

The Maximum Length of Circuit Codes With Long Bit Runs and a New Characterization Theorem

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Abstract

We study circuit codes with long bit runs (sequences of distinct transitions) and derive a formula for the maximum length for an infinite class of symmetric circuit codes with long bit runs. This formula also results in an improved lower bound on the maximum length for an infinite class of circuit codes without restrictions on symmetry or bit run length. We also present a new characterization of circuit codes of spread k based on a theorem of Deimer.

Keywords: Circuit Code, Snake in the Box, Coil in the Box, k-Coil, Error Correcting Code

1 Introduction

Let $I(d)$ denote the graph of the d -dimensional hypercube. A simple cycle $C = (x_1, \dots, x_N)$ in $I(d)$ is called a *circuit*. A circuit C has three important characteristics: its ambient dimension d , its *spread* k which is the minimum distance in $I(d)$ two vertices x_i and $x_j \in C$ can have if they are not “close” in C , and its length N . Let G be a subgraph of $I(d)$ and let x and y be vertices of G . Define $d_G(x, y)$ as the minimum number of edges that need to be traversed in G to travel from x to y (with $d_G(x, y) = \infty$ if no such path exists). A circuit code of spread k can then be formally defined as follows.

Definition 1.1. *A subgraph C of $I(d)$ is a circuit code of spread k (a (d, k) circuit code) if:*

1. C is a circuit.
2. If x and y are vertices of C with $d_{I(d)}(x, y) < k$ then $d_C(x, y) = d_{I(d)}(x, y)$.

A useful alternate characterization of a spread k circuit code was given by Klee.

Lemma 1.2 (Klee [14] Lemma 2). *A d -dimensional circuit code C of length $N \geq 2k$ has spread k if and only if for all vertices $x, y \in C$, $d_C(x, y) \geq k \Rightarrow d_{I(d)}(x, y) \geq k$.*

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Circuit codes were first introduced in [13]. Since then they have been extensively studied as both combinatorial objects (generalizing the well-known Snake in the Box problem [15]) and as a type of error-correcting code ([17, 12, 3, 2, 21, 4]).¹

In this note we study circuit codes with long bit runs (sequences of distinct transitions between cyclically consecutive vertices). We develop structural results and upper and lower bounds on the maximum length of such codes when they are symmetric (Lemmas 3.5 - 3.8). These results lead to an exact formula for the maximum length of a symmetric (d, k, r) circuit code when the spread k is odd and the bit run length r is maximum relative to d and k (Theorem 3.9). Furthermore, Theorem 3.9 results in an improved lower bound on the maximum length of a (d, k) circuit code, without restriction on symmetry or bit run length, when d and k satisfy the same assumptions as in Theorem 3.9 (Corollary 3.10). We also prove a new characterization theorem for circuit codes with spread k based upon a necessary condition of [5] (Theorem 4.4).

2 Transition Sequences

Each vertex of $I(d)$ corresponds to a binary vector of length d , so for every circuit $C = (x_1, \dots, x_N)$ of $I(d)$ we can define a *transition sequence* $T = (\tau_1, \dots, \tau_N)$ where τ_i denotes the position in which x_i and x_{i+1} (or x_N and x_1) differ. Using the convention that $x_1 = \vec{0}$ for any circuit, we see that the transition sequence corresponds uniquely to the edges in C . Since $I(d)$ is bipartite this implies $|T|$ is even [10].

Define a *segment* of a sequence $T = (\tau_1, \dots, \tau_N)$ as a subsequence of cyclically consecutive elements. For any $x_i, x_j \in C = (x_1, \dots, x_N)$ with $i < j$ there are exactly two segments in T between x_i and x_j , corresponding to the two paths in C traversing the edges: $x_i x_{i+1}, \dots, x_{j-1} x_j$ and $x_j x_{j+1}, \dots, x_{N-1} x_N, x_N x_1, \dots, x_{i-1} x_i$. These segments are $(\tau_i, \tau_{i+1}, \dots, \tau_{j-1})$ and $(\tau_j, \tau_{j+1}, \dots, \tau_N, \tau_1, \dots, \tau_{i-1})$. If $i = j$ then the two segments are \emptyset and T . These segments are called complements because they partition T . If \hat{T} is a segment in T , its complement is denoted \hat{T}^c , and $(\hat{T}^c)^c = \hat{T}$.

The set of *transition elements* $\{t_1, \dots, t_m\}$ ($m \leq d$) of T are the unique elements of T . When T is the transition sequence of a circuit each $t_i \in \{t_1, \dots, t_m\}$ must appear in T an even number of times. Without loss of generality, we assume $m = d$, otherwise the code can be embedded in a lower dimension.

Definition 2.1. For a segment \hat{T} of T , let $\delta(\hat{T})$ denote the number of transition elements in \hat{T} that appear with odd parity.

Observe that if \hat{T} corresponds to a path in C between vertices $x, y \in C$, then $d_{I(d)}(x, y) = \delta(\hat{T})$. For notational convenience (especially in Lemmas 3.5-3.7) we sometimes assume that a certain segment \hat{T} of T constitutes its first m transitions or that the first m transitions of T are $(1, \dots, m)$ for some appropriate value of m . This is done without loss of generality since we can cyclically shift the indices of vertices in C (and thus of the transitions in T) and are free to label $\{t_1, \dots, t_d\}$ according to any permutation of $\{1, \dots, d\}$.

¹For a survey of circuit codes, see [9] chapter 17.

3 Circuit Codes with Long Bit Runs

Given a (d, k) circuit code C with transition sequence T , the *maximum bit run*, $\phi(C)$, denotes the length of the longest segment of T that does not repeat a transition element. Similarly, the *minimum bit run*, $\xi(C)$, denotes the maximum length an arbitrary segment of T can have without repeating a transition. It is easy to see that $\phi(C) \leq d$, and if $N = |C| > 2k$ then $\xi(C) \geq k + 1$. The special case where $k = 1$ and C is a Hamiltonian circuit has been extensively studied [8, 7, 20], and there exist such C where $\xi(C) \geq d - 3 \log_2 d$ [7]. An upper bound on $\phi(C)$ (in terms of d) is given by the following theorem.

Theorem 3.1 (Singleton [19] Theorem 3). *If C is a (d, k) circuit code with $N = |C| > 2d$, then $\phi(C) \geq k + 2$ implies $d \geq k + 1 + \lfloor \frac{\phi(C)}{2} \rfloor$.*

Let $\mathcal{F}(d, k, r)$ denote the set of spread k circuit codes in dimension d with $\phi(C) \geq r$. Note that $\mathcal{F}(d, k, r + 1) \subseteq \mathcal{F}(d, k, r)$ and $\mathcal{F}(d, k + 1, r) \subseteq \mathcal{F}(d, k, r)$. Let $L(d, k, r)$ denote the maximum length of an element of $\mathcal{F}(d, k, r)$, i.e. $K(d, k)$ for $C \in \mathcal{F}(d, k, r)$. Douglas [6] proved the following results.

Proposition 3.2 (Douglas [6] Remark (6)). *For k even and l odd, with $k \geq 2l - 2$, $L(\frac{3k}{2} + \frac{l+1}{2}, k, k + l) \leq 4k + 3l - 1$.*

Proposition 3.3 (Douglas [6] Remark (7)). *For k odd and l even, with $k \geq 2l + 1$, $L(\frac{3k}{2} + \frac{l+1}{2}, k, k + l) \leq 4k + 3l + 2$.*

In addition [6] established exact values for $K(d, k)$ for the following cases (which can be interpreted as tight upper bounds for the length of $C \in \mathcal{F}(d, k, k + 1)$). Notice that the upper bounds of Propositions 3.2 and 3.3 are not tight for the cases below.

Theorem 3.4.

- (i) $K(\frac{3k}{2} + 2, k) = 4k + 6$ for k even. ([6] Theorem 3).
- (ii) $K(\lfloor \frac{3k}{2} \rfloor + 2, k) = 4k + 4$ for k odd. ([6] Theorem 4).
- (iii) $K(\lfloor \frac{3k}{2} \rfloor + 3, k) = 4k + 8$ for k odd and ≥ 9 . ([6] Theorem 5).

Formulas for the exact value of the maximum length of a circuit code are extremely rare, in fact the only non-trivial formulas known are those in Theorem 3.4. The main result of this section (Theorem 3.9) is a new formula for the maximum length of symmetric $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k + l)$ when k is odd and l is even ≥ 2 with $k \geq 2l + 1$, and a new lower bound on $K(\frac{3k}{2} + \frac{l+1}{2}, k)$ that improves upon the best known lower bound when k and l satisfy these conditions (Corollary 3.10).

We begin with a technical lemma showing that codes with the longest bit run possible in their dimension (per Theorem 3.1) have $\xi(C)$ minimum. Our argument is a generalization of an approach used by [19].

Lemma 3.5. *Let C be a $(\frac{3k}{2} + \frac{l+1}{2}, k, k + l)$ circuit code where: k is even and l odd, or k odd and l even, and having length $N > 2d = 3k + (l + 1)$. Then $\xi(C) = k + 1$.*

Furthermore, any segment $\hat{T} = (\tau_{r+1}, \dots, \tau_{r+2k+l+1})$ of the transition sequence T of C where $\tau_{r+1}, \dots, \tau_{r+k+l}$ are all distinct has $\delta(\hat{T}) = k$ and all $\frac{3k}{2} + \frac{l+1}{2}$ transition elements appear in \hat{T} .

Proof. We assume that the transition sequence T of C begins with the segment:

$$\hat{T} = \underbrace{\tau_1, \dots, \tau_{k+l}}_{\omega_1}, \underbrace{\tau_{k+l+1}, \dots, \tau_{2k+l+1}}_{\omega_2}, \alpha$$

where ω_1 consists of $k+l$ distinct transitions and all the transitions in ω_2 are distinct. Note that since $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$ its transition sequence T contains a segment \hat{T} of $k+l$ distinct transitions, and the next $k+1$ transitions must all be distinct since $\xi(C) \geq k+1$. Without loss of generality these $2k+l+1$ transitions are the first $2k+l+1$ transitions of T .

Let n_1 denote the number of transitions appearing once in (ω_1, ω_2) and let n_2 denote the number of transitions appearing twice. Then $n_1 \geq k$ (as $k < |(\omega_1, \omega_2)| < N - k$, so $\delta((\omega_1, \omega_2)) \geq k$), $n_1 + n_2 \leq d$, and $n_1 + 2n_2 = 2k+l+1$. Thus $2d \geq 3k+l+1$ with equality implying $n_1 = k$. Since $d = \frac{1}{2}(3k + (l+1))$ we see that $\delta((\omega_1, \omega_2)) = n_1 = k$ and $n_2 = \frac{1}{2}(k + (l+1)) = d - n_1$. Hence all transitions occur in (ω_1, ω_2) and thus in \hat{T} .

Now $\delta((\omega_1, \omega_2)) = n_1 = k$, so $\delta(\hat{T}) = \delta((\omega_1, \omega_2, \alpha)) = \delta((\omega_1, \omega_2)) \pm 1$ must equal $k+1$ (as $N > 2d$). Thus α occurs twice in (ω_1, ω_2) , so $\alpha \in \omega_2$. Hence (ω_2, α) is a segment of T of size $k+2$ with a repeated transition, so $\xi(C) = k+1$. \square

Recall that a *symmetric* code is one whose transition sequence $T = (\tau_1, \dots, \tau_N)$ satisfies $\tau_i = \tau_{i+N/2}$ for $1 \leq i \leq N/2$. In the next two results we use the (ω_1, ω_2) structure from Lemma 3.5 and related observations to derive new upper bounds on the length of symmetric $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$, substantially improving upon the general case upper bounds of Propositions 3.2 and 3.3.

Lemma 3.6. *Let k be even and l be odd ≥ 3 with $k \geq 2l-2$, or k is odd and l even ≥ 2 with $k \geq 2l+1$. Let $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$ be symmetric, then $|C| \leq 4k + 2(l+1)$.*

Proof. Assume not, then without loss of generality the transition sequence T of C contains a segment

$$\hat{T} = \underbrace{\tau_1, \dots, \tau_{k+l}}_{\omega_1}, \underbrace{\tau_{k+l+1}, \dots, \tau_{2k+l+1}}_{\omega_2}, \alpha$$

where all $k+l$ transitions in ω_1 are distinct and all of the $k+1$ transitions in ω_2 are distinct. We know (by Lemma 3.5) that $\delta((\omega_1, \omega_2)) = k$, thus α must appear an even number of times in (ω_1, ω_2) . Since all transitions appear in (ω_1, ω_2) it follows that $\alpha \in \omega_2$, so $\alpha = \tau_{k+l+1}$, and $\alpha \in \omega_1$, so $\tau_{k+l+1} \in \{\tau_1, \dots, \tau_l\}$ (as $\xi(C) = k+1$).

Now consider the segment $\beta = (\alpha, \dots, \tau_1, \dots, \tau_l)$ of T . If k is even then $|\beta| \leq \frac{l}{2} - \frac{3}{2} + l = \frac{3(l-1)}{2}$ by Proposition 3.2. If k is odd, then $|\beta| \leq \frac{3l}{2}$ by Proposition 3.3. In the even case, $\frac{3(l-1)}{2} < k$ since $k \geq 2l-2$, in the odd case $\frac{3l}{2} < k$ since $k \geq 2l+1$. Hence by Definition 1.1 we require $\delta(\beta) = |\beta|$ in both cases, i.e. all transitions in β must be distinct. But since $\alpha \in \{\tau_1, \dots, \tau_l\}$ this is impossible and we reach a contradiction. \square

Lemma 3.7. *Let k be odd and l even ≥ 2 with $k \geq 2l + 1$. Then for $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$ and symmetric, $|C| \leq 4k + 2l$.*

Proof. From Lemma 3.6 we have $|C| \leq 4k + 2(l + 1)$. Assume for contradiction that $|C| = 4k + 2(l + 1)$, then without loss of generality the transition sequence T of C has the form:

$$T = \underbrace{1, 2, \dots, l}_{\omega_1}, \underbrace{l+1, \dots, k+l}_{\omega_2}, \underbrace{\alpha_1, \dots, \alpha_k, \alpha_{k+1}}_{\omega_3}, 1, 2, \dots, l, l+1, \dots, k+l, \alpha_1, \dots, \alpha_k, \alpha_{k+1}.$$

Let \hat{T} denote the segment $(\omega_1, \omega_2, \omega_3, \alpha_{k+1})$. From the Lemma 3.5 we know that all $d = \frac{3k}{2} + \frac{l+1}{2}$ transition elements are used in \hat{T} , that each transition appears once or twice in \hat{T} , and that $\delta(\hat{T}) = k$. From this we have $k \leq \delta((\omega_1, \omega_2, \omega_3)) = \delta(\hat{T}) \pm 1$, so $\delta((\omega_1, \omega_2, \omega_3)) = k + 1$ and $\alpha_{k+1} \in (\omega_1, \omega_2, \omega_3)$. Hence all transition elements appear in $(\omega_1, \omega_2, \omega_3)$. Furthermore, since $\xi(C) = k + 1$, $\alpha_{k+1} \notin \omega_3$, so $\alpha_{k+1} \in (\omega_1, \omega_2)$.

Define β as (ω_2, ω_3) . Clearly $|\beta| = 2k$, and since $|C| = 4k + 2(l + 1)$ this means $\delta(\beta) \geq k$. A precise value for $\delta(\beta)$ is given by

$$\delta(\beta) = (d - (k + l)) + s_1 + (k - s_2) \quad (1)$$

where $d - (k + l)$ is the number of transition elements in $\{1, \dots, d\}$ that have not appeared in (ω_1, ω_2) , s_1 is the number of transitions in ω_3 that appear in ω_1 , and s_2 is the number of transitions in ω_3 that appear in ω_2 . Furthermore

$$|\omega_3| = k = (d - (k + l)) + s_1 + s_2. \quad (2)$$

From (1) and (2) we deduce the following relationships:

$$(1/2)(k + l - 1) = s_1 + s_2 \quad (3)$$

$$s_1 \geq \lceil (l - 1)/2 \rceil = l/2 \quad (4)$$

$$s_2 \leq \lfloor (1/2)k \rfloor = (1/2)(k - 1). \quad (5)$$

Now consider the segment $\gamma = (\omega_3, \alpha_{k+1}, \omega_1)$ of T . This segment has size $k + 1 + l$ and so $k \leq \delta(\gamma)$. An upper bound on $\delta(\gamma)$ is given by $(d - (k + l)) + (l - s_1) + s_2 + 1$. Thus we require

$$2k - d + s_1 \leq s_2 + 1. \quad (6)$$

The only values of s_1 and s_2 consistent with (3) – (6) are $s_1 = \frac{l}{2}$ and $s_2 = \frac{k-1}{2}$. Plugging these into (1) we get $\delta(\beta) = k + 1$. Recall that we have shown $\alpha_{k+1} \in (\omega_1, \omega_2)$. Clearly $\alpha_{k+1} \notin \omega_1$ since (α_{k+1}, ω_1) is a segment of T of size $< k$ and thus all transitions in (α_{k+1}, ω_1) must be distinct since C has spread k .

Therefore $\alpha_{k+1} \in \omega_2$ and so $\delta(\beta, \alpha_{k+1}) = \delta(\beta) - 1 = k$. Thus $\delta(\beta, \alpha_{k+1}, 1)$ must equal $k + 1$ and hence $1 \notin \{\alpha_1, \dots, \alpha_{k+1}\}$. But since $\delta(\hat{T}) = k$ and 1 occurs only once in \hat{T} , this implies $\delta((2, \dots, l, \omega_2, \alpha_1, \dots, \alpha_{k+1})) = k - 1$ a contradiction.

Therefore we reach a contradiction and conclude that $|C| \leq 4k + 2l$. \square

Unlike the case with the upper bounds of Propositions 3.2 and 3.3, the upper bound of $4k + 2l$ can be achieved. This is shown in the following example which is based on a canonical augmentation approach similar to [16].

Example 1. *We wish to see if a symmetric $(16, 9, 9 + 4)$ circuit code of length 44 exists. If such a code C exists then, without loss of generality, it has transition sequence*

$$T = \underbrace{1, \dots, 4}_{\omega_1}, \underbrace{5, \dots, 13}_{\omega_2}, \underbrace{\alpha_1, \dots, \alpha_9}_{\omega_3}, 1, \dots, 4, 5, \dots, 13, \alpha_1, \dots, \alpha_9.$$

Since T is symmetric, all 16 transition elements appear in $(\omega_1, \omega_2, \omega_3)$. By construction and the fact that $\xi(C) = 10$, each transition element appears once or twice in $(\omega_1, \omega_2, \omega_3)$. Using $\delta((\omega_2, \omega_3)) \geq 9$ and $\delta((\omega_3, 1, \dots, 4)) \geq 9$ and the proof approach of Lemma 3.5 we find that: 2 members of ω_3 are in ω_1 , 4 members of ω_3 are in ω_2 , and 3 members of ω_3 are in $\{14, 15, 16\}$. From the structure of T we also deduce that $\alpha_i \notin \{1, \dots, i\}$ for $1 \leq i \leq 9$. Hence 2 members of $\{2, 3, 4\}$ are in $\{\alpha_1, \alpha_2, \alpha_3\}$. This greatly reduces the search space for possible $(16, 9, 9 + 4)$ codes.

Using canonical augmentation of (ω_1, ω_2) plus the refinements mentioned above, we find the following $(16, 9, 9 + 4)$ code of length 44:

$$T = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 2, 4, 6, 14, 8, 15, 10, 16, 12,$$

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 2, 4, 6, 14, 8, 15, 10, 16, 12.$$

Example 1 suggests a general structure for symmetric $(\frac{3k}{2} + \frac{l+1}{2}, k, k + l)$ codes with k odd and l even ≥ 2 having length $4k + 2l$. Define the code C by the symmetric transition sequence whose first half is as follows:

$$\underbrace{1, \dots, k + l}_{\omega_1 \text{ length} = k+l}, \underbrace{2, 4, \dots, 2l - 2}_{\omega_2 \text{ length} = l-1}, \underbrace{\gamma_1, \beta_1, \gamma_2, \beta_2, \dots, \gamma_{d-(k+l)}, \beta_{d-(k+l)}}_{\omega_3 \text{ length} = k+1-l}. \quad (7)$$

Here β_1 through $\beta_{d-(k+l)}$ are $2l, 2l + 2, \dots, k + l - 1$ and γ_1 through $\gamma_{d-(k+l)}$ are $k + l + 1, k + l + 2, \dots, d$.

Lemma 3.8. *The symmetric transition sequence in (7) defines a circuit code $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k + l)$ for k odd and l even ≥ 2 having length $4k + 2l$.*

Proof. Let T be the symmetric transition sequence whose first half is defined as in (7) and let C be the attendant $d(= \frac{3k}{2} + \frac{l+1}{2})$ -dimensional circuit code defined by T . Evidently $\phi(C) \geq k + l$ and $|C| = 4k + 2l$, so all that needs to be verified is the spread.

Partition T as $(\underbrace{\omega_1, \omega_2, \omega_3}_A, \underbrace{\omega_1, \omega_2, \omega_3}_B)$ where $\omega_1 = 1, \dots, k + l$, $\omega_2 = 2, 4, \dots, 2l - 2$, and $\omega_3 = \gamma_1, \dots, \beta_{d-(k+l)}$. Select x and y arbitrarily from all pairs of vertices $u, v \in C$ with $d_C(u, v) \geq k$. We will prove that $d_{I(d)}(x, y) \geq k$, hence by Lemma 1.2 C has spread $\geq k$. Let $\hat{T} = (\tau_i, \dots, \tau_{j-1})$ denote a shortest segment in T between x and y .

Case I. \hat{T} is a segment of A .

Hence $\tau_i \in \omega_m$ and $\tau_{j-1} \in \omega_n$ with $m \leq n$ (else $|\hat{T}| > 2k + l$).

Subcase I.1. τ_i and τ_{j-1} are both in ω_s for $s \in \{1, 2, 3\}$.

Each ω_s contains no repeated transition elements, hence $\delta(\hat{T}) = |\hat{T}| = d_C(x, y) \geq k$.

Subcase I.2 $\tau_i \in \omega_1$ and $\tau_{j-1} \in \omega_2$.

Then $\tau_i = a \in \{1, \dots, k+l\}$ and $\tau_{j-1} = 2b \in \{2, 4, \dots, 2l-2\}$. We may assume $2b \geq a$ or else no transitions occur more than once (and hence $\delta(\hat{T}) = |\hat{T}| \geq k$). Only those transitions in $D = \{2\lceil \frac{a}{2} \rceil, 2\lceil \frac{a}{2} \rceil + 2, \dots, 2b\}$ are repeated and thus $\delta(\hat{T}) = (k+l-a+1)+b-2|D|$. The size of D is $b - \lceil \frac{a}{2} \rceil + 1$, so $\delta(\hat{T}) = (k+l-a+1)+b-2(b - \lceil \frac{a}{2} \rceil + 1) \geq k+l-b-1 \geq k$.

Subcase I.3. $\tau_i \in \omega_1$ and $\tau_{j-1} \in \omega_3$.

Then $\tau_i = a \in \omega_1$ and $\tau_{j-1} = \beta_s$ or $\gamma_s \in \omega_3$. Consider $\tilde{T} = (\tau_i = a, \dots, k+l, 2, 4, \dots, \beta_u)$ where $u = s$ if $\tau_{j-1} = \beta_s$ and $u = s-1$ if $\tau_{j-1} = \gamma_s$ (with $\beta_0 = 2l-2$). Since each γ_u occurs only once in A it is clear that $\delta(\tilde{T}) \geq \delta(\hat{T})$. We may assume $2\lceil \frac{a}{2} \rceil \leq \beta_u$ since otherwise no transitions are repeated in \tilde{T} and hence in \hat{T} , implying $\delta(\hat{T}) \geq k$. Only the transitions in $D = \{2\lceil \frac{a}{2} \rceil, 2\lceil \frac{a}{2} \rceil + 2, \dots, \beta_u = 2(l+u-1)\}$ are repeated. Since $|D| = (l+u-1 - \lceil \frac{a}{2} \rceil) + 1$ we have $\delta(\tilde{T}) = (k+l-a+1) + (l-1) + 2u - 2|D| \geq k$.

Subcase I.4. $\tau_i \in \omega_2$ and $\tau_{j-1} \in \omega_3$.

Since all transitions in (ω_2, ω_3) are distinct, $\delta(\hat{T}) = |\hat{T}| = d_C(x, y) \geq k$.

Clearly the analysis is the same if both τ_i and $\tau_{j-1} \in B$.

Case II. $\tau_i \in A$ and $\tau_{j-1} \in B$.

In this case we have $\tau_i \in \omega_m$ and $\tau_{j-1} \in \omega_n$ where $m \geq n$ (else $|\hat{T}| > 2k+l$ and \hat{T} is not a shortest segment in T between x and y .)

Subcase II.1. τ_i and τ_{j-1} are both in ω_s for $s \in \{1, 2, 3\}$.

Then τ_i is the a th element of ω_s in A and τ_{j-1} is the b th element of ω_s in B , and since \hat{T} is a shortest segment between x and y in T we have $b < a$. Since $\delta(\hat{T})$ does not depend on the ordering of the transitions in \hat{T} , rearrange \hat{T} as: $\hat{T} = 1, \dots, \underbrace{\tau_{j-1}}_{\text{was in } B}, \tau_i, \dots, \underbrace{\beta_{d-(k+l)}}_{\text{was in } A}$.

Define m as $j - (2k+l)$, since C is symmetric the sequence of values of the transitions in \hat{T} is the same as in $T' = (\tau_1, \tau_2, \dots, \tau_{m-1}, \tau_i, \tau_{i+1}, \dots, \tau_{2k+l})$ (note since $b < a$ we have $m-1 < i$) and thus $\delta(\hat{T}) = \delta(T')$. Furthermore T' is a subsequence of $\tilde{T} = (\tau_1 = 1, \dots, \tau_{2k+l} = \beta_{d-(k+l)})$ so the only transitions occurring twice are those even elements in $\{1, \dots, k+l\}$ and no transition occurs three times or more. Observe that by construction of T , $\delta(\tilde{T}) = k+1$. Also note that the only transitions in \tilde{T} absent from T' are $(\tau_m, \tau_{m+1}, \dots, \tau_{i-1})$.

Suppose $s = 1$, then $(\tau_m, \tau_{m+1}, \dots, \tau_{i-1}) = (\tau_{m-1} + 1, \tau_{m-1} + 2, \dots, \tau_i - 1)$. Let $D = \{\tau_{m-1} + 1, \dots, \tau_i - 1\}$, then $\delta(\hat{T}) = \delta(\tilde{T}) - |\{t \in D \mid t \text{ odd}\}| + |\{t \in D \mid t \text{ even}\}| \geq k+1 - |\{t \in D \mid t \text{ odd}\}| + (|\{t \in D \mid t \text{ odd}\}| - 1) \geq k$.

Suppose $s = 2$, then trivially $\delta(\hat{T}) \geq \delta(\tilde{T}) = k+1$ as we are only removing single instances of transitions that appear twice in $1, \dots, \beta_{d-(k+l)}$.

Finally, suppose $s = 3$. Each transition in $D_1 = \{\tau_m, \tau_{m+1}, \dots, \tau_{i-1}\} \cap \{1, \dots, k+l\}$ increases $\delta(\hat{T})$ by 1 relative to $\delta(\tilde{T})$ as we are removing from \tilde{T} a single instance of a duplicated transition. Each transition in $D_2 = \{\tau_m, \tau_{m+1}, \dots, \tau_{i-1}\} - \{1, \dots, k+l\}$ decreases $\delta(\hat{T})$ by 1 relative to $\delta(\tilde{T})$, as we are removing from \tilde{T} a transition that only occurred once. Since the elements in D_1 and D_2 alternate in $(\tau_m, \tau_{m+1}, \dots, \tau_{i-1})$ (as the

β_u 's and γ_u 's) we see that $\delta(\hat{T}) \geq \delta(\tilde{T}) - 1 = k$.

Subcase II.2. $\tau_i \in \omega_3$ and $\tau_{j-1} \in \omega_1$.

Then τ_i is the a th element of ω_3 in A and τ_{j-1} is the b th element of ω_1 in B . If $b < \beta_{\lceil \frac{a}{2} \rceil} = 2(l + \lceil \frac{a}{2} \rceil - 1)$ then all transitions in \hat{T} are distinct, so $\delta(\hat{T}) \geq k$. Otherwise, only those transitions in $D = \{\beta_{\lceil \frac{a}{2} \rceil} = 2(l + \lceil \frac{a}{2} \rceil - 1), 2(l + \lceil \frac{a}{2} \rceil), \dots, 2\lfloor \frac{b}{2} \rfloor\}$ are repeated. Since $|D| \leq (b - 2(l + \lceil \frac{a}{2} \rceil - 1))/2 + 1$ we have $\delta(\hat{T}) \geq (k + 1 - l - a + 1) + b - 2|D| \geq k + l - 2 \geq k$.

Subcase II.3. $\tau_i \in \omega_3$ and $\tau_{j-1} \in \omega_2$.

Then τ_i is the a th element of ω_3 and $\tau_{j-1} = 2b \in \{2, 4, \dots, 2l - 2\}$. Adding each element in $\{2b + 2, 2b + 4, \dots, 2l - 2\}$ to \hat{T} to get a new segment $\tilde{T} = (\tau_i, \dots, \beta_{d-(k+l)}, \omega_1, \omega_2)$ results in $\delta(\hat{T}) \geq \delta(\tilde{T})$ since each of the added transitions occurred exactly once in \hat{T} . If $a = 1$ then \tilde{T} can be rearranged as $(1, \dots, \beta_{d-(k+l)})$ and $\delta(\hat{T}) \geq \delta(\tilde{T}) > k$. If $a > 1$ and $|\tilde{T}| = 2k + l - 1$, update $\tilde{T} \rightarrow (\tilde{T}, \gamma_1)$. Now $\delta(\hat{T}) \geq \delta(\tilde{T}) - 1$ and $\delta(\tilde{T}) = \delta(1, \dots, \beta_{d-(k+l)})$ (after rearrangement) $= k + 1$. Otherwise ($a > 1$ and $|\tilde{T}| \leq 2k + l - 2$), update $\tilde{T} \rightarrow (\tilde{T}, \gamma_1, \beta_1)$. Then $\delta(\hat{T}) = \delta(\tau_i, \dots, \beta_{d-(k+l)}, \omega_1, \omega_2) + 1$ (from γ_1) $- 1$ (from β_1) $\leq \delta(\hat{T})$. Now \tilde{T} falls under subcase II.1 (with $s = 3$), hence $k \leq \delta(\tilde{T}) \leq \delta(\hat{T})$. In both $a > 1$ cases we have $\delta(\hat{T}) \geq k$.

Subcase II.4. $\tau_i \in \omega_2$ and $\tau_{j-1} \in \omega_1$.

Then $\tau_i = 2a \in \{2, 4, \dots, 2l - 2\}$ and τ_{j-1} is the b th element of ω_1 . If $2a > b$ then no transitions are repeated in \hat{T} and $\delta(\hat{T}) = |\hat{T}| = d_C(x, y) \geq k$. Otherwise ($2a \leq b$) add $2, 4, \dots, 2a - 2$ to the beginning of \hat{T} to get a new segment $\tilde{T} = (\omega_2, \omega_3, 1, \dots, b = \tau_{j-1})$ with $\delta(\hat{T}) \geq \delta(\tilde{T})$. If $b = k + l$ then \tilde{T} can be rearranged as $(1, \dots, \beta_{d-(k+l)})$ and $\delta(\hat{T}) \geq \delta(\tilde{T}) > k$. If $b < k + l$ and $|\tilde{T}| = 2k + l - 1$, update $\tilde{T} \rightarrow (k + l, \tilde{T})$. Now $\delta(\hat{T}) \geq \delta(\tilde{T}) - 1$ and $\delta(\tilde{T}) = \delta(1, \dots, \beta_{d-(k+l)}) = k + 1$. Otherwise ($b < k + l$ and $|\tilde{T}| \leq 2k + l - 2$), update $\tilde{T} \rightarrow (k + l - 1, k + l, \tilde{T})$. Then $\delta(\hat{T}) = \delta(\omega_2, \omega_3, 1, \dots, b) + 1$ (from $k + l$) $- 1$ (from $k + l - 1$) $\leq \delta(\hat{T})$. Now \tilde{T} falls under subcase II.1 (with $s = 1$), hence $\delta(\tilde{T}) \geq k$. In both $b < k + l$ cases we have $\delta(\hat{T}) \geq k$.

Thus in all cases $d_{I(d)}(x, y) = \delta(\hat{T}) \geq k$, proving the claim. \square

Following from Lemmas 3.7 and 3.8 we have a formula for the maximum length for a certain class of symmetric (d, k, r) circuit codes.

Theorem 3.9. *Let k be odd and let l be even ≥ 2 with $k \geq 2l + 1$. Then the maximum length of a symmetric code $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k + l)$ is exactly $4k + 2l$.*

The best known lower bound on $K(d, k)$ for general d and odd k was given in [19] (stronger bounds are known for $k = 2, 3, 4$ [1, 19, 2] or for fixed k as $d \rightarrow \infty$ [18]):

$$K(d, k) \geq (k + 1)2^{\lfloor \frac{2d}{k+1} \rfloor - 1}, \text{ when } k \text{ odd and } \left\lfloor \frac{2d}{k+1} \right\rfloor \geq 2. \quad (8)$$

For k and l satisfying the conditions of Theorem 3.9 and $d = \frac{3k}{2} + \frac{l+1}{2}$ we have $\lfloor \frac{2d}{k+1} \rfloor = 3$ and thus (8) implies $K(d, k) \geq 4k + 4$. Clearly any symmetric $C \in \mathcal{F}(d, k, r)$ is a (d, k) circuit code and $4k + 4 \leq 4k + 2l$ for $l \geq 2$. Thus Theorem 3.9 implies an improved lower bound on $K(\frac{3k}{2} + \frac{l+1}{2}, k)$.

Corollary 3.10. *Let k be odd and let l be even ≥ 2 with $k \geq 2l+1$. Then $K(\frac{3k}{2} + \frac{l+1}{2}, k) \geq 4k + 2l$.*

Remark 3.11. *In the proof of Theorem 3.4 (ii) in [6] it is shown that all maximum length $(\lfloor \frac{3k}{2} \rfloor + 2, k)$ codes (k odd) are isomorphic. This is not true for $K(\lfloor \frac{3k}{2} \rfloor + 3, k)$ codes with k odd. Consider the $(16, 9)$ code C defined by the transition sequence T of Example 1, it has length $44 = K(16, 9)$ by Theorem 3.4 (iii). Another $(16, 9)$ code of length 44 is C' given by*

$$T' = 1, 11, 2, 12, 3, 13, 4, 14, 5, 16, 15, 6, 11, 7, 12, 8, 13, 9, 14, 16, 10, 15, \\ 1, 11, 2, 12, 3, 13, 4, 14, 5, 16, 15, 6, 11, 7, 12, 8, 13, 9, 14, 16, 10, 15.$$

Because $\phi(C) = 13$ and $\phi(C') = 12$ the two codes are not isomorphic.

An interesting implication of Theorem 3.9 and Theorem 3.4 (ii) and (iii) is that $L(\frac{3k}{2} + \frac{l+1}{2}, k, k+l) = K(\frac{3k}{2} + \frac{l+1}{2}, k)$ for odd $k \geq 9$ and $l = 2$ or 4 . For such k and l , the additional constraints that $\phi(C) \geq k+l$ or even that C be symmetric do not affect the maximum code length. Since (Remark 3.11) not all such maximum length codes are isomorphic or satisfy $\phi(C) \geq k+l$, this implication appears non-trivial and leads us to conjecture the following generalization of Theorem 3.4.

Conjecture 3.12. *Let k be odd ≥ 9 and l even ≥ 2 with $k \geq 2l+1$. Then there exists symmetric $C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$ attaining length $K(d, k)$. Hence $K(\frac{3k}{2} + \frac{l+1}{2}, k) = 4k + 2l$ for such (k, l) pairs.*

4 A Characterization Theorem for Circuit Codes of Spread k

Let C be a (d, k) circuit code of length N with transition sequence $T = (\tau_1, \dots, \tau_N)$ and transition elements $\{t_1, \dots, t_d\}$. For each $i \in \{1, \dots, d\}$ let T^i define the subsequence of T resulting from removing t_i .

Definition 4.1. *Given a (d, k) circuit code C with transition sequence T , the i th sub-circuit code C^i is the walk in $I(d)$ induced by the sequence T^i for $i \in \{1, \dots, d\}$*

Although it may not be apparent from Definition 4.1, as long as C^i is sufficiently long it is a $(d-1, k-1)$ circuit code, as shown by the following result.

Theorem 4.2 (Deimer [5] Theorem 1). *Let C be an (d, k) circuit code of length N with transition sequence $T = (\tau_1, \dots, \tau_N)$ and transition elements $\{t_1, \dots, t_d\}$. Let n_i denote the number of times t_i occurs in T for $i \in \{1, \dots, d\}$. If $|C^i| \geq 2(k-1)$ then C^i is a $(d-1, k-1)$ circuit code of length $N - n_i$.*

If $k \geq 1$ and $|C| > 4(k-1)$ (as will be the case in Theorem 4.4) then the requirement on $|C^i|$ is easily satisfied. To see this, note that when $k \geq 1$ that C contains no repeated vertices and hence no transition element t_i can appear twice consecutively. Thus $|C^i| \geq N/2 > 2(k-1)$ for each $i \in \{1, \dots, d\}$.

Theorem 4.2, in conjunction with our results from Section 3 yields a corollary which appears to fill a ‘‘gap’’ in the results of Theorem 3.4.

Corollary 4.3. *Let k be even and l odd ≥ 3 with $k \geq 2l - 2$. Then $K(\frac{3k}{2} + \frac{l+1}{2}, k) \geq 4k + 2l$. In particular this implies $K(\frac{3k}{2} + 3, k) \geq 4k + 10$ for k even ≥ 8 .*

Proof. Define $k' = k + 1$ and $l' = l - 1$, then k' is odd and l' is even ≥ 2 with $k' \geq 2l' + 1$. Let $d' = \frac{3k'}{2} + \frac{l'+1}{2}$ and let $C \in \mathcal{F}(d', k', k' + l')$ be defined by the transition sequence T given in (7). Then $|C| = 4k' + 2l' = 4k + 2l + 2$. Observe that $t_{d'}$ occurs only twice in T , so by Theorem 4.2, $C^{t_{d'}}$ is a $(d' - 1, k' - 1) = (\frac{3k}{2} + \frac{l+1}{2}, k)$ circuit code of length $4k + 2l$, thus $K(\frac{3k}{2} + \frac{l+1}{2}, k) \geq 4k + 2l$. Taking $l = 5$ yields $K(\frac{3k}{2} + 3, k) \geq 4k + 10$ for $k \geq 8$. \square

The converse of Theorem 4.2 also holds, giving an alternate (to Lemma 1.2) characterization for circuit codes of spread k as we will now show.

Theorem 4.4. *Let $k \geq 2$, $d \geq k$, and let C be a d -dimensional circuit code with length $N > 4(k - 1)$. Then C has spread k if and only if C^i is a $(d - 1, k - 1)$ circuit code for $i = 1, \dots, d$.*

Proof. Suppose C has spread k , it immediately follows that C^i is a $(d - 1, k - 1)$ circuit code for $i = 1, \dots, d$ by Theorem 4.2. Now suppose that C^i is a $(d - 1, k - 1)$ circuit code for $i = 1, \dots, d$, and let $x, y \in C$ such that $d_C(x, y) \geq k$. We will show that $d_{I(d)}(x, y) \geq k$, by Lemma 1.2 it follows that C has spread k .

We begin with some definitions. Let T be the transition sequence of C , and let \hat{T} be the shortest segment of T between x and y and let \hat{T}^c denote its complement. Also let $\{t_{\alpha(1)}, \dots, t_{\alpha(m)}\} \subseteq \{t_1, \dots, t_d\}$ denote the transition elements appearing in \hat{T} . For $i = 1, \dots, m$ let $T^{\alpha(i)}$ denote the subsequence of T formed by deleting $t_{\alpha(i)}$, let $C^{\alpha(i)}$ be the attendant subcircuit code, and let $x^{\alpha(i)}$ and $y^{\alpha(i)}$ be the projections of x and y onto this $d - 1$ dimensional space.

Next, we make a crucial observation: within any segment of T of length $\leq k$ the transition element t_i ($i \in 1, \dots, d$) can appear at most once, otherwise C^j would violate Definition 1.1 (for spread $k - 1$) for each $j \neq i$.

Since $x \neq y$ we may assume that $t_{\alpha(1)}$ occurs an odd number of times in \hat{T} . Then $d_{I(d)}(x, y) \geq d_{I(d-1)}(x^{\alpha(1)}, y^{\alpha(1)}) + 1 \geq \min\{d_{C^{\alpha(1)}}(x^{\alpha(1)}, y^{\alpha(1)}), k - 1\} + 1$. Since $|\hat{T}| \geq k$ (and $|\hat{T}^c| \geq |\hat{T}|$) and $t_{\alpha(1)}$ can occur at most once per k cyclically consecutive elements of T we have $d_{C^{\alpha(1)}}(x^{\alpha(1)}, y^{\alpha(1)}) \geq k - 1$ and hence $d_{I(d)}(x, y) \geq k$, proving the claim². \square

We remark that Theorem 4.4 may be of practical interest since it suggests circuit codes have a decomposition structure that might be exploitable by parallelized algorithms, for example when verifying the spread.

5 Conclusions

In this note we presented several new results on circuit codes. In Section 3 we investigated circuit codes with long bit runs, establishing the exact value of $K(d, k)$ for symmetric

²In $C^{\alpha(1)}$ the direction of the shortest segment of $T^{\alpha(1)}$ between $x^{\alpha(1)}$ and $y^{\alpha(1)}$ (e.g. from $x^{\alpha(1)}$ to $y^{\alpha(1)}$ or from $y^{\alpha(1)}$ to $x^{\alpha(1)}$) may be reversed from the direction of \hat{T} . But N is sufficiently large so that $d_{C^{\alpha(1)}}(x^{\alpha(1)}, y^{\alpha(1)})$ remains $\geq k - 1$.

$C \in \mathcal{F}(\frac{3k}{2} + \frac{l+1}{2}, k, k+l)$ when k is odd and l is even ≥ 2 with $k \geq 2l+1$ (Theorem 3.9) and an improved lower bound on $K(\frac{3k}{2} + \frac{l+1}{2}, k)$ for such (k, l) pairs (Corollary 3.10). In Section 4 we proved a new characterization of circuit codes of spread k (Theorem 4.4) that is a converse to Deimer's Theorem, and improved the lower bound on $K(\frac{3k}{2} + \frac{l+1}{2}, k)$ when k even and l odd ≥ 3 with $k \geq 2l-2$ (Corollary 4.3).

Several interesting questions remain open for investigation. Proving Conjecture 3.12 even for the case $l = 6$ would represent notable progress in computing exact values for $K(d, k)$. Furthermore, although the structural and upper and lower bounds presented here were developed for proving Theorem 3.9 it would be interesting to see if they could be adapted to other types of circuit codes (e.g. single track codes [11]) or different (d, k) ranges. Finally, it would be interesting to see if Theorem 4.4 could be further developed to lead to an efficient parallel algorithm.

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