# Do non-free LCD codes over finite commutative Frobenius rings exist? 

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#### Abstract

In this paper, we clarify some aspects on LCD codes in the literature. We first prove that a non-free LCD code does not exist over finite commutative Frobenius local rings. We then obtain a necessary and sufficient condition for the existence of LCD code over finite commutative Frobenius rings. We later show that a free constacyclic code over finite chain ring is LCD if and only if it is reversible, and also provide a necessary and sufficient condition for a constacyclic code to be reversible over finite chain rings. We illustrate the minimum Lee-distance of LCD codes over some finite commutative chain rings and demonstrate the results with examples. We also got some new optimal $\mathbb{Z}_{4}$ codes of different lengths which are cyclic LCD codes over $\mathbb{Z}_{4}$.


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## 1. Introduction

Codes over finite rings is quite a popular topic of interest. A linear code $C$ a su is called Linear Complementary Dual (LCD) code if $C$ meets its dual $C^{\perp}$ trivially. LCD codes were first investigated by Massey, he showed there a characterization of LCD codes and non-LCD codes over finite fields and demonstrated that asymptotically good LCD codes exist [20]. LCD codes have been widely applied in data storage, communications systems, consumer electronics, and cryptography. Carlet and Guilley shown an application of LCD codes against side-channel attacks and fault injection attacks, and presented several constructions of LCD codes [4]. Cyclic LCD codes over finite fields are also referred as reversible codes, Yang and Massey gave a necessary and sufficient condition for a cyclic code to have a complementary dual [26] and proved that reversible cyclic codes over finite fields are LCD codes. In [21], Massey showed that some cyclic LCD codes over finite fields are BCH codes, and also constructed reversible convolutional codes which are in fact LCD codes. Tzeng and Hartmann [25] proved that the minimum distance of a class of LCD codes is greater than that given by the BCH bound. Using the hull dimension spectra of linear codes, Sendrier showed that LCD codes meet the asymptotic GilbertVarshamov bound [24]. Dougherty et. al. developed a linear programming bound on the largest size

[^0]of an LCD code of given length and minimum distance [9]. Guneri et. al. studied quasi-cyclic complementary dual codes using their concatenated structure in [14] and [13]. Ding et al. constructed several families of cyclic LCD codes over finite fields and analyzed their parameters [7]. In [17], Li et al. studied a class of LCD BCH codes. Jin showed that some Reed-Solomon codes are equivalent to LCD codes [16]. In [6], the authors proved that any MDS code is equivalent to an LCD code and constructed LCD Maximum distance Separable codes. Jitman et. al. studied Complementary dual subfield linear codes over finite fields [2].

Recently, in [18], Liu and Liu studied LCD codes over finite chain rings and provided a necessary and sufficient condition for a free linear code to be a LCD code over finite chain ring. They also gave a sufficient condition for a linear code (not necessarily free) over a finite chain ring to be LCD code, which says "A linear code $C$ over a finite chain ring with generator matrix $G$ is LCD code if $\mathrm{GG}^{T}$ is invertible, where $G^{T}$ is the transpose of $G "[18$, Theorem 3.5]. They provided an example [18, Example 2] to state that the converse of [18, Theorem 3.5] is not in general true. However there is a mistake in their example. In this paper, we prove that the converse of [18, Theorem 3.5] is indeed true. This lead to the main result (see Theorem 2) of this paper, it proves that there are no non-free LCD codes over finite commutative local Frobenius rings by showing that any LCD code over a finite commutative Frobenius ring is the Chinese product of LCD codes over finite commutative local Frobenius rings (see Theorem 5). The other contributions are the characterizations of projection of LCD codes (see Theorem 3 ) and lift LCD codes (see Theorem 4) over a finite commutative local Frobenius ring. We also show that a free constacyclic code $C$ over finite chain ring is LCD code if and only if $C$ is reversible. We also prove a necessary and sufficient condition for a constacyclic code $C$ of length $n$ over finite chain rings to be reversible when $n$ is relatively prime to the characteristic of the finite chain ring.

The paper is organized as follows: In Section 2, we provide some basic tools which are required to understand the results of further sections. In Section 3, we discuss LCD codes over finite commutative Frobenius rings. Finally, Section 4 studies the complementary dual constacyclic codes over finite chain rings in more general setting by a uniform method.

## 2. Some notations and basic results of codes over finite commutative Frobenius rings

Throughout this section $R$ is a commutative finite ring with multiplicative unity 1 distinct to 0 . A commutative finite ring $R$ is Frobenius if $R$ as $R$-module is injective. Alternatively, we can say a finite ring is Frobenius if $R / J(R)$ is isomorphic to $\operatorname{soc}(R)$ (as $R$-modules), where $J(R)$ is the Jacobson radical and $\operatorname{soc}(R)$ is the socle of the ring $R$. Recall that the Jacobson radical is the intersection of all maximal ideals in the ring and the socle of the ring is the sum of the minimal $R$-submodules. A ring is a local ring if it has unique maximal ideal. A principal ideal ring is a ring such that each of its ideal is generated by a single element.

Let $R$ be a finite ring with maximal ideals $\mathfrak{m}_{1}, \cdots, \mathfrak{m}_{u}$ and $s_{1}, \cdots, s_{u}$ their indices of stability, respectively. Clearly $R / \mathrm{m}_{j}^{s_{j}}$ is a finite local ring with maximal ideal $\mathrm{m}_{j} / \mathrm{m}_{j}^{s_{j}}$. Then we have the ring epimorphisms

$$
\begin{array}{rllc}
\Phi_{j}: & R & \rightarrow & R_{j} \\
& a & \mapsto & a+\mathfrak{m}_{j} \tag{1}
\end{array}
$$

and $\operatorname{Ker}\left(\Phi_{j}\right)=m_{j}^{s_{j}}(1 \leq j \leq u)$. The ring epimorphisms $\Phi_{j}(1 \leq j \leq u)$ induce the following ring homo-
morphisms

$$
\begin{align*}
\Phi: & R
\end{align*} \rightarrow \begin{gathered}
R_{1} \times \cdots \times R_{u} \\
a \tag{2}
\end{gathered} \mapsto\left(\Phi_{1}(a), \cdots, \Phi_{u}(a)\right) .
$$

Since the maximal ideals $\mathrm{m}_{1}, \cdots, \mathrm{~m}_{u}$ of $R$ are coprime and $\bigcap_{j=1}^{u} \mathrm{~m}_{j}^{s_{j}}=\left\{0_{R}\right\}$, the ring homomorphism (4) is a ring isomorphism, by the Chinese remainder theorem, see [22, p.224]. We denote the inverse of this map by CRT and we say that $R$ is the Chinese product of rings $\left\{R_{j}\right\}_{j=1}^{u}$.
Theorem 1. [22, p.224] Let $R$ be a Frobenius ring, then

$$
R=C R T\left(R_{1}, R_{2}, \cdots, R_{u}\right),
$$

where $R_{j}$ is a local Frobenius ring.
That is $R_{j}:=R / \mathrm{m}_{j}^{S_{j}}$ is a local Frobenius ring for each $j$. The following is an example of a ring that is local Frobenius ring but not a chain ring. We shall use this ring to exhibit several of the results of the paper.

Example 2.1. [10] Let $R_{m}=\mathbb{Z}_{2}\left[u_{1}, u_{2}, \cdots, u_{m}\right] /\left\langle u_{1}^{2}, u_{2}^{2}, \cdots, u_{m}^{2}\right\rangle$, where $\left\langle u_{1}^{2}, u_{2}^{2}, \cdots, u_{m}^{2}\right\rangle$ denotes the ideal generated by $u_{1}^{2}, u_{2}^{2}, \cdots, u_{m}^{2}$, and $m$ is a positive integer. Then

$$
\mathcal{J}\left(R_{m}\right)=\left\langle u_{1}, u_{2}, \cdots, u_{m}\right\rangle \text { and } \operatorname{Soc}\left(R_{m}\right)=\left\langle\prod_{i=1}^{k} u_{i}\right\rangle .
$$

Thus $R_{m} / \mathcal{J}\left(R_{m}\right) \cong \operatorname{SoC}\left(R_{m}\right)$ (as $R_{m}$-modules), so $R_{m}$ is a finite commutative local Frobenius ring. However $R_{m}$ is non-chain if $m>1$.

We shall use the previous decomposition of rings to understand codes defined over finite commutative Frobenius rings. The zero element in $R^{n}$ will be denoted as $\mathbf{0}$. A linear code $C$ of length $n$ over a finite ring $R$ is an $R$-submodule of $R^{n}$. Let $C_{j}$ be a code of length $n$ over $R_{j}$, and extend the map $\Phi$ coordinatewise to $R^{n}$ as

$$
\begin{align*}
\Phi: \quad R^{n} & \rightarrow\left(R_{1}\right)^{n} \times \cdots \times\left(R_{u}\right)^{n} \\
\mathbf{a} & \mapsto\left(\Phi_{1}(\mathbf{a}), \cdots, \Phi_{u}(\mathbf{a})\right), \tag{3}
\end{align*}
$$

where

$$
\begin{array}{cccc}
\Phi_{j}: & R^{n} & \rightarrow & \left(R_{j}\right)^{n} \\
& \left(a_{1}, a_{2}, \cdots, a_{n}\right) & \mapsto & \left(\Phi_{j}\left(a_{1}\right), \Phi_{j}\left(a_{2}\right), \cdots, \Phi_{j}\left(a_{n}\right)\right) . \tag{4}
\end{array}
$$

Then $C=\operatorname{CRT}\left(C_{1}, C_{2}, \cdots, C_{u}\right)=\Phi^{-1}\left(C_{1} \times C_{2} \times \cdots \times C_{u}\right)$, where $\Phi_{j}(C)=C_{j}$ for $1 \leq j \leq u$. We say that $C$ is the Chinese product of codes $C_{1}, C_{2}, \cdots, C_{u}$. This allows us to reduce the study of codes over finite commutative Frobenius rings to that of codes over finite commutative local Frobenius rings. The rank of a linear code $C$ over $R$ of length $n$, is defined by

$$
\operatorname{rank}_{R}(C):=\min \left\{i \in \mathbb{N}: \text { there exists a monomorphism } C \hookrightarrow R^{i} \text { as } R \text {-modules }\right\} .
$$

We say that a linear code $C$ over $R$ is free if $C$ is isomorphic (as a module) to $R^{t}$ for some $t$. It is immediate that if $C$ is free then $\operatorname{rank}_{R}(C)=t$, where $C \cong R^{t}$. A linear $[n, k]$-code over $R$ is an $R$-submodule of $R^{n}$ of rank $k$. Note that a standard generator matrix for any free linear $[n, k]$-code $C$ over $R$ is of the form $\left[\mathrm{I}_{k} \mid \mathrm{M}\right] \mathrm{U}$, where $M$ is a matrix over $R, \mathrm{U}$ is a permutation matrix and $k=\operatorname{rank}_{R}(C)$.

Lemma 1. [8, Theorem 2.4] Let $C_{j}$ be a linear code over $R_{j}$ for $i=1,2, \ldots, u$, and $C=C R T\left(C_{1}, C_{2}, \cdots, C_{u}\right)$. Then

1. $|C|=\prod_{i=1}^{u}\left|C_{j}\right|$;
2. $\operatorname{rank}_{R}(C)=\max \left\{\operatorname{rank}_{R_{j}}\left(C_{j}\right): 1 \leq j \leq u\right\}$;
3. $C$ is a free code if and only if each $C_{j}$ is a free code with the same rank $\operatorname{ran} k_{R}(C)$.

We attach the standard inner-product to $R^{n}$, that is

$$
\begin{equation*}
[\mathbf{v}, \mathbf{w}]=\sum_{j=1}^{n} v_{j} w_{j} \tag{5}
\end{equation*}
$$

where $\mathbf{v}:=\left(v_{1}, v_{2}, \cdots, v_{n}\right)$ and $\mathbf{w}:=\left(w_{1}, w_{2}, \cdots, w_{n}\right)$ are elements in $R^{n}$. For a code $C$, its dual code is defined as follows:

$$
\begin{equation*}
C^{\perp}=\left\{\mathbf{u} \in R^{n}:[\mathbf{u}, \mathbf{c}]=0_{R}, \text { for all } \mathbf{c} \text { in } C\right\} \tag{6}
\end{equation*}
$$

It is well known that for codes over Frobenius rings, $|C|\left|C^{\perp}\right|=|R|^{n}$, (see [27] for a proof).
Lemma 2. [8, Theorem 2.7] If $C=C R T\left(C_{1}, C_{2}, \cdots, C_{u}\right)$ is a code over $R$, then $C^{\perp}=C R T\left(C_{1}^{\perp}, C_{2}^{\perp}, \cdots, C_{u}^{\perp}\right)$.
For the rest of the paper $R$ will denote the Chinese product of finite commutative local Frobenius rings $R_{1}, R_{2}, \ldots, R_{u}$ unless otherwise is specified. Let $\mathrm{M}_{k \times n}\left(R_{j}\right)$ be the set of all $k \times n$-matrices over $R_{j}$. For $\mathrm{A} \in \mathrm{M}_{k \times n}\left(R_{j}\right)$, the transpose of the matrix A is denoted by $\mathrm{A}^{T}$. We also let $\mathbf{0}$ denote the zero matrix, where the size will either be obvious from the context or specified whenever necessary. Similarly, we denote the $k \times k$ identity matrix by $I_{k}$. The elements $\mathbf{c}_{1}, \mathbf{c}_{2}, \ldots, \mathbf{c}_{k} \in R^{n}$ are said to be linearly independent over $R_{j}$ if for all $\left(x_{1}, x_{2}, \cdots, x_{k}\right)$ in the set $\left(R_{j}\right)^{k}$ such that $x_{1} \mathbf{c}_{1}+x_{2} \mathbf{c}_{2}+\cdots+x_{k} \mathbf{c}_{k}=\mathbf{0}$ implies that $x_{1}=$ $x_{2}=\cdots=x_{k}=0$. If the rows of a $k \times n$-matrix A over $R_{j}$ are linearly independent, then we say that A is a full-row-rank matrix. If there is an $k \times n$-matrix B over $R_{j}$ such that $\mathrm{AB}=\mathrm{I}_{k}$, then we say that A is rightinvertible and B is a right inverse of A . When $k=n$, we say that A is non-singular, if the determinant $\operatorname{det}(\mathrm{A})$ is a unit of $R_{j}$. Otherwise, A is said to singular. Note that a matrix A is invertible over $R_{j}$, if and only if A is nonsingular over $R_{j}$. The following two results about full-row-rank matrices over finite commutative Frobenius rings appear in [11].

Lemma 3. Let $R_{j}$ be a finite commutative Frobenius rings. A $k \times n$-matrix A is full-row-rank, if and only if A is right-invertible.

Lemma 4. Let A be an $k \times k$-matrix over a finite commutative Frobenius ring $R$. The following statements are equivalent:

1. A is invertible.
2. A is non-singular.
3. A is full-row-rank.

The next corollary follows from a typical linear algebra argument.
Corollary 1. The $k \times n$-matrixA over $R_{j}$ is singular, if and only if there is a nonzero vector $X:=\left(x_{1}, \cdots, x_{k}\right)^{T}$ in $R^{k}$ such that $\mathrm{A} X=\mathbf{0}$.

## 3. Characterization of LCD codes over finite commutative Frobenius rings

In [18, Theorem 3.5], it is proved that any linear code $C_{j}$ over $R_{j}$ with a generator matrix $G_{j}$ is LCD if, $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is invertible, and other hand it is also stated that the converse of [18, Theorem 3.5] is not true in general with an example [18, Example 2]. However there is a mistake in that example (as $(2,0,0,2,0)$ is in $C \cap C^{\perp}$ ). From [18, Corollary 3.6.], if $C_{j}$ is free then the converse of [18, Theorem 3.5] is true. Therefore, it is enough to prove that any LCD code over a finite commutative local Frobenius ring $R_{j}$ is free.

Definition 3.1. An $R_{j}$-module $C_{j}$ of rank $k$ is projective if there is an $R_{j}$-module $M$ such that $\left(R_{j}\right)^{k}$ and $C_{j} \oplus M$ are isomorphic (as $R_{j}$-modules).

Remark 1. Let $A_{j}$ and $B_{j}$ be $R_{j}$-modules. If $A_{j} \oplus B_{j}$ is free, then $A_{j}$ and $B_{j}$ are projective.
Lemma 5. [15, Theorem 2.] Any projective module over a local ring is free.
In the following result, we prove that there does not exist non-free LCD code over finite commutative local Frobenius rings.

Theorem 2. Over finite commutative local Frobenius rings, any LCD code is free.
Proof. Let $C_{j}$ be an LCD code over a commutative local Frobenius ring $R_{j}$ and $n$ is the length of $C_{j}$. Then $C_{j} \oplus C_{j}^{\perp}$ is a direct summand in $\left(R_{j}\right)^{n}$. Since $R$ is Frobenius, by the results in [27], $C$ satisfies $\left|C_{j}\right| \times\left|C_{j}^{\perp}\right|=\left|R_{j}\right|^{n}$. Thereby $C_{j} \oplus C_{j}^{\perp}=\left(R_{j}\right)^{n}$. So the $R$-module $C_{j} \oplus C_{j}^{\perp}$ is free, and by Remark 1, it follows that $C_{j}$ is projective. Now $C_{j}$ is a finitely generated projective $R_{j}$-module and $R_{j}$ is a local ring and by Lemma $5, C_{j}$ is free.

It follows from Theorem 2 and [18, Corollary 3.6] that there does not exist non-free LCD codes over finite commutative local Frobenius rings. We now show that the converse of Theorem 2 does not hold in general. To show this we cite the following example.

Example 3.1. Let $C$ be a linear code over $\mathbb{Z}_{4}$ with generator matrix

$$
\mathrm{G}=\left[\begin{array}{lllllll}
1 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right]
$$

Clearly $C$ is free. But $C$ is not $L C D$, as $(0,0,0,2,2,0,0) \in C \cap C^{\perp}$.
Proposition 1. A linear code $C_{j}$ over $R_{j}$ with generator matrix $\mathrm{G}_{j}$. If $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is nonsingular, then $C_{j}$ is free.

Proof. Suppose that $C_{j}$ is not free. Then $\mathrm{G}_{j}$ is not full-row-rank. From Lemma 3, it follows that $\mathrm{G}_{j}$ is not right-invertible. Hence $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is singular.
Corollary 2. A linear code $C_{j}$ over $R_{j}$ with generator matrix $\mathrm{G}_{j}$ is $L C D$, if and only if $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is nonsingular.
Proof. Suppose that $C_{j}$ is LCD with rank $k$, and $\mathbf{c} \in C_{j}$. From Theorem 2, $C_{j}$ is free and $\mathbf{c}$ can be written as $\mathbf{c}=\mathbf{v G}$ for some $\mathbf{v}$ in $\left(R_{j}\right)^{k}$. If $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is singular, by Corollary 1 there is a nonzero vector $\mathbf{u}$ in $\left(R_{j}\right)^{k}$ such that $\mathbf{u G}{ }_{j} \mathrm{G}_{j}^{T}=\mathbf{0}$. Now $\mathbf{u G}$ is a nonzero vector in $C_{j}$. So that

$$
(\mathbf{u G}) \mathbf{c}^{T}=(\mathbf{u G})(\mathbf{v G})^{T}=\mathbf{u} G_{j} \mathrm{G}_{j}^{T} \mathbf{v}^{T}=\mathbf{0} \mathbf{v}^{T}=\mathbf{0}
$$

and hence $\mathbf{u G}$ is also a vector in $C_{j}^{\perp}$. It follows that $C_{j} \cap C_{j}^{\perp} \neq\{\mathbf{0}\}$, i.e., that $C$ is not LCD. Absurd, therefore $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is nonsingular.

Suppose that $\mathrm{G}_{j} \mathrm{G}_{j}^{T}$ is nonsingular. Let $\mathbf{c} \in C_{j} \cap C_{j}^{\perp}$, by Proposition $1, C_{j}$ is free. On the one hand, $\mathbf{c} \in C_{j}$ implies that there is $\mathbf{u} \in\left(R_{j}\right)^{k}$ such that $\mathbf{c}=\mathbf{u} \mathrm{G}_{j}$. It follows that

$$
\begin{equation*}
\mathbf{c} \mathrm{G}_{j}^{T}\left(\mathrm{G}_{j} \mathrm{G}_{j}^{T}\right)^{-1} \mathrm{G}_{j}=\mathbf{u} \mathrm{G}_{j} \mathrm{G}_{j}^{T}\left(\mathrm{G}_{j} \mathrm{G}_{j}^{T}\right)^{-1} \mathrm{G}_{j}=\mathbf{u} \mathrm{G}_{j}=\mathbf{c} \tag{7}
\end{equation*}
$$

and the other hand, $\mathbf{c} \in C_{j}^{\perp}$, it follows that $\mathbf{c G}{ }_{j}^{T}=\mathbf{0}$. So

$$
\begin{equation*}
\mathbf{c} \mathrm{G}_{j}^{T}\left(\mathrm{G}_{j} \mathrm{G}_{j}^{T}\right)^{-1} \mathrm{G}_{j}=\mathbf{0}\left(\mathrm{G}_{j} \mathrm{G}_{j}^{T}\right)^{-1} \mathrm{G}_{j}=\mathbf{0} . \tag{8}
\end{equation*}
$$

From (7) and (8), it follows that $\mathbf{c}=\mathbf{0}$. Whence $C_{j}$ is LCD.
Example 3.2. The linear code $C$ of length 8 generated by $G=\left[\begin{array}{cccccccc}1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 & 1\end{array}\right]$ over $\mathbb{Z}_{4}$ is LCD code whose minimum Lee distance is 4 and has free rank $4\left(\left[8,4^{4}, 4\right]\right.$-code). The Gay image of $C$ is a non-linear binary code of length 16 and minim Hamming distance 4 .

A linear [ $n, k$ ]-code $C^{\prime}$ over $R_{j}$ is a lift of a linear $[n, k]$-code $C$ over $S$ by ring epimorphism $f_{j}: S \rightarrow$ $R_{j}$, if $C^{\prime}=f_{j}(C)$, where

$$
f_{j}(C):=\left\{\left(f_{j}\left(c_{1}\right), f_{j}\left(c_{2}\right), \cdots, f_{j}\left(c_{n}\right)\right):\left(c_{1}, c_{2}, \cdots, c_{n}\right) \in C\right\}
$$

We call $C^{\prime}$ the projection of $C$ by $f_{j}$.
Lemma 6. Let $S$ and $R_{j}$ be finite commutative local Frobenius rings with $S^{\times}$and $\left(R_{j}\right)^{\times}$the unit group of $S$ and $R$, respectively. Then $f_{j}\left(S^{\times}\right)=\left(R_{j}\right)^{\times}$, for any ring epimorphism $f_{j}: S \rightarrow R_{j}$.

The following result is a generalization of [18, Theorem 3.9] to any finite commutative local Frobenius ring $S$ and any ring epimorphism $f_{j}: S \rightarrow R_{j}$.

Theorem 3. Let $S$ and $R_{j}$ be finite commutative local Frobenius rings. The projection of any $\operatorname{LCD}[n, k]-$ code over $S$ by ring epimorphism $f_{j}: S \rightarrow R_{j}$, is also an $L C D[n, k]$-code over $R_{j}$.

Proof. Let $C$ be an LCD $[n, k]$-code over $S$ with a generator matrix $G$. From Theorem 2, $C$ is free. Therefore the projection $f_{j}(C)$ of $C$ by $f_{j}$ is a free $[n, k]$-code over $R_{j}$ with a generator matrix $f_{j}(\mathrm{G})$. Now

$$
f_{j}\left(\operatorname{det}\left(\mathrm{GG}^{T}\right)\right)=\operatorname{det}\left(f_{j}(\mathrm{G}) f_{j}\left(\mathrm{G}^{T}\right)\right) .
$$

From Lemma 6 and Theorem 2, it follows that, $\operatorname{det}\left(f_{j}(\mathrm{G}) f_{j}\left(\mathrm{G}^{T}\right)\right)$ is a unit in $R_{j}$. Whence $f_{j}(C)$ is a LCD [ $n, k$ ]-code over $R_{j}$.

The result revisits and extends [18, Theorem 3.10] to any finite commutative local Frobenius ring $S$ and any ring epimorphism $f_{j}: S \rightarrow R_{j}$.

Theorem 4. Let $S$ and $R_{j}$ be finite commutative local Frobenius rings. Any lift of an $\operatorname{LCD}[n, k]$-code over $R_{j}$ by ring epimorphism $f_{j}: S \rightarrow R_{j}$, is also an $L C D[n, k]$-code over $S$.

Proof. Let $C^{\prime}$ be an LCD $[n, k]$-code over $R_{j}$ with a generator matrix $\mathrm{G}_{j}$. Since $f_{j}: S \rightarrow R_{j}$ is a ringepimorphism, there is a full-row-rank matrix G over $S$ such that $\mathrm{G}_{j}=f_{j}(\mathrm{G})$. Consider the free $[n, k]$ code $C$ over $S$ with generator matrix G. Now

$$
f_{j}\left(\operatorname{det}\left(\mathrm{GG}^{T}\right)\right)=\operatorname{det}\left(f_{j}(\mathrm{G}) f_{j}\left(\mathrm{G}^{T}\right)\right)
$$

By Lemma 6 and Theorem 2, it follows that $\mathrm{GG}^{T}$ is nonsingular. Consequently, $C$ is LCD.
The map

$$
\pi_{m}: \quad \begin{array}{clc}
R_{m} & \rightarrow \mathbb{F}_{2} \\
\sum_{A \subseteq\{1,2, \cdots, m\}} c_{A} \prod_{i \in A} u_{i} & \mapsto & c_{\emptyset}
\end{array}
$$

is a ring-epimorphism. From Theorems 3 and 4, a linear code $C$ over $R_{m}$ is LCD, if and only if $\pi_{m}(C)$ is a binary LCD code. From [5, Theorem 1], if $(1,1, \cdots, 1) \notin \pi_{m}(C)^{\perp}$ then $C$ is LCD if and only if there exists a basis $\mathbf{c}_{1}, \mathbf{c}_{2}, \cdots, \mathbf{c}_{k}$ of $C$ such that $\left[\mathbf{c}_{i}, \mathbf{c}_{j}\right]=\delta_{i, j}$, for all $1 \leq i, j \leq k$.

Example 3.3. Consider the linear $[n, k]$-code $C$ over $R_{m}$ with generator matrix

$$
\left(\begin{array}{cccccccc}
1 & 0 & 0 & \cdots & 0 & \lambda_{1,1} & \cdots & \lambda_{1, n-k} \\
0 & 1 & 0 & \cdots & 0 & \lambda_{2,1} & \cdots & \lambda_{2, n-k} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & 0 & \lambda_{k-1,1} & \cdots & \lambda_{k-1, n-k} \\
0 & \cdots & 0 & 0 & 1 & \lambda_{k, 1} & \cdots & \lambda_{k, n-k}
\end{array}\right),
$$

where $n-k$ is an even integer and $\pi_{m}\left(\lambda_{i, j}\right)=1$, for all $1 \leq i \leq k, 1 \leq j \leq n-k$. From Theorem 4 , the code $C$ is $L C D$, since $\pi_{m}(C)$ is a binary $L C D$ code, by [5, Theorem 1].
Theorem 5. A linear code $C=C R T\left(C_{1}, C_{2}, \cdots, C_{u}\right)$ over $R$ is LCD if and only if, the linear code $C_{j}$ over $R_{j}$ is $L C D$, for all $1 \leq j \leq u$.
Proof. The map $\Phi: R \rightarrow R_{1} \times \cdots \times R_{u}$ is a ring-isomorphism, and by Lemma 2 , it follows that

$$
C \cap C^{\perp}=\operatorname{CRT}\left(C_{1} \cap C_{1}^{\perp}, C_{2} \cap C_{2}^{\perp}, \cdots, C_{u} \cap C_{u}^{\perp}\right) .
$$

Thus $C$ is LCD over $R$ if and only if, $C_{j}$ is LCD over $R_{j}$ for all $1 \leq j \leq u$.
Remark 2. From Lemma 7 and Theorem 5, it is easy to see that an LCD code $C=C R T\left(C_{1}, C_{2}, \cdots, C_{u}\right)$ over $R$ is non-free, if and only if there are $1 \leq j_{1}<j_{2} \leq u$ such that $\operatorname{rank}_{R_{j_{1}}}\left(C_{j_{1}}\right) \neq \operatorname{rank} k_{R_{j_{2}}}\left(C_{j_{2}}\right)$.
Example 3.4. Let $C_{1}$ be an $L C D$ code over $\mathbb{Z}_{3}$ with generator matrix $G_{1}:=\left(\begin{array}{lllll}1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1\end{array}\right)$, and $C_{2}$ be an LCD code over $\mathbb{Z}_{5}$ with generator matrix $G_{2}:=\left(\begin{array}{lllll}1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4 & 2\end{array}\right)$. From Remark 2, the Chinese product of $C_{1}$ and $C_{2}$ is the non-free LCD code $C$ over $\mathbb{Z}_{15}$ with generator matrix

$$
\mathrm{G}:=\left(\begin{array}{ccccc}
1 & 0 & 6 & 1 & 1 \\
0 & 1 & 0 & 4 & 7 \\
0 & 0 & 10 & 10 & 10
\end{array}\right)
$$

since $\operatorname{rank}_{\mathbb{Z}_{3}}\left(C_{1}\right)=3 \neq 2=\operatorname{rank}_{\mathbb{Z}_{5}}\left(C_{2}\right)$. But $C$ is a projective module over $\mathbb{Z}_{15}$.
We now are ready to answer the question: "Do non-free LCD codes over finite commutative Frobenius ring $R$ exist?". It is evident from Example 3.4 that "non-free $L C D$ codes over finite commutative Frobenius rings exist and they are projective modules over R."

## 4. Constacyclic LCD codes over finite chain rings

Throughout this section $R$ will denote a finite chain ring (and hence a Frobenius ring) with residue field $\mathbb{F}_{q}, \gamma$ a unit in $R$, and $n$ a positive integer relatively prime to $q$. The projection $\pi: R \rightarrow \mathbb{F}_{q}$ extends naturally to a projection $R[X] \rightarrow \mathbb{F}_{q}[X]$ as follows: $\pi(f)=\sum_{i} \pi\left(f_{i}\right) X^{i}$ for $f=\sum_{i} f_{i} X^{i}$; also a projection $R^{n} \rightarrow\left(\mathbb{F}_{q}\right)^{n}$ as follows: $\pi(\mathbf{c})=\left(\pi\left(c_{0}\right), \pi\left(c_{1}\right), \cdots, \pi\left(c_{n-1}\right)\right)$ for $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{n-1}\right)$. Thus for any nonempty subset $C$ of $R^{n}, \pi(C)=\{\pi(\mathbf{c}): \mathbf{c} \in C\}$.

Recall that a linear code $C$ of length $n$ over $R$ is $\gamma$-constacyclic if $\left(\gamma c_{n-1}, c_{0}, c_{1}, \cdots, c_{n-2}\right) \in C$, whenever $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in C . C$ is called cyclic and negacyclic, respectively, when $\gamma$ is 1 and -1 . A constacyclic code of length $n$ over $R$ is non-repeated root if $n$ and $q$ are coprime. It is known that the $\gamma$-constacyclic codes over $R$ are identified to ideals of $R[X] /\left\langle X^{n}-\gamma\right\rangle$ via the $R$-module isomorphism

$$
\begin{array}{cccc}
\Upsilon: & R^{n} & \rightarrow & R[X] /\left\langle X^{n}-\gamma\right\rangle \\
& \left(c_{0}, c_{1}, \cdots, c_{n-1}\right) & \mapsto & c_{0}+c_{1} x+\cdots+c_{n-1} x^{n-1}
\end{array}
$$

where $x:=X+\left\langle X^{n}-\gamma\right\rangle$. In this section, we deal with non-repeated root $\gamma$-constacyclic LCD codes of length $n$ over $R$.

Let $k \in\{0,1,2, \ldots, n\}$ and $f:=f_{0}+f_{1} X+\cdots+f_{k} X^{k}$ be a polynomial of degree $\operatorname{deg}(f):=k$ over $R$ with $k<n$, we denote by $\mathrm{M}_{k}(f)$, the $(n-k) \times n$-matrix defined by:

$$
\left(\begin{array}{cccccccc}
f_{0} & f_{1} & \cdots & f_{k} & 0 & 0 & \cdots & 0  \tag{9}\\
0 & f_{0} & f_{1} & \cdots & f_{k} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & f_{0} & f_{1} & \cdots & f_{k} & 0 \\
0 & \cdots & 0 & 0 & f_{0} & f_{1} & \cdots & f_{k}
\end{array}\right)
$$

Obviously, if $f_{0}$ is a unit in $R$, then the rank of $\mathrm{M}_{k}(f)$ is $n-k$. Note that for any free $\gamma$-constacyclic code $C$ over $R$ of rank $n-k$, there is an only monic polynomial $g$ of degree $k$ dividing $X^{n}-\gamma$ in $R[X]$ whose $\mathrm{M}_{k}(g)$ is a generator matrix for $C$. This polynomial $g$ is called the generator polynomial of $C$ and the free $\gamma$-constacyclic code over $R$ with generator polynomial $g$ of length $n$, is denoted $\mathscr{P}(R ; n ; g)$. Conventionally, $\mathscr{P}(R ; n ; g)=\{\mathbf{0}\}$, if deg $(g) \geq n$. Thus $X^{n}-\gamma$ is the generator polynomial of $\{\mathbf{0}\}$.

From now on, $g$ denotes a monic polynomial over $R$ with $g(0)$ is a unit in $R$, and the nonzero element $\gamma$ in $\Gamma(R)$ is the remainder of the Euclidian division of $X^{n}$ by $g$.

From [19, Theorem 5.2.], the quotient ring $R[X] /\left\langle X^{n}-\gamma\right\rangle$ is a principal ideal ring, if either $R$ is a field, or $X^{n}-\pi(\gamma)$ is free-square. Recall that a polynomial over a finite field is called square-free, if it has no multiple irreducible factors in its decomposition. Of course, $X^{n}-\pi(\gamma)$ is free-square since $\operatorname{gcd}(n, q)=1$. From [23, Theorem 2.7], if $g \in R[X]$ is monic and $\pi(g)$ is square-free, then $g$ factors uniquely into monic, coprime basic-irreducible. For any polynomial $f$ in $\mathbb{F}_{q}[X]$ dividing $X^{n}-\gamma$ [22, Theorem XIII.4] implies the existence and unicity of a polynomial $g \in R[X]$ such that $\pi(g)=f$ and $g$ divides $X^{n}-\gamma$, since $X^{n}-\gamma$ is square-free in $\mathbb{F}_{q}[X]$. The polynomial $g$ will be called the Hensel lift of $f$.

Lemma 7. [12, Lemma 3.1 (3)] Let $g_{1}$ and $g_{2}$ be monic polynomials over $R$ dividing $X^{n}-\gamma$. Then

$$
\begin{equation*}
\mathscr{P}\left(R ; n ; g_{1}\right) \cap \mathscr{P}\left(R ; n ; g_{2}\right)=\mathscr{P}\left(R ; n ; \mu\left(g_{1}, g_{2}\right)\right), \tag{10}
\end{equation*}
$$

where $\mu\left(g_{1}, g_{2}\right)$ denotes the Hensel lift of $\operatorname{lcm}\left(\pi\left(g_{1}\right), \pi\left(g_{2}\right)\right)^{1}$ to $R[X]$.

[^1]For a polynomial $f(X)$ of degree $r, f^{*}(X)$ denotes its reciprocal polynomial and is given by $f^{*}(X)=$ $X^{r} f\left(\frac{1}{X}\right)$. A polynomial $f(X)$ is self-reciprocal, if $f^{*}(X)=f(X)$. Consider the permutation $\Phi: R^{n} \rightarrow R^{n}$ defined as follows: $\Phi\left(\left(a_{0}, a_{1}, \ldots, a_{n-1}\right)\right)=\left(a_{n-1}, a_{n-2}, \ldots, a_{0}\right)$. Recall that a linear code $C$ of length $n$ over $R$ is reversible if $\Phi(C)=C$. Obviously,

$$
\begin{equation*}
\Phi(\mathscr{P}(R ; n ; g))=\mathscr{P}\left(R ; n ; g^{*}\right) . \tag{11}
\end{equation*}
$$

On the other hand, for any $\gamma$-constacyclic code $C=\mathscr{P}(R ; n ; g)$, it is well-know that

$$
\begin{equation*}
C^{\perp}=\Phi(\mathscr{P}(R ; n ; \widehat{g})) \tag{12}
\end{equation*}
$$

where $\widehat{g}(X) g(X)=X^{n}-\gamma$. This leads to the following result.
Proposition 2. Let $g$ be a monic polynomial $g$ of degree $k$ dividing $X^{n}-\gamma$. If $C=\mathscr{P}(R ; n ; g)$, then $C^{\perp}=\mathscr{P}\left(R ; n ; \widehat{g^{*}}\right)$, where $\widehat{g}(X) g(X)=X^{n}-\gamma$.

From the precedent result, we have $C^{\perp}=\mathscr{P}\left(R ; n ; \widehat{g^{*}}\right)$ and $\widehat{g^{*}}(X)$ divides $X^{n}-\gamma^{-1}$. For this, we have the following result.

Proposition 3. The dual of any $\gamma$-constacyclic code over $R$ is $\gamma^{-1}$-constacyclic.
Obviously, both $\{\mathbf{0}\}$ and $R^{n}$ are $\gamma$-constacyclic codes for any unit $\gamma$ in $R$. Inversely, we have the following result.

Lemma 8. Let $C$ be a free code of length $n$ over $R$. If $C$ is both $\alpha$-constacyclic and $\beta$-constacyclic for $\alpha, \beta$ units in $R$ with $\pi(\alpha) \neq \pi(\beta)$, then either $C=\{\boldsymbol{0}\}$ or $C=R^{n}$.

Proof. Assume that $C \neq\{\mathbf{0}\}$. There exists a polynomial $g:=g_{0}+g_{1} X+\cdots+g_{k-1} X^{k-1}+X^{k}$ with $k<$ $n$ such that $C=\mathscr{P}(R ; n ; g)$. Then the word $\mathbf{c}:=\left(0, \cdots, 0, g_{0}, g_{1}, \cdots, g_{k-1}, 1\right)$ belongs to $C$. Since $C$ is both $\alpha$-constacyclic and $\beta$-constacyclic, it follows that $\mathbf{c}_{\alpha}:=\left(\alpha, 0, \cdots, 0, g_{0}, g_{1}, \cdots, g_{k-1}\right) \in C$ and $\mathbf{c}_{\beta}:=$ $\left(\beta, 0, \cdots, 0, g_{0}, g_{1}, \cdots, g_{k-1}\right) \in C$. Thus $\alpha \mathbf{c}_{\beta}-\beta \mathbf{c}_{\alpha}=(\alpha-\beta)\left(0,0, \cdots, 0, g_{0}, g_{1}, \cdots, g_{k-1}\right)$. Now $\pi(\alpha) \neq \pi(\beta)$ and $C$ is linear over $R$, therefore $\mathbf{c}^{\prime}:=\left(0,0, \cdots, 0, g_{0}, g_{1}, \cdots, g_{k-1}\right) \in C$. And so on, we have $(0, \cdots, 0,1) \in C$ since $g_{0}$ is a unit in $R$. By constacyclicity of $C$, it follows that $C=R^{n}$.

Corollary 3. If $\pi\left(\gamma^{2}\right) \neq 1$, then any free $\gamma$-constacyclic code of length $n$ over $R$ is LCD.
Proof. Assume that $\pi\left(\gamma^{2}\right) \neq 1$, and let $C$ be a free $\gamma$-constacyclic code of length $n$ over $R$. Then by Proposition 3, $C^{\perp}$ is a $\gamma^{-1}$-constacyclic code. Thus, $C \cap C^{\perp}$ is both $\gamma$-constacyclic and $\gamma^{-1}$-constacyclic. Therefore, by Lemma $8, C \cap C^{\perp}=\{0\}$, i.e., $C$ is an LCD code as because $C \cap C^{\perp}$ can not be $R^{n}$, when $\pi\left(\gamma^{2}\right) \neq 1$.

Thus, in order to obtain all $\gamma$-constacyclic LCD codes, we need to consider only the case when $\pi\left(\gamma^{2}\right)=1$. Moreover, the dual code of any $\pi(\gamma)$-constacyclic code over $\mathbb{F}_{q}$ is still a $\pi(\gamma)$-constacyclic code over $\mathbb{F}_{q}$ when $\pi\left(\gamma^{2}\right)=1$.

Lemma 9. Let $C$ be an $\alpha$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ with $\alpha^{2}=1$. The following assertions are equivalent.

1. $C$ is $L C D$;
2. $g$ is self-reciprocal;
3. $C$ is reversible.

Proof. Let $C=\mathscr{P}\left(\mathbb{F}_{q} ; n ; f\right)$. From Proposition $2, C^{\perp}=\mathscr{P}\left(\mathbb{F}_{q} ; n ; \widehat{f^{*}}\right)$ and since $\alpha^{2}=1$, it follows from Proposition 7 that

$$
C \cap C^{\perp}=\mathscr{P}\left(\mathbb{F}_{q} ; n ; l \mathrm{~cm}\left(f, \widehat{f^{*}}\right)\right)
$$

So $C$ is LCD if and only if $\operatorname{lcm}\left(f, \widehat{f^{*}}\right)=X^{n}-\alpha$. Since $\operatorname{deg}(\widehat{f})=n-\operatorname{deg}(f)$ and $\operatorname{deg}(f)=\operatorname{deg}(f)^{*}$, it follows that $f$ and $\widehat{f^{*}}$ are coprime. Hence $\operatorname{lcm}\left(f, \widehat{f^{*}}\right)=f \widehat{f^{*}}$. As $f \widehat{f}=X^{n}-\alpha$ and $\alpha^{2}=1$, it follows that

$$
f^{*} \widehat{f^{*}}=X^{n}-\alpha^{-1}=X^{n}-\alpha=f \widehat{f^{*}}
$$

which is equivalent to saying $f=f^{*}$. From Eq. (11), $C$ is reversible if and only if $f=f^{*}$.
Remark 3. Let $g$ be a monic polynomial in $R[X]$. Since $\mathscr{P}(R ; n ; g)=\left\{\boldsymbol{c} \in R^{n}: g\right.$ divides $\left.\Psi(\boldsymbol{c})\right\}$, it follows that $\pi(\mathscr{P}(R ; n ; g))=\mathscr{P}\left(\mathbb{F}_{q} ; n ; \pi(g)\right)$.

From Remark 3, Theorems 3 and 4, we have
Lemma 10. Let $C$ be a $\gamma$-constacyclic code of length $n$ over $R$. Then $C$ is $L C D$ if and only if $\pi(C)$ is both $\pi(\gamma)$-constacyclic and LCD.

Theorem 6. Let $C$ be a $\gamma$-constacyclic code of length $n$ over $R$ and $g$ its generator polynomial. Then $C$ is $L C D$ and $\gamma^{2}=1$ if and only if $C$ is reversible.

Proof. Let $C=\mathscr{P}(R ; n ; g)$. From Proposition $2, C^{\perp}=\mathscr{P}\left(R ; n ; \widehat{g^{*}}\right)$. Since $\gamma^{2}=1$, it follows that $g^{*}$ divides $X^{n}-\gamma$. It can use Lemma 7 and we have $C \cap C^{\perp}=\mathscr{P}(R ; n ; \mu(g, \widehat{g *}))$. Then $C$ is LCD and $\gamma^{2}=1$ if and only if $\mu\left(g, \widehat{g^{*}}\right)=X^{n}-\gamma$, this implies that $\mu\left(g, \widehat{g^{*}}\right)=g \widehat{g^{*}}$. Since $g \widehat{g}=X^{n}-\gamma$, it follows that $g^{*}=g$. By Equality (11), $C$ is reversible.

Conversely, if $C$ is reversible, then $\pi(C)$ is also reversible. From Lemmas 9 and 10, $C$ is LCD. Moreover if $C$ is reversible, by Equality (11), we have $g^{*}=g$. But $g \widehat{g}=X^{n}-\gamma$ and $g^{*} \widehat{g^{*}}=X^{n}-\gamma^{-1}$. So $X^{n}-\gamma=X^{n}-\gamma^{-1}$, because $g^{*}=g$. Whence $\gamma^{2}=1$.

We now will provide some examples to demonstrate our results. We used the Magma Computer Algebra System [3] in our computations. We have got some good codes, some optimal known codes and some new optimal codes over $\mathbb{Z}_{4}[1]$.

Example 4.1. The factorization of $X^{7}-1$ over $\mathbb{Z}_{4}$ into a product of basic irreducible polynomials over $\mathbb{Z}_{4}$ is given by

$$
X^{7}-1=(X-1)\left(X^{3}+2 X^{2}+X+7\right)\left(X^{3}+3 X^{2}+2 X+7\right)
$$

Let $f(X)=X^{3}+2 X^{2}+X+7$ and $g(X)=X^{3}+3 X^{2}+2 X+7$. From Theorem 6 , we have

- The cyclic code $\mathscr{P}\left(\mathbb{Z}_{4} ; 7 ;(X-1)\right)$ is LCD and reversible. This is $\left[7,4^{6}, 2\right]$ optimal code.
- The cyclic code $\mathscr{P}\left(\mathbb{Z}_{4} ; 7 ; f(X)\right)$ is not LCD, since $f(X)$ is not self-reciprocal.
- The cyclic code $\mathscr{P}\left(\mathbb{Z}_{4} ; 7 ; f(X) g(X)\right)$ is LCD, since $f(X) g(X)$ is self-reciprocal. This code has minimum Lee distance 7 but has only 4 codewords.

Note that if $C$ is $\gamma$-constacyclic of odd length over $\mathbb{Z}_{4}$, then $C$ is LCD if and only if $C$ is reversible.

Example 4.2. The factorization of $X^{15}-1$ over $\mathbb{Z}_{4}$ into a product of basic irreducible polynomials over $\mathbb{Z}_{4}$ is given by

$$
X^{15}-1=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)
$$

The self-reciprocal polynomials and the LCD codes generated by those seld-reciprocal polynomials are shown in the following table:

| Generators (self-reciprocal) of LCD code C | $\left[n, 4^{k_{1}}, d_{L}\right]$ | Remarks |
| :---: | :---: | :---: |
| $g_{1}=X-1$ | $\left[15,4^{14}, 2\right]$ |  |
| $g_{2}=X^{2}+X+1$ | $\left[15,4^{13}, 2\right]$ |  |
| $g_{3}=(X-1)\left(X^{2}+X+1\right)$ | $\left[15,4^{12}, 2\right]$ |  |
| $g_{4}=X^{4}+X^{3}+X^{2}+X+1$ | $\left[15,4^{11}, 2\right]$ |  |
| $g_{5}=(X-1)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{10}, 2\right]$ |  |
| $g_{6}=\left(X^{2}+X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{9}, 4\right]$ | Good |
| $g_{7}=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{8}, 4\right]$ |  |
| $g_{8}=\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)$ | $\left[15,4^{7}, 3\right]$ |  |
| $g_{9}=(X-1)\left(X^{4}+3\right)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)$ | $\left[15,4^{6}, 6\right]$ | Good |
| $g_{10}=\left(X^{2}+X+1\right)$ | $\left[15,4^{5}, 3\right]$ |  |
| $g_{11}=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)$ | $\left[15,4^{4}, 6\right]$ |  |
| $g_{12}=\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{3}, 5\right]$ |  |
| $g_{13}=(x-1)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{2}, 10\right]$ | Good |
| $g_{14}=\left(X^{2}+X+1\right)\left(X^{4}+2 X^{2}+3 X+1\right)\left(X^{4}+3 X^{2}+2 X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)$ | $\left[15,4^{1}, 15\right]$ |  |

Example 4.3. The factorization of $X^{9}-1, X^{17}-1, X^{31}-1$ and $X^{63}-1$ over $\mathbb{Z}_{4}$ into a product of basic irreducible polynomials are given by

$$
\begin{gathered}
X^{9}-1=(X-1)\left(X^{2}+X+1\right)\left(X^{6}+X^{3}+1\right) \\
X^{17}-1=(X-1)\left(X^{8}+2 X^{6}+3 X^{5}+X^{4}+3 X^{3}+2 X^{2}+1\right)\left(X^{8}+X^{7}+3 X^{6}+3 X^{4}+3 X^{2}+X+1\right) \\
X^{31}-1=h_{1} h_{2} h_{3} h_{4} h_{5} h_{6} h_{7}
\end{gathered}
$$

where $h_{1}=(X-1), h_{2}=\left(X^{5}+3 X^{2}+2 X+3\right), h_{3}=\left(X^{5}+2 X^{4}+X^{3}+3\right), h_{4}=\left(X^{5}+2 X^{4}+3 X^{3}+X^{2}+3 X+3\right)$, $\left.h_{5}=\left(X^{5}+3 X^{4}+X^{2}+3 X+3\right), h_{6}=X^{5}+X^{4}+3 X^{3}+X+3\right), h_{7}=\left(X^{5}+X^{4}+3 X^{3}+X^{2}+2 X+3\right)$, and

$$
X^{63}-1=g_{1} g_{2} \ldots g_{13}
$$

where $_{1}=(X-1), g_{2}=\left(X^{2}+X+1\right), g_{3}=\left(X^{3}+2 X^{2}+X+3\right), g_{4}=\left(X^{3}+3 X^{2}+2 X+3\right), g_{5}=\left(X^{6}+2 X^{3}+3 X+1\right)$, $g_{6}=\left(X^{6}+X^{3}+1\right), g_{7}=\left(X^{6}+2 X^{5}+3 X^{4}+3 X^{2}+X+1\right), g_{8}=\left(X^{6}+2 X^{5}+X^{4}+X^{3}+3 X+1\right) ; g_{9}=$ $\left(X^{6}+3 X^{5}+2 X^{3}+1\right) ; g_{10}=\left(X^{6}+3 X^{5}+2 X^{4}+X^{2}+X+1\right), g_{11}=\left(X^{6}+3 X^{5}+X^{3}+X^{2}+2 X+1\right)$, $g_{12}=\left(X^{6}+X^{5}+X^{4}+2 X^{2}+3 X+1\right), g_{13}=\left(X^{6}+X^{5}+3 X^{4}+3 X^{2}+2 X+1\right)$. In the following table, we list cyclic LCD codes over $\mathbb{Z}_{4}$ of different lengths and their generators. It is noted that some of the codes (which are LCD) are good known codes and some are new optimal codes over $\mathbb{Z}_{4}$ [1].

| Generators ofC | $\left[n, 4^{k_{1}}, d_{L}\right]$ | Remarks |
| :---: | :---: | :---: |
| $(X-1)\left(X^{6}+X^{3}+1\right)$ | $\left[9,4^{2}, 6\right]$ | Good |
| $\left(X^{6}+X^{3}+1\right)$ | $\left[9,4^{3}, 3\right]$ | Good |
| $(X-1)\left(X^{2}+X+1\right)$ | $\left[9,4^{6}, 2\right]$ | Good |
| $\left(X^{8}+X^{7}+3 X^{6}+3 X^{4}+3 X^{2}+X+1\right)$ | $\left[17,4^{9}, 7\right]$ | Optimal |
| $(X-1)\left(X^{8}+X^{7}+3 X^{6}+3 X^{4}+3 X^{2}+X+1\right)$ | $\left[17,4^{8}, 8\right]$ | Optimal |
| $h_{1} h_{2} h_{3} h_{4} h_{4} h_{7}$ | $\left[31,4^{10}, 16\right]$ | Optimal |
| $h_{2} h_{3} h_{5} h_{6}$ | $\left[31,4^{11}, 12\right]$ | Optimal |
| $h_{1} h_{5} h_{6}$ | $\left[31,4^{20}, 8\right]$ | Optimal |
| $h_{2} h_{3}$ | $\left[31,4^{21}, 6\right]$ | Optimal |
| $g_{2} g_{3} g_{4} g_{5} g_{6} g_{7} g_{9} g_{10} g_{12} g_{13}$ | $\left[63,4^{13}, 36\right]$ | Optimal |
| $g_{1} g_{3} g_{4} g_{5} g_{6} g_{7} g_{9} g_{10} g_{12} g_{13}$ | $\left[63,4^{14}, 34\right]$ | Optimal |
| $g_{3} g_{4} g_{6} g_{7} g_{8} g_{10} g_{11} g_{12} g_{13}$ | $\left[63,4^{15}, 21\right]$ | Optimal |
| $g_{1} g_{6} g_{7} g_{10} g_{11} g_{12} g_{13}$ | $\left[63,4^{20}, 18\right]$ | Optimal |
| $g_{1} g_{5} g_{6} g_{7} g_{9} g_{13}$ | $\left[63,4^{22}, 16\right]$ | Optimal |
| $g_{1} g_{2} g_{7} g_{8} g_{10} g_{11} g_{12} g_{13}$ | $\left[63,4^{43}, 14\right]$ | Optimal |
| $g_{2} g_{3} g_{4} g_{6} g_{10} g_{12}$ | $\left[63,4^{\left.4^{77}, 12\right]}\right.$ | Optimal |
| $g_{1} g_{2} g_{3} g_{4} g_{10} g_{12}$ | $\left[63,4^{22}, 10\right]$ | Optimal |
| $g_{2} g_{5} g_{6} g_{7} g_{9} g_{13}$ | $\left[63,4^{31}, 9\right]$ | Optimal |
| $g_{3} g_{4} g_{7} g_{8} g_{11} g_{13}$ | $\left[63,4^{33}, 7\right]$ | Optimal |
| $g_{1} g_{8} g_{11}$ | $\left[63,4^{50}, 6\right]$ | Optimal |

Example 4.4. The factorization of $X^{15}-1$ over $\mathbb{Z}_{8}$ into a product of basic irreducible polynomials over $\mathbb{Z}_{8}$ is given by
$X^{15}-1=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+X^{3}+X^{2}+X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right)$.
Out of 1 and $X^{15}-1$, there are 14 self-reciprocal polynomials dividing $X^{15}-1$ in $\mathbb{Z}_{8}[X]$ and they are:

$$
\begin{gathered}
g_{1}=X-1 ; \\
g_{2}=X^{2}+X+1 ; \\
g_{3}=(X-1)\left(X^{2}+X+1\right) ; \\
g_{4}=X^{4}+3 X^{3}+6 X^{2}+4 X+1 ; \\
g_{5}=(X-1)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right) ; \\
g_{6}=\left(X^{2}+X+1\right)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right) ; \\
g_{7}=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right) ; \\
g_{8}=\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{9}=(X-1)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{10}=\left(X^{2}+X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{11}=(X-1)\left(X^{2}+X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{12}=\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{13}=(X-1)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) ; \\
g_{14}=\left(X^{2}+X+1\right)\left(X^{4}+3 X^{3}+6 X^{2}+4 X+1\right)\left(X^{4}+4 X^{3}+6 X^{2}+3 X+1\right)\left(X^{4}+3 X^{3}+2 X^{2}+1\right) .
\end{gathered}
$$

From Theorem 6, the nontrivial cyclic code $\mathscr{P}\left(\mathbb{Z}_{8} ; 15 ; g_{i}\right)$ over $\mathbb{Z}_{8}$, is LCD, for all $1 \leq i \leq 14$. Moreover $\mathscr{P}\left(\mathbb{Z}_{8} ; 15 ; g_{i}(\gamma X)\right)$ is a nontrivial $\gamma$-constacyclic LCD code over $\mathbb{Z}_{8}$, where $\gamma \in\{3 ; 5 ; 7\}$, for all $1 \leq i \leq 14$. Hence there are 56 nontrivial constacyclic LCD codes of length 15 over $\mathbb{Z}_{8}$.

Example 4.5. The factorization of $X^{9}-1$ over $\mathbb{Z}_{8}$ into a product of basic irreducible polynomials over $\mathbb{Z}_{8}$ is given by

$$
X^{9}-1=(X-1)\left(X^{2}+X+1\right)\left(X^{6}+X^{3}+1\right) .
$$

All three factors of $X^{9}-1$ over $\mathbb{Z}_{8}$ are self-reciprocal polynomials in $\mathbb{Z}_{8}[X]$ and hence all cyclic codes of length 9 over $\mathbb{Z}_{8}$ are LCD and so reversible.

| Generators of $C$ | $\left[n, 4^{k_{1}}, d_{H}\right]$ |
| :---: | :---: |
| $(X-1)$ | $\left[9,8^{8}, 2\right]$ |
| $\left(X^{6}+X^{3}+1\right)$ | $\left[9,8^{3}, 3\right]$ |
| $\left(X^{2}+X+1\right)$ | $\left[9,4^{7}, 2\right]$ |
| $(X-1)\left(X^{6}+X^{3}+1\right)$ | $\left[9,8^{2}, 6\right]$ |
| $(X-1)\left(X^{2}+X+1\right)$ | $\left[9,8^{6}, 3\right]$ |
| $\left(X^{2}+X+1\right)\left(X^{6}+X^{3}+1\right)$ | $\left[9,8^{1}, 9\right]$ |

Example 4.6. The Cyclic code $C$ of length 5 generated by $g(X)=X^{2}+(3 w+2) X+1$ over $G R(4,2)$, where $G R(4,2)$ is the Galois Extension of $\mathbb{Z}_{4}$ order 2 and $w$ is a root of the basic primitive polynomial $X^{2}+X+1$, is LCD code and its minimum Hamming distance is $\left.3\left(5,16^{3}, 3\right]\right)$.

## 5. Conclusion

In paper, we have done an extensive study of LCD codes over finite commutative Frobenius rings. We have first corrected a wrong result given in [18] which in deed led to the claim that "there do not exist non-free LCD codes over finite commutative local Frobenius rings". We also answered the question posed in the title that there exists non-free LCD codes over finite commutative Frobenius rings but not over finite commutative local Frobenius rings. We have also obtained a necessary and sufficient condition for any linear code over a finite commutative Frobenius ring to be LCD. We also characterized non-repeated root constacyclic LCD codes and revercible over finite chain rings and we found some new optimal codes over $\mathbb{Z}_{4}$ which are infact cyclic LCD codes over $\mathbb{Z}_{4}$.

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[^1]:    ${ }^{1} 1 \mathrm{~cm}\left(\pi\left(g_{1}\right), \pi\left(g_{2}\right)\right):$ the least common multiple of $\pi\left(g_{1}\right), \pi\left(g_{2}\right)$.

