59 | 60 | 60 | 61 | 61 | 62 | 63 | 64 | 65 | 66 | 66 | 67 | 67 | 68 |
| $\delta_{\infty}-\delta_{0}$ | 7 | 6 | 7 | 6 | 7 | 6 | 1 | 1 | 1 | 1 | 1 | 4 | 6 | 7 | 6 |
| $\ell_{0}$ | 68 | 68 | 69 | 77 | 77 | 78 | 88 | 88 | 89 | 89 | 90 | 90 | 91 | 91 |  |
| $\ell_{\infty}$ | 68 | 69 | 69 | 77 | 78 | 78 | 88 | 89 | 89 | 90 | 90 | 91 | 91 | 92 |  |
| $\delta_{\infty}-\delta_{0}$ | 5 | 4 | 5 | 4 | 4 | 4 | 2 | 3 | 4 | 3 | 2 | 3 | 4 | 3 |  |
| $\ell_{0}$ | 92 | 92 | 93 | 93 | 94 | 99 | 99 | 100 | 100 | 101 | 101 | 102 | 110 | 110 |  |
| $\ell_{\infty}$ | 92 | 93 | 93 | 94 | 94 | 99 | 100 | 100 | 101 | 101 | 102 | 102 | 110 | 111 |  |
| $\delta_{\infty}-\delta_{0}$ | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |  |

## 6 Quantum codes from one-point AG codes on the GGS curves

In this section we use families of one-point AG codes from the GGS curve to construct quantum codes. The main ingredient is the so called CSS contruction which enables to construct quantum codes from classical linear codes; see [29, Lemma 2.5].

We denote by $q$ a prime power. A $q$-ary quantum code $Q$ of length $N$ and dimension $k$ is defined to be a $q^{k}$-dimensional Hilbert subspace of a $q^{N}$-dimensional Hilbert space $\mathbb{H}=\left(\mathbb{C}^{q}\right)^{\otimes n}=\mathbb{C}^{q} \otimes \cdots \otimes \mathbb{C}^{q}$. If $Q$ has minimum distance $D$, then $Q$ can correct up to $\left\lfloor\frac{D-1}{2}\right\rfloor$ quantum errors. The notation $[[N, k, D]]_{q}$ is used to denote such a quantum code $Q$. For a $[[N, k, D]]_{q}$-quantum code the quantum Singleton bound holds, that is, the minimum distance satisfies $D \leq 1+(N-k) / 2$. The quantum Singleton defect is $\delta^{Q}:=N-k-2 D+2 \geq 0$, and the relative quantum Singleton defect is $\Delta^{Q}:=\delta^{Q} / N$. If $\delta^{Q}=0$, then the code is said to be quantum MDS. For a detailed introduction on quantum codes see [29] and the references therein.

Lemma 6.1. (CSS construction) Let $C_{1}$ and $C_{2}$ denote two linear codes with parameters $\left[N, k_{i}, d_{i}\right]_{q}, i=1,2$, and assume that $C_{1} \subset C_{2}$. Then there exists an $\left[\left[N, k_{2}-k_{1}, D\right]\right]_{q}$ code with $D=\min \left\{w t(c) \mid c \in\left(C_{2} \backslash C_{1}\right) \cup\left(C_{1}^{\perp} \backslash C_{2}^{\perp}\right)\right\}$, where $w t(c)$ is the weight of $c$.

We consider the following general t-point construction due to La Guardia and Pereira;
see [29, Theorem 3.1]. It is a direct application of Lemma 6.1] to AG codes.
Lemma 6.2. (General t-point construction) Let $\mathcal{X}$ be a nonsingular curve over $\mathbb{F}_{q}$ with genus $g$ and $N+t$ distinct $\mathbb{F}_{q}$-rational points, for some $N, t>0$. Assume that $a_{i}, b_{i}, i=1, \ldots, t$, are positive integers such that $a_{i} \leq b_{i}$ for all $i$ and $2 g-2<\sum_{i=1}^{t} a_{i}<\sum_{i=1}^{t} b_{i}<N$. Then there exists a quantum code with parameters $[[N, k, D]]_{q}$ with $k=\sum_{i=1}^{t} b_{i}-\sum_{i=1}^{t} a_{i}$ and $D \geq \min \left\{N-\sum_{i=1}^{t} b_{i}, \sum_{i=1}^{t} a_{i}-(2 g-2)\right\}$.

Let $n \geq 5$ be an odd integer. We apply Lemma 6.2 to one-point codes on the GGS curve.
Proposition 6.3. Let $a, b \in \mathbb{N}$ be such that

$$
(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)-2<a<b<q^{2 n+2}-q^{n+3}+q^{n+2}
$$

Then there exists a quantum code with parameters $[[N, b-a, D]]_{q^{2 n}}$, where

$$
\begin{gathered}
N=q^{2 n+2}-q^{n+3}+q^{n+2} \\
D \geq \min \left\{q^{2 n+2}-q^{n+3}+q^{n+2}-b, a-(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)+2\right\}
\end{gathered}
$$

Proof. Let $G G S(q, n)$ be the GGS curve with equations (1), genus $g$, and infinite point $P_{\infty}$. Consider the divisors $\bar{D}$ as in (2), $G_{1}=a P_{\infty}$, and $G_{2}=b P_{\infty}$. Note that $\operatorname{supp}\left(G_{1}\right) \cap$ $\operatorname{supp}(\bar{D})=\operatorname{supp}\left(G_{2}\right) \cap \operatorname{supp}(\bar{D})=\emptyset$. From Lemma 6.2, there exists a quantum code with parameters $[[N, b-a, D]]_{q^{2 n}}$, where $D \geq \min \{N-b, a-(2 g-2)\}=\min \left\{q^{2 n+2}-q^{n+3}+\right.$ $\left.q^{n+2}-b, a-(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)-2\right\}$.

Another application of the CSS construction can be obtained looking at the dual codes of the one-point codes from the GGS curve. Let $P \in G G S(q, n)$. Fix $a=\rho_{\ell} \in H(P)$ and $b=\rho_{\ell+s} \in H(P)$ with $C_{2}=C_{\ell}(P)=C_{\ell}$ and $C_{1}=C_{\ell+s}(P)=C_{\ell+s}$, where $s \geq 1$. Clearly $C_{1} \subset C_{2}$, as $C_{\ell} \subsetneq C_{\ell+s}$ for every $s \geq 1$. The dimensions of $C_{2}$ and $C_{1}$ are $k_{2}=N-h_{\ell}$ and $k_{1}=N-h_{\ell+s}=N-h_{\ell}-s$ respectively, where $h_{i}$ denotes the number of non-gaps at $P$ which are smaller than or equal to $i$. Thus, $k_{2}-k_{1}=s$. According to the CSS construction, these choices induce an $[[N, s, D]]_{q^{2 n}}$ quantum code, where $N=q^{2 n+2}-q^{n+3}+q^{n+2}$ and $D=$ $\min \left\{w t(c) \mid c \in\left(C_{2} \backslash C_{1}\right) \cup\left(C_{1}^{\perp} \backslash C_{2}^{\perp}\right)\right\}=\min \left\{w t(c) \mid c \in\left(C_{\ell} \backslash C_{\ell+s}\right) \cup\left(C\left(D, G_{1}\right) \backslash C\left(D, G_{2}\right)\right)\right\}$, with $G_{2}=\rho_{\ell} P$ and $G_{1}=\rho_{\ell+s} P$. In particular,

$$
\begin{equation*}
D \geq \min \left\{d_{O R D}\left(C_{\ell}\right), d_{1}\right\} \tag{9}
\end{equation*}
$$

where $d_{1}$ denotes the minimum distance of the code $C\left(D, G_{1}\right)$. Following this construction and using an improvement of Inequality (9), the next theorem is obtained.

Theorem 6.4. Let $g=(q-1)\left(q^{n+1}+q^{n}-q^{2}\right) / 2$ and $N=q^{2 n+2}-q^{n+3}+q^{n+2}$. For every $\ell \in[3 g-1, N-g]$ and $s \in[1, N-2 \ell]$, there exists a quantum code with parameters $[[N, s, D]]_{q^{2 n}}$, where $D \geq \ell+1-g$.

Proof. Since $\ell \geq 3 g-1$, we have $\rho_{\ell+s}=g-1+\ell+s$, and hence $d_{1} \geq N-\operatorname{deg}\left(G_{1}\right)=$ $N-\rho_{\ell+s}=N-\ell-s-g+1$. From Theorem 3.15, $d_{O R D}\left(C_{\ell}\right)=\ell+1-g$. Thus, $D \geq$ $\min \left\{d_{\text {ORD }}\left(C_{\ell}\right), d_{1}\right\}=\ell+1-g$. The claim follows.

For fixed $q$, we can construct as a direct consequence of Theorem 6.4 families of quantum codes depending on $n$ such that their relative quantum Singleton defect goes to zero as $n$ goes to infinity. An example is the following.
Corollary 6.5. Let $g=(q-1)\left(q^{n+1}+q^{n}-q^{2}\right) / 2$ and $N=q^{2 n+2}-q^{n+3}+q^{n+2}$. For every $\ell \in$ $[3 g-1, N-g]$, fix $s=N-2 \ell$. Then there exists a quantum code with parameters $[[N, s, D]]_{q^{2 n}}$ with $D \geq \ell+1-g$, whose relative quantum Singleton defect $\Delta_{n}^{Q}=(N-s-2 D+2) / N$ satisfies

$$
\Delta_{n}^{Q} \leq \frac{2 g}{N}=\frac{(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)}{q^{2 n+2}-q^{n+3}+q^{n+2}}
$$

Hence, $\lim _{n \rightarrow \infty} \Delta_{n}^{Q}=0$.
Using the computation of $d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ in Section 3, we produce infinite families of quantum codes in which the lower bound in (9) is explicitely determined. We look at those cases for which (9) reads $D \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)>\ell+1-g$ and this bound is better than the one stated in Theorem 6.4. According to Proposition 3.14. we choose $\rho_{\ell} \in H\left(P_{\infty}\right)$ such that $\rho_{\ell+1}=(0,1, k)$ for some $k \in[m, 2 m)$.
Proposition 6.6. Let $q=2^{n}$ for $n \geq 5$ odd, $g=(q-1)\left(q^{n+1}+q^{n}-q^{2}\right) / 2$, and $N=$ $q^{2 n+2}-q^{n+3}+q^{n+2}$. Let $\ell \in[g, 3 g-1]$ be such that $\rho_{\ell+1} \in H\left(P_{\infty}\right)$ is of type $(0,1, k)$ for some $k \in[m, 2 m)$. Let $s \in[1, N-2 \ell-5]$. Then there exists a quantum code with parameters $[[N, s, D]]_{q^{2 n}}$ where

$$
D \geq \ell+1-g+ \begin{cases}5, & \text { if } k<m \text { or } m \leq k<\frac{9 m-11}{8} \\ 3, & \text { if } \frac{9 m-11}{8} \leq k<\frac{11 m-9}{8} \\ 1, & \text { if } \frac{11 m-9}{8} \leq k\end{cases}
$$

Proof. Arguing as in the proof of Theorem 6.4, we have that $d_{1} \geq N-\ell-s-g+1$. Thus, from Proposition 3.16 and Lemma 3.9. Inequality (9) reads

$$
D \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)= \begin{cases}8 k-7 m+13, & \text { if } k<m \text { or } m \leq k<\frac{9 m-11}{8} \\ 8 k-7 m+11, & \text { if } \frac{9 m-11}{8} \leq k<\frac{11 m-9}{8} \\ 8 k-7 m+9, & \text { if } \frac{11 m-9}{8} \leq k\end{cases}
$$

Since $\ell+1-g=\rho_{\ell+1}-2 g+1=2 m+8 k-(9 m-7)+1=8 k-7 m+8$, the claim follows.

## 7 Convolutional codes from one-point AG codes on the GGS curves

In this section we use a result due to De Assis, La Guardia, and Pereira 6] which allows to construct unit-memory convolutional codes with certain parameters $\left(N, k, \gamma ; m, d_{f}\right)_{q}$ starting from AG codes.

Consider the polynomial ring $R=\mathbb{F}_{q}[X]$. A convolutional code $C$ is an $R$-submodule of rank $k$ of the module $R^{N}$. Let $G(X)=\left(g_{i j}(X)\right) \in \mathbb{F}_{q}[X]^{k \times N}$ be a generator matrix of $C$ over $\mathbb{F}_{q}[X], \gamma_{i}=\max \left\{\operatorname{deg} g_{i j}(X) \mid 1 \leq j \leq N\right\}, \gamma=\sum_{i=1}^{k} \gamma_{i}, m=\max \left\{\gamma_{i} \mid 1 \leq i \leq k\right\}$, and $d_{f}$ be the minimum weight of a word $c \in C$. Then we say that $C$ has length $N$, dimension $k$, degree $\gamma$, memory $m$, and free distance. If $m=1, C$ is said to be a unit-memory convolutional code. In this case we use for $C$ the notation $\left(N, k, \gamma ; m, d_{f}\right)_{q}$. For a detailed introduction on convolutional codes see [6, 35] and the references therein.
Lemma 7.1. ([6, Theorem 3]) Let $\mathcal{X}$ be a nonsingular curve over $\mathbb{F}_{q}$ with genus $g$. Consider an $A G$ code $C^{\perp}(D, G)$ with $2 g-2<\operatorname{deg}(G)<N$. Then there exists a unit-memory convolutional code with parameters $\left(N, k-\ell, \ell ; 1, d_{f} \geq d\right)_{q}$, where $\ell \leq k / 2, k=\operatorname{deg}(G)+1-g$ and $d \geq N-\operatorname{deg}(G)$.

We apply Lemma 7.1 to one-point AG codes from the GGS curve.
 such that $(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)-2<\rho_{\ell}<N$, where $N=q^{2 n+2}-q^{n+3}+q^{n+2}$. Then there exists a unit-memory convolutional code with parameters $\left(N, k-s, s ; 1, d_{f} \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)\right.$ ), where $k=\rho_{\ell}+1-\frac{(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)}{2}$ and $s \leq k / 2$.

Proof. The result follows from Lemma [7.1. The inequality $d_{f} \geq d_{O R D}\left(C_{\ell}\left(P_{\infty}\right)\right)$ follows from $d_{f} \geq d$ and Theorem 3.15 applied to the dual code $C_{\ell}\left(P_{\infty}\right)$.

In particular, Theorem 3.15 yields the following corollary.
Corollary 7.3. Consider the $\mathbb{F}_{q^{2 n}}$-maximal $G G S$ curve $G G S(q, n)$ and let $\rho_{\ell} \in H\left(P_{\infty}\right)$ be such that $(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)-2<\rho_{\ell}<N$, where $N=q^{2 n+2}-q^{n+3}+q^{n+2}$ and $\ell \geq 3 \frac{(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)}{2}$. Then there exists a unit-memory convolutional code with parameters $\left(N, k-s, s ; 1, d_{f}\right)$, where $k=\rho_{\ell}+1-\frac{(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)}{2}, s \leq k / 2$, and $d_{f} \geq$ $\ell+1-\frac{(q-1)\left(q^{n+1}+q^{n}-q^{2}\right)}{2}$.

## 8 The Automorphism group of $C\left(\bar{D}, \ell P_{\infty}\right)$

In this section we investigate the automorphism group of the code $C\left(\bar{D}, \ell P_{\infty}\right)$, where $\bar{D}$ is as in (2).

Lemma 8.1. The automorphism group $\operatorname{Aut}(G G S(q, n))$ has exactly two short orbits on $G G S(q, n)$; one consists of $P_{\infty}$, the other consists of the $q^{3} \mathbb{F}_{q^{2}}$-rational points other than $P_{\infty}$.

Proof. From [15, 16], $\operatorname{Aut}(G G S(q, n))=Q \rtimes \Sigma$, where $Q=\left\{Q_{a, b} \mid a, b \in \mathbb{F}_{q^{2}}, a^{q}+a=b^{q+1}\right\}$ and $\Sigma=\left\langle g_{\zeta}\right\rangle$, with

$$
Q_{a, b}=\left(\begin{array}{cccc}
1 & b^{q} & 0 & a  \tag{10}\\
0 & 1 & 0 & b \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad g_{\zeta}=\left(\begin{array}{cccc}
\zeta^{q^{n}+1} & 0 & 0 & 0 \\
0 & \zeta^{\frac{q^{n}+1}{q+1}} & 0 & 0 \\
0 & 0 & \zeta & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

$\zeta$ a primitive $\left(q^{n}+1\right)(q-1)$-th root of unity. Therefore, $\operatorname{Aut}(G G S(q, n))$ fixes $P_{\infty}$. Also, Aut $(G G S(q, n))$ acts transitively on the $q^{3}$ affine points of $G G S(q, n)$ having zero $Z$-coordinate, which coincide with the $\mathbb{F}_{q^{2}}$-rational points of $G G S(q, n)$ other than $P_{\infty}$.

Suppose $\operatorname{Aut}(G G S(q, n))$ has another short orbit $\mathcal{O}$. Since $G G S(q, n)$ has zero $p$-rank and $\operatorname{Aut}(G G S(q, n))$ fixes $P_{\infty}, \mathcal{O}$ is tame. Hence, by Schur-Zassenhaus Theorem [34, Theorem 9.19], the stabilizer of a point $P \in \mathcal{O}$ is contained up to conjugation in $\Sigma$. This is a contradiction, as $\Sigma$ acts semiregularly out of the plane $Z=0$.

Note from (10) that $\operatorname{Aut}(G G S(q, n))$ is defined over $\mathbb{F}_{q^{2 n}}$. Let $\pi_{a}$ be the plane $Z=a$. The points of $\pi_{0} \cap G G S(q, n)$ are exactly the $q^{3}+1 \mathbb{F}_{q^{2}}$-rational points of $G G S(q, n)$, while all coordinates of any point of $G G S(q, n) \backslash \pi_{0}$ are not in $\mathbb{F}_{q^{2}}$. The group $\Sigma$ fixes all points in $\pi_{0} \cap G G S(q, n)$ and acts semiregularly on the planes $\pi_{a}$, while the group Q acts transitively on $\pi_{0} \cap G G S(q, n)$ and fixes $G G S(q, n) \cap \pi_{a}$ for all $a$. Also, $Q$ acts faithfully on the Hermitian curve $\mathcal{H}_{q}: Y^{q+1}=X^{q}+X$ by $(X, Y, T) \mapsto \bar{Q} \cdot(X, Y, T)$, where $\bar{Q}$ is obtained from $Q$ deleting the third row and column.
Proposition 8.2. The automorphism group of $C\left(\bar{D}, \ell P_{\infty}\right)$ contains a subgroup isomorphic to

$$
\left(\operatorname{Aut}(G G S(q, n)) \rtimes \operatorname{Aut}\left(\mathbb{F}_{q^{2 n}}\right)\right) \rtimes \mathbb{F}_{q^{2 n}}^{*}
$$

Proof. The set $S_{\sigma}$ of points of $G G S(q, n)$ fixed by a non-trivial automorphism $\sigma$ of $\operatorname{Aut}_{\mathbb{F}_{q^{2 n}}}(G G S(q, n))=$ $\operatorname{Aut}(G G S(q, n))$ has size $N_{\sigma} \leq q^{3}+1$. In fact, if $\sigma \notin Q$, then $S_{\sigma} \subseteq \pi_{0}$. If $\sigma \in Q$, then from $\sigma\left(P_{\infty}\right)=P_{\infty}$ we have that the induced automorphism $\bar{\sigma} \in \operatorname{Aut}\left(\mathcal{H}_{q}\right)$ fixes only $\mathbb{F}_{q^{2}}$-rational
points of $\mathcal{H}_{q}$; hence, $\sigma$ fixes only $\mathbb{F}_{q^{2}}$-rational points of $G G S(q, n)$, that is, $S_{\sigma} \subseteq \pi_{0}$. Since $\left|G G S(q, n) \cap \pi_{0}\right|=q^{3}+1, N_{\sigma} \leq q^{3}+1$. Now the claim follows from [1, Proposition 2.3].
Proposition 8.3. If $q^{n}+1 \leq \ell \leq q^{n+2}-q^{3}$ and $\{\ell, \ell-1\} \subset H\left(P_{\infty}\right)$, then

$$
\operatorname{Aut}\left(C\left(\bar{D}, \ell P_{\infty}\right)\right) \cong\left(\operatorname{Aut}(G G S(q, n)) \rtimes \operatorname{Aut}\left(\mathbb{F}_{q^{2 n}}\right)\right) \rtimes \mathbb{F}_{q^{2 n}}^{*}
$$

Proof. We apply [11, Theorem 3.4].

- The divisor $G=\ell P_{\infty}$ is effective.
- A plane model of degree $q^{n}+1$ for $G G S(q, n)$ is

$$
\begin{equation*}
\Pi(G G S(q, n)): \quad Z^{q^{n}+1}=X^{q^{3}}+X-\left(X^{q}+X\right)^{q^{2}-q+1} . \tag{11}
\end{equation*}
$$

In fact, $Z^{m(q+1)}=Y^{q+1} h(X)^{q+1}=X^{q^{3}}+X-\left(X^{q}+X\right)^{q^{2}-q+1}$; also, Equation (11) is irreducible since it defines a Kummer extension $\mathbb{K}(x, z) / \mathbb{K}(x)$ totally ramified over the pole of $x$. Therefore, $\mathbb{K}(G G S(q, n))=\mathbb{K}(x, z)$, and $x, z \in \mathcal{L}(G)$ from the assumption $\ell \geq q^{n}+1$.

- The support of $D$ is preserved by the Frobenius morphism $\varphi:(x, z) \mapsto\left(x^{p}, z^{p}\right)$, since $\varphi\left(P_{\infty}\right)=P_{\infty}$ and $\operatorname{supp}(D)=G G S(q, n)\left(\mathbb{F}_{q^{2 n}}\right) \backslash\left\{P_{\infty}\right\}$.
- Let $N$ be the length of $C\left(\bar{D}, \ell P_{\infty}\right)$. Then the condition $N>\operatorname{deg}(G) \cdot \operatorname{deg}(\Pi(G G S(q, n)))$ reads

$$
q^{2 n+2}-q^{n+3}+q^{n+2}>\ell\left(q^{n}+1\right)
$$

which is implied by the assumption $\ell \leq q^{n+2}-q^{3}$.

-     - If $P=P_{\infty}$, then $\mathcal{L}(G) \neq \mathcal{L}(G-P)$ since $\ell \in H\left(P_{\infty}\right)$.
- If $P \neq P_{\infty}$, then $1 \in \mathcal{L}(G) \backslash \mathcal{L}(G-P)$.
- If $P=Q=P_{\infty}$, then $\mathcal{L}(G-P) \neq \mathcal{L}(G-P-Q)$ since $\ell-1 \in H\left(P_{\infty}\right)$.
- If $P=P_{\infty}$ and $Q \neq P_{\infty}$, then $1 \in \mathcal{L}(G-P) \backslash \mathcal{L}(G-P-Q)$.
- If $P \neq P_{\infty}$ and $Q=P_{\infty}$, then $f-\mu \in \mathcal{L}(G-P) \backslash \mathcal{L}(G-P-Q)$, where $f \in \mathcal{L}(G)$ has pole divisor $\ell P_{\infty}$ and $\mu=f(P)$.
- If $P, Q \neq P_{\infty}$ and $P \neq Q$, choose $f=z-z(P)$ or $f=x-x(P)$ according to $z(P) \neq z(Q)$ or $x(P) \neq x(Q)$; then $f \in \mathcal{L}(G-P) \backslash \mathcal{L}(G-P-Q)$.
- If $P=Q \neq P_{\infty}$, then $z-z(P) \in \mathcal{L}(G-P) \backslash \mathcal{L}(G-P-Q)$.

Thus we can apply [11, Theorem 3.4] to prove the claim.

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