

DENSE FAMILIES OF MODULAR CURVES, PRIME NUMBERS AND UNIFORM SYMMETRIC TENSOR RANK OF MULTIPLICATION IN CERTAIN FINITE FIELDS

STÉPHANE BALLEZ AND ALEXEY ZYKIN

ABSTRACT. We obtain new uniform bounds for the symmetric tensor rank of multiplication in finite extensions of any finite field \mathbb{F}_p or \mathbb{F}_{p^2} where p denotes a prime number ≥ 5 . In this aim, we use the symmetric Chudnovsky-type generalized algorithm applied on sufficiently dense families of modular curves defined over \mathbb{F}_{p^2} attaining the Drinfeld–Vladuts bound and on the descent of these families to the definition field \mathbb{F}_p . These families are obtained thanks to prime number density theorems of type Hoheisel, in particular a result due to Dudek (2016).

1. INTRODUCTION

1.1. **Notation.** Let $q = p^s$ be a prime power, \mathbb{F}_q be the finite field with q elements and \mathbb{F}_{q^n} be the degree n extension of \mathbb{F}_q . The multiplication of two elements of \mathbb{F}_{q^n} is an \mathbb{F}_q -bilinear application from $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ onto \mathbb{F}_{q^n} . It can be considered as an \mathbb{F}_q -linear application from the tensor product $\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ onto \mathbb{F}_{q^n} . Consequently it can be also viewed as an element T of $(\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, namely an element of $\mathbb{F}_{q^n}^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. More precisely, when T is written

$$(1) \quad T = \sum_{i=1}^r x_i^* \otimes y_i^* \otimes c_i,$$

where the r elements x_i^* and the r elements y_i^* are in the dual $\mathbb{F}_{q^n}^*$ of \mathbb{F}_{q^n} and the r elements c_i are in \mathbb{F}_{q^n} , the following holds for any $x, y \in \mathbb{F}_{q^n}$:

$$x \cdot y = \sum_{i=1}^r x_i^*(x) y_i^*(y) c_i.$$

Date: November 6, 2018.

Key words and phrases. Algebraic function field, tower of function fields, tensor rank, algorithm, finite field, modular curve, Shimura curve.

The second author was partially supported by ANR Globes ANR-12-JS01-0007-01 and by the Russian Academic Excellence Project '5-100'.

Definition 1. *The minimal number of summands in a decomposition of the multiplication tensor T is called the rank of the tensor of the multiplication in the extension field \mathbb{F}_{q^n} (or bilinear complexity of the multiplication) and is denoted by $\mu_q(n)$:*

$$\mu_q(n) = \min \left\{ r \mid T = \sum_{i=1}^r x_i^* \otimes y_i^* \otimes c_i \right\}.$$

It is known that the tensor T can have a symmetric decomposition:

$$(2) \quad T = \sum_{i=1}^r x_i^* \otimes x_i^* \otimes c_i.$$

Definition 2. *The minimal number of summands in a symmetric decomposition of the multiplication tensor T is called the symmetric tensor rank of the multiplication (or the symmetric bilinear complexity of the multiplication) and is denoted by $\mu_q^{\text{sym}}(n)$:*

$$\mu_q^{\text{sym}}(n) = \min \left\{ r \mid T = \sum_{i=1}^r x_i^* \otimes x_i^* \otimes c_i \right\}.$$

From an asymptotical point of view, let us define the following

$$(3) \quad M_q^{\text{sym}} = \limsup_{k \rightarrow \infty} \frac{\mu_q^{\text{sym}}(k)}{k},$$

$$(4) \quad m_q^{\text{sym}} = \liminf_{k \rightarrow \infty} \frac{\mu_q^{\text{sym}}(k)}{k}.$$

Let F/\mathbb{F}_q be a function field of genus g over the finite field \mathbb{F}_q and $N_k(F)$ be the number of places of degree k of F/\mathbb{F}_q .

Let us define:

$$N_q(g) = \max \{ N_1(F) \mid F \text{ is a function field over } \mathbb{F}_q \text{ of genus } g \}$$

and

$$A(q) = \limsup_{g \rightarrow +\infty} \frac{N_q(g)}{g}.$$

We know that (Drinfeld–Vladuts bound):

$$A(q) \leq q^{\frac{1}{2}} - 1,$$

the bound being attained if q is a square.

1.2. Known results. The original algorithm of D.V. and G.V. Chudnovsky introduced in [10] is symmetric by definition and leads to the two following results from [3], [8] and [7]:

Theorem 3. *Let q be a prime power and let $n > 1$ be an integer. Let F/\mathbb{F}_q be an algebraic function field of genus g and N_k be the number of places of degree k in F/\mathbb{F}_q . If F/\mathbb{F}_q is such that $2g + 1 \leq q^{\frac{n-1}{2}}(q^{\frac{1}{2}} - 1)$ then:*

1) *if $N_1 > 2n + 2g - 2$, then*

$$\mu_q^{\text{sym}}(n) \leq 2n + g - 1,$$

2) *if $N_1 + 2N_2 > 2n + 2g - 2$ and there exists a non-special divisor of degree $g - 1$, then*

$$\mu_q^{\text{sym}}(n) \leq 3n + 2g.$$

Theorem 4. *Let q be a power of a prime p and let n be an integer. Then the symmetric tensor rank $\mu_q^{\text{sym}}(n)$ of multiplication in any finite field \mathbb{F}_{q^n} is linear with respect to the extension degree; more precisely, there exists a constant C_q such that for any integer $n > 1$,*

$$\mu_q^{\text{sym}}(n) \leq C_q n.$$

From different versions of symmetric algorithms of Chudnovsky type applied to good towers of algebraic function fields of type Garcia–Stichtenoth attaining the Drinfeld–Vladuts bounds of order one, two or four, different authors have obtained uniform bounds for the tensor rank of multiplication, namely general expressions for C_q , such as the following best currently published estimates:

Theorem 5. *Let $q = p^r$ be a power of a prime p and let n be an integer > 1 . Then:*

(i) *If $q = 2$, then $\mu_q^{\text{sym}}(n) \leq 15.46n$ (cf. [6, Corollary 29] and [9])*

(ii) *If $q = 3$, then $\mu_q^{\text{sym}}(n) \leq 7.732n$ (cf. [6, Corollary 29] and [9])*

(iii) *If $q \geq 4$, then $\mu_q^{\text{sym}}(n) \leq 3 \left(1 + \frac{\frac{4}{3}p}{q - 3 + 2(p-1)\frac{q}{q+1}} \right) n$ (cf. [7])*

(iv) *If $p \geq 5$, then $\mu_p^{\text{sym}}(n) \leq 3 \left(1 + \frac{8}{3p - 5} \right) n$ (cf. [7])*

(v) *If $q \geq 4$, then $\mu_{q^2}^{\text{sym}}(n) \leq 2 \left(1 + \frac{p}{q - 3 + (p-1)\frac{q}{q+1}} \right) n$ (cf. [1]*

and [7])

(vi) *If $p \geq 5$, then $\mu_{p^2}^{\text{sym}}(n) \leq 2 \left(1 + \frac{2}{p - \frac{33}{16}} \right) n$ (cf. [7])*

1.3. New results. The main goal of the paper is to improve the upper bounds for $\mu_q^{\text{sym}}(n)$ from the previous theorem for the assertions concerning the extensions of finite fields \mathbb{F}_{p^2} and \mathbb{F}_p where p is a prime number. Note that one of main ideas used in this paper was introduced in [4] by the first author thanks to the use of the Chebyshev Theorem (or also called the Bertrand Postulat) to bound the gaps between prime numbers in order to construct families of modular curves as dense as possible. Later, motivated by [4], the approach of using such bounds on gaps between prime numbers (e.g. Baker-Harman-Pintz) was also used in the preprint [12] in order to improve the upper bounds of $\mu_{p^2}^{\text{sym}}(n)$ where p is a prime number. In our paper, we improve all the known uniform upper bounds for $\mu_{p^2}^{\text{sym}}(n)$ and $\mu_p^{\text{sym}}(n)$ for $p \geq 5$.

2. NEW UPPER BOUNDS

In this section, we give new better upper bounds for the symmetric tensor rank of multiplication in certain extensions of finite fields \mathbb{F}_{p^2} and \mathbb{F}_p . In order to do that, we construct suitable families of modular curves defined over \mathbb{F}_{p^2} and \mathbb{F}_p .

Theorem 6. *Let l_k be the k -th prime number. Then there exists a real number $\alpha < 1$ such that the difference between two consecutive prime numbers l_k and l_{k+1} satisfies*

$$l_{k+1} - l_k \leq l_k^\alpha$$

for any prime $l_k \geq x_\alpha$.

In particular, one can take $\alpha = \frac{21}{40}$ with the value of x_α that can in principle be determined effectively, or $\alpha = \frac{2}{3}$ with $x_\alpha = \exp(\exp(33.3))$.

Proof. It is known that for all $x > x_\alpha$, the interval $[x - x^\alpha, x]$ with $\alpha = \frac{21}{40}$ contains prime numbers by a result of Baker, Harman and Pintz [2, Theorem 1]. Moreover, the value of x_α can in principle be determined, according to the authors. However, to our knowledge, this computation has not been realized yet.

For a bigger $\alpha = \frac{2}{3}$, Dudek obtained recently in [11] an explicit bound $x_\alpha \geq \exp(\exp(33.3))$.

□

□

2.1. The case of the quadratic extensions of prime fields.

Proposition 7. *Let $p \geq 5$ be a prime number, and let x_α be the constant from Theorem 6.*

(1) *If $p \neq 11$, then for any integer $n \geq \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$ we have*

$$\mu_{p^2}^{\text{sym}}(n) \leq 2 \left(1 + \frac{1 + \epsilon_p(n)}{p-3} \right) n - \frac{(1 + \epsilon_p(n))(p+1)}{p-3} - 1,$$

$$\text{where } \epsilon_p(n) = \left(\frac{2n}{p-3} \right)^{\alpha-1}.$$

(2) For $p = 11$ and $n \geq (p-3)x_\alpha + p - 1 = 8x_\alpha + 10$ we have

$$\mu_{p^2}^{\text{sym}}(n) \leq 2 \left(1 + \frac{1 + \epsilon_p(n)}{p-3} \right) n - \frac{2(1 + \epsilon_p(n))(p-1)}{p-3},$$

$$\text{where } \epsilon_p(n) = \left(\frac{n}{p-3} \right)^{\alpha-1}.$$

(3) Asymptotically the following inequality holds for any $p \geq 5$:

$$M_{p^2}^{\text{sym}} \leq 2 \left(1 + \frac{1}{p-3} \right).$$

Proof. First, let us consider the characteristic p such that $p \neq 11$. Then it is known ([15, Corollary 4.1.21] and [14, proof of Theorem 3.9]) that the modular curve $X_k = X_0(11l_k)$, where l_k is the k -th prime number, is of genus $g_k = l_k$ and satisfies $N_1(X_k(\mathbb{F}_{p^2})) \geq (p-1)(g_k+1)$, where $N_1(X_k(\mathbb{F}_{p^2}))$ denotes the number of rational points over \mathbb{F}_{p^2} of the curve X_k . Let us consider an integer $n > 1$. Then there exist two consecutive prime numbers l_k and l_{k+1} such that

$$(5) \quad (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2$$

and

$$(6) \quad (p-1)(l_k+1) \leq 2n + 2l_k - 2$$

(here we use the fact that $p \geq 5$). Let us consider the algebraic function field F_{k+1}/\mathbb{F}_{p^2} associated to the curve X_{k+1} of genus l_{k+1} defined over \mathbb{F}_{p^2} . Denoting by $N_i(F_k/\mathbb{F}_{p^2})$ the number of places of degree i of F_k/\mathbb{F}_{p^2} , we get

$$N_1(F_{k+1}/\mathbb{F}_{p^2}) \geq (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2.$$

We also know that $l_{k+1} - l_k \leq l_k^\alpha$, when $l_k \geq x_\alpha$ by Theorem 6. Thus $l_{k+1} \leq (1 + \epsilon(l_k))l_k$, with $\epsilon(l_k) = l_k^{\alpha-1}$.

It is easy to check that the inequality $2g+1 \leq q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ of Theorem 3 holds for any prime power $q \geq 5$. Indeed, it is enough to verify that

$$q^{l_k \frac{p-3}{4} + \frac{p-1}{4}} (q^{\frac{1}{2}} - 1) \geq 2(1 + \epsilon(l_k))l_k + 1,$$

which is true since

$$q^{x \frac{p-3}{4} + \frac{p-1}{4}} (q^{\frac{1}{2}} - 1) - 4x - 1 \geq 0$$

for any $x \geq 0$.

Thus, for any integer $n \geq \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$ the function field F_{k+1}/\mathbb{F}_{p^2} satisfies Theorem 3, so

$$\mu_{p^2}^{\text{sym}}(n) \leq 2n + l_{k+1} - 1 \leq 2n + (1 + \epsilon(l_k))l_k - 1,$$

with $l_k \leq \frac{2n}{p-3} - \frac{p+1}{p-3}$ by (6).

Let us remark that, as $l_k \leq \frac{2n}{p-3}$, $\epsilon(l_k) \leq \epsilon_p(n) = (\frac{2n}{p-3})^{\alpha-1}$, which gives the first inequality.

Now, let us consider the characteristic $p = 11$. Take the modular curve $X_k = X_0(23l_k)$, where l_k is the k -th prime number. By [15, Proposition 4.1.20], we easily compute that the genus of X_k is $g_k = 2l_k + 1$. It is also known that the curve X_k has good reduction modulo p outside 23 and l_k . Moreover, by using [15, Proof of Theorem 4.1.52], we obtain that the number of \mathbb{F}_{p^2} -rational points over of the reduction X_k/p modulo p satisfies

$$N_1(X_k(\mathbb{F}_{p^2})) \geq \frac{\mu_N(p-1)/12}{\deg \lambda_N} \geq 2(p-1)(l_k+1)$$

in the notation of loc. cit.

Let us take an integer $n > 1$. There exist two consecutive prime numbers l_k and l_{k+1} such that

$$2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2$$

and

$$2(p-1)(l_k+1) \leq 2n + 2(2l_k+1) - 2,$$

i.e.

$$(7) \quad (p-1)(l_{k+1}+1) > n + 2l_{k+1}$$

and

$$(8) \quad (p-1)(l_k+1) \leq n + 2l_k.$$

Let us consider the algebraic function field F_{k+1}/\mathbb{F}_{p^2} associated to the curve X_{k+1} of genus $g_{k+1} = 2l_{k+1} + 1$ defined over \mathbb{F}_{p^2} . We have

$$N_1(F_{k+1}/\mathbb{F}_{p^2}) \geq 2(p-1)(l_{k+1}+1) > 2n + 4l_{k+1}.$$

As before $l_{k+1} \leq (1 + \epsilon(l_k))l_k$, with $\epsilon(l_k) = l_k^{\alpha-1}$.

It is also easy to check that the inequality $2g + 1 \leq q^{\frac{n-1}{2}}(q^{\frac{1}{2}} - 1)$ of Theorem 3 holds when q is a power of 11, which follows from the fact that

$$11^{4l_k + \frac{9}{2}}(11^{\frac{1}{2}} - 1) \geq 8l_k + 3.$$

Thus, for any integer $n \geq (p-3)x_\alpha + p - 1$, the algebraic function field F_{k+1}/\mathbb{F}_{p^2} satisfies Theorem 3, so

$$\mu_{p^2}^{\text{sym}}(n) \leq 2n + 2l_{k+1} \leq 2n + 2(1 + \epsilon(l_k))l_k$$

with $l_k \leq \frac{n}{p-3} - \frac{p-1}{p-3}$ by (8).

We remark that as $l_k \leq \frac{n}{p-3}$, $\epsilon(l_k) \leq \epsilon_p(n) = (\frac{n}{p-3})^{\alpha-1}$, which gives the second inequality of the proposition.

Finally, when $n \rightarrow +\infty$, the prime numbers $l_k \rightarrow +\infty$, thus both for $p \neq 11$ and $p = 11$ the corresponding $\epsilon_p(n) \rightarrow 0$. So in the two cases we obtain

$$M_{p^2}^{\text{sym}} \leq 2 \left(1 + \frac{1}{p-3} \right).$$

□

□

Remark 8. *It is easy to see that the bounds obtained in Proposition 7 are generally better than the published best known bounds (v) and (vi) recalled in Theorem 5. Indeed, it is sufficient to consider the asymptotic bounds which are deduced from them and to see that for any prime $p \geq 5$ we have $\frac{1}{p-3} < \frac{p}{p-3+(p-1)\frac{p}{p+1}}$ and $\frac{1}{p-3} < \frac{2}{p-\frac{33}{16}}$ respectively.*

Remark 9. *Note that the bounds obtained in [12, Corollary 28] also concern the symmetric tensor rank of multiplication in the finite fields even if it is not mentioned. Indeed, the distinction between $\mu_q^{\text{sym}}(n)$ and $\mu_q(n)$ was exploited only from [13]. So, we can compare our proposition 7 with Corollary 8 there. Firstly, note that the bounds in [12, Corollary 28] are only valid for $p \geq 7$. Moreover, the only bound which is best than our bounds is the asymptotic bound [12, Corollary 28, Bound (vi)] given for an unknown sufficiently large n , contrary to our uniform bound with $\alpha = \frac{2}{3}$ for $n \geq \exp(\exp(33.3))$.*

2.2. The case of prime fields.

Proposition 10. *Let $p \geq 5$ be a prime number, let x_α be defined as in Lemma 6, and $\epsilon_p(n)$ as in Proposition 7.*

(1) *If $p \neq 11$, then for any integer $n \geq \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$ we have*

$$\mu_p^{\text{sym}}(n) \leq 3 \left(1 + \frac{\frac{4}{3}(1 + \epsilon_p(n))}{p-3} \right) n - \frac{2(1 + \epsilon_p(n))(p+1)}{p-3}.$$

(2) *For $p = 11$ and $n \geq (p-3)x_\alpha + p - 1 = 8x_\alpha + 10$ we have*

$$\mu_p^{\text{sym}}(n) \leq 3 \left(1 + \frac{\frac{4}{3}(1 + \epsilon_p(n))}{p-3} \right) n - \frac{4(1 + \epsilon_p(n))(p-1)}{p-3} + 1.$$

(3) *Asymptotically the following inequality holds for any $p \geq 5$:*

$$M_p^{\text{sym}} \leq 3 \left(1 + \frac{\frac{4}{3}}{p-3} \right).$$

Proof. It suffices to consider the same families of curves as in the proof of Proposition 7.

When $p \neq 11$ we take $X_k = X_0(11l_k)$, where l_k is the k -th prime number. These curves are defined over \mathbb{F}_p , hence, we can consider the associated algebraic function fields F_k/\mathbb{F}_p defined over \mathbb{F}_p and we have $N_1(F_k/\mathbb{F}_{p^2}) = N_1(F_k/\mathbb{F}_p) + 2N_2(F_k/\mathbb{F}_p) \geq (p-1)(l_k+1)$, since $F_k/\mathbb{F}_{p^2} = F_k/\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ for any k . Note that the genus of the algebraic function fields F_k/\mathbb{F}_p is also $g_k = l_k$, since the genus is preserved under descent.

Given an integer $n > 1$, there exist two consecutive prime numbers l_k and l_{k+1} such that

$$(9) \quad (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2$$

and

$$(10) \quad (p-1)(l_k+1) \leq 2n + 2l_k - 2.$$

Let us consider the algebraic function field F_{k+1}/\mathbb{F}_p associated to the curve X_{k+1} of genus l_{k+1} defined over \mathbb{F}_p . We get

$$N_1(F_{k+1}/\mathbb{F}_p) + 2N_2(F_{k+1}/\mathbb{F}_p) \geq (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2.$$

As before $l_{k+1} \leq (1 + \epsilon(l_k))l_k$, with $\epsilon(l_k) = l_k^{\alpha-1}$, and from the proof of the previous proposition we know that the inequality $2g+1 \leq q^{\frac{n-1}{2}}(q^{\frac{1}{2}} - 1)$ of Theorem 3 holds. Consequently, for any integer $n \geq \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$, the algebraic function field F_{k+1}/\mathbb{F}_p satisfies Theorem 3, 2) since by [5, Theorem 11 (i)] there always exists a non-special divisor of degree $g_{k+1} - 1$ for $p \geq 5$. So

$$\mu_p^{\text{sym}}(n) \leq 3n + 2l_{k+1} \leq 3n + 2(1 + \epsilon(l_k))l_k$$

with $l_k \leq \frac{2n}{p-3} - \frac{p+1}{p-3}$ by (10). As before, $\epsilon(l_k) \leq \epsilon_p(n) = \left(\frac{2n}{p-3}\right)^{\alpha-1}$.

When $p = 11$ we use once again the family of curves $X_k = X_0(23l_k)$. They are defined over \mathbb{F}_p , hence we can consider the associated algebraic function fields F_k/\mathbb{F}_p over \mathbb{F}_p and we have $N_1(F_k/\mathbb{F}_{p^2}) = N_1(F_k/\mathbb{F}_p) + 2N_2(F_k/\mathbb{F}_p) \geq (p-1)(l_k+1)$. The genus of the algebraic function fields F_k/\mathbb{F}_p defined over \mathbb{F}_p is also $g_k = 2l_k + 1$ since the genus is preserved under descent.

Given an integer $n > 1$, there exist two consecutive prime numbers l_k and l_{k+1} such that

$$2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2$$

and

$$2(p-1)(l_k+1) \leq 2n + 2(2l_k+1) - 2,$$

i.e.

$$(11) \quad (p-1)(l_{k+1}+1) > n + 2l_{k+1}$$

and

$$(12) \quad (p-1)(l_k+1) \leq n + 2l_k.$$

Let us consider the algebraic function field F_{k+1}/\mathbb{F}_p associated to the curve X_{k+1} of genus $g_{k+1} = 2l_{k+1} + 1$ defined over \mathbb{F}_p . We get

$$N_1(F_{k+1}/\mathbb{F}_p) + 2N_2(F_{k+1}/\mathbb{F}_p) \geq 2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2.$$

As above $l_{k+1} \leq (1 + \epsilon(l_k))l_k$, with $\epsilon(l_k) = l_k^{\alpha-1}$, and the inequality $2g+1 \leq q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ of Theorem 3 holds. Consequently, for any integer $n \geq (p-3)x_\alpha + p - 1$, the algebraic function field F_{k+1}/\mathbb{F}_p satisfies Theorem 3, 2) since, as before, there exists a non-special divisor of degree $g_{k+1} - 1$ by [5, Theorem 11 (i)]. So,

$$\mu_p^{\text{sym}}(n) \leq 3n + 2g_{k+1} \leq 3n + 2(2l_{k+1} + 1) \leq 3n + 2(1 + \epsilon)l_k$$

with $l_k \leq \frac{n}{p-3} - \frac{p-1}{p-3}$ by (12). We can also bound $\epsilon(l_k) \leq \epsilon_p(n) = \left(\frac{n}{p-3}\right)^{\alpha-1}$.

Finally, when $n \rightarrow +\infty$, the prime numbers $l_k \rightarrow +\infty$, thus both for $p \neq 11$ and $p = 11$, $\epsilon_p(n) \rightarrow 0$. So we obtain $M_p^{\text{sym}} \leq 3 \left(1 + \frac{4}{p-3}\right)$. \square

\square

Remark 11. *It is easy to see that the bounds obtained in Proposition 10 are generally better than the best known bounds (iii) and (iv) recalled in Theorem 5. Indeed, it is sufficient to consider the asymptotic bounds which are deduced from them and to see that for any prime $p \geq 5$ we have $\frac{4}{3} < \frac{\frac{4}{3}p}{p-3+\frac{2(p-1)p}{p+1}}$ and $\frac{4}{3} < \frac{8}{3p-5}$ respectively.*

ACKNOWLEDGMENTS

The first author wishes to thank Sary Drappeau, Olivier Ramaré, Hugues Randriambololona, Joël Rivat and Serge Vladuts for valuable discussions.

REFERENCES

- [1] Nicolas Arnaud. *Évaluations dérivés, multiplication dans les corps finis et codes correcteurs*. PhD thesis, Université de la Méditerranée, Institut de Mathématiques de Luminy, 2006.

- [2] Roger Baker, Glyn Harman, and János Pintz. The difference between consecutive primes, II. *Proceedings of the London Mathematical Society*, 83(3):532–562, 2001.
- [3] Stéphane Ballet. Curves with Many Points and Multiplication Complexity in Any Extension of \mathbb{F}_q . *Finite Fields and Their Applications*, 5:364–377, 1999.
- [4] Stéphane Ballet. On the tensor rank of the multiplication in the finite fields. *Journal of Number Theory*, 128:1795–1806, 2008.
- [5] Stéphane Ballet and Dominique Le Brigand. On the existence of non-special divisors of degree g and $g - 1$ in algebraic function fields over \mathbb{F}_q . *Journal of Number Theory*, 116:293–310, 2006.
- [6] Stéphane Ballet and Julia Pieltant. Tower of algebraic function fields with maximal Hasse-Witt invariant and tensor rank of multiplication in any extension of \mathbb{F}_2 and \mathbb{F}_3 . *Journal of Pure and Applied Algebra*, To appear.
- [7] Stéphane Ballet, Julia Pieltant, Matthieu Rambaud, and Jeroen Sijssling. On some bounds for symmetric tensor rank of multiplication in finite fields. *Contemporary Mathematics, Amer. Math. Soc.*, (686):93–121, 2017.
- [8] Stéphane Ballet and Robert Rolland. Multiplication algorithm in a finite field and tensor rank of the multiplication. *Journal of Algebra*, 272(1):173–185, 2004.
- [9] Murat Cenk and Ferruh Özbudak. On multiplication in finite fields. *Journal of Complexity*, pages 172–186, 2010.
- [10] David Chudnovsky and Gregory Chudnovsky. Algebraic complexities and algebraic curves over finite fields. *Journal of Complexity*, 4:285–316, 1988.
- [11] Adrian W. Dudek. An explicit result for primes between cubes. *Functiones and Approximatio Commentarii Mathematici*, 55(2):177–197, 2016.
- [12] Hugues Randriambololona. Divisors of the form $2d-g$ without sections and bilinear complexity of multiplication in finite fields. *ArXiv e-prints*, 2011.
- [13] Hugues Randriambololona. Bilinear complexity of algebras and the Chudnovsky-Chudnovsky interpolation method. *Journal of Complexity*, 28(4):489–517, 2012.
- [14] Igor Shparlinski, Michael Tsfasman, and Serguei Vlăduț. Curves with many points and multiplication in finite fields. In H. Stichtenoth and M.A. Tsfasman, editors, *Coding Theory and Algebraic Geometry*, number 1518 in Lectures Notes in Mathematics, pages 145–169, Berlin, 1992. Springer-Verlag. Proceedings of AGCT-3 conference, June 17-21, 1991, Luminy.
- [15] Michael Tsfasman and Serguei Vlăduț. *Algebraic-Geometric Codes*. Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

STÉPHANE BALLE

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE,
FRANCE. INSTITUT DE MATHÉMATIQUES DE MARSEILLE, CASE 907, 163 AV-
ENUE DE LUMINY, F-13288 MARSEILLE CEDEX 9, FRANCE.

E-mail address: `stephane.ballet@univ-amu.fr`

ALEXEY ZYKIN

LABORATOIRE GAATI, UNIVERSITÉ DE LA POLYNÉSIE FRANÇAISE
BP 6570 — 98702 FAA'A, TAHITI, POLYNÉSIE FRANÇAISE
NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS
AG LABORATORY NRU HSE
INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS OF THE RUSSIAN
ACADEMY OF SCIENCES

E-mail address: `alzykin@gmail.com`