

The Linear Complexity of a Class of Binary Sequences With Optimal Autocorrelation

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Abstract Binary sequences with optimal autocorrelation and large linear complexity have important applications in cryptography and communications. Very recently, a class of binary sequences of period $4p$ with optimal autocorrelation was proposed by interleaving four suitable Ding-Helleseth-Lam sequences (Des. Codes Cryptogr., DOI 10.1007/s10623-017-0398-5), where p is an odd prime with $p \equiv 1 \pmod{4}$. The objective of this paper is to determine the minimal polynomial and the linear complexity of this class of binary optimal sequences via a sequence polynomial approach. It turns out that this class of sequences has quite good linear complexity.

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1 Introduction

Let $\mathbf{a} = (a(0), a(1), \dots, a(N-1))$ be a binary sequence of period N , its periodic autocorrelation is defined by

$$R_{\mathbf{a}}(\tau) = \sum_{i=0}^{N-1} (-1)^{a(i)+a(i+\tau)}.$$

Herein and hereafter the addition $i + \tau$ is performed modulo N . Let \mathbb{Z}_N denote the ring of integers modulo N . The set

$$C_{\mathbf{a}} = \{t \in \mathbb{Z}_N : a(t) = 1\}$$

is called the support of \mathbf{a} , and \mathbf{a} is called the characteristic sequence of the set $C_{\mathbf{a}}$. It is easily verified that

$$R_{\mathbf{a}}(\tau) = N - 4|(C_{\mathbf{a}} + \tau) \cap C_{\mathbf{a}}|, \tau \in \mathbb{Z}_N. \quad (1)$$

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By (1), one has $R_{\mathbf{a}}(\tau) \equiv N \pmod{4}$ for each $1 \leq \tau < N$. Therefore, the optimal value of out-of-phase autocorrelation of binary sequences can be classified into the following four types:

- (A) $R_{\mathbf{a}}(\tau) = 0$ for $N \equiv 0 \pmod{4}$;
- (B) $R_{\mathbf{a}}(\tau) \in \{1, -3\}$ for $N \equiv 1 \pmod{4}$;
- (C) $R_{\mathbf{a}}(\tau) \in \{\pm 2\}$ for $N \equiv 2 \pmod{4}$; and
- (D) $R_{\mathbf{a}}(\tau) = -1$ for $N \equiv 3 \pmod{4}$.

The sequences in Types (A) and (D) are called perfect sequences and ideal sequences with two-level autocorrelation, respectively. The only known perfect binary sequence up to equivalence is the $(0, 0, 0, 1)$. It is conjectured that there is no perfect binary sequences of period $N > 4$ which is widely believed to be true in both mathematical and engineer societies. Hence, it is natural to consider the next smallest value for the out-of-phase autocorrelation of a binary sequence of period $N \equiv 0 \pmod{4}$. That is, $R_{\mathbf{a}}(\tau) \in \{0, \pm 4\}$. If $R_{\mathbf{a}}(\tau) \in \{0, -4\}$ when τ ranges from 1 to $N - 1$, then \mathbf{a} is referred to as a sequence with *optimal autocorrelation value* (with respect to the values) [11]. If $R_{\mathbf{a}}(\tau) \in \{0, \pm 4\}$ when τ ranges from 1 to $N - 1$, then \mathbf{a} is referred to as a sequence with *optimal autocorrelation magnitude* (with respect to the magnitude of the autocorrelation values) [13].

Binary sequences with optimal autocorrelation value/magnitude have important applications in many areas of cryptography, communication and radar [5]. Finding new binary sequences with optimal autocorrelation value/magnitude has been an interesting research topic in sequence design. During the last four decades, numerous constructions of binary sequences with optimal autocorrelation have been reported in the literature (see [9], [6], [3], [1], [13], [2], [11] and the references therein).

The linear complexity of a sequence is often defined in terms of the shortest linear feedback shift register that can generate the sequence. In order to resist the well-known Berlekamp-Massey algorithm [8], the employed sequences should have large linear complexity from the view point of cryptography. A well-rounded treatment of the linear complexity of sequences with optimal autocorrelation was given in [12] and [7].

Very recently, a new class of binary sequences with optimal autocorrelation magnitude was proposed in [10]. This construction is given as follows. Let p be an odd prime with $p \equiv 1 \pmod{4}$, $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be four binary sequences of period p and $\mathbf{b} = (b(0), b(1), b(2), b(3))$ be a binary sequence of period 4. Then a binary sequence of period $4p$ can be obtained as below:

$$\mathbf{u} = I(\mathbf{a}_0 + b(0), L^d(\mathbf{a}_1) + b(1), L^{2d}(\mathbf{a}_2) + b(2), L^{3d}(\mathbf{a}_3) + b(3)), \quad (2)$$

where I and L denote the interleaved operator and the left cyclic shift operator respectively, and d is a positive integer satisfying $4d \equiv 1 \pmod{p}$. It was shown in [10] that the sequence \mathbf{u} obtained from (2) is optimal with respect to the autocorrelation magnitude, i.e., $R_{\mathbf{u}}(\tau) \in \{0, \pm 4\}$ for all $0 < \tau < 4p$, if the sequences $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are chosen to be some Ding-Helleseth-Lam sequences and the sequence \mathbf{b} satisfies $b(0) = b(2)$ and $b(1) = b(3)$.

The objective of this paper is to determine the minimal polynomial and linear complexity of the optimal sequences proposed in [10] based on the sequence polynomial approach. It turns out that this class of sequences has quite good linear complexity.

2 Preliminaries

In this section, we present some basic notation and results on sequences which will be needed in the sequel.

2.1 Interleaved structure

Let $\{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{T-1}\}$ be a set of T sequences of period N . An $N \times T$ matrix U is formed by placing the sequence \mathbf{a}_i on the i -th column, where $0 \leq i \leq T-1$. Then one can obtain an interleaved sequence \mathbf{u} of period NT by concatenating the successive rows of the matrix U . For simplicity, the interleaved sequence \mathbf{u} can be written as

$$\mathbf{u} = I(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{T-1}),$$

where I denotes the interleaved operator. For more details on interleaved structure, the reader is referred to [5].

2.2 Linear complexity via the sequence polynomial approach

Let $\mathbf{s} = (s(i))_{i=0}^{\infty}$ be a sequence over a field \mathbf{F} of period N . A polynomial of the form $f(x) = 1 + c_1x + c_2x^2 + \dots + c_rx^r \in \mathbf{F}[x]$ is called the characteristic polynomial of the sequence \mathbf{s} if $s(i) = c_1s(i-1) + c_2s(i-2) + \dots + c_rs(i-r)$ holds for any $i \geq r$, where $\mathbf{F}[x]$ denotes the set of all the polynomials in x over \mathbf{F} . The minimal polynomial $\mathbb{M}_{\mathbf{s}}(x)$ of the sequence \mathbf{s} is the monic polynomial with the lowest degree in all characteristic polynomials of \mathbf{s} , and the linear complexity of \mathbf{s} is then defined by the degree of $\mathbb{M}_{\mathbf{s}}(x)$, that is $\text{LC}(\mathbf{s}) = \deg(\mathbb{M}_{\mathbf{s}}(x))$. The sequence polynomial of \mathbf{s} , denoted by $\mathbb{P}_{\mathbf{s}}(x)$, is defined as

$$\mathbb{P}_{\mathbf{s}}(x) = \sum_{i=0}^{N-1} s(i)x^i \in \mathbf{F}[x].$$

There are a few ways to determine the linear span and minimal polynomial of a periodic sequence. One of them is given in the following lemma via the sequence polynomial approach.

Lemma 1 ([4], p. 87, Theorem 5.3) *Let \mathbf{s} be a sequence over a finite field of period N . Then*

- 1) *the minimal polynomial of \mathbf{s} is $\mathbb{M}_{\mathbf{s}}(x) = \frac{x^N - 1}{\gcd(x^N - 1, \mathbb{P}_{\mathbf{s}}(x))}$; and*
- 2) *the linear complexity of \mathbf{s} is $\text{LC}(\mathbf{s}) = N - \deg(\gcd(x^N - 1, \mathbb{P}_{\mathbf{s}}(x)))$.*

The following result gives relations of the sequence polynomials of some related sequences.

Lemma 2 ([12],[7]) *Let \mathbf{a} be a binary sequence of period N . Then*

- 1) $\mathbb{P}_{\mathbf{b}}(x) = x^{N-\tau} \mathbb{P}_{\mathbf{a}}(x)$ if $\mathbf{b} = L^{\tau}(\mathbf{a})$;
- 2) $\mathbb{P}_{\mathbf{b}}(x) = \mathbb{P}_{\mathbf{a}}(x) + \frac{x^N - 1}{x - 1}$ if \mathbf{b} is the complement sequence of \mathbf{a} (i.e., $b(t) = a(t) + 1$ for all t); and
- 3) $\mathbb{P}_{\mathbf{u}}(x) = \mathbb{P}_{\mathbf{a}_0}(x^4) + x \mathbb{P}_{\mathbf{a}_1}(x^4) + x^2 \mathbb{P}_{\mathbf{a}_2}(x^4) + x^3 \mathbb{P}_{\mathbf{a}_3}(x^4)$ if $\mathbf{u} = I(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

Let \mathbf{s} and \mathbf{v} be two binary sequences of period N . Let r be a positive integer with $\gcd(r, N) = 1$. The sequence \mathbf{v} is said to be a r -decimation of \mathbf{s} if $v(t) = s(rt \pmod N)$ for all $0 \leq t < N$. Two sequences \mathbf{s} and \mathbf{v} are said to be equivalent if \mathbf{v} is a cyclic shift version of the decimation of \mathbf{s} or its complement. Otherwise, they are said to be inequivalent. The following result gives the relationship between the minimal polynomials of two binary equivalent sequences.

Lemma 3 (Lemma 5, [12]) *Let $M_{\mathbf{s}}(x)$ and $M_{\mathbf{v}}(x)$ be the minimal polynomials of two binary sequences \mathbf{s} and \mathbf{v} of period N , respectively. Then we have*

- 1) $M_{\mathbf{v}}(x) = M_{\mathbf{s}}(x)$ if the sequence \mathbf{v} can be obtained from \mathbf{s} by a cyclic shift.
- 2) $\deg(M_{\mathbf{v}}(x)) = \deg(M_{\mathbf{s}}(x))$ if the sequence \mathbf{v} can be obtained from \mathbf{s} by a decimation r with $\gcd(r, N) = 1$.
- 3) If the sequence \mathbf{v} is a complement of \mathbf{s} , then

$$M_{\mathbf{v}}(x) = \begin{cases} M_{\mathbf{s}}(x)(x-1), & \text{if } (x-1) \nmid M_{\mathbf{s}}(x), \\ M_{\mathbf{s}}(x)/(x-1), & \text{if } (x-1) \mid M_{\mathbf{s}}(x) \text{ and } (x-1)^2 \nmid M_{\mathbf{s}}(x), \\ M_{\mathbf{s}}(x), & \text{if } (x-1)^2 \mid M_{\mathbf{s}}(x). \end{cases}$$

Lemma 3 implies that two binary sequences \mathbf{s} and \mathbf{v} of the same period are inequivalent if $|\deg(M_{\mathbf{s}}(x)) - \deg(M_{\mathbf{v}}(x))| \geq 2$. This fact will be used to judge when the optimal sequences obtained in [10] (see Theorems 2 and 4 in Section 3) are inequivalent.

2.3 Ding-Helleseth-Lam sequences

Let $p = 4f + 1$ be an odd prime, where f is a positive integer, and let θ be a generator of the multiplicative group of the field \mathbb{Z}_p , then the cyclotomic classes D_i of order 4 are defined as $D_i = \{\theta^{i+4j} : 0 \leq j \leq f-1\}$ for $0 \leq i \leq 3$. Using the cyclotomic classes of order 4, Ding, Hellesteth and Lam constructed several classes of optimal binary sequences with period p as follows.

Theorem 1 ([3]) *Let $p = 4f + 1 = x^2 + 4y^2$ be an odd prime, where f, x, y are positive integers. Then all the sequences of period p with supports $D_0 \cup D_1$, $D_1 \cup D_2$, $D_2 \cup D_3$, $D_0 \cup D_3$ respectively are optimal sequences with autocorrelation values 1 and -3 if and only if f is odd and $y = \pm 1$.*

Let m be the order of 2 modulo p and β be a primitive p -th root of unity over the finite field \mathbb{F}_{2^m} , that is, \mathbb{F}_{2^m} is the splitting field of $x^p - 1$. Define

$$\mathbb{S}(x) = \sum_{i \in D_0 \cup D_1} x^i, \quad \mathbb{T}(x) = \sum_{i \in D_1 \cup D_2} x^i. \quad (3)$$

With the help of the properties of the polynomials $\mathbb{S}(x)$ and $\mathbb{T}(x)$, the linear complexity of the Ding-Helleseth-Lam sequences was determined in [3]. In the sequel, we need some basic facts about the values of $\mathbb{S}(x)$ and $\mathbb{T}(x)$ at the point β used in the proof of Theorem 12 in [3], which can be easily verified and will play an important role in proving our main results.

Lemma 4 ([3]) *With the notation above, we have*

$$\mathbb{S}(\beta^k) = \mathbb{S}(\beta), \mathbb{T}(\beta), \mathbb{S}(\beta) + 1, \mathbb{T}(\beta) + 1$$

when $k \in D_0, D_1, D_2, D_3$, respectively.

3 The linear complexity of the optimal sequences obtained from (2)

From now on, we adopt the following notation unless otherwise stated:

- $p = 4f + 1$ is an odd prime with f being odd.
- d is a positive integer satisfying $4d \equiv 1 \pmod{p}$.
- θ is a generator of the multiplicative group of \mathbb{Z}_p .
- $D_i = \{\theta^{i+4j} : 0 \leq j \leq f-1\}$ for $0 \leq i \leq 3$ are the cyclotomic classes of order 4.
- $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ are the Ding-Helleseth-Lam sequences of period p with the supports $D_0 \cup D_1, D_0 \cup D_3, D_1 \cup D_2, D_2 \cup D_3$, respectively.
- $\mathbb{S}(x)$ and $\mathbb{T}(x)$ are two polynomials given in (3).

The following result was proved in [10].

Theorem 2 (Theorem 1, [10]) *Let $\mathbf{b} = (b(0), b(1), b(2), b(3))$ be a binary sequence with $b(0) = b(2)$ and $b(1) = b(3)$, and $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (\mathbf{s}_3, \mathbf{s}_2, \mathbf{s}_1, \mathbf{s}_1)$. Then the binary sequence \mathbf{u} constructed from (2) is optimal with respect to the autocorrelation magnitude, i.e., $R_{\mathbf{u}}(\tau) \in \{0, \pm 4\}$ for all $0 < \tau < 4p$.*

In what follows, we determine the linear complexity of the optimal sequences in Theorem 2 with the help of Lemmas 1 and 4. We always assume that \mathbb{F}_{2^m} is the splitting field of $x^p - 1$ and β is a primitive p -th root of $x^p - 1$ in \mathbb{F}_{2^m} . Then the set $\{\beta^i : i = 0, 1, 2, \dots, p-1\}$ of roots of $x^p - 1$ is a cyclic group of order p with respect to the multiplication in \mathbb{F}_{2^m} . Let \mathbf{u} be the sequence obtained in Theorem 2 and $\mathbb{P}_{\mathbf{u}}(x)$ be its sequence polynomial. It then follows from Lemma 1 that

$$\text{LC}(\mathbf{u}) = 4p - \deg(\gcd(x^{4p} - 1, \mathbb{P}_{\mathbf{u}}(x))) = 4p - \sum_{i=0}^{p-1} N_i. \quad (4)$$

where $N_i = \min\{k_i, 4\}$ and k_i denotes the multiplicity of β^i as a root of $\mathbb{P}_{\mathbf{u}}(x)$.

The following lemmas will be needed to prove the main result of this paper in the sequel.

Lemma 5 *Let symbols be the same as before. Then we have*

$$\mathbb{P}_{\mathbf{s}_1}(x) = \mathbb{S}(x), \mathbb{P}_{\mathbf{s}_2}(x) = \mathbb{S}(x^{\theta^3}); \mathbb{P}_{\mathbf{s}_3}(x) = \mathbb{S}(x^{\theta}); \text{ and } \mathbb{P}_{\mathbf{s}_4}(x) = \mathbb{S}(x^{\theta^2}).$$

Proof The conclusion follows from the definitions of $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$, and the fact that

$$\sum_{i \in D_j \cup D_{j+1}} x^i = \sum_{i \in D_0 \cup D_1} x^{\theta^j \cdot i} = \mathbb{S}(x^{\theta^j})$$

for any $0 \leq j \leq 3$.

Lemma 6 *For the sequence \mathbf{u} in Theorem 2, we have*

$$\mathbb{P}_{\mathbf{u}}(x) = \mathbb{S}(x^{4\theta}) + x^p \mathbb{S}(x^{4\theta^3}) + x^{2p} \mathbb{S}(x^4) + x^{3p} \mathbb{S}(x^4) + \mathbb{P}_{\mathbf{b}}(x) \cdot \frac{x^{4p} - 1}{x^4 - 1}. \quad (5)$$

Proof Observe that $p = 4f + 1$ and $4d \equiv 1 \pmod{p}$ lead to

$$d \equiv -f \equiv p - f \equiv 3f + 1 \pmod{p}$$

which further implies that

$$2d \equiv 6f + 2 \equiv 2f + 1 \pmod{p},$$

and

$$3d \equiv 9f + 3 \equiv f + 1 \pmod{p}.$$

This together with (2) implies that

$$\mathbf{u} = I(\mathbf{s}_3 + b(0), L^{3f+1}(\mathbf{s}_2) + b(1), L^{2f+1}(\mathbf{s}_1) + b(2), L^{f+1}(\mathbf{s}_1) + b(3)).$$

According to Lemmas 5 and 2, the sequence polynomials of the following sequences

$$\mathbf{s}_3 + b(0), L^{3f+1}(\mathbf{s}_2) + b(1), L^{2f+1}(\mathbf{s}_1) + b(2), L^{f+1}(\mathbf{s}_1) + b(3)$$

are respectively given by

$$\mathbb{S}(x^\theta) + b(0) \cdot \frac{x^p - 1}{x - 1},$$

$$x^f \mathbb{S}(x^{\theta^3}) + b(1) \cdot \frac{x^p - 1}{x - 1},$$

$$x^{2f} \mathbb{S}(x) + b(2) \cdot \frac{x^p - 1}{x - 1},$$

and

$$x^{3f} \mathbb{S}(x) + b(3) \cdot \frac{x^p - 1}{x - 1}.$$

It follows from Lemma 2 again that

$$\begin{aligned} \mathbb{P}_{\mathbf{u}}(x) &= \left(\mathbb{S}(x^{4\theta}) + b(0) \cdot \frac{x^{4p} - 1}{x^4 - 1} \right) + x \cdot \left(x^{4f} \mathbb{S}(x^{4\theta^3}) + b(1) \cdot \frac{x^{4p} - 1}{x^4 - 1} \right) + \\ &\quad x^2 \cdot \left(x^{8f} \mathbb{S}(x^4) + b(2) \cdot \frac{x^{4p} - 1}{x^4 - 1} \right) + x^3 \cdot \left(x^{12f} \mathbb{S}(x^4) + b(3) \cdot \frac{x^{4p} - 1}{x^4 - 1} \right) \\ &= \mathbb{S}(x^{4\theta}) + x^p \mathbb{S}(x^{4\theta^3}) + x^{2p} \mathbb{S}(x^4) + x^{3p} \mathbb{S}(x^4) + \mathbb{P}_{\mathbf{b}}(x) \cdot \frac{x^{4p} - 1}{x^4 - 1}. \end{aligned}$$

This completes the proof of this lemma.

According to (4), to determine the linear complexity of the sequence \mathbf{u} , it suffices to determine N_i for each $0 \leq i \leq p - 1$. This can be done based on Lemma 6. Specifically, we have the following results.

Lemma 7 $N_i = 0$ for each $1 \leq i \leq p - 1$.

Proof By (5) in Lemma 6, $\mathbb{P}_{\mathbf{u}}(\beta^i) = 0$ if and only if $\mathbb{S}(\beta^{4i\theta}) + \mathbb{S}(\beta^{4i\theta^3}) = 0$ due to $\beta^p = 1$ and $\beta \neq 1$. The fact $p = 4f + 1$ with f being odd implies that 2 is a non-square element in \mathbb{Z}_p since the Legendre symbol $(\frac{2}{p}) = (-1)^{(p^2-1)/8} = -1$. This means $2 \in D_1 \cup D_3$ and then $4 \in D_2$. Thus, we have $4\theta \in D_3$ and $4\theta^3 \in D_1$. Then, by Lemma 4 we have

$$\mathbb{S}(\beta^{4i\theta}) + \mathbb{S}(\beta^{4i\theta^3}) = \mathbb{T}(\beta) + \mathbb{T}(\beta) + 1 = 1,$$

if $i \in D_0 \cup D_2$, and

$$\mathbb{S}(\beta^{4i\theta}) + \mathbb{S}(\beta^{4i\theta^3}) = \mathbb{S}(\beta) + \mathbb{S}(\beta) + 1 = 1,$$

if $i \in D_1 \cup D_3$. That is, $\mathbb{S}(\beta^{4i\theta}) + \mathbb{S}(\beta^{4i\theta^3}) \neq 0$ for any $1 \leq i \leq p-1$. This means that β^i cannot be a root of $\mathbb{P}_{\mathbf{u}}(x)$ for each $1 \leq i \leq p-1$, which finishes the proof of this lemma.

Lemma 8 *Let symbols be the same as before. Then we have*

- 1) $\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1) = x^4 - 1$ and $N_0 = 4$ if $\mathbf{b} = (0, 0, 0, 0)$;
- 2) $\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1) = x^3 + x^2 + x + 1$ and $N_0 = 3$ if $\mathbf{b} = (1, 1, 1, 1)$; and
- 3) $\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1) = x^2 - 1$ and $N_0 = 2$ if $\mathbf{b} = (1, 0, 1, 0)$ or $\mathbf{b} = (0, 1, 0, 1)$.

Proof We only need to calculate $\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1)$, since N_0 is equal to the degree of the polynomial $\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1)$. Let $\mathbb{E}(x) = \mathbb{S}(x^{4\theta}) + x^p \mathbb{S}(x^{4\theta^3}) + x^{2p} \mathbb{S}(x^4) + x^{3p} \mathbb{S}(x^4)$. It follows from (3) that $\mathbb{S}(1) = 0$ since $\frac{p-1}{2}$ is even. Thus $(x-1) | \mathbb{S}(x^k)$ and therefore $(x^4 - 1) | \mathbb{S}(x^{4k})$ for any nonzero integer k . It then follows that $(x^4 - 1) | \mathbb{E}(x)$. This together with the fact $\gcd(x^4 - 1, \frac{x^{4p}-1}{x^4-1}) = 1$ means that

$$\gcd(\mathbb{P}_{\mathbf{u}}(x), x^4 - 1) = \gcd(\mathbb{P}_{\mathbf{b}}(x), x^4 - 1),$$

which completes the proof of this lemma.

Now, we are in a position to present the main result of this paper.

Theorem 3 *Let \mathbf{u} be the optimal sequence of period $4p$ in Theorem 2. Then the minimal polynomial of the sequence \mathbf{u} is $\mathbb{M}_{\mathbf{u}}(x) = (x^{4p} - 1)/g(x)$ and the linear complexity of \mathbf{u} is $\text{LC}(u) = 4p - \epsilon$, where*

- 1) $g(x) = x^4 - 1$ and $\epsilon = 4$ if $\mathbf{b} = (0, 0, 0, 0)$;
- 2) $g(x) = x^3 + x^2 + x + 1$ and $\epsilon = 3$ if $\mathbf{b} = (1, 1, 1, 1)$; and
- 3) $g(x) = x^2 - 1$ and $\epsilon = 2$ if $\mathbf{b} = (1, 0, 1, 0)$ or $\mathbf{b} = (0, 1, 0, 1)$.

Proof The conclusions follow directly from (4), and Lemmas 7 and 8.

The following example computed by Magma confirms the results in Theorem 3.

Example 1 Let $p = 29 = 4f + 1 = x^2 + 4y^2$ for $x = 5$, $y = -1$, and $f = 7$. Let $\alpha = 2$ be a primitive element of \mathbb{Z}_p . Then four cyclotomic classes of order 4 with respect to \mathbb{Z}_p are given by

$$D_0 = \{1, 7, 16, 20, 23, 24, 25\},$$

$$D_1 = \{2, 3, 11, 14, 17, 19, 21\},$$

$$D_2 = \{4, 5, 6, 9, 13, 22, 28\},$$

$$D_3 = \{8, 10, 12, 15, 18, 26, 27\}$$

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