AG codes and AG quantum codes from the GGS curve

D. Bartoli, M. Montanucci, G. Zini

Abstract

In this paper, algebraic-geometric (AG) codes associated with the GGS maximal curve are investigated. The Weierstrass semigroup at all \mathbb{F}_{q^2} -rational points of the curve is determined; the Feng-Rao designed minimum distance is computed for infinite families of such codes, as well as the automorphism group. As a result, some linear codes with better relative parameters with respect to one-point Hermitian codes are discovered. Classes of quantum and convolutional codes are provided relying on the constructed AG codes.

Keywords: GGS curve, AG code, quantum code, convolutional code, code automorphisms.

MSC Code: 94B27. ¹

1 Introduction

In [13, 14] Goppa used algebraic curves to construct linear error correcting codes, the so called algebraic geometric codes (AG codes). The construction of an AG code with alphabet a finite field \mathbb{F}_q requires that the underlying curve is \mathbb{F}_q -rational and involves two \mathbb{F}_q -rational divisors D and G on the curve.

In general, to construct a "good" AG code over \mathbb{F}_q we need a curve \mathcal{X} with low genus g with respect to its number of \mathbb{F}_q -rational points. In fact, from the Goppa bounds on the parameters of the code it follows that the relative Singleton defect is upper bounded by the ratio g/N, where N can be as large as the number of \mathbb{F}_q -rational points of \mathcal{X} not in the

¹ This research was partially supported by Ministry for Education, University and Research of Italy (MIUR) (Project PRIN 2012 "Geometrie di Galois e strutture di incidenza" - Prot. N. 2012XZE22K_005) and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM).

URL: Daniele Bartoli (daniele.bartoli@dmi.unipg.it), Maria Montanucci (maria.montanucci@unibas.it), Giovanni Zini (gzini@math.unifi.it).

support of G. Maximal curves over \mathbb{F}_q attain the Hasse-Weil upper bound for the number of \mathbb{F}_q -rational points with respect to their genus and for this reason they have been used in a number of works. Examples of such curves are the Hermitian curve, the GK curve [12], the GGS curve [10], the Suzuki curve [7], the Klein quartic when $\sqrt{q} \equiv 6 \pmod{7}$ [33], together with their quotient curves. Maximal curves often have large automorphism groups which in many cases can be inherited by the code: this can bring good performances in encoding [25] and decoding [17].

Good bounds on the parameters of one-point codes, that is AG codes arising from divisors G of type nP for a point P of the curve, have been obtained by investigating the Weierstrass semigroup at P. These results have been later generalized to codes and semigroups at two or more points; see e.g. [4,5,20,21,27,30,31].

AG codes from the Hermitian curve have been widely investigated; see [8,22–24,37,39,40] and the references therein. Other constructions based on the Suzuki curve and the curve with equation $y^q + y = x^{q^r+1}$ can be found in [32] and [36]. More recently, AG Codes from the GK curve have been constructed in [1,3,9].

In the present work we investigate one-point AG codes from the $\mathbb{F}_{q^{2n}}$ -maximal GGS curve, $n \geq 5$ odd. The GGS curve has more short orbits under its automorphism group than other maximal curves, see [15], and hence more possible structures for the Weierstrass semigroups at one point. On the one hand this makes the investigation more complicated; on the other hand it gives more chances of finding one-point AG codes with good parameters. One achievement of this work is the determination of the Weierstrass semigroup at any \mathbb{F}_{q^2} -rational point.

We show that the one-point codes at the infinite point P_{∞} inherit a large automorphism group from the GGS curve; for many of such codes, the full automorphism group is obtained. Moreover, for q = 2, we compute explicitly the Feng-Rao designed minimum distance, which improves the Goppa designed minimum distance. As an application, we provide families of codes with q = 2 whose relative Singleton defect goes to zero as n goes to infinity. We were not able to produce analogous results for an \mathbb{F}_{q^2} -rational affine point P_0 , because of the more complicated structure of the Weierstrass semigroup. In a comparison between onepoint codes from P_{∞} and one-point codes from P_0 , it turns out that the best codes come sometimes from P_{∞} , other times from P_0 ; we give evidence of this fact with tables for the case q = 2, n = 5.

Note that in general, many of our codes are better than the comparable one-point Hermitian codes on the same alphabet. In fact, let C_1 be a code from a one-point divisor G_1 on the $\mathbb{F}_{q^{2n}}$ -maximal GGS curve with genus g_1 , with alphabet $\mathbb{F}_{q^{2n}}$, length N_2 , designed dimension $k_1^* = \deg G_1 - g_1 + 1$, and designed minimum distance $d_1^* = \deg G_1 - (2g_1 - 2)$. In the same way, let C_2 be a code from a one-point divisor G_2 on the $\mathbb{F}_{q^{2n}}$ -maximal Hermitian curve with genus g_2 , with the same alphabet $\mathbb{F}_{q^{2n}}$ and length $N_2 = N_1$ as C_1 , designed dimension $k_2^* = \deg G_2 - g_2 + 1$, and designed minimum distance $d_2^* = \deg G_2 - (2g_2 - 2)$. In order to compare C_1 and C_2 , we can choose G_1 and G_2 such that $k_1^* = k_2^*$. Then the difference $d_1^* - d_2^*$, like the difference $\delta_2^* - \delta_1^*$ between the designed Singleton defects, is equal to $g_2 - g_1 = \frac{1}{2}(q^{2n} - q^{n+2} + q^3 - q^2) \gg 0$.

Finally, we apply our results on AG codes to construct families of quantum codes and convolutional codes.

2 Preliminaries

2.1 Curves and codes

Let \mathcal{X} be a projective, geometrically irreducible, nonsingular algebraic curve of genus gdefined over the finite field \mathbb{F}_q of size q. The symbols $\mathcal{X}(\mathbb{F}_q)$ and $\mathbb{F}_q(\mathcal{X})$ denote the set of \mathbb{F}_q -rational points and the field of \mathbb{F}_q -rational functions, respectively. A divisor D on \mathcal{X} is a formal sum $n_1P_1 + \cdots + n_rP_r$, where $P_i \in \mathcal{X}(\mathbb{F}_q)$, $n_i \in \mathbb{Z}$, $P_i \neq P_j$ if $i \neq j$. The divisor Dis \mathbb{F}_q -rational if it coincides with its image $n_1P_1^q + \cdots + n_rP_r^q$ under the Frobenius map over \mathbb{F}_q . For a function $f \in \mathbb{F}_q(\mathcal{X})$, div(f) and $(f)_{\infty}$ indicate the divisor of f and its pole divisor. Also, the Weierstrass semigroup at P will be indicated by H(P). The Riemann-Roch space associated with an \mathbb{F}_q -rational divisor D is

$$\mathcal{L}(D) := \{ f \in \mathcal{X}(\mathbb{F}_q) \setminus \{0\} : div(f) + D \ge 0 \} \cup \{0\}$$

and its dimension over \mathbb{F}_q is denoted by $\ell(D)$.

Let $P_1, \ldots, P_N \in \mathcal{X}(\mathbb{F}_q)$ be pairwise distinct points and consider the divisor $D = P_1 + \cdots + P_N$ and another \mathbb{F}_q -rational divisor G whose support is disjoint from the support of D. The AG code C(D,G) is the image of the linear map $\eta : \mathcal{L}(G) \to \mathbb{F}_q^N$ given by $\eta(f) = (f(P_1), f(P_2), \ldots, f(P_N))$. The code has length N and if $N > \deg(G)$ then η is an embedding and the dimension k of C(D,G) is equal to $\ell(G)$. The minimum distance dsatisfies $d \ge d^* = N - \deg(G)$, where d^* is called the designed minimum distance of C(D,G); if in addition $\deg(G) > 2g - 2$, then by the Riemann-Roch Theorem $k = \deg(G) - g + 1$; see [19, Th. 2.65]. The dual code $C^{\perp}(D,G)$ is an AG code with dimension $k^{\perp} = N - k$ and minimum distance $d^{\perp} \ge \deg G - 2g + 2$. If $G = \alpha P, \alpha \in \mathbb{N}, P \in \mathcal{X}(\mathbb{F}_q)$, the AG codes C(D,G) and $C^{\perp}(D,G)$ are referred to as one-point AG codes. Let H(P) be the Weierstrass semigroup associated with P, that is

$$H(P) := \{ n \in \mathbb{N}_0 \mid \exists f \in \mathbb{F}_q(\mathcal{X}), (f)_{\infty} = nP \} = \{ \rho_1 = 0 < \rho_2 < \rho_3 < \cdots \}.$$

Denote by $f_{\ell} \in \mathbb{F}_q(\mathcal{X}), \ \ell \geq 1$, a rational function such that $(f_{\ell})_{\infty} = \rho_{\ell} P$. For $\ell \geq 0$,

define the Feng-Rao function

$$\nu_{\ell} := |\{(i,j) \in \mathbb{N}_0^2 : \rho_i + \rho_j = \rho_{\ell+1}\}|.$$

Consider $C_{\ell}(P) = C^{\perp}(P_1 + P_2 + \cdots + P_N, \rho_{\ell}P), P, P_1, \ldots, P_N$ pairwise distint points in $\mathcal{X}(\mathbb{F}_q)$. The number

$$d_{ORD}(C_{\ell}(P)) := \min\{\nu_m : m \ge \ell\}$$

is a lower bound for the minimum distance $d(C_{\ell}(P))$ of the code $C_{\ell}(P)$, called the order bound or the Feng-Rao designed minimum distance of $C_{\ell}(P)$; see [19, Theorem 4.13]. Also, by [19, Theorem 5.24], $d_{ORD}(C_{\ell}(P)) \ge \ell + 1 - g$ and equality holds if $\ell \ge 2c - g - 1$, where $c = \max\{m \in \mathbb{Z} : m - 1 \notin H(P)\}.$

A numerical semigroup is called telescopic if it is generated by a sequence (a_1, \ldots, a_k) such that

- $gcd(a_1,\ldots,a_k)=1;$
- for each $i = 2, ..., k, a_i/d_i \in \langle a_1/d_{i-1}, ..., a_{i-1}/d_{i-1} \rangle$, where $d_i = \gcd(a_1, ..., a_i)$;

see [28]. The semigroup H(P) is called symmetric if $2g - 1 \notin H(P)$. The property of being symmetric for H(P) gives rise to useful simplifications of the computation of $d_{ORD}(C_{\ell}(P))$, when $\rho_{\ell} > 2g$. The following result is due to Campillo and Farrán; see [2, Theorem 4.6].

Proposition 2.1. Let \mathcal{X} be an algebraic curve of genus g and let $P \in \mathcal{X}(\mathbb{F}_q)$. If H(P) is a symmetric Weierstrass semigroup then one has

$$d_{ORD}(C_{\ell}(P)) = \nu_{\ell},$$

for all $\rho_{\ell+1} = 2g - 1 + e$ with $e \in H(P) \setminus \{0\}$.

2.2 The automorphism group of an AG code C(D,G)

In the following we use the same notation as in [11, 26]. Let $\mathcal{M}_{N,q} \leq \operatorname{GL}(N,q)$ be the subgroup of matrices having exactly one non-zero element in each row and column. For $\gamma \in \operatorname{Aut}(\mathbb{F}_q)$ and $M = (m_{i,j})_{i,j} \in \operatorname{GL}(N,q)$, let M^{γ} be the matrix $(\gamma(m_{i,j}))_{i,j}$. Let $\mathcal{W}_{N,q}$ be the semidirect product $\mathcal{M}_{N,q} \rtimes \operatorname{Aut}(\mathbb{F}_q)$ with multiplication $M_1\gamma_1 \cdot M_2\gamma_2 := M_1M_2^{\gamma} \cdot \gamma_1\gamma_2$. The *automorphism group* $\operatorname{Aut}(C(D,G))$ of C(D,G) is the subgroup of $\mathcal{W}_{N,q}$ preserving C(D,G), that is,

$$M\gamma(x_1,\ldots,x_N) := ((x_1,\ldots,x_N) \cdot M)^{\gamma} \in C(D,G) \text{ for any } (x_1,\ldots,x_N) \in C(D,G).$$

Let $\operatorname{Aut}_{\mathbb{F}_q}(\mathcal{X})$ denote the \mathbb{F}_q -automorphism group of \mathcal{X} . Also, let

$$\operatorname{Aut}_{\mathbb{F}_q,D,G}(\mathcal{X}) = \{ \sigma \in \operatorname{Aut}_{\mathbb{F}_q}(\mathcal{X}) \mid \sigma(D) = D, \, \sigma(G) \approx_D G \},\$$

where $G' \approx_D G$ if and only if there exists $u \in \mathbb{F}_q(\mathcal{X})$ such that G' - G = (u) and $u(P_i) = 1$ for $i = 1, \ldots, N$, and

$$\operatorname{Aut}_{\mathbb{F}_q,D,G}^+(\mathcal{X}) := \{ \sigma \in \operatorname{Aut}_{\mathbb{F}_q}(\mathcal{X}) \mid \sigma(D) = D, \, \sigma(|G|) = |G| \},\$$

where $|G| = \{G + (f) \mid f \in \overline{\mathbb{F}}_q(\mathcal{X})\}$ is the linear series associated with G. Note that $\operatorname{Aut}_{\mathbb{F}_q,D,G}(\mathcal{X}) \subseteq \operatorname{Aut}^+_{\mathbb{F}_q,D,G}(\mathcal{X}).$

Remark 2.2. Suppose that $\operatorname{supp}(D) \cup \operatorname{supp}(G) = \mathcal{X}(\mathbb{F}_q)$ and each point in $\operatorname{supp}(G)$ has the same weight in G. Then

$$\operatorname{Aut}_{\mathbb{F}_q,D,G}(\mathcal{X}) = \operatorname{Aut}_{\mathbb{F}_q,D,G}^+(\mathcal{X}) = \{ \sigma \in \operatorname{Aut}_{\mathbb{F}_q}(\mathcal{X}) \mid \sigma(\operatorname{supp}(G)) = \operatorname{supp}(G) \}.$$

In [11] the following result was proved.

Theorem 2.3. ([11, Theorem 3.4]) Suppose that the following conditions hold:

- G is effective;
- $\ell(G-P) = \ell(G) 1$ and $\ell(G-P-Q) = \ell(G) 2$ for any $P, Q \in \mathcal{X}$;
- \mathcal{X} has a plane model $\Pi(\mathcal{X})$ with coordinate functions $x, y \in \mathcal{L}(G)$;
- \mathcal{X} is defined over \mathbb{F}_p ;
- the support of D is preserved by the Frobenius morphism $(x, y) \mapsto (x^p, y^p)$;
- $N > \deg(G) \cdot \deg(\Pi(\mathcal{X})).$

Then

$$\operatorname{Aut}(C(D,G)) \cong (\operatorname{Aut}^+_{\mathbb{F}_q,D,G}(\mathcal{X}) \rtimes \operatorname{Aut}(\mathbb{F}_q)) \rtimes \mathbb{F}_q^*.$$

If any non-trivial element of $\operatorname{Aut}_{\mathbb{F}_q}(\mathcal{X})$ fixes at most N-1 \mathbb{F}_q -rational points of \mathcal{X} then $\operatorname{Aut}(C(D,G))$ contains a subgroup isomorphic to $(\operatorname{Aut}_{\mathbb{F}_q,D,G}(\mathcal{X}) \rtimes \operatorname{Aut}(\mathbb{F}_q)) \rtimes \mathbb{F}_q^*$; see [1, Proposition 2.3].

2.3 The GGS curve

Let q be a prime power and consider an odd integer n. The GGS curve GGS(q, n) is defined by the equations

$$\begin{cases} X^{q} + X = Y^{q+1} \\ Y^{q^{2}} - Y = Z^{m} \end{cases},$$
(1)

where $m = (q^n + 1)/(q + 1)$; see [10]. The genus of GGS(q, n) is $\frac{1}{2}(q - 1)(q^{n+1} + q^n - q^2)$, and GGS(q, n) is $\mathbb{F}_{q^{2n}}$ -maximal.

Let $P_0 = (0, 0, 0)$, $P_{(a,b,c)} = (a, b, c)$, and let P_{∞} be the unique ideal point of GGS(q, n). Note that GGS(q, n) is singular, being P_{∞} its unique singular point. Yet, there is only one place of GGS(q, n) centered at P_{∞} ; therefore, we can actually construct AG codes from GGS(q, n) as described in Section 2.1 (see [38, Appendix B] and [18, Chapter 8] for an introduction to the concept of place of a curve). The divisors of the functions x, y, zsatisfying $x^q + x = y^{q+1}$, $y^{q^2} - y = z^m$ are

$$(x) = m(q+1)P_0 - m(q+1)P_{\infty},$$

$$(y) = m \sum_{\substack{\alpha^q + \alpha = \beta \\ \beta \in \mathbb{F}_{q^2}}} P_{(\alpha,0,0)} - mqP_{\infty},$$

$$(z) = \sum_{\substack{\alpha^q + \alpha = \beta \\ \beta \in \mathbb{F}_{q^2}}} P_{(\alpha,\beta,0)} - q^3 P_{\infty}.$$

Throughout the paper we indicate by \overline{D} and D the divisors

$$\overline{D} = \sum_{P \in GGS(q,n)(\mathbb{F}_{q^{2n}}), \ P \neq P_{\infty}} P, \qquad \tilde{D} = \sum_{P \in GGS(q,n)(\mathbb{F}_{q^{2n}}), \ P \neq P_{0}} P.$$
(2)

2.4 Structure of the paper

The paper is organized as follows. In Section 3 the value of $d_{ORD}(C_{\ell}(P_{\infty}))$ for q = 2 and $n \geq 5$ is obtained, where $C_{\ell}(P_{\infty}) = C^{\perp}(\overline{D}, \ell P_{\infty})$; this is applied in Section 3.5 to two families of codes with q = 2 whose relative Singleton defect goes to zero as n goes to infinity. In Section 4 we determine the Weierstrass semigroup at P_0 , and hence at any \mathbb{F}_{q^2} -rational affine point of GGS(q, n). The tables in Section 5 describe the parameters of $C_{\ell}P_{\infty}$ and $C_{\ell}(P_0)$ in the particular case $q^{2n} = 2^{10}$. Sections 6 and 7 provide families of quantum codes and convolutional codes constructed from $C_{\ell}(P_{\infty})$ and $C_{\ell}(P_0)$. Finally, we compute in Section 8 the automorphism group of the AG code $C(\overline{D}, \ell P_{\infty})$ for $q^n + 1 \leq \ell \leq q^{n+2} - q^3$.

3 The computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ for q = 2

In this section we deal with the codes $C_{\ell}(P_{\infty}) = C^{\perp}(\overline{D}, \rho_{\ell}P_{\infty})$, where \overline{D} is as in (2). Our purpose is to exhibit the exact value of $d_{ORD}(C_{\ell}(P_{\infty}))$ for the case q = 2. First of all we determine the values of ν_{ℓ} (Subsection 3.1); in Subsections 3.2, 3.3, 3.4 we compute $d_{ORD}(C_{\ell}(P_{\infty}))$.

3.1 The Feng-Rao function ν_{ℓ} for q = 2

Assume that q = 2 and $n \ge 5$ is odd. Let $m = \frac{2^{n+1}}{3}$. Then, from [15, Corollary 3.5],

$$H(P_{\infty}) = \left\{ i(2^{n}+1) + 2j\frac{2^{n}+1}{3} + 8k \mid i \in \{0,1\}, \ j \in \{0,1,2,3\}, \ k \ge 0 \right\}.$$

Remark 3.1. Let $\rho_{\ell} = i(2^n + 1) + 2j\frac{2^n+1}{3} + 8k \in H(P_{\infty})$. Then ρ_{ℓ} is uniquely determined by the triple (i, j, k).

Proof. Assume that $i(2^n+1)+2j\frac{2^n+1}{3}+8k = i'(2^n+1)+2j'\frac{2^n+1}{3}+8k'$. Then $i \equiv i' \pmod{2}$ and since i, i' < 2 we have that i = i'. Thus, $2j\frac{2^n+1}{3}+8k = 2j'\frac{2^n+1}{3}+8k'$. Since this implies that $j \equiv j' \pmod{4}$ and j, j' < 4, we have that j = j' and k = k' and the claim follows. \Box

According to Remark 3.1, the notation (i, j, k) is used to indicate the non-gap at P_{∞} associated with the choices of the parameters i, j, k. In order to compute $d_{ORD}(C_{\ell}(P_{\infty}))$ the following definition is required. Let $\rho_{\ell} \in H(P_{\infty})$ be fixed. Assume that $\rho_{\ell+1} = (i, j, k)$. Then,

$$\nu_{\ell} = \left| \{ (i_r, j_r, k_r), \ r = 1, 2 \mid (i, j, k) = (i_1, j_1, k_1) + (i_2, j_2, k_2) \} \right|.$$

In the following lemmas we determine the value of ν_{ℓ} .

Lemma 3.2. Let $\rho_{\ell} \in H(P_{\infty})$ be fixed. Assume that $\rho_{\ell+1} = (1, j, k)$ for some j = 0, 1, 2, 3and $k \ge 0$. Then,

$$\nu_{\ell} = \begin{cases} 2(j+1)(k+1), & \text{if } k < m, \\ 2(j+1)(k+1) + 2(3-j)(k-m+1), & \text{otherwise.} \end{cases}$$

Proof. Let i_1, i_2, j_1, j_2, k_1 , and $k_2 \in \mathbb{N}$ be such that

$$(2^{n}+1)+2jm+8k = (i_{1}+i_{2})(2^{n}+1)+2(j_{1}+j_{2})m+8(k_{1}+k_{2}) = 3(i_{1}+i_{2})m+2(j_{1}+j_{2})m+8(k_{1}+k_{2}) = 3(i_{1}+i_{2})m+2(i_{1}+i_{2})m+8(k_{1}+k_{2}) = 3(i_{1}+i_{2})m+2(i$$

Then $i_1 + i_2 \equiv 1 \pmod{2}$ and since $i_1 + i_2 \leq 2$ we have that $i_1 + i_2 = 1$. This implies that

$$3m + 2jm + 8k = 3m + 2(j_1 + j_2)m + 8(k_1 + k_2),$$

and hence

$$jm + 4k = (j_1 + j_2)m + 4(k_1 + k_2).$$
(3)

Assume that j = 0. Then from (3), $(j_1 + j_2)m \equiv 0 \pmod{4}$ and so, $j_1 + j_2 = 4h$ for some integer h. Since $0 \leq j_1 + j_2 \leq 6$ we have that h = 0 or h = 1. In the first case $k_1 + k_2 = k$, in the second case $k_1 + k_2 = k - m$. Since $k_1 + k_2 \geq 0$, if k < m the second case cannot occur. Thus, if k < m, since we have 2 possible choices for i_1 and (k+1) choices for k_1 (while i_2 and k_2 are determined according to the choices of i_1 and k_1 , respectively), then $\nu_{\ell} = 2(k+1)$. Also, if $k \geq m$ we have that

$$\nu_{\ell} = 2(k+1) + 2 \cdot |\{(j_1, j_2) : 0 \le j_1, j_2 \le 3, \ j_1 + j_2 = 4\}| \cdot (k-m+1) = 2(k+1) + 6(k-m+1) = 2(k+1) + 6(k-m+1) + 2(k+1) + 6(k-m+1) + 6(k-1) + 6(k-1)$$

and the claim follows by direct checking.

Assume that j = 1. Then from (3), $(j_1 + j_2)m \equiv m \pmod{4}$ and so $j_1 + j_2 = 1 + 4h$ for some integer h. Since $0 \leq j_1 + j_2 \leq 6$ we have that h = 0 or h = 1. In the first case $k_1 + k_2 = k$, in the second case $k_1 + k_2 = k - m$. Since $k_1 + k_2 \geq 0$ if k < m the second case cannot occur. Thus, if k < m, since we have 2 possible choices for i_1 , 2 possible choices for j_1 and (k + 1) choices for k_1 , then $\nu_{\ell} = 4(k + 1)$. Also, if $k \geq m$ we have that,

 $\nu_{\ell} = 4(k+1) + 2 \cdot |\{(j_1, j_2) : 0 \le j_1, j_2 \le 3, \ j_1 + j_2 = 5\}| \cdot (k-m+1) = 4(k+1) + 4(k-m+1),$

and the claim follows by direct checking.

Assume that j = 2. Then from (3), $(j_1 + j_2)m \equiv 2m \pmod{4}$ and so $j_1 + j_2 = 2 + 4h$, for some integer h. Since $0 \leq j_1 + j_2 \leq 6$ we have that h = 0 or h = 1. In the first case $k_1 + k_2 = k$, in the second case $k_1 + k_2 = k - m$. Since $k_1 + k_2 \geq 0$, if k < m the second case cannot occur. Thus, if k < m, since we have 2 possible choices for i_1 , 3 possible choices for j_1 and (k + 1) choices for k_1 , then $\nu_{\ell} = 6(k + 1)$. Also, if $k \geq m$ we have that

$$\nu_{\ell} = 6(k+1) + 2(k-m+1) \cdot \left| \{ (j_1, j_2) : 0 \le j_1, j_2 \le 3, \ j_1 + j_2 = 6 \} \right| = 6(k+1) + 2(k-m+1),$$

and the claim follows by direct checking.

Assume that j = 3. Then from (3), $(j_1 + j_2)m \equiv 3m \pmod{4}$ and so $j_1 + j_2 = 3 + 4h$, for some integer h. Since $0 \le j_1 + j_2 \le 6$ we have that h = 0. Since this implies that $k_1 + k_2 = k$, we have that $\nu_{\ell} = 8(k+1)$.

Using a similar approach we can prove the following.

Lemma 3.3. Let $\rho_{\ell} \in H(P_{\infty})$ be fixed. Assume that $\rho_{\ell+1} = (0, j, k)$ for some j = 0, 1, 2, 3 and $k \ge 0$. Then,

$$\nu_{\ell} = \begin{cases} (j+1)(k+1) + \lfloor \frac{j}{3} \rfloor (k+1), & \text{if } k < m, \\ (j+1)(k+1) + \lfloor \frac{j}{3} \rfloor (k+1) + (5-2\max\{0, j-2\})(k-m+1), & \text{if } m \le k < 2m, \\ (j+1)(k+1) + \lfloor \frac{j}{3} \rfloor (k+1) + \\ + (5-2\max\{0, j-2\})(k-m+1) + \max\{0, 2-j\}(k-2m+1), & \text{otherwise.} \end{cases}$$

3.2 Computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ for $\rho_{\ell+1} = (1, j, k)$ and $\rho_{\ell} \leq 2g$

Let $\rho_{\ell} \in H(P_{\infty})$. Assume that $\rho_{\ell+1} = (1, j, k)$ for some j = 0, 1, 2, 3 and $k \ge 0$. Recall that $C_{\ell}(P_{\infty})$ is the dual code of the AG code $C(\overline{D}, \rho_{\ell}P_{\infty})$, where \overline{D} is as in (2).

Lemma 3.4. If $\rho_{\ell+1} = (1, 0, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 2, & if \ k = 0, \\ 3, & if \ k \leq \lfloor \frac{m}{8} \rfloor, \\ 4, & if \ \frac{m}{8} < k \leq \lfloor \frac{m}{4} \rfloor, \\ 5, & if \ \frac{m}{4} < k \leq \lfloor \frac{3m}{8} \rfloor, \\ 6, & if \ \frac{3m}{8} < k \leq \lfloor \frac{m}{2} \rfloor, \\ 8, & if \ \frac{m}{2} < k \leq \lfloor \frac{3m}{4} \rfloor, \\ 8(\lceil k - \frac{3m}{4} \rceil + 1), & if \ \frac{3m}{4} \leq k \leq m - 2, \\ \nu_{\ell} = 2m, & if \ k = m - 1. \end{cases}$$

Proof. For $\rho_s \in H(P_{\infty})$ the following system of inequalities is considered:

$$\begin{cases} \rho_{s+1} \ge \rho_{\ell+1}, \\ \nu_s \le \nu_{\ell}. \end{cases}$$

$$\tag{4}$$

In order to compute $d_{ORD}(C_{\ell}(P_{\infty}))$ we take the minimum value of ν_s such that System (4) is satisfied. Also, a case-by-case analysis with respect to $a \in \{0, 1\}$ is required. Assume that $\rho_{s+1} = (a, b, c)$ for some $a \in \{0, 1\}, b \in \{0, 1, 2, 3\}$ and $c \ge 0$. From Lemma 3.2, System (4) reads,

$$\begin{cases} 3am + 2bm + 8c \ge 3m + 8k, \\ \nu_s \le 2(k+1). \end{cases}$$
(5)

Case 1: a = 1 and c < m. From Lemma 3.2, System (5) reads

$$\begin{cases} 2bm + 8c \ge 8k, \\ 2(b+1)(c+1) \le 2(k+1). \end{cases}$$

- If b = 0 then c = k and so the unique solution is ν_{ℓ} itself.
- If b = 1 then $c \ge \lceil k \frac{m}{4} \rceil$ and $c \le \lfloor \frac{k-1}{2} \rfloor$. Such a c exists if and only if $\lceil k \frac{m}{4} \rceil \le \lfloor \frac{k-1}{2} \rfloor$. Assume that k is odd. Then $k - \lfloor \frac{m}{4} \rfloor = \lceil k - \frac{m}{4} \rceil \le \lfloor \frac{k-1}{2} \rfloor = \frac{k-1}{2}$ if and only if $k \le 2\lfloor \frac{m}{4} \rfloor - 1$. Similarly if k is even then c exists if and only if $k - \lfloor \frac{m}{4} \rfloor \le \frac{k-2}{2}$, that is $k \le 2\lfloor \frac{m}{4} \rfloor - 2$. For these cases the minimum is obtained taking $c = \max\{0, \lceil k - \frac{m}{4} \rceil\}$ and hence $\nu_s = 4(\max\{0, \lceil k - \frac{m}{4} \rceil\} + 1)$.
- If b = 2 then $c \ge \lceil k \frac{m}{2} \rceil$ and $c \le \lfloor \frac{k-2}{3} \rfloor$. As before, such a c exists if and only if $\lceil k \frac{m}{2} \rceil \le \lfloor \frac{k-2}{3} \rfloor$. This is equivalent to $k \le \frac{3}{2} \lfloor \frac{m}{2} \rfloor 1$ if $k \equiv 0 \pmod{3}$, to $k \le \frac{3}{2} \lfloor \frac{m}{2} \rfloor 2$ if $k \equiv 1 \pmod{3}$, to $k \le \frac{3}{2} \lfloor \frac{m}{2} \rfloor 1$ if $k \equiv 2 \pmod{3}$. For these cases the minimum is obtained taking $c = \max\{0, \lceil k \frac{m}{2} \rceil\}$ and hence $\nu_s = 6(\max\{0, \lceil k \frac{m}{2} \rceil\} + 1)$.
- If b = 3 then $c \ge \lceil k \frac{3m}{4} \rceil$ and $c \le \lfloor \frac{k-3}{4} \rfloor$. As before, such a c exists if and only if $\lceil k \frac{3m}{4} \rceil \le \lfloor \frac{k-3}{4} \rfloor$. By direct checking, this is equivalent to $k \le m 2$. Here the minimum is obtained taking $c = \max\{0, \lceil k \frac{3m}{4} \rceil\}$ and hence $\nu_s = 8(\max\{0, \lceil k \frac{3m}{4} \rceil\} + 1)$.

When $k > \frac{3m}{4}$ and $k \le m-2$ the minimum value above is obtained as $\nu_s = 8(\lceil k - \frac{3m}{4} \rceil + 1)$. We observe that if k = m-1 then $\nu_{\ell} = 2(k+1) = 2m$ and $8(\max\{0, \lceil k - \frac{3m}{4} \rceil\} + 1) = 8(\lceil m-1 - \frac{3m}{4} \rceil\} + 1) > 2m$. This implies that if k = m-1 then the minimum value is $\nu_{\ell} = 2m$ itself. Thus, combining the previous results we obtain

$$\min\{\nu_s \mid a = 1 \text{ and } c < m\} = \begin{cases} 2, & \text{if } k = 0, \\ 4 & \text{if } 1 \le k \le \lfloor \frac{m}{4} \rfloor, \\ 6, & \text{if } \frac{m}{4} < k \le \lfloor \frac{m}{2} \rfloor, \\ 8, & \text{if } \frac{m}{2} < k \le \lfloor \frac{3m}{4} \rfloor, \\ 8(\lceil k - \frac{3m}{4} \rceil + 1), & \text{if } \frac{3m}{4} < k \le m - 2, \\ 2m = \nu_\ell, & \text{if } k = m - 1. \end{cases}$$
(6)

Case 2: a = 1 and $c \ge m$. From Lemma 3.2 System (5) reads,

$$\begin{cases} 2bm + 8c \ge 8k, \\ 2(b+1)(c+1) + 2(3-b)(c-m+1) \le 2(k+1) \end{cases}$$

Since $2(b+1)(c+1) + 2(3-b)(c-m+1) \ge 2(c+1)$ and c > k this case cannot occur.

Case 3: a = 0 and c < m. From Lemma 3.3 (5) reads,

$$\begin{cases} 2bm + 8c \ge 3m + 8k, \\ (b+1)(c+1) + \lfloor \frac{b}{3} \rfloor (c+1) \le 2(k+1). \end{cases}$$

- If b = 0 then $c \ge \lfloor k + \frac{3m}{8} \rfloor$ and $c \le 2k + 1$. Such a c exists if and only if $k \ge \lfloor \frac{3m}{8} \rfloor$. For these cases, the minimum is obtained taking $c = \lfloor k + \frac{3m}{8} \rfloor$ and hence $\nu_s = (\lfloor k + \frac{3m}{8} \rfloor + 1)$.
- The case b = 1 cannot occur. In fact we have $c \ge \lfloor k + \frac{m}{8} \rfloor$ and $2(c+1) \le 2(k+1)$, a contradiction.
- If b = 2 then $c \ge \lceil k \frac{m}{8} \rceil$ and $c \le \lfloor \frac{2k-1}{3} \rfloor$. Such a c exists if and only if $k + \lceil -\frac{m}{8} \rceil \le \lfloor \frac{2k+1}{3} \rfloor$. This is equivalent to $k \le 3 \lfloor \frac{m}{8} \rfloor 1$ if $2k \equiv 1 \pmod{3}$, to $k \le 3 \lfloor \frac{m}{8} \rfloor + 1$ if $2k \equiv 2 \pmod{3}$, to $k \le 3 \lfloor \frac{m}{8} \rfloor 3$ if $2k \equiv 0 \pmod{3}$. For these cases, the minimum is obtained taking $c = \max\{0, \lceil k \frac{m}{8} \rceil\}$ and hence $\nu_s = 3(\max\{0, \lceil k \frac{m}{8} \rceil\} + 1)$.
- If b = 3 then $c \ge \lceil k \frac{3m}{8} \rceil$ and $c \le \lfloor \frac{2k-3}{5} \rfloor$. Such a c exists if and only if $k + \lceil -\frac{3m}{8} \rceil \le \lfloor \frac{2k-3}{5} \rfloor$. This is equivalent to $k \le \frac{5}{3} \lfloor \frac{3m}{8} \rfloor \frac{5}{3}$ if $2k \equiv 0 \pmod{5}$, to $k \le \frac{5}{3} \lfloor \frac{3m}{8} \rfloor 2$ if $2k \equiv 1 \pmod{5}$, to $k \le \frac{5}{3} \lfloor \frac{3m}{8} \rfloor \frac{7}{3}$ if $2k \equiv 2 \pmod{5}$, to $k \le \frac{5}{3} \lfloor \frac{3m}{8} \rfloor 1$ if $2k \equiv 3 \pmod{5}$, to $k \le \frac{5}{3} \lfloor \frac{3m}{8} \rfloor \frac{4}{3}$ if $2k \equiv 4 \pmod{5}$. In these cases, the minimum is obtained taking $c = \max\{0, \lceil k \frac{3m}{8} \rceil\}$ and hence $\nu_s = 5(\max\{0, \lceil k \frac{3m}{8} \rceil\} + 1)$.

Thus, we obtain

$$\min\{\nu_{s} \mid a = 0 \text{ and } c < m\} = \begin{cases} 3, \text{ if } k \leq \lfloor \frac{m}{8} \rfloor, \\ 5, \text{ if } \frac{m}{8} < k \leq \lfloor \frac{3m}{8} \rfloor, \\ 5(\lceil k - \frac{3m}{8} \rceil + 1), \text{ if } \lceil \frac{3m}{8} \rceil \leq k \leq \lfloor \frac{5}{3} \lfloor \frac{3m}{8} \rfloor - \frac{7}{3} \rfloor, \\ (\lceil k + \frac{3m}{8} \rceil + 1), \text{ otherwise.} \end{cases}$$
(7)

Case 4: a = 0 and $m \le c < 2m$. From Lemma 3.3 System (5) reads,

$$\begin{cases} 2bm + 8c \ge 3m + 8k, \\ (b+1)(c+1) + \lfloor \frac{b}{3} \rfloor (c+1) + (5 - 2\max\{0, b-2\})(c-m+1) \le 2(k+1). \end{cases}$$

Since $(b+1)(c+1) + \lfloor \frac{b}{3} \rfloor (c+1) + (5-2\max\{0, b-2\})(c-m+1) \ge (b+1)(c+1)$ and c > k, cases b = 1, 2, 3 cannot occur. Thus b = 0 and

$$\begin{cases} 8c \ge 3m + 8k, \\ (c+1) + 5(c-m+1) \le 2(k+1). \end{cases}$$

Hence $c \ge \lfloor k + \frac{3m}{8} \rfloor$ and $c \le \lfloor \frac{2k+5m-4}{6} \rfloor$. Since $c \ge m$ then $k \ge \frac{m+4}{2}$. The minimum value is obtained (when it is possible) for $c = \lfloor k + \frac{3m}{8} \rfloor$. By direct checking the minimum ν_{ℓ} is bigger than the one obtained in (6), and hence we can discard this case.

Case 5: a = 0 and $c \ge 2m$. From Lemma 3.3 System (5) reads,

$$\begin{cases} 2bm + 8c \ge 3m + 8k, \\ (b+1)(c+1) + \lfloor \frac{b}{3} \rfloor (c+1) + (5-2\max\{0, b-2\})(c-m+1) + \max\{0, 2-b\}(c-2m+1) \le 2(k+1). \end{cases}$$

 $(b+1)(c+1) \ge 2m+1$ and $2(k+1) \le 2m$ this case cannot occur.

Taking the minimum of the values in (6) and (7) the claim follows.

Using the same arguments the following results are obtained.

Lemma 3.5. If $\rho_{\ell+1} = (1, 3, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \nu_{\ell}.$$

Lemma 3.6. If $\rho_{\ell+1} = (1, 1, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 4, & \text{if } k = 0, \\ 5, & \text{if } k \leq \lfloor \frac{m}{8} \rfloor, \\ 6 & \text{if } \frac{m}{8} < k \leq \lfloor \frac{m}{4} \rfloor, \\ 8, & \text{if } \frac{m}{4} < k \leq \lfloor \frac{m}{2} \rfloor, \\ 8(\lceil k - \frac{m}{2} \rceil + 1), & \text{if } \lceil \frac{m}{2} \rceil \leq k \leq \lfloor \frac{3m}{4} \rfloor - 2, \\ 2(\lceil \frac{m}{4} + k \rceil + 1) + 6(\lceil \frac{m}{4} + k \rceil - m + 1), & \text{if } \lfloor \frac{3m}{4} \rfloor - 1 \leq k \leq m - 2, \\ 4m, & \text{if } k = m - 1. \end{cases}$$

Lemma 3.7. If $\rho_{\ell+1} = (1, 2, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 6, \ if \ k = 0, \\ 8, \ if \ k \leq \lfloor \frac{m}{4} \rfloor, \\ 8(\lceil k - \frac{m}{4} \rceil + 1), \ if \ \lceil \frac{m}{4} \rceil \leq k \leq \lfloor \frac{m}{2} \rfloor - 2, \\ 2(\lceil k + \frac{m}{2} \rceil + 1) + 6(\lceil k + \frac{m}{2} \rceil - m + 1), \ if \ \lfloor \frac{m}{2} \rfloor - 1 \leq k \leq \lfloor \frac{3m}{4} \rfloor - 2, \\ 4(\lceil k + \frac{m}{4} \rceil + 1) + 4(\lceil k + \frac{m}{4} \rceil - m + 1), \ if \ \lfloor \frac{3m}{4} \rfloor - 1 \leq k \leq m - 2, \\ \nu_{\ell} = 6m, \ if \ k = m - 1. \end{cases}$$

3.3 Computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ for $\rho_{\ell+1} = (0, j, k)$ and $\rho_{\ell} \leq 2g$

Using the same arguments as above we obtain the following results in the case $\rho_{\ell} \leq 2g$. Lemma 3.8. If $\rho_{\ell+1} = (0, 0, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 2, & if \ k \leq \lfloor \frac{3m}{8} \rfloor, \\ 3, & if \ \lceil \frac{3m}{8} \rceil \leq k \leq \lfloor \frac{m}{2} \rfloor, \\ 4, & if \ \lceil \frac{m}{2} \rceil \leq k \leq \lfloor \frac{5m}{8} \rfloor, \\ 5, & if \ \lceil \frac{5m}{8} \rceil \leq k \leq \lfloor \frac{3m}{4} \rfloor, \\ 6, & if \ \lceil \frac{3m}{4} \rceil \leq k \leq \lfloor \frac{7m}{8} \rfloor, \\ 8, & if \ \lceil \frac{7m}{8} \rceil \leq k \leq m-1 \end{cases}$$

Lemma 3.9. If $\rho_{\ell+1} = (0, 1, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 2, & if \ k \leq \lfloor \frac{m}{8} \rfloor, \\ 3, & if \ \lceil \frac{m}{8} \rceil \leq k \leq \lfloor \frac{m}{4} \rfloor, \\ 4, & if \ \lceil \frac{m}{4} \rceil \leq k \leq \lfloor \frac{3m}{8} \rfloor, \\ 5, & if \ \lceil \frac{3m}{8} \rceil \leq k \leq \lfloor \frac{m}{2} \rfloor, \\ 6, & if \ \lceil \frac{m}{2} \rceil \leq k \leq \lfloor \frac{5m}{8} \rfloor, \\ 8(\max\{0, \lceil k - \frac{7m}{8} \rceil\} + 1), & if \ \lceil \frac{5m}{8} \rceil \leq k \leq m - 1. \end{cases}$$

Lemma 3.10. If $\rho_{\ell+1} = (0, 3, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 6, & \text{if } k \leq \lfloor \frac{m}{8} \rfloor, \\ 8(\max\{0, \lceil \frac{3m}{8} \rceil\} + 1), & \text{if } \lceil \frac{m}{8} \rceil \leq k \leq m - 2, \\ \nu_{\ell} = 5(k+1), & \text{if } k = m - 1. \end{cases}$$

Lemma 3.11. If $\rho_{\ell+1} = (0, 2, k)$ for some k < m then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 4, & \text{if } k \leq \lfloor \frac{m}{8} \rfloor, \\ 5, & \text{if } \lceil \frac{m}{8} \rceil \leq k \leq \lfloor \frac{m}{4} \rfloor, \\ 6, & \text{if } \lceil \frac{m}{4} \rceil \leq k \leq \lfloor \frac{3m}{8} \rfloor, \\ 8(\max\{0, \lceil k - \frac{5m}{8} \rceil\} + 1), & \text{if } \lceil \frac{3m}{8} \rceil \leq k \leq \lfloor \frac{7m}{8} \rfloor - 2, \\ 2(\lceil k + \frac{m}{8} \rceil + 1), & \text{if } \lfloor \frac{7m}{8} \rfloor - 1 \leq k \leq m - 3, \\ 3(k+1) = \nu_{\ell}, & \text{if } k \in \{m-2, m-1\}. \end{cases}$$

Lemma 3.12. If $\rho_{\ell+1} = (0, 0, k)$ for some $m \le k < 2m$ then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 8(\lceil k - \frac{9m}{8} \rceil + 1), & \text{if } m \le k < \lfloor \frac{11m}{8} - 1 \rfloor, \\ 2(\lceil k - \frac{3m}{8} \rceil + 1) + \max\{0, 6(\lceil k - \frac{3m}{8} \rceil - m + 1)\}, & \text{if } \lfloor \frac{11m}{8} - 1 \rfloor \le k < 2m \end{cases}$$

3.4 Computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ for $\rho_{\ell} > 2g$

Proposition 3.13. The Weierstrass semigroup $H(P_{\infty}) = \langle q^3, mq, q^n + 1 \rangle$ is telescopic.

Proof. Let $a_1 = q^3$, $a_2 = mq$, $a_3 = q^n + 1$, $d_0 = 0$, $d_1 = q^3$, $d_2 = \gcd(q^3, mq) = q$, $d_3 = \gcd(q^3, mq, q^n + 1) = 1$. Then $a_i/d_i \in \langle a_1/d_{i-1}, \ldots, a_{i-1}/d_{i-1} \rangle$ for i = 2, 3; that is, $H(P_{\infty})$ is telescopic.

Proposition 3.13 implies that $H(P_{\infty})$ is symmetric, from [28, Lemma 6.5]. This also follows from the fact that the divisor $(2g-2)P_{\infty}$ is canonical; see [15, Lemma 3.8] and [28, Remark 4.4].

In the following, Proposition 2.1 is used to reduce the direct computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ with $\rho_{\ell} > 2g$, only to those cases for which $\rho_{\ell+1} \neq 2g - 1 + e$ for any $e \in H(P_{\infty}) \setminus \{0\}$. Since the cases in which $\rho_{\ell+1} = (0, 0, k)$ for k < 2m or $\rho_{\ell+1} = (i, j, k)$ for k < m have been already studied, they can be excluded.

Proposition 3.14. Let $\rho_{\ell} \in H(P_{\infty})$ with $\rho_{\ell} > 2g$ and $\rho_{\ell+1} = (i, j, k)$ and $k \ge m$. If $\rho_{\ell+1} \ne (0, 0, k)$ for any $k \in [m, 2m)$, then $\rho_{\ell+1} - 2g + 1 \notin H(P_{\infty})$ if and only if $\rho_{\ell+1} = (0, 1, k)$ for some $k \in [m, 2m)$.

Proof. Write k = m + s for some $s \ge 0$. We prove the claim using a case-by-case analysis with respect to the values of i and j. We recall that $2g-1 = (2^{n+1}+2^n-4)-1 = 9m-8$.

Case 1: i = 1. Clearly, $\rho_{\ell+1} = 3m + 2jb + 8m + 8s$.

- If j = 0, then $\rho_{\ell+1} = 3m + 8m + 8s = (9m 8) + (2m + 8(s + 1)) = 2g 1 + e$. Writing e = (0, 1, s + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.
- If j = 1, then $\rho_{\ell+1} = 3m + 2m + 8m + 8s = (9m 8) + (4m + 8(s + 1)) = 2g 1 + e$. Writing e = (0, 2, s + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.
- If j = 2, then $\rho_{\ell+1} = 3m + 4m + 8m + 8s = (9m 8) + (6m + 8(s + 1)) = 2g 1 + e$. Writing e = (0, 3, s + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.
- If j = 3, then $\rho_{\ell+1} = 3m + 6m + 8k = (9m 8) + (8(k + 1)) = 2g 1 + e$. Writing e = (0, 0, k + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.

Case 2: i = 0. Clearly, $\rho_{\ell+1} = 2jb + 8k = 2jb + 8m + 8s$.

- If j = 0, then in particular we can write k = 2m + t for $t \ge 0$, since $k \ge 2m$. Thus, $\rho_{\ell+1} = 16m + 8t = (9m - 8) + (7m + 8(t + 1)) = 2g - 1 + e$. Writing e = (1, 2, t + 1)we have that $e \in H(P_{\infty})$, so this case cannot occur.
- If j = 1, then $\rho_{\ell+1} = 2m + 8k$. We first assume that $k \ge 2m$ and so that k = 2m + t for some $t \ge 0$. In this case $\rho_{\ell+1} = 2m + 16m + 8t = (9m - 8) + (9m + 8(t+1)) = 2g - 1 + e$. Writing e = (1, 3, t+1) we have that $e \in H(P_{\infty})$, so this case cannot occur. Thus, $k \in [m, 2m)$. In this case, $\rho_{\ell+1} = 2m + 8m + 8s = (9m - 8) + (m + 8(s+1)) = 2g - 1 + e$. By direct computation $e \notin H(P_{\infty})$ and the claim follows.
- If j = 2, then $\rho_{\ell+1} = 4m + 8m + 8s = (9m 8) + (3m + 8(s + 1)) = 2g 1 + e$. Writing e = (1, 0, s + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.
- If j = 3, then $\rho_{\ell+1} = 6m + 8m + 8s = (9m 8) + (5m + 8(s + 1)) = 2g 1 + e$. Writing e = (1, 1, s + 1) we have that $e \in H(P_{\infty})$, so this case cannot occur.

Since from Proposition 3.13 the Weierstrass semigroup $H(P_{\infty})$ is symmetric, its conductor is c = 2g; equivalently, its largest gap is 2g - 1. The following theorem shows that the exact value of $d_{ORD}(C_{\ell}(P_{\infty}))$ is known for $\rho_{\ell+1} \ge 4g$; see [2, Proposition 4.2 (iii)].

Theorem 3.15. Let H(P) be a Weierstrass semigroup. Then $d_{ORD}(C_{\ell}(P)) \ge \ell + 1 - g$ and equality holds if $\rho_{\ell+1} \ge 4g$.

According to the results obtained in the previous sections, Remark 3.14, and Theorem 3.15, to complete the computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ for every $\rho_{\ell} \in H(P_{\infty})$, only the case $\rho_{\ell} \in [2g, 4g-1)$ with $\rho_{\ell+1} = (0, 1, k)$ and $k \in [m, 2m)$ has to be considered.

Proposition 3.16. Let $\rho_{\ell} \in H(P_{\infty})$ be such that $\rho_{\ell} > 2g$ and $\rho_{\ell+1} = (0, 1, k) < 4g$ for $k \in [m, 2m)$. Then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} \nu_{\ell+5} = 8k - 7m + 13, & \text{if } k < \frac{9m - 11}{8}, \\ \nu_{\ell+3} = 8k - 7m + 11, & \text{if } \frac{9m - 11}{8} \le k < \frac{11m - 9}{8}, \\ \nu_{\ell+1} = 8k - 7m + 9, & \text{if } k \ge \frac{11m - 9}{8}. \end{cases}$$

Proof. Arguing as in the previous propositions one can prove that the value of $d_{ORD}(C_{\ell}(P_{\infty}))$

is obtained by $\nu_{\ell+5}$, $\nu_{\ell+3}$, and $\nu_{\ell+1}$, if $k < \frac{9m-11}{8}$, $\frac{9m-11}{8} \leq k < \frac{11m-9}{8}$, and $k \leq \frac{11m-9}{8}$ respectively. Since $\rho_{\ell+1} \geq 2g$, we have that $\rho_{\ell+t} = \rho_{\ell+1} + (t-1)$ for every $t \geq 1$. Assume that $k < \frac{9m-11}{8}$. By direct checking $\rho_{\ell+6} = \rho_{\ell+1} + 5 = (1,3,\tilde{k})$, where $\tilde{k} = k - \frac{7m-5}{8}$. Hence from Lemma 3.2, $d_{ORD}(C_{\ell}(P_{\infty})) = 8(k - \frac{7m-5}{8} + 1) = 8k - 7m + 13$, as $\tilde{k} < \frac{9m-11}{8} - \frac{7m-5}{8} < m$.

Assume that $\frac{9m-11}{8} \leq k < \frac{11m-9}{8}$. By direct checking $\rho_{\ell+4} = \rho_{\ell+1} + 3 = (1,0,\tilde{k})$ where $\tilde{k} = k - \frac{m-3}{8}$. Hence $\tilde{k} \ge m-1$ and from Lemma 3.2, $d_{ORD}(C_{\ell}(P_{\infty})) = 2m = 8k - 7m + 11$ if $\tilde{k} = m - 1$, while $d_{ORD}(C_{\ell}(P_{\infty})) = 2(\tilde{k} + 1) + 6(\tilde{k} - m + 1) = 8k - 7m + 11$ if $\tilde{k} \ge m$. Assume that $k \ge \frac{11m-9}{8}$. By direct checking $\rho_{\ell+2} = \rho_{\ell+1} + 1 = (1, 1, \tilde{k})$ where $\tilde{k} = k - \frac{3m-1}{8}$.

Hence $\tilde{k} \ge m-1$ and from Lemma 3.2, $d_{ORD}(C_{\ell}(P_{\infty})) = 4m = 8k - 7m + 9$ if $\tilde{k} = m - 1$, while $d_{ORD}(C_{\ell}(P_{\infty})) = 4(\tilde{k}+1) + 4(\tilde{k}-m+1) = 8k - 7m + 9$ if $\tilde{k} \ge m$.

For $q \neq 2$, we cannot determine $d_{ORD}(C_{\ell}(P_{\infty}))$ for all ℓ . Yet, this is possible for certain ℓ , as shown in the following propositions.

Proposition 3.17. If $\rho_{\ell+1} \leq (q-1)(q^n+1)$, then

$$d_{ORD}(C_{\ell}(P_{\infty})) = j+1,$$

where $j \le q - 1$ satisfies $(j - 1)(q^n + 1) < \rho_{\ell+1} \le j(q^n + 1)$.

Proof. Since $H(P_{\infty})$ is telescopic from Proposition 3.13, we can apply [28, Theorem 6.11]. The claim then follows because $q^n + 1 = \max\{\frac{q^3}{1}, \frac{mq}{1}, \frac{q^n+1}{1}\}$.

Proposition 3.18. If $\frac{3}{2}(q-1)(q^{n+1}+\frac{1}{3}q^n-q^2-\frac{2}{3})-2 < \ell \leq \frac{3}{2}(q-1)(q^{n+1}+q^n-q^2)-2$, then

$$d_{ORD}(C_{\ell}(P_{\infty})) = \min\{\rho_t \mid \rho_t \ge \ell + 1 - g\}.$$

Proof. This is the claim of [28, Theorem 6.10].

3.5Application for q = 2: families of AG codes with relative Singleton defect going to zero

In this section, we assume that q = 2 and provide two families of codes of type $C_{\ell}(P_{\infty})$ in the cases $\rho_{\ell} = 9m$ and $\rho_{\ell} = 9m + 8$, with relative Singleton defect going to zero as n goes to infinity. We denote by δ and Δ the Singleton defect and the relative Singleton defect of $C_{\ell}(P_{\infty})$, respectively.

Lemma 3.19. Fix $n \ge 5$ odd. Then 9m - 1, 9m, $9m + 1 \in H(P_{\infty})$.

Proof. A direct computation shows that $9m - 1 = (0, 3, \frac{2^n}{8}), 9m = (1, 3, 0), \text{ and } 9m + 1 = (0, 2, \frac{5 \cdot 2^{n-3} + 1}{3})$, thus the claim follows.

We now assume that $\rho_{\ell} = 9m$. Since $\rho_{\ell+1} = 9m + 1 = (0, 2, \frac{5 \cdot 2^{n-3} + 1}{3})$ the following result follows from Lemma 1.3.

Corollary 3.20. Assume that $\rho_{\ell} = (1, 3, 0)$. Then $\nu_{\ell} = 3 \cdot \left(\frac{5 \cdot 2^{n-3} + 1}{3} + 1\right) \ge 24$.

Proposition 3.21. The code $C_{\ell}(P_{\infty})$ is an $[N, k, d]_{2^{2n}}$ -linear code with

- $N = (3m-1)^2 + (3m-1)(9m-7),$
- $k = N \frac{9m+9}{2}$,
- $d \ge d_{ORD}(C_{\ell}(P_{\infty})) = 16,$
- $\delta \le N k + 1 d_{ORD}(C_{\ell}(P_{\infty})) = \frac{9m 21}{2},$
- $\Delta = \frac{\delta}{N} \leq \frac{9m-21}{2(3m-1)(12m-8)}$; hence, Δ goes to zero as n goes to infinity.

Proof. Since GGS(q, n) is $\mathbb{F}_{2^{2n}}$ -maximal, we have

$$N = (2^{2n} + 1 + 2g(GGS(q, n))2^n) - 1 = 2^{2n} + 2^n(9m - 7) = (3m - 1)^2 + (3m - 1)(9m - 7);$$

the last equality follows from $m = (2^n + 1)/3$. Since $C_{\ell}(P_{\infty}) = C^{\perp}(\overline{D}, \rho_{\ell}P_{\infty}), k = N - \tilde{k}$ where \tilde{k} is the dimension of $C(\overline{D}, \rho_{\ell}P_{\infty})$. As $\deg(\rho_{\ell}P_{\infty}) > 2g(GGS(q, n)) - 2$, from the Riemann-Roch Theorem follows

$$k = N - \tilde{k} = N - (\deg(\rho_{\ell}P_{\infty}) + 1 - g(GGS(q, n))) = N - \left(9m + 1 - \frac{9m - 7}{2}\right) = r - \frac{9m + 9}{2}$$

By Lemma 3.11, $d_{ORD}(C_{\ell}(P_{\infty})) \ge 16$. To prove the claim is sufficient to show that there exists $\rho_s \ge \rho_{\ell}$ such that $\nu_s = 16$. To this end we take $\rho_{s+1} = (1,3,1) = 9m + 8 > 9m + 1$. From Lemma 1.1, $\nu_s = 2(b+1)(c+1) = 2(3+1)(1+1) = 16$ and the claim follows. Now the claim on δ and Δ follows by direct computation.

We now assume that $\rho_{\ell} = 9m+8$, so that $\rho_{\ell+1} = 9m+9 = (0, 2, \frac{5\cdot 2^{n-3}+1}{3}+1) = (0, 2, \frac{5m+9}{8})$. Arguing as in the proof of Proposition 3.21 and using Lemma 3.11, the following result is obtained.

Proposition 3.22. The code $C_{\ell}(P_{\infty})$ is an $[r, k, d]_{2^{2n}}$ -linear code with

• $r = (3m-1)^2 + (3m-1)(9m-7),$

• $k = r - \frac{9m+25}{2}$,

•
$$d \ge d_{ORD}(C_{\ell}(P_{\infty})) = 2(\lceil \frac{6m+9}{8} \rceil + 1),$$

- $\delta \le r k + 1 d_{ORD}(C_{\ell}(P_{\infty})) = \frac{9m + 25}{2} 2\left(\left\lceil \frac{6m + 9}{8} \right\rceil + 1\right) < \frac{6m + 21}{2},$
- $\Delta = \frac{\delta}{r} < \frac{6m+21}{2(3m-1)(12m-8)}$; hence, Δ goes to zero as n goes to infinity.

4 Weierstrass semigroup at P_0

In this section we describe the Weierstrass semigroup at P_0 , and hence at any \mathbb{F}_{q^2} -rational affine point by Lemma 8.1. Consider the functions

$$\frac{y^r z^t}{x^s}, \qquad s \in [0, q^2 - 1], r \in [0, s], t \in \left[0, \left\lfloor \frac{sm(q+1) - rqm}{q^3} \right\rfloor\right].$$
(8)

All these functions belong to $H(P_0)$. In fact,

$$\left(\frac{y^r z^t}{x^s}\right) = (mr + t - m(q+1)s)P_0 + (m(q+1)s - mqr - tq^3)P_\infty$$

and by assumption

$$m(q+1)s - mqr - tq^{3} \ge 0.$$
Proposition 4.1. Let $t \in \left[0, \min\left(\left\lfloor \frac{sm(q+1) - rqm}{q^{3}}\right\rfloor, m - 1\right)\right]$ and $s \in [0, q], r \in [0, s]$

or

$$s \in [q+1, q^2 - 1], r \in [0, q].$$

Then all the integers mr + t - m(q+1)s are distinct.

Proof. Suppose $mr + t - m(q+1)s = m\overline{r} + \overline{t} - m(q+1)\overline{s}$. Then $t \equiv \overline{t} \pmod{m}$, which implies $t = \overline{t}$. Now, from $mr - m(q+1)s = m\overline{r} - m(q+1)\overline{s}$, $r \equiv \overline{r} \pmod{q+1}$, which yields $r = \overline{r}$ and $s = \overline{s}$.

Proposition 4.2. Consider the following sets

$$\mathcal{L}_1 := \left\{ -t - rm + m(q+1)s \mid s \in [0,q], r \in [0,s], \\ t \in \left[0, ((s-r)q+s)\frac{m-q^2+q-1}{q^3} + s - r \right] \right\};$$

$$\mathcal{L}_{2} := \left\{ -t - rm + m(q+1)s \mid s \in [q+1, q^{2} - q], r \in [0, q], \\ t \in \left[0, ((s-r)q + s)\frac{m - q^{2} + q - 1}{q^{3}} + s - r \right] \right\};$$

$$\mathcal{L}_3 := \left\{ -t - rm + m(q+1)s \mid s \in [q^2 - q + 1, q^2 - 2], \\ r \in [0, q + s - q^2 - 1], t \in [0, m - 1] \right\};$$

$$\mathcal{L}_4 := \left\{ -t - rm + m(q+1)s \mid s \in [q^2 - q + 1, q^2 - 2], r \in [q + s - q^2, q], \\ t \in \left[0, ((s-r)q + s)\frac{m - q^2 + q - 1}{q^3} + s - r \right] \right\};$$

$$\mathcal{L}_{5} := \{-t + m(q+1)(q^{2}-1) \mid t \in [q^{3}, m-1]\};$$

$$\mathcal{L}_{6} := \{-t - rm + m(q+1)(q^{2}-1) \mid r \in [1, q-2], t \in [0, m-1]\};$$

$$\mathcal{L}_{7} := \left\{-t - rm + m(q+1)(q^{2}-1) \mid r \in [q-1, q], \\ t \in \left[0, ((q^{2}-1-r)q + q^{2}-1)\frac{m-q^{2}+q-1}{q^{3}} + q^{2}-1 - r\right]\right\}.$$

Then each \mathcal{L}_i is contained in $\mathcal{L}((2g-1)P_0)$.

Proof. By direct computations.

Finally, we can give the description of the Weierstrass semigroup $H(P_0)$. **Proposition 4.3.**⁷

$$\bigcup_{i=1}^{l} \mathcal{L}_i = H(P_0) \cap \{0, \dots, 2g-1\}.$$

Proof. By direct computations, since

$$|\mathcal{L}_1| = \left(\frac{q^4 + 5q^3 + 8q^2 + 4q}{6}\right) \left(\frac{m - q^2 + q - 1}{q^3}\right) + \frac{(q+1)(q+2)(q+3)}{6},$$

$$\begin{aligned} |\mathcal{L}_2| &= \left(\frac{q^6 - q^5 - q^4 - 3q^2 - 2q}{2}\right) \left(\frac{m - q^2 + q - 1}{q^3}\right) + \frac{q^5 - 2q^4 + 2q^3 - q^2 - 6q}{2}, \\ |\mathcal{L}_4| &= \left(\frac{3q^5 + 2q^4 - 20q^3 + q^2 + 8q + 12}{6}\right) \left(\frac{m - q^2 + q - 1}{q^3}\right) + \frac{3q^4 - q^3 - 18q^2 + 22q - 12}{6}, \\ |\mathcal{L}_3| &= \frac{m(q - 2)(q - 1)}{2}, \qquad |\mathcal{L}_5| = m - q^3, \qquad |\mathcal{L}_6| = \sum_{r=1}^{q-2} m = (q - 2)m, \\ |\mathcal{L}_7| &= \frac{m - q^2 + q - 1}{q^3}(2q^3 - q - 2) + 2q^2 - 2q + 1 \\ \text{and } \mathcal{L}_i \cap \mathcal{L}_i = \emptyset \text{ if } i \neq j. \end{aligned}$$

and $\mathcal{L}_i \cap \mathcal{L}_j = \emptyset$ if $i \neq j$.

Let $C_{\ell}(P_0) = C^{\perp}(\tilde{D}, \tilde{\rho}_{\ell}P_0)$, where \tilde{D} is as in (2) and $\tilde{\rho}_{\ell}$ is the ℓ -th positive non-gap at P_0 . In this case it has not been possible to determine $d_{ORD}(C_{\ell}(P_0))$ for any n since we do not have a basis of the Weierstrass semigroup at P_0 . Nevertheless, Tables 1, 2, and ?? give evidence that for some specific values of ℓ the AG codes and AG quantum codes from $C_{\ell}(P_0)$ are better than $C_{\ell}(P_{\infty})$, since the designed relative Singleton defect of $C_{\ell}(P_0)$ is smaller than the one of $C_{\ell}(P_{\infty})$.

$\mathbf{5}$ AG codes on the GGS curve for q = 2 and n = 5

In this section a more detailed description of the results obtained in the previous sections is given for the particular case q = 2, n = 5. Recall that in this case

$$H(P_{\infty}) = \{0, 8, 16, 22, 24, 30, 32, 33, 38, 40, 41, 44, 46, 48, 49, 52\} \cup \{54, \dots, 57\}$$

 $\cup \{60\} \cup \{62, \dots, 66\} \cup \{68\} \cup \{70, \dots, 74\} \cup \{76, \dots, 82\} \cup \{84, \dots, 90\} \cup \{92, \dots\}.$

For the point P_0 (and hence for any \mathbb{F}_{q^2} -rational point), we have from Proposition 4.3

 $H(P_0) = \{0, 21, 22\} \cup \{29, \dots, 33\} \cup \{42, 43, 44\} \cup \{50, \dots, 55\} \cup \{58, \dots, 66\} \cup \{71, \dots, 77\} \cup \{79, \dots\}.$

Table 1 contains the parameters of the codes $C_{\ell_{\infty}}(P_{\infty})$ and $C_{\ell_0}(P_0)$; in particular, their common length N=3968 and dimension k, their Feng-Rao designed minimum distance d_{ORD}^∞ and d_{ORD}^0 , their designed Singleton defects $\delta_{\infty} = N + 1 - k - d_{ORD}^{\infty}$ and $\delta_0 = N + 1 - k - d_{ORD}^0$, and their designed relative Singleton defects $\Delta_{\infty} = \frac{\delta_{\infty}}{N}$ and $\Delta_0 = \frac{\delta_0}{N}$.

N	$_{k}$	ρ_{ℓ_∞}	d^∞_{ORD}	$\delta_{\infty} \leq$	$\Delta_{\infty} \leq$	$ ho_{\ell_0}$	d^0_{ORD}	$\delta_0 \leq$	$\Delta_0 \leq$
3968	3966	8	2	1	0,0003	21	2	1	0,0003
3968	3965	16	2	2	0,0006	22	2	2	0,0006
3968	3964	22	2	3	0,0008	29	2	3	0,0008
3968	3963	24	2	4	0.0011	30	2	4	0.0011
3968	3962	30	2	5	0.0013	31	2	5	0,0013
3968	3961	32	2	6	0,0016	32	2	6	0,016
3968	3960	33	3	6	0,0016	33	3	6	0,016
3968	3959	38	3	7	0,0018	42	3	7	0,018
3968	3958	40	3	8	0,0021	43	3	8	0,021
3968	3957	41	3	9	0,0023	44	3	9	0,023
3968	3956	44	4	9	0.0023	50	3	10	0.026
3968	3955	46	4	10	0,0026	51	3	11	0,028
3968	3954	48	4	11	0,0028	52	3	12	0,031
3968	3953	49	4	12	0.0031	53	3	13	0.033
3968	3952	52	4	13	0,0033	54	3	14	0.036
3968	3951	54	4	14	0,0036	55	3	15	0,038
3968	3950	55	5	14	0,0036	58	4	15	0,038
3968	3949	56	5	15	0,0038	59	5	15	0.038
3968	3948	57	5	16	0,0041	60	5	16	0.041
3968	3947	60	5	17	0,0043	61	5	17	0.043
3968	3946	62	5	18	0,0046	62	5	18	0.046
3968	3945	63	5	19	0,0048	63	5	19	0.048
3968	3944	64	5	20	0,0051	64	5	20	0.051
3968	3943	65	5	21	0.0053	65	5	21	0.053
3968	3942	66	6	21	0,0053	66	6	21	0.053
3968	3941	68	6	22	0,0056	71	6	22	0,056
3968	3940	70	6	23	0,0058	72	6	23	0,058
3968	3939	71	6	24	0,0061	73	6	24	0.061
3968	3938	72	6	25	0,0064	74	6	25	0,064
3968	3937	73	6	26	0,0066	75	6	26	0,066
3968	3936	74	6	27	0,0069	76	6	27	0,069
3968	3935	76	6	28	0,0071	77	6	28	0,0071
3968	3934	77	8	27	0,0069	79	8	27	0,069
3968	3933	78	8	28	0,0071	80	8	28	0,0071
3968	3932	79	8	29	0,0074	81	8	29	0,0074
3968	3931	80	8	30	0,0076	82	8	30	0,0076
3968	3930	81	8	31	0,0079	83	8	31	0,0079
3968	3929	82	8	32	0,0081	84	8	32	0,0081
3968	3928	84	8	33	0,0084	85	8	33	0,0084
3968	3927	85	8	34	0,0086	86	8	34	0,0086
3968	3926	86	8	35	0,0089	87	8	35	0,0089
3968	3925	87	8	36	0,0091	88	8	36	0,0091
3968	3924	88	8	37	0,0094	89	8	37	0,0094
3968	3923	89	8	38	0,0096	90	8	38	0,0096
3968	3922	90	8	39	0,0099	91	8	39	0,0099
3968	3921	92	8	40	0,0101	92	8	40	0,0101
3968	3920	93	8	41	0,0104	93	8	41	0,0104
3968	3919	94	8	42	0,0106	94	8	42	0,0106
3968	3918	95	8	43	0,0109	95	8	43	0,0109
3968	3917	96	8	44	0,0111	96	8	44	0,0111
3968	3916	97	8	45	0,0114	97	8	45	0,0114

Table 1: Codes $C_{\ell_{\infty}}(P_{\infty})$ and $C_{\ell_0}(P_0)$, $q^n = 2^5$

n	k	$ ho_{\ell_{\infty}}$	d_{ORD}^{∞}	$\delta_{\infty} \leq$	$\Delta_{\infty} \leq$	$ ho_{\ell_0}$	d^0_{ORD}	$\delta_o \leq$	$\Delta_0 \leq$
3968	3915	98	8	46	0.0116	98	8	46	0.0116
3968	3914	99	16	39	0.0099	99	9	46	0.0116
3968	3913	100	16	40	0.0101	100	16	40	0.0101
3968	3912	101	16	41	0.0104	101	21	36	0.0091
3968	3911	102	16	42	0.0106	102	22	36	0.0091
3968	3910	102	16	43	0.0109	103	22	37	0.0094
3968	3909	104	16	44	0.0111	104	22	38	0.0096
3968	3908	105	16	45	0.0114	105	22	39	0.0099
3968	3907	106	16	46	0.0116	106	22	40	0.0101
3968	3906	107	22	41	0.0104	107	22	41	0.0104
3968	3905	108	22	42	0.0106	108	22	42	0.0106
3968	3904	109	22	43	0.0109	109	22	43	0.0109
3968	3903	110	22	44	0.0111	110	22	44	0.0111
3968	3902	111	22	45	0.0114	111	26	41	0.0104
3968	3901	112	22	46	0.0116	112	29	39	0.0099
3968	3900	113	24	45	0.0114	113	29	40	0.0101
3968	3899	114	24	46	0.0116	114	29	41	0.0104
3968	3898	115	30	41	0,0104	115	29	42	0.0106
3968	3897	116	30	42	0.0106	116	29	43	0.0109
3968	3896	117	30	43	0.0109	117	29	44	0.0111
3968	3895	118	30	44	0,0111	118	29	45	0.0114
3968	3894	119	30	45	0,0114	119	29	46	0,0116
3968	3893	120	30	46	0.0116	120	30	46	0.0116
3968	3892	121	32	45	0,0114	121	31	46	0,0116
3968	3891	122	32	46	0,0116	122	36	42	0,0106
3968	3890	123	33	46	0,0116	123	37	42	0,0106
3968	3889	124	38	42	0,0106	124	37	43	0,0109
3968	3888	125	38	43	0,0109	125	37	44	0,0111
3968	3887	126	38	44	0,0111	126	37	45	0,0114
3968	3886	127	38	45	0,0114	127	37	46	0,0116
3968	3885	128	38	46	0,0116	128	38	46	0,0116
3968	3884	129	40	45	0,0114	129	39	46	0,0116
3968	3883	130	40	46	0,0116	130	40	46	0,0116
3968	3882	131	41	46	0,0116	131	41	46	0,0116
3968	3881	132	44	44	0,0111	132	42	46	0,0116
3968	3880	133	44	45	0,0114	133	46	43	0,0109
3968	3879	134	44	46	0,0116	134	48	42	0,0106
3968	3878	135	46	45	0,0114	135	48	43	0,0109
3968	3877	136	46	46	0,0116	136	50	42	0,0106
3968	3876	137	48	45	0,0114	137	50	43	0,0109
3968	3875	138	48	46	0,0116	138	50	44	0,0111
3968	3874	139	49	46	0,0116	139	50	45	0,0114
3968	3873	140	52	44	0,0111	140	50	46	0,0116
3968	3872	141	52	45	0,0114	141	51	46	0,0116
3968	3871	142	52	46	$0,0\overline{116}$	142	52	46	$0,0\overline{116}$
3968	3870	143	54	45	$0,0\overline{114}$	143	53	46	$0,0\overline{116}$
3968	3869	144	54	46	$0,0\overline{116}$	144	56	44	$0,0\overline{111}$
3968	3868	145	55	46	$0,0\overline{116}$	145	57	44	$0,0\overline{111}$
3968	3867	146	56	46	0,0116	146	58	44	$0,011\overline{1}$
3968	3866	147	57	46	$0,0\overline{116}$	147	58	45	$0,0\overline{114}$
3968	3865	148	60	44	0,0111	148	58	46	$0,011\overline{6}$
3968	3864	149	60	45	0,0114	149	59	46	$0,011\overline{6}$

Table 1 : continued from previous page

n	k	$ ho_{\ell_{\infty}}$	d^{∞}_{ORD}	$\delta_{\infty} \leq$	$\Delta_{\infty} \leq$	$ ho_{\ell_0}$	d^0_{ORD}	$\delta_o \leq$	$\Delta_0 \leq$
3968	3863	150	60	46	0,0116	150	60	46	0,0116
3968	3862	151	62	45	0,0114	151	61	46	0,0116
3968	3861	152	62	46	0,0116	152	62	46	0,0116
3968	3860	153	63	46	0,0116	153	63	46	0,0116
3968	3859	154	64	46	0,0116	154	64	46	0,0116
3968	3858	155	65	46	0,0116	155	66	45	0,0114
3968	3857	156	66	46	0,0116	156	66	46	0,0116
3968	3856	157	68	45	0,0114	157	67	46	0,0116
3968	3855	158	68	46	0,0116	158	68	46	0,0116
3968	3854	159	70	45	0,0114	159	69	46	0,0116
3968	3853	160	70	46	0,0116	160	70	46	0,0116
3968	3852	161	71	46	0,0116	161	71	46	0,0116
3968	3851	162	72	46	0,0116	162	72	46	0,0116
3968	3850	163	73	46	0,0116	163	73	46	0,0116
3968	3849	164	74	46	0,0116	164	74	46	0,0116
3968	3848	165	76	45	0,0114	165	75	46	0,0116
3968	3847	166	76	46	0,0116	166	76	46	0,0116
3968	3846	167	77	46	0,0116	167	77	46	0,0116
3968	3845	168	78	46	0,0116	168	78	46	0,0116
3968	3844	169	79	46	0,0116	169	79	46	0,0116
3968	3843	170	80	46	0,0116	170	80	46	0,0116
3968	3842	171	81	46	0,0116	171	81	46	0,0116
3968	3841	172	82	46	0,0116	172	82	46	0,0116
3968	3840	173	84	45	0,0114	173	83	46	0,0116
3968	3839	174	84	46	0,0116	174	84	46	0,0116
3968	3838	175	85	46	0,0116	175	85	46	0,0116
3968	3837	176	86	46	0,0116	176	86	46	0,0116
3968	3836	177	87	46	0,0116	177	87	46	0,0116
3968	3835	178	88	46	0,0116	178	88	46	0,0116
3968	3834	179	89	46	$0,011\overline{6}$	179	89	46	0,0116
3968	3833	180	90	46	$0,011\overline{6}$	180	90	46	0,0116
3968	3832	181	92	45	$0,011\overline{4}$	181	91	46	0,0116
3968	3831	182	92	46	$0,011\overline{6}$	182	92	46	0,0116
3968	$39\overline{68} - \ell_{\infty}$	$\rho_{\ell_{\infty}} \ge 183$	$\ell_{\infty} - 45$	46	0,0116	$\rho_{\ell_0} \ge 183$	$\ell_0 - 45$	46	0,0116

Table 1 : continued from previous page

Table 2 provides some examples in which codes of type $C_{\ell_0}(P_0)$ have better parameters than codes of type $C_{\ell_{\infty}}(P_{\infty})$. In particular, the length n of the two codes is 3968, the dimension k_0 and the Feng-Rao designed minimum distance d_{ORD}^0 of $C_{\ell_0}(P_0)$ are greater than or equal to the corresponding parameters k_{∞} and d_{ORD}^{∞} of $C_{\ell_{\infty}}(P_{\infty})$, and the designed Singleton defect $\delta_0 = n + 1 - k_0 - d_{ORD}^0$ of $C_{\ell_0}(P_0)$ is strictly smaller than the designed Singleton defect $\delta_{\infty} = n + 1 - k_{\infty} - d_{ORD}^\infty$ of $C_{\ell_{\infty}}(P_{\infty})$.

ℓ_0	3	4	5	6	8	9	10	19	20	21	22	23	24	26	27
ℓ_{∞}	4	5	6	7	9	10	11	20	21	22	23	24	25	27	28
$\delta_{\infty} - \delta_0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ℓ_0	28	29	30	31	32	34	35	36	37	38	39	40	41	42	43
ℓ_{∞}	29	30	31	32	33	35	36	37	38	39	40	41	42	43	44
$\delta_{\infty} - \delta_0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ℓ_0	44	45	46	47	48	49	50	51	52	55	56	56	57	57	58
ℓ_{∞}	45	46	47	48	49	50	51	52	53	56	56	57	57	58	58
$\delta_{\infty} - \delta_0$	1	1	1	1	1	1	1	1	1	1	5	6	6	7	6
*	-	-	-	-	-	-	-	-	-	-	ÿ	Ů	Ŭ	•	÷
ℓ_0	58	59	59	60	60	61	61	62	63	64	65	66	66	67	67
$\frac{\ell_0}{\ell_\infty}$	58 59	59 59	59 60	60 60	60 61	61 61	61 62	62 63	63 64	64 65	65 66	66 66	66 67	67 67	67 68
$\frac{\ell_0}{\ell_\infty}$ $\frac{\delta_\infty - \delta_0}{\delta_\infty - \delta_0}$	58 59 7	59 59 6	59 60 7	60 60 6	60 61 7	61 61 6	61 62 1	62 63 1	63 64 1	64 65 1	65 66 1	66 66 4	66 67 6	67 67 7	67 68 6
$ \begin{array}{c} \ell_0 \\ \ell_\infty \\ \delta_\infty - \delta_0 \\ \hline \ell_0 \end{array} $	58 59 7 68	59 59 6 68	59 60 7 69	60 60 6 77	60 61 7 77	61 61 6 78	61 62 1 88	62 63 1 88	63 64 1 89	64 65 1 89	65 66 1 90	66 66 4 90	66 67 6 91	67 67 7 91	67 68 6
$ \begin{array}{c} \ell_{0} \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \hline \ell_{0} \\ \ell_{\infty} \\ \end{array} $	58 59 7 68 68	59 59 6 68 69	59 60 7 69 69	60 60 6 77 77	60 61 7 77 78	61 61 6 78 78	61 62 1 88 88	62 63 1 88 89	63 64 1 89 89	64 65 1 89 90	65 66 1 90 90	66 66 4 90 91	66 67 6 91 91	67 67 7 91 92	67 68 6
$\begin{array}{c} \ell_{0} \\ \ell_{0} \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \hline \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \end{array}$	58 59 7 68 68 5	59 59 6 68 69 4	$59 \\ 60 \\ 7 \\ 69 \\ 69 \\ 5 \\ 5$		60 61 7 77 78 4	61 61 6 78 78 4	61 62 1 88 88 2	62 63 1 88 89 3	63 64 1 89 89 4	64 65 1 89 90 3	65 66 1 90 90 2	66 66 4 90 91 3	$ \begin{array}{c} 66\\ 67\\ 6\\ 91\\ 91\\ 4 \end{array} $	67 67 7 91 92 3	67 68 6
$ \begin{array}{c} \ell_{0} \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \hline \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \hline \ell_{0} \end{array} $	58 59 7 68 68 5 92	59 59 6 68 69 4 92	59 60 7 69 69 5 93			61 61 6 78 78 4 99	61 62 1 88 88 2 99	62 63 1 88 89 3 100	63 64 1 89 89 4 100	64 65 1 89 90 3 101	65 66 1 90 90 2 101	66 66 4 90 91 3 102	$ \begin{array}{c} 66\\ 67\\ 6\\ 91\\ 91\\ 4\\ 110\\ \end{array} $	67 67 7 91 92 3 110	67 68 6
$ \begin{array}{c} \ell_{0} \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \end{array} $ $ \begin{array}{c} \ell_{0} \\ \ell_{\infty} \\ \delta_{\infty} - \delta_{0} \\ \end{array} $ $ \begin{array}{c} \ell_{0} \\ \ell_{\infty} \\ \ell_{\infty} \\ \end{array} $	58 59 7 68 68 5 92 92	59 59 6 68 69 4 92 93	59 60 7 69 69 5 93 93	60 60 6 77 77 4 93 94	60 61 7 77 78 4 94 94	61 61 6 78 78 78 4 99 99	61 62 1 888 88 2 999 100	62 63 1 88 89 3 100 100	63 64 1 89 89 4 100 101	64 65 1 89 90 3 101 101	65 66 1 90 90 2 101 102	66 66 4 90 91 3 102 102	66 67 6 91 91 4 110 110	67 67 7 91 92 3 110 111	67 68 6

Table 2: Designed Singleton defect of $C_{\ell_0}(P_0)$ and $C_{\ell_{\infty}}(P_{\infty})$, $q^n = 2^5$

6 Quantum codes from one-point AG codes on the GGS curves

In this section we use families of one-point AG codes from the GGS curve to construct quantum codes. The main ingredient is the so called *CSS contruction* which enables to construct quantum codes from classical linear codes; see [29, Lemma 2.5].

We denote by q a prime power. A q-ary quantum code Q of length N and dimension k is defined to be a q^k -dimensional Hilbert subspace of a q^N -dimensional Hilbert space $\mathbb{H} = (\mathbb{C}^q)^{\otimes n} = \mathbb{C}^q \otimes \cdots \otimes \mathbb{C}^q$. If Q has minimum distance D, then Q can correct up to $\lfloor \frac{D-1}{2} \rfloor$ quantum errors. The notation $[[N, k, D]]_q$ is used to denote such a quantum code Q. For a $[[N, k, D]]_q$ -quantum code the quantum Singleton bound holds, that is, the minimum distance satisfies $D \leq 1 + (N-k)/2$. The quantum Singleton defect is $\delta^Q := N - k - 2D + 2 \geq 0$, and the relative quantum Singleton defect is $\Delta^Q := \delta^Q/N$. If $\delta^Q = 0$, then the code is said to be quantum MDS. For a detailed introduction on quantum codes see [29] and the references therein.

Lemma 6.1. (CSS construction) Let C_1 and C_2 denote two linear codes with parameters $[N, k_i, d_i]_q$, i = 1, 2, and assume that $C_1 \subset C_2$. Then there exists an $[[N, k_2 - k_1, D]]_q$ code with $D = \min\{wt(c) \mid c \in (C_2 \setminus C_1) \cup (C_1^{\perp} \setminus C_2^{\perp})\}$, where wt(c) is the weight of c.

We consider the following general t-point construction due to La Guardia and Pereira;

see [29, Theorem 3.1]. It is a direct application of Lemma 6.1 to AG codes.

Lemma 6.2. (General t-point construction) Let \mathcal{X} be a nonsingular curve over \mathbb{F}_q with genus g and N + t distinct \mathbb{F}_q -rational points, for some N, t > 0. Assume that $a_i, b_i, i = 1, \ldots, t$, are positive integers such that $a_i \leq b_i$ for all i and $2g - 2 < \sum_{i=1}^t a_i < \sum_{i=1}^t b_i < N$. Then there exists a quantum code with parameters $[[N, k, D]]_q$ with $k = \sum_{i=1}^t b_i - \sum_{i=1}^t a_i$ and $D \geq \min \{N - \sum_{i=1}^t b_i, \sum_{i=1}^t a_i - (2g - 2)\}$.

Let $n \ge 5$ be an odd integer. We apply Lemma 6.2 to one-point codes on the GGS curve. **Proposition 6.3.** Let $a, b \in \mathbb{N}$ be such that

$$(q-1)(q^{n+1}+q^n-q^2)-2 < a < b < q^{2n+2}-q^{n+3}+q^{n+2}.$$

Then there exists a quantum code with parameters $[[N, b - a, D]]_{q^{2n}}$, where

$$N = q^{2n+2} - q^{n+3} + q^{n+2},$$

$$D \ge \min \left\{ q^{2n+2} - q^{n+3} + q^{n+2} - b, a - (q-1)(q^{n+1} + q^n - q^2) + 2 \right\}.$$

Proof. Let GGS(q, n) be the GGS curve with equations (1), genus g, and infinite point P_{∞} . Consider the divisors \overline{D} as in (2), $G_1 = aP_{\infty}$, and $G_2 = bP_{\infty}$. Note that $\operatorname{supp}(G_1) \cap \operatorname{supp}(\overline{D}) = \operatorname{supp}(G_2) \cap \operatorname{supp}(\overline{D}) = \emptyset$. From Lemma 6.2, there exists a quantum code with parameters $[[N, b - a, D]]_{q^{2n}}$, where $D \ge \min \{N - b, a - (2g - 2)\} = \min \{q^{2n+2} - q^{n+3} + q^{n+2} - b, a - (q-1)(q^{n+1} + q^n - q^2) - 2\}$.

Another application of the CSS construction can be obtained looking at the dual codes of the one-point codes from the GGS curve. Let $P \in GGS(q, n)$. Fix $a = \rho_{\ell} \in H(P)$ and $b = \rho_{\ell+s} \in H(P)$ with $C_2 = C_{\ell}(P) = C_{\ell}$ and $C_1 = C_{\ell+s}(P) = C_{\ell+s}$, where $s \ge 1$. Clearly $C_1 \subset C_2$, as $C_{\ell} \subsetneq C_{\ell+s}$ for every $s \ge 1$. The dimensions of C_2 and C_1 are $k_2 = N - h_{\ell}$ and $k_1 = N - h_{\ell+s} = N - h_{\ell} - s$ respectively, where h_i denotes the number of non-gaps at Pwhich are smaller than or equal to i. Thus, $k_2 - k_1 = s$. According to the CSS construction, these choices induce an $[[N, s, D]]_{q^{2n}}$ quantum code, where $N = q^{2n+2} - q^{n+3} + q^{n+2}$ and D = $\min\{wt(c) \mid c \in (C_2 \setminus C_1) \cup (C_1^{\perp} \setminus C_2^{\perp})\} = \min\{wt(c) \mid c \in (C_\ell \setminus C_{\ell+s}) \cup (C(D, G_1) \setminus C(D, G_2))\}$, with $G_2 = \rho_{\ell}P$ and $G_1 = \rho_{\ell+s}P$. In particular,

$$D \ge \min\{d_{ORD}(C_\ell), d_1\},\tag{9}$$

where d_1 denotes the minimum distance of the code $C(D, G_1)$. Following this construction and using an improvement of Inequality (9), the next theorem is obtained. **Theorem 6.4.** Let $g = (q-1)(q^{n+1} + q^n - q^2)/2$ and $N = q^{2n+2} - q^{n+3} + q^{n+2}$. For every $\ell \in [3g-1, N-g]$ and $s \in [1, N-2\ell]$, there exists a quantum code with parameters $[[N, s, D]]_{q^{2n}}$, where $D \ge \ell + 1 - g$.

Proof. Since $\ell \geq 3g - 1$, we have $\rho_{\ell+s} = g - 1 + \ell + s$, and hence $d_1 \geq N - \deg(G_1) = N - \rho_{\ell+s} = N - \ell - s - g + 1$. From Theorem 3.15, $d_{ORD}(C_\ell) = \ell + 1 - g$. Thus, $D \geq \min\{d_{ORD}(C_\ell), d_1\} = \ell + 1 - g$. The claim follows.

For fixed q, we can construct as a direct consequence of Theorem 6.4 families of quantum codes depending on n such that their relative quantum Singleton defect goes to zero as n goes to infinity. An example is the following.

Corollary 6.5. Let $g = (q-1)(q^{n+1}+q^n-q^2)/2$ and $N = q^{2n+2}-q^{n+3}+q^{n+2}$. For every $\ell \in [3g-1, N-g]$, fix $s = N-2\ell$. Then there exists a quantum code with parameters $[[N, s, D]]_{q^{2n}}$ with $D \ge \ell + 1 - g$, whose relative quantum Singleton defect $\Delta_n^Q = (N - s - 2D + 2)/N$ satisfies

$$\Delta_n^Q \le \frac{2g}{N} = \frac{(q-1)(q^{n+1}+q^n-q^2)}{q^{2n+2}-q^{n+3}+q^{n+2}}.$$

Hence, $\lim_{n\to\infty} \Delta_n^Q = 0$.

Using the computation of $d_{ORD}(C_{\ell}(P_{\infty}))$ in Section 3, we produce infinite families of quantum codes in which the lower bound in (9) is explicitly determined. We look at those cases for which (9) reads $D \ge d_{ORD}(C_{\ell}(P_{\infty})) > \ell + 1 - g$ and this bound is better than the one stated in Theorem 6.4. According to Proposition 3.14, we choose $\rho_{\ell} \in H(P_{\infty})$ such that $\rho_{\ell+1} = (0, 1, k)$ for some $k \in [m, 2m)$.

Proposition 6.6. Let $q = 2^n$ for $n \ge 5$ odd, $g = (q-1)(q^{n+1} + q^n - q^2)/2$, and $N = q^{2n+2} - q^{n+3} + q^{n+2}$. Let $\ell \in [g, 3g-1]$ be such that $\rho_{\ell+1} \in H(P_{\infty})$ is of type (0, 1, k) for some $k \in [m, 2m)$. Let $s \in [1, N - 2\ell - 5]$. Then there exists a quantum code with parameters $[[N, s, D]]_{q^{2n}}$ where

$$D \ge \ell + 1 - g + \begin{cases} 5, & \text{if } k < m \text{ or } m \le k < \frac{9m - 11}{8}, \\ 3, & \text{if } \frac{9m - 11}{8} \le k < \frac{11m - 9}{8}, \\ 1, & \text{if } \frac{11m - 9}{8} \le k. \end{cases}$$

Proof. Arguing as in the proof of Theorem 6.4, we have that $d_1 \ge N - \ell - s - g + 1$. Thus, from Proposition 3.16 and Lemma 3.9, Inequality (9) reads

$$D \ge d_{ORD}(C_{\ell}(P_{\infty})) = \begin{cases} 8k - 7m + 13, & \text{if } k < m \text{ or } m \le k < \frac{9m - 11}{8}, \\ 8k - 7m + 11, & \text{if } \frac{9m - 11}{8} \le k < \frac{11m - 9}{8}, \\ 8k - 7m + 9, & \text{if } \frac{11m - 9}{8} \le k. \end{cases}$$

Since $\ell + 1 - g = \rho_{\ell+1} - 2g + 1 = 2m + 8k - (9m - 7) + 1 = 8k - 7m + 8$, the claim follows.

7 Convolutional codes from one-point AG codes on the GGS curves

In this section we use a result due to De Assis, La Guardia, and Pereira [6] which allows to construct unit-memory convolutional codes with certain parameters $(N, k, \gamma; m, d_f)_q$ starting from AG codes.

Consider the polynomial ring $R = \mathbb{F}_q[X]$. A convolutional code C is an R-submodule of rank k of the module R^N . Let $G(X) = (g_{ij}(X)) \in \mathbb{F}_q[X]^{k \times N}$ be a generator matrix of C over $\mathbb{F}_q[X], \gamma_i = \max\{\deg g_{ij}(X) \mid 1 \leq j \leq N\}, \gamma = \sum_{i=1}^k \gamma_i, m = \max\{\gamma_i \mid 1 \leq i \leq k\}, \text{ and } d_f$ be the minimum weight of a word $c \in C$. Then we say that C has length N, dimension k, degree γ , memory m, and free distance. If m = 1, C is said to be a unit-memory convolutional code. In this case we use for C the notation $(N, k, \gamma; m, d_f)_q$. For a detailed introduction on convolutional codes see [6,35] and the references therein.

Lemma 7.1. ([6, Theorem 3]) Let \mathcal{X} be a nonsingular curve over \mathbb{F}_q with genus g. Consider an AG code $C^{\perp}(D,G)$ with $2g - 2 < \deg(G) < N$. Then there exists a unit-memory convolutional code with parameters $(N, k - \ell, \ell; 1, d_f \ge d)_q$, where $\ell \le k/2$, $k = \deg(G) + 1 - g$ and $d \ge N - \deg(G)$.

We apply Lemma 7.1 to one-point AG codes from the GGS curve.

Proposition 7.2. Consider the $\mathbb{F}_{q^{2n}}$ -maximal GGS curve GGS(q, n) and let $\rho_{\ell} \in H(P_{\infty})$ be such that $(q-1)(q^{n+1}+q^n-q^2)-2 < \rho_{\ell} < N$, where $N = q^{2n+2}-q^{n+3}+q^{n+2}$. Then there exists a unit-memory convolutional code with parameters $(N, k-s, s; 1, d_f \ge d_{ORD}(C_{\ell}(P_{\infty})))$, where $k = \rho_{\ell} + 1 - \frac{(q-1)(q^{n+1}+q^n-q^2)}{2}$ and $s \le k/2$.

Proof. The result follows from Lemma 7.1. The inequality $d_f \ge d_{ORD}(C_\ell(P_\infty))$ follows from $d_f \ge d$ and Theorem 3.15 applied to the dual code $C_\ell(P_\infty)$.

In particular, Theorem 3.15 yields the following corollary.

Corollary 7.3. Consider the $\mathbb{F}_{q^{2n}}$ -maximal GGS curve GGS(q,n) and let $\rho_{\ell} \in H(P_{\infty})$ be such that $(q-1)(q^{n+1}+q^n-q^2)-2 < \rho_{\ell} < N$, where $N = q^{2n+2}-q^{n+3}+q^{n+2}$ and $\ell \geq 3\frac{(q-1)(q^{n+1}+q^n-q^2)}{2}$. Then there exists a unit-memory convolutional code with parameters $(N, k - s, s; 1, d_f)$, where $k = \rho_{\ell} + 1 - \frac{(q-1)(q^{n+1}+q^n-q^2)}{2}$, $s \leq k/2$, and $d_f \geq \ell + 1 - \frac{(q-1)(q^{n+1}+q^n-q^2)}{2}$.

8 The Automorphism group of $C(\overline{D}, \ell P_{\infty})$

In this section we investigate the automorphism group of the code $C(\overline{D}, \ell P_{\infty})$, where \overline{D} is as in (2).

Lemma 8.1. The automorphism group $\operatorname{Aut}(GGS(q,n))$ has exactly two short orbits on GGS(q,n); one consists of P_{∞} , the other consists of the $q^3 \mathbb{F}_{q^2}$ -rational points other than P_{∞} .

Proof. From [15, 16], Aut $(GGS(q, n)) = Q \rtimes \Sigma$, where $Q = \{Q_{a,b} \mid a, b \in \mathbb{F}_{q^2}, a^q + a = b^{q+1}\}$ and $\Sigma = \langle g_{\zeta} \rangle$, with

$$Q_{a,b} = \begin{pmatrix} 1 & b^{q} & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_{\zeta} = \begin{pmatrix} \zeta^{q^{n}+1} & 0 & 0 & 0 \\ 0 & \zeta^{\frac{q^{n}+1}{q+1}} & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
(10)

 ζ a primitive $(q^n + 1)(q - 1)$ -th root of unity. Therefore, $\operatorname{Aut}(GGS(q, n))$ fixes P_{∞} . Also, $\operatorname{Aut}(GGS(q, n))$ acts transitively on the q^3 affine points of GGS(q, n) having zero Z-coordinate, which coincide with the \mathbb{F}_{q^2} -rational points of GGS(q, n) other than P_{∞} .

Suppose $\operatorname{Aut}(GGS(q, n))$ has another short orbit \mathcal{O} . Since GGS(q, n) has zero *p*-rank and $\operatorname{Aut}(GGS(q, n))$ fixes P_{∞} , \mathcal{O} is tame. Hence, by Schur-Zassenhaus Theorem [34, Theorem 9.19], the stabilizer of a point $P \in \mathcal{O}$ is contained up to conjugation in Σ . This is a contradiction, as Σ acts semiregularly out of the plane Z = 0.

Note from (10) that $\operatorname{Aut}(GGS(q, n))$ is defined over $\mathbb{F}_{q^{2n}}$. Let π_a be the plane Z = a. The points of $\pi_0 \cap GGS(q, n)$ are exactly the $q^3 + 1 \mathbb{F}_{q^2}$ -rational points of GGS(q, n), while all coordinates of any point of $GGS(q, n) \setminus \pi_0$ are not in \mathbb{F}_{q^2} . The group Σ fixes all points in $\pi_0 \cap GGS(q, n)$ and acts semiregularly on the planes π_a , while the group Q acts transitively on $\pi_0 \cap GGS(q, n)$ and fixes $GGS(q, n) \cap \pi_a$ for all a. Also, Q acts faithfully on the Hermitian curve $\mathcal{H}_q : Y^{q+1} = X^q + X$ by $(X, Y, T) \mapsto \overline{Q} \cdot (X, Y, T)$, where \overline{Q} is obtained from Q deleting the third row and column.

Proposition 8.2. The automorphism group of $C(\overline{D}, \ell P_{\infty})$ contains a subgroup isomorphic to

$$(\operatorname{Aut}(GGS(q,n)) \rtimes \operatorname{Aut}(\mathbb{F}_{q^{2n}})) \rtimes \mathbb{F}_{q^{2n}}^*.$$

Proof. The set S_{σ} of points of GGS(q, n) fixed by a non-trivial automorphism σ of $\operatorname{Aut}_{\mathbb{F}_{q^{2n}}}(GGS(q, n)) = \operatorname{Aut}(GGS(q, n))$ has size $N_{\sigma} \leq q^3 + 1$. In fact, if $\sigma \notin Q$, then $S_{\sigma} \subseteq \pi_0$. If $\sigma \in Q$, then from $\sigma(P_{\infty}) = P_{\infty}$ we have that the induced automorphism $\bar{\sigma} \in \operatorname{Aut}(\mathcal{H}_q)$ fixes only \mathbb{F}_{q^2} -rational

points of \mathcal{H}_q ; hence, σ fixes only \mathbb{F}_{q^2} -rational points of GGS(q, n), that is, $S_{\sigma} \subseteq \pi_0$. Since $|GGS(q, n) \cap \pi_0| = q^3 + 1$, $N_{\sigma} \leq q^3 + 1$. Now the claim follows from [1, Proposition 2.3]. \Box

Proposition 8.3. If $q^n + 1 \leq \ell \leq q^{n+2} - q^3$ and $\{\ell, \ell - 1\} \subset H(P_{\infty})$, then

$$\operatorname{Aut}(C(\overline{D}, \ell P_{\infty})) \cong (\operatorname{Aut}(GGS(q, n)) \rtimes \operatorname{Aut}(\mathbb{F}_{q^{2n}})) \rtimes \mathbb{F}_{q^{2n}}^*$$

Proof. We apply [11, Theorem 3.4].

- The divisor $G = \ell P_{\infty}$ is effective.
- A plane model of degree $q^n + 1$ for GGS(q, n) is

$$\Pi(GGS(q,n)): \quad Z^{q^{n}+1} = X^{q^{3}} + X - (X^{q} + X)^{q^{2}-q+1}.$$
(11)

In fact, $Z^{m(q+1)} = Y^{q+1}h(X)^{q+1} = X^{q^3} + X - (X^q + X)^{q^2 - q + 1}$; also, Equation (11) is irreducible since it defines a Kummer extension $\mathbb{K}(x, z)/\mathbb{K}(x)$ totally ramified over the pole of x. Therefore, $\mathbb{K}(GGS(q, n)) = \mathbb{K}(x, z)$, and $x, z \in \mathcal{L}(G)$ from the assumption $\ell \ge q^n + 1$.

- The support of D is preserved by the Frobenius morphism $\varphi : (x, z) \mapsto (x^p, z^p)$, since $\varphi(P_{\infty}) = P_{\infty}$ and $\operatorname{supp}(D) = GGS(q, n)(\mathbb{F}_{q^{2n}}) \setminus \{P_{\infty}\}.$
- Let N be the length of $C(\overline{D}, \ell P_{\infty})$. Then the condition $N > \deg(G) \cdot \deg(\Pi(GGS(q, n)))$ reads

$$q^{2n+2} - q^{n+3} + q^{n+2} > \ell(q^n + 1),$$

which is implied by the assumption $\ell \leq q^{n+2} - q^3$.

- - If $P = P_{\infty}$, then $\mathcal{L}(G) \neq \mathcal{L}(G P)$ since $\ell \in H(P_{\infty})$.
 - If $P \neq P_{\infty}$, then $1 \in \mathcal{L}(G) \setminus \mathcal{L}(G P)$.

- If
$$P = Q = P_{\infty}$$
, then $\mathcal{L}(G - P) \neq \mathcal{L}(G - P - Q)$ since $\ell - 1 \in H(P_{\infty})$.

- If $P = P_{\infty}$ and $Q \neq P_{\infty}$, then $1 \in \mathcal{L}(G P) \setminus \mathcal{L}(G P Q)$.
- If $P \neq P_{\infty}$ and $Q = P_{\infty}$, then $f \mu \in \mathcal{L}(G P) \setminus \mathcal{L}(G P Q)$, where $f \in \mathcal{L}(G)$ has pole divisor ℓP_{∞} and $\mu = f(P)$.
- If $P, Q \neq P_{\infty}$ and $P \neq Q$, choose f = z z(P) or f = x x(P) according to $z(P) \neq z(Q)$ or $x(P) \neq x(Q)$; then $f \in \mathcal{L}(G P) \setminus \mathcal{L}(G P Q)$.

- If
$$P = Q \neq P_{\infty}$$
, then $z - z(P) \in \mathcal{L}(G - P) \setminus \mathcal{L}(G - P - Q)$.

Thus we can apply [11, Theorem 3.4] to prove the claim.

References

- D. Bartoli, M. Montanucci, G. Zini, "Multi-Point AG Codes on the GK Maximal Curve," Des. Codes Cryptogr., to appear. DOI 10.1007/s10623-017-0333-9
- [2] A. Campillo and J. I. Farrán, "Computing Weierstrass semigroups and the Feng-Rao distance from singular plane models," preprint, arXiv: 9910155.
- [3] A. S. Castellanos and G. C. Tizziotti, "Two-Point AG Codes on the GK Maximal Curves," *IEEE Trans. Inf. Theory*, vol. 62, no. 2, pp. 681–686, 2016.
- [4] C. Carvalho and T. Kato, "On Weierstrass semigroups and sets: A review with new results," *Geometriae Dedicata*, vol. 139, no. 1, pp. 195–210, 2009.
- [5] C. Carvalho and F. Torres, "On Goppa Codes and Weierstrass Gaps at Several Points," Des. Codes Cryptogr., vol. 35, no. 2, pp. 211–225, 2005.
- [6] F. M. de Assis, G. G. La Guardia and F. R. F. Pereira, "New Convolutional Codes Derived from Algebraic Geometry Codes," preprint, arXiv: 1612.07157.
- [7] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields," Ann. Math., vol. 103, pp. 103–161, 1976.
- [8] I. Duursma and R. Kirov, "Improved Two-Point Codes on Hermitian Curves," IEEE Trans. Inform. Theory, vol. 57, no. 7, pp. 4469–4476, 2011.
- [9] S. Fanali and M. Giulietti, "One-Point AG Codes on the GK Maximal Curves," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 202–210, 2010.
- [10] A. Garcia, C. Güneri, and H. Stichtenoth, "A generalization of the Giulietti-Korchmáros maximal curve," Advances in Geometry, vol. 10, no. 3, pp. 427–434, 2010.
- [11] M. Giulietti and G. Korchmáros, "On automorphism groups of certain Goppa codes," Des. Codes Cryptogr., vol 47, pp. 177–190, 2008.
- [12] M. Giulietti and G. Korchmáros, "A new family of maximal curves over a finite field," Math. Ann., vol. 343, pp. 229–245, 2009.
- [13] V. D. Goppa, "Codes on algebraic curves," Dokl. Akad. NAUK, SSSR, vol. 259, no. 6, pp. 1289–1290, 1981.
- [14] V. D. Goppa, "Algebraico-geometric codes," Izv. Akad. Nauk SSSR Ser. Mat., vol. 46, no. 4, pp. 75–91, 1982.

- [15] C. Güneri, M. Özdemir, and H. Stichtenoth," The automorphism group of the generalized Giulietti-Korchmáros function field," Adv. Geom., vol. 13, pp. 369–380, 2013.
- [16] R. Guralnick, B. Malmskog, and R. Pries, "The automorphism group of a family of maximal curves, J. Algebra, vol. 361, pp. 92–106, 2012.
- [17] C. Heegard, J. Little, and K. Saints, "Systematic encoding via Gröbner bases for a class of algebraic-geometric Goppa codes," *IEEE Trans. Inf. Theory*, vol. 41, pp. 1752–1761, 1995.
- [18] J.W.P. Hirschfeld, G. Korchmáros, and F. Torres, Algebraic curves over a finite field, Princeton Univ. Press, 2008.
- [19] T. Høholdt, J. H. van Lint, and R. Pellikaan, "Algebraic geometry codes," in Handbook of Coding Theory, V. S. Pless, W. C. Huffman, and R. A. Brualdi, Eds. Amsterdam, The Netherlands: Elsevier, 1998, vol. 1, pp. 871–961.
- [20] M. Homma, "The Weierstrass semigroup of a pair of points on a curve," Arch. Math., vol. 67, pp. 337–348, 1996.
- [21] M. Homma and S. J. Kim, "Goppa codes with Weierstrass pairs," J. Pure Appl. Algebra, vol. 162, pp. 273–290, 2001.
- [22] M. Homma and S. J. Kim, "Toward the Determination of the Minimum Distance of Two-Point Codes on a Hermitian Curve," *Des. Codes Cryptogr.*, vol. 37, no. 1, pp. 111–132, 2005.
- [23] M. Homma and S. J. Kim, "The complete determination of the minimum distance of two-point codes on a Hermitian curve," *Des. Codes Cryptogr.*, vol. 40, no. 1, pp. 5–24, 2006.
- [24] M. Homma and S. J. Kim, "The Two-Point Codes on a Hermitian Curve with the Designed Minimum Distance," Des. Codes Cryptogr., vol. 38, no. 1, pp. 55–81, 2006.
- [25] D. Joyner, "An error-correcting codes package," SIGSAM Comm. Computer Algebra, vol. 39, no. 2, pp. 65–68, 2005.
- [26] D. Joyner and A. Ksir, "Automorphism groups of some AG codes," IEEE Trans. Inf. Theory, vol. 52, no. 7, pp. 3325–3329, 2006.
- [27] S. J. Kim, "On the index of the Weierstrass semigroup of a pair of points on a curve," Arch. Math., vol. 62, no. 1, pp. 73–82, 1994.

- [28] C. Kirfel and R. Pellikaan, "The minimum distance of codes in an array coming from telescopic semigroups," *IEEE Trans. Inf. Theory*, vol. 41, pp. 1720–1732, 1995.
- [29] G. G. La Guardia, F. R. F. Pereira, "Good and asymptotically good quantum codes derived from Algebraic geometry codes," preprint, arXiv:1612.07150.
- [30] B. Lundell and J. McCullough, "A generalized floor bound for the minimum distance of geometric Goppa codes," J. Pure Appl. Algebra, vol. 207, pp. 155–164, 2006.
- [31] G. L. Matthews, "Weierstrass Pairs and Minimum Distance of Goppa Codes," Des. Codes Cryptogr., vol. 22, pp. 107–121, 2001.
- [32] G. L. Matthews, "Codes from the Suzuki function field," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3298–3302, 2004.
- [33] S. Meagher and J. Top, "Twists of genus three curves over finite fields," *Finite Fields Appl.*, vol 16, pp. 347–368, 2010.
- [34] H.E. Rose, A Course on Finite Groups, Springer Science and Business Media, London (2009).
- [35] J. Rosenthal and R. Smarandache, "Maximum Distance Separable Convolutional Codes," Appl. Algebra Engrg. Comm. Comput., vol. 10, pp. 15–32, 1999.
- [36] A. Sepúlveda and G. Tizziotti, "Weierstrass semigroup and codes over the curve $y^q + y = x^{q^r} + 1$," Adv. Math. Commun., vol. 8, no. 1, pp. 67–72, 2014.
- [37] H. Stichtenoth, "A note on Hermitian codes over $GF(q^2)$," *IEEE Trans. Inform. Theory*, vol. 34, no. 5, pp. 1345–1348, 1988.
- [38] H. Stichtenoth, Algebraic function fields and codes, Springer, 2009.
- [39] H. J. Tiersma, "Remarks on codes from Hermitian curves," IEEE Trans. Inform. Theory, vol. 33, pp. 605–609, 1987.
- [40] K. Yang and P. V. Kumar, "On the true minimum distance of Hermitian codes," in Coding theory and algebraic Geometry (Luminy, 1991), vol. 1518 of Lecture Notes in Math., pp. 99–107, Berlin: Springer, 1992.