# On optimal weak algebraic manipulation detection codes and weighted external difference families

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### **Abstract**

This paper provides a combinatorial characterization of weak algebraic manipulation detection (AMD) codes via a kind of generalized external difference families called bounded standard weighted external difference families (BSWEDFs). By means of this characterization, we improve a known lower bound on the maximum probability of successful tampering for the adversary's all possible strategies in weak AMD codes. We clarify the relationship between weak AMD codes and BSWEDFs with various properties. We also propose several explicit constructions for BSWEDFs, some of which can generate new optimal weak AMD codes.

### **Index Terms**

Algebraic manipulation detection code, difference family, weighted external difference family.

## I. INTRODUCTION

Algebraic manipulation detection (AMD) codes were first introduced by Cramer *et al.* [5] to convert linear secret sharing schemes into robust secret sharing schemes and build nearly optimal robust fuzzy extractors. For those cryptographic applications, AMD codes received much attention and were further studied in [1], [6], [7]. Generally speaking, for AMD codes, we consider two different settings: the adversary has full knowledge of the source (the *strong model*) and the adversary has no knowledge about the source (the *weak model*). In the viewpoint of combinatorics, AMD codes were proved to be closely related with various kinds of external difference families for both strong and weak models by Paterson and Stinson [20]. In the literature, optimal AMD codes in the strong model and their corresponding generalized external difference families received the most attention (see [2], [12], [15], [17], [20], [18], [21], [22], and the references therein), while relatively little was known about AMD codes under the weak model.

In this paper, we focus on weak AMD codes. In [20], Paterson and Stinson first derived a theoretic bound on the maximum probability of successful tampering for weak AMD codes. Very recently, Huczynska and Paterson [13] characterized the optimal weak AMD codes with respect to the Paterson-Stinson bound by weighted external difference families. Natural questions arise from the Paterson-Stinson bound and the corresponding characterization are: (i) Whether the Paterson-Stinson bound is always tight; (ii) If not, what are the equivalent combinatorial structures for those optimal weak AMD codes not having been characterized by the characterization in [13].

To answer these questions, in this paper, we further study the relationship between weak AMD codes and weighted external difference families. Firstly, we define a new type of weighted external difference families which are proved equivalent with weak AMD codes. By means of this combinatorial characterization of weak AMD codes: (1) We improve the known lower bound on the maximum probability of successful tampering for the adversary's all possible strategies; (2) We derive a necessary condition for the Paterson-Stinson bound to be achieved; (3) We determine the exact combinatorial structure for a weak AMD code to be optimal, when the Paterson-Stinson bound is not achievable. In this way, some weak AMD codes which have not been identified to be *R*-optimal previously now can be identified to be in fact *R*-optimal. Secondly, we show the relationships between this new type of weighted external difference families and other types of external difference families. Finally, we exhibit several explicit constructions of optimal weighted external difference families to generate optimal weak AMD codes.

This paper is organized as follows. In Section II, we introduces some preliminaries about AMD codes. In Section III, we investigate the relationship between AMD codes and external difference families. In Section IV, we describe several explicit constructions for bounded standard weighted external difference families, which are combinatorial equivalents of weak AMD codes. Conclusion is drawn in Section V.

## II. PRELIMINARIES

In this section we describe some notation and definitions about AMD codes.

- Let (G, +) be an Abelian group of order n with identity 0;
- For a positive integer n, let  $\mathbb{Z}_n$  be the residue class ring of integers modulo n;
- For a multi-set B and a positive integer k, let  $k \boxtimes B$  denote the multi-set, where each element of B repeated k times;

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- For a subset  $B \subseteq G$ , D(B) denotes the multi-set  $\{a b \in G : a, b \in B, a \neq b\}$ ;
- For subsets  $B_1, B_2 \subseteq G$ ,  $D(B_1, B_2)$  denotes the multi-set  $\{a b \in G : a \in B_1, b \in B_2\}$ ;
- For a multi-set B, let  $\sharp(a,B)$  denote the number of times that a appears in B;
- For positive integers  $k_1, k_2, \ldots, k_m$ , let  $lcm(k_1, k_2, \ldots, k_m)$  denote the least common multiple of  $k_1, k_2, \ldots, k_m$ .

Let S be the source space, i.e., the set of plaintext messages with size m, and G be the encoded message space. An encoding function E maps  $s \in S$  to some  $g \in G$ . Let  $A_s \subseteq G$  denote the set of valid encodings of  $s \in S$ , where  $A_s \cap A_{s'} = \emptyset$  is required for any  $s \neq s'$  so that any message  $g \in A_s$  can be correctly decoded as D(g) = s. Denote  $\mathcal{A} \triangleq \{A_s : s \in S\}$ .

**Definition 1** ([20]): For given (S, G, A, E), let

- The value  $\Delta \in G \setminus \{0\}$  be chosen according to the adversary's strategy  $\sigma$ ;
- The source message  $s \in S$  be chosen uniformly at random by the encoder, i.e., we assume equiprobable sources;
- The message s be encoded into  $g \in A_s$  using the encoding function E;
- The adversary wins (a successful tampering) if and only if  $g + \Delta \in A_{s'}$  with  $s' \neq s$ .

The probability of successful tampering is denoted by  $\rho_{\sigma}$  for strategy  $\sigma$  of the adversary. The code  $(S,G,\mathcal{A},E)$  is called an  $(n,m,a,\rho)$  algebraic manipulation detection code (or an  $(n,m,a,\rho)$ -AMD code for short) under the weak model, where  $a=\sum_{s\in S}|A_s|$  and  $\rho$  denotes the maximum probability of successful tampering for all possible strategies, i.e.,

$$\rho = \max_{\sigma} \rho_{\sigma}.$$

Specially, if E encodes s to an element of  $A_s$  uniformly, i.e.,  $Pr(E(s) = g) = \frac{1}{|A_s|}$  for any  $s \in S$  and  $g \in A_s$ , then we use  $(S, G, A, E_u)$  to distinguish this kind of AMD codes under the weak model, which were also termed as weak AMD codes in [13].

For weak AMD codes, the following Paterson-Stinson bound was derived in [20].

**Lemma 1** ([20]): For any weak  $(n, m, a, \rho)$ -AMD code, the probability  $\rho$  satisfies

$$\rho \ge \frac{a(m-1)}{m(n-1)}.$$

**Definition 2** ([20]): A weak AMD code that meets the bound of Lemma 1 with equality is said to be R-optimal with respect to the bound in Lemma 1, where R is used to indicate that random choosing  $\Delta$  is an optimal strategy for the adversary.

# III. ALGEBRAIC MANIPULATION DETECTION CODES AND EXTERNAL DIFFERENCE FAMILIES

In this section, we study the relationship between algebraic manipulation detection codes and external difference families. Before doing this, we first introduce some notation and definitions about difference families and their generalizations.

**Definition 3** ([4]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of subsets of G. Then  $\mathcal{B}$  is called a *difference family* (DF) if each nonzero element of G appears exactly  $\lambda$  times in the multi-set  $\bigcup_{1 \leq i \leq m} D(B_i)$ . Let  $K = (|B_1|, |B_2|, \ldots, |B_m|)$ . One briefly says that  $\mathcal{B}$  is an  $(n, K, \lambda)$ -DF.

When m=1 the set  $B_1$  is also called an  $(n,k=|B_1|,\lambda)$  difference set. If  $\mathcal{B}$  forms a partition of G, then  $\mathcal{B}$  is called a partitioned difference family (PDF) [9] and denoted as an  $(n,K,\lambda)$ -PDF.

**Definition 4** ([20]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G. Then  $\mathcal{B}$  forms an external difference family (EDF) if each nonzero element of G appears exactly  $\lambda$  times in the union of multi-sets  $D(B_i, B_j)$  for  $1 \leq i \neq j \leq m$ , i.e.,

$$\bigcup_{1 \le i \ne j \le m} D(B_i, B_j) = \lambda \boxtimes (G \setminus \{0\}).$$

We briefly denote  $\mathcal{B}$  as an  $(n, m, K, \lambda)$ -EDF, where  $K = (|B_1|, |B_2|, \dots, |B_m|)$ . An EDF is *regular* if  $|B_1| = |B_2| = \dots = |B_m| = k$ , denoted as an  $(n, m, k, \lambda)$ -EDF, which is also named as a perfect difference system of sets (refer to [16], [11], [10] for instances).

**Definition 5** ([20]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G. Then  $\mathcal{B}$  is a bounded external difference family (BEDF) if each nonzero element of G appears at most  $\lambda$  times in the union of multi-sets  $D(B_i, B_j)$  for  $1 \leq i \neq j \leq m$ , i.e.,

$$\bigcup_{1 \le i \ne j \le m} D(B_i, B_j) \subseteq \lambda \boxtimes (G \setminus \{0\}).$$

We briefly denote  $\mathcal{B}$  as an  $(n, m, K, \lambda)$ -BEDF, where  $K = (|B_1|, |B_2|, \dots, |B_m|)$ .

To construct AMD codes, in [20], the following generalizations of EDF were also introduced.

**Definition 6** ([20]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G.  $\mathcal{B}$  is called an  $(n, m; k_1, k_2, \cdots, k_m; \lambda_1, \lambda_2, \cdots, \lambda_m)$ -generalized strong external difference family (GSEDF) if for any given  $1 \leq i \leq m$ , each nonzero element of G appears exactly  $\lambda_i$  times in the union of multi-sets  $D(B_i, B_j)$  for  $1 \leq j \neq i \leq m$ , i.e.,

$$\bigcup_{\{j:1\leq j\leq m,\,j\neq i\}} D(B_i,B_j) = \lambda_i \boxtimes (G\setminus\{0\}),\tag{1}$$

where  $k_i = |B_i|$  for  $1 \le i \le m$ .

**Definition 7** ([20]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G. Then  $\mathcal{B}$  forms an  $(n, m; k_1, k_2, \dots, k_m; \lambda_1, \lambda_2, \dots, \lambda_m)$ -bounded generalized strong external difference family (BGSEDF) if for any given  $1 \leq i \leq m$ , each nonzero element of G appears at most  $\lambda_i$  times in the union of multi-sets  $D(B_i, B_j)$  for  $1 \leq j \neq i \leq m$ , i.e.,

$$\bigcup_{\{j:1\leq j\leq m,\ j\neq i\}} D(B_i,B_j) \subseteq \lambda_i \boxtimes (G\setminus\{0\}), \tag{2}$$

where  $k_i = |B_i|$  for  $1 \le i \le m$ .

**Definition 8** ([20]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G. Then  $\mathcal{B}$  is an  $(n, m; c_1, c_2, \dots, c_l; w_1, w_2, \dots, w_l; \lambda_1, \lambda_2, \dots, \lambda_l)$ -partitioned external difference family (PEDF) if for any given  $1 \leq t \leq l$ ,

$$\bigcup_{\{i:|B_i|=w_t\}} \bigcup_{\{j:1\leq j\leq m,\,j\neq i\}} D(B_i,B_j) = \lambda_t \boxtimes (G\setminus\{0\}),\tag{3}$$

where  $c_t = |\{i : |B_i| = w_t, 1 \le i \le m\}|$  for  $1 \le t \le l$ 

To characterize weak AMD codes, we further generalize external difference families to weighted external differences families.

**Definition 9:** Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of disjoint subsets of G. Let  $K = (k_1, k_2, \dots, k_m)$  with  $k_i = |B_i|$  for  $1 \leq i \leq m$  and  $\widetilde{k} = \operatorname{lcm}(k_1, k_2, \dots, k_m)$ . Define  $\widetilde{\mathcal{B}} = \{\widetilde{B}_i : B_i \in \mathcal{B}\}$  as the *standard weighted multi-sets* of  $\mathcal{B}$ , where

$$\widetilde{B}_i \triangleq \frac{\widetilde{k}}{|B_i|} \boxtimes B_i = \frac{\widetilde{k}}{k_i} \boxtimes B_i.$$

Then  $\mathcal{B}$  is called an  $(n, m, K, a, \lambda)$ -bounded standard weighted external difference family (BSWEDF) if  $\lambda$  is the smallest positive integer such that

$$\bigcup_{1 \le i \ne j \le m} D(B_i, \widetilde{B}_j) \subseteq \lambda \boxtimes (G \setminus \{0\}),$$

where  $a = \sum_{1 \le i \le m} k_i$ . Furthermore, if  $\mathcal{B}$  satisfies

$$\bigcup_{1 \le i \ne j \le m} D(B_i, \widetilde{B}_j) = \lambda \boxtimes (G \setminus \{0\}),$$

then it is named as a standard weighted external difference family, also denoted as an  $(n, m, K, a, \lambda)$ -SWEDF for short.

For BSWEDFs and SWEDFs, we have the following facts on their parameters.

**Lemma 2:** Let  $\mathcal{B}$  be an  $(n, m, K, a, \lambda)$ -BSWEDF. Then we have

$$\lambda \ge \left| \frac{\widetilde{k}a(m-1)}{n-1} \right| . \tag{4}$$

Specially, if  $\mathcal{B}$  is an  $(n, m, K, a, \lambda)$ -SWEDF, then  $(n-1) \mid (\widetilde{k}a(m-1))$  and

$$\lambda = \frac{\widetilde{k}a(m-1)}{n-1}.$$

*Proof.* Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$ . The fact

$$\bigcup_{1 \leq i \neq j \leq m} D(B_i, \widetilde{B}_j) = \bigcup_{1 \leq i \neq j \leq m} \bigcup_{b \in B_i} D(\{b\}, \widetilde{B}_j)$$

means that

$$\left| \bigcup_{1 \le i \ne j \le m} D(B_i, \widetilde{B}_j) \right| = \sum_{1 \le i \le m} \sum_{\substack{1 \le j \le m \\ i \ne i}} \sum_{b \in B_i} |D(\{b\}, \widetilde{B}_j)| = \sum_{1 \le i \le m} \sum_{\substack{1 \le j \le m \\ i \ne i}} \sum_{b \in B_i} \widetilde{k} = \widetilde{k}a(m-1).$$
 (5)

Thus, we have  $\lambda \geq \lceil \frac{\widetilde{k}a(m-1)}{n-1} \rceil$ .

Similarly, for the case of SWEDFs, by Definition 9 and (5), we have  $\lambda(n-1) = \widetilde{k}a(m-1)$ , i.e.,  $\lambda = \frac{\widetilde{k}a(m-1)}{n-1}$ , which also means  $(n-1) \mid (\widetilde{k}a(m-1))$ .

**Definition 10:** An  $(n, m, K, a, \lambda)$ -BSWEDF is said to be *optimal* if  $\lambda$  takes the smallest possible value for given n, m, and K.

Specially, an  $(n, m, K, a, \lambda)$ -BSWEDF is optimal if  $\lambda$  achieves the lower bound given by (4) with equality, i.e.,  $\lambda = \lceil \frac{\tilde{k}a(m-1)}{n-1} \rceil$ .

For  $\Delta \in G \setminus \{0\}$ , let  $\rho_{\Delta}$  denote the probability that the adversary wins by modifying  $g \in A_s$  into  $g + \Delta \in A_{s'}$  for some  $s' \neq s$ . Thus, we have  $\rho = \max\{\rho_{\Delta} : \Delta \in G \setminus \{0\}\}$ .

**Theorem 1:** There exists a weak  $(n,m,a,\rho)$ -AMD code  $(S,G,\mathcal{A},E_u)$  if and only if there exists an  $(n,m,K,a,\lambda)$ -BSWEDF, where  $|G|=n,\ a=\sum_{1\leq i\leq m}|A_{s_i}|,\ K=(|A_{s_1}|,|A_{s_2}|,\cdots,|A_{s_m}|),\ s_i\in S,$  and  $\rho=\frac{\lambda}{\tilde{k}m}.$ 

*Proof.* If  $(S, G, A, E_u)$  is a weak  $(n, m, a, \rho)$ -AMD code, then for any  $\Delta \in G \setminus \{0\}$ , we have

$$\rho_{\Delta} \le \rho = \frac{\lambda}{\widetilde{k}m},$$

that is,

$$\frac{\lambda}{\widetilde{k}m} \ge \rho_{\Delta} = \sum_{s \in S} Pr(s) \sum_{g \in A_s} Pr(E_u(s) = g) \left( \sum_{s' \ne s, s' \in S} Pr(g + \Delta \in A_{s'}) \right)$$

$$= \sum_{s \in S} \frac{1}{m} \sum_{g \in A_s} \frac{1}{|A_s|} \left( \sum_{s' \ne s, s' \in S} Pr(g + \Delta \in A_{s'}) \right)$$

$$= \sum_{s \in S} \frac{1}{m} \frac{1}{|A_s|} \left( \sum_{s' \ne s, s' \in S} \sum_{g \in A_s} Pr(g + \Delta \in A_{s'}) \right),$$
(6)

where the second equality holds by the fact that  $E_u$  encodes s to elements of  $A_s$  with uniform probability. Note that for given  $\Delta$ , s,  $g \in A_s$  and  $s' \neq s$ ,

$$Pr(g + \Delta \in A_{s'}) = \begin{cases} 1, & \Delta \in D(A_{s'}, \{g\}), \\ 0, & \Delta \notin D(A_{s'}, \{g\}). \end{cases}$$

Thus, Inequality (6) implies that

$$\frac{\lambda}{m} \ge \widetilde{k} \rho_{\Delta} = \sum_{s \in S} \frac{1}{m} \frac{\widetilde{k}}{|A_{s}|} \left( \sum_{s' \ne s, s' \in S} \sum_{g \in A_{s}} Pr(g + \Delta \in A_{s'}) \right)$$

$$= \sum_{s \in S} \frac{1}{m} \frac{\widetilde{k}}{|A_{s}|} \left( \sum_{s' \ne s, s' \in S} \sharp (\Delta, D(A_{s'}, A_{s})) \right)$$

$$= \sum_{s \in S} \frac{1}{m} \left( \sum_{s' \ne s, s' \in S} \frac{\widetilde{k}}{|A_{s}|} \sharp (\Delta, D(A_{s'}, A_{s})) \right)$$

$$= \sum_{s \in S} \frac{1}{m} \left( \sum_{s' \ne s, s' \in S} \sharp \left( \Delta, D(A_{s'}, \widetilde{A}_{s}) \right) \right)$$

$$= \frac{1}{m} \sharp \left( \Delta, \bigcup_{s, s' \in S, S} D(A_{s'}, \widetilde{A}_{s}) \right),$$
(7)

where  $\sharp(\Delta,B)$  denotes the number of times that  $\Delta$  appears in the multi-set B. This means that any  $\Delta \in G \setminus \{0\}$  appears at most  $\lambda$  times in the multi-set  $\bigcup_{\substack{s,s' \in S, \\ s' \neq s}} D(A_{s'},\widetilde{A}_s)$ , i.e.,

$$\bigcup_{\substack{s,s'\in S,\\s',s'}} D(A_{s'},\widetilde{A}_s) \subseteq \lambda \boxtimes (G\setminus\{0\}).$$

Note that  $\rho = \max\{\rho_{\Delta}: \Delta \in G \setminus \{0\}\}$  means there exists at least one  $\Delta \in G \setminus \{0\}$  such that the equality in (7) holds. Then  $\{A_s: s \in S\}$  forms an  $(n, m, (|A_{s_1}|, |A_{s_2}|, \cdots, |A_{s_m}|), a, \lambda)$ -BSWEDF by Definition 9.

Conversely, suppose that there exists an  $(n, m, K, a, \lambda)$ -BSWEDF  $\mathcal{B} = \{B_i : 1 \le i \le m\}$  over G. Let  $S = \{s_i : 1 \le i \le m\}$  and  $A_{s_i} = B_i$  for  $1 \le i \le m$ . Then we can define a weak AMD code, where  $E_u(s_i) = g \in B_i$  with equiprobability. For any  $\Delta \in G \setminus \{0\}$ , similarly as (6), we have

$$\rho_{\Delta} = \sum_{s \in S} \frac{1}{m} \frac{1}{|A_s|} \left( \sum_{\substack{s' \neq s, s' \in S \\ g \neq s, s' \in S}} \sum_{g \in A_s} Pr(g + \Delta \in A_{s'}) \right)$$

$$= \sum_{1 \leq i \leq m} \frac{1}{m} \frac{1}{|B_i|} \left( \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \sharp(\Delta, D(B_j, B_i)) \right)$$

$$= \sum_{1 \leq i \leq m} \frac{1}{\widetilde{k}m} \left( \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \sharp(\Delta, D(B_j, \widetilde{B}_i)) \right)$$

$$= \frac{1}{\widetilde{k}m} \left( \sum_{1 \leq j \neq i \leq m} \sharp(\Delta, D(B_j, \widetilde{B}_i)) \right)$$

$$\leq \frac{\lambda}{\widetilde{k}m},$$

where the last inequality holds by the fact that  $\mathcal{B}$  is an  $(n, m, K, a, \lambda)$ -BSWEDF. According to Definition 9, the equality is achieved for at least one  $\Delta \in G \setminus \{0\}$  in the preceding inequality. Thus, the weak  $(n, m, a, \rho)$ -AMD code defined based on the BSWEDF  $\mathcal{B}$  satisfies

$$\rho = \max\{\rho_{\Delta} : \Delta \in G \setminus \{0\}\} = \frac{\lambda}{\widetilde{k}m},$$

which completes the proof.

When we consider the optimality of BSWEDF, the size-distribution  $K=(k_1,k_2,\ldots,k_m)$  is given. However, the R-optimality of weak AMD codes only relates with  $a=\sum_{1\leq i\leq m}k_i$  as defined in [20] but disregards the exact size-distribution K of  $\mathcal{A}$ . There may exist several BSWEDFs with different K which correspond to weak AMD codes with exactly the same parameter a. Thus, although the BSWEDF gives a characterization of the weak AMD code, in general, the optimal BSWEDF for a given K does not necessarily correspond to an K-optimal weak AMD code for a given K.

**Definition 11:** For given n, m and a, an  $(n, m, K, a, \lambda)$ -BSWEDF is said to be *strongly optimal* if  $\frac{\lambda}{\tilde{k}m} = \rho_{(n,m,a)}$ , where

$$\rho_{(n,m,a)} = \min_{K'} \left\{ \frac{\lambda'}{\widetilde{k'}m} : \exists (n,m,K',a,\lambda') \text{-BSWEDF } s.t. \sum_{1 \le i \le m} k'_i = a \right\}.$$
 (8)

By Theorem 1 and Lemma 2, we have

**Corollary 1:** For any weak  $(n, m, a, \rho)$ -AMD code  $(S, G, A, E_u)$ , we have

$$\rho \ge \rho_{(n,m,a)} \ge \min_{K} \left\{ \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil \frac{1}{\widetilde{k}m} : \sum_{1 \le i \le m} k_i = a \right\},\,$$

where  $|A_i| = k_i$  for any  $A_i \in \mathcal{A}$ .

*Proof.* Let  $(S, G, A, E_u)$  be a weak  $(n, m, a, \rho)$ -AMD code. By Theorem 1, there exists an  $(n, m, K, a, \lambda)$ -BSWEDF with  $\lambda = \widetilde{k} m \rho$ . Then by Lemma 2 and (8),

$$\rho = \frac{\lambda}{\widetilde{k}m} \ge \rho_{(n,m,a)} \ge \min_{K} \left\{ \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil \frac{1}{\widetilde{k}m} : \sum_{1 \le i \le m} k_i = a \right\}.$$

**Definition 12:** A weak AMD code with  $\rho = \rho_{(n,m,a)}$  is said to be *R-optimal* with respect to the bound in Corollary 1. When  $(n-1) \mid (\widetilde{k}a(m-1))$ , the bound in Corollary 1 is exactly the same as the one given in Lemma 1. However, when  $(n-1) \nmid (\widetilde{k}a(m-1))$ , our bound in Corollary 1 can improve the known one in Lemma 1. The following is an easy example.

Corollary 2: For any weak  $(n, m, a, \rho)$ -AMD code  $(S, G, A, E_u)$ , if n-1 is a prime and a < n-1, then we have

$$\rho \ge \min_{K} \left\{ \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil \frac{1}{\widetilde{k}m} : \sum_{1 \le i \le m} k_i = a \right\} > \frac{a(m-1)}{m(n-1)}.$$

*Proof.* The lemma follows from the facts that  $k_i \le a < n-1$  for  $1 \le i \le m$ ,  $m \le a < n-1$ , and n-1 is a prime. In this case,  $(n-1) \nmid (\widetilde{k}a(m-1))$ .

A more concrete example is listed below.

**Example 1:** Let n=10, m=3, and a=5. Let  $B=\{\{5\},\{2\},\{0,4,6\}\}$  be a family of disjoint subsets of  $\mathbb{Z}_{10}$ , which corresponding to a weak  $(10,3,5,\rho)$ -AMD code, where  $\rho=\frac{1}{3}\cdot\frac{1}{1}\cdot 1+\frac{1}{3}\cdot\frac{1}{1}\cdot 0+\frac{1}{3}\cdot\frac{1}{3}\cdot 1=\frac{4}{9}$ . According to Lemma 1 and definition 2, this is not an R-optimal weak AMD code. However, R-optimality should mean that random choosing  $\Delta$  is an optimal strategy for the adversary. Clearly, according to Corollary 1, the parameter  $\rho$  cannot be smaller then

$$\min_{K} \left\{ \left\lceil \frac{\tilde{k}5(3-1)}{10-1} \right\rceil \frac{1}{3\tilde{k}} : \sum_{1 \le i \le 3} k_i = 5 \right\} \\
= \min_{K} \left\{ \left\lceil \frac{\text{lcm}(1,1,3) \cdot 5 \cdot 2}{9} \right\rceil \frac{1}{3 \text{lcm}(1,1,3)}, \left\lceil \frac{\text{lcm}(1,2,2) \cdot 5 \cdot 2}{9} \right\rceil \frac{1}{3 \text{lcm}(1,2,2)} \right\} \\
= \min_{K} \left\{ \frac{4}{9}, \frac{1}{2} \right\} = \frac{4}{9}.$$

Therefore, this example should be an R-optimal weak  $(10, 3, 5, \rho)$ -AMD code. This trouble is due to the fact that the known bound in Lemma 1 is not always tight.

Relationships between optimal weak AMD codes and optimal BSWEDFs are described below.

Corollary 3: Let n and m be positive integers.

- (I) For given  $K=(k_1,k_2,\ldots,k_m)$ , let  $\rho_{(n,m,K)}$  denote the smallest possible  $\rho$  for weak  $(n,m,\sum_{1\leq i\leq m}k_i,\rho)$ -AMD codes. Then a weak  $(n,m,a,\rho)$ -AMD code  $(S,G,A,E_u)$  has the smallest  $\rho$ , i.e.,  $\rho=\rho_{(n,m,K)}$  if and only if its corresponding BSWEDF with parameters  $(n,m,K,a,\lambda=\widetilde{k}m\rho)$  is optimal, where  $S=\{s_i:1\leq i\leq m\},\ \mathcal{A}=\{A_{s_i}:1\leq i\leq m\},\ k_i=|A_{s_i}|\ \text{for}\ 1\leq i\leq m,\ K=(k_1,k_2,\ldots,k_m),\ \text{and}\ a=\sum_{1\leq i\leq m}k_i.$  (II) For given a, there exists an R-optimal weak  $(n,m,a,\rho)$ -AMD code  $(S,G,\overline{\mathcal{A}},\overline{E}_u)$  with respect to the bound in Corollary
- (II) For given a, there exists an R-optimal weak  $(n, m, a, \rho)$ -AMD code  $(S, G, \overline{\mathcal{A}}, \overline{E}_u)$  with respect to the bound in Corollary 1 if and only if there exists a strongly optimal  $(n, m, K, a, \lambda)$ -BSWEDF, where |G| = n,  $a = \sum_{s \in S} |A_s|$ ,  $\rho = \rho_{(n, m, a)}$ , and  $\lambda = \widetilde{k} m \rho_{(n, m, a)}$ .
- (III) There exists an R-optimal weak  $(n, m, a, \rho)$ -AMD code  $(S, G, \mathcal{A}, E_u)$  with respect to the bound in Lemma 1 if and only if there exists an  $(n, m, K, a, \lambda)$ -SWEDF, where  $\rho = \frac{a(m-1)}{m(n-1)}$ , and  $\lambda = \frac{\tilde{k}a(m-1)}{n-1}$ .

*Proof.* By Theorem 1, for given n, m, K (or a, resp.), a weak AMD code with the smallest  $\rho$  is equivalent to a BSWEDF with the smallest  $\lambda$ , i.e., an optimal (or strongly optimal, resp.) BSWEDF. The third part of the result follows directly from Theorem 1 and Lemma 2.

**Example 2:** Let n = 10, m = 3, and a = 5. Let  $\mathcal{B}^{(1)} = \{B_1^{(1)} = \{5\}, B_2^{(1)} = \{4, 6\}, B_3^{(1)} = \{2, 8\}\}$  and  $\mathcal{B}^{(2)} = \{5\}, B_2^{(2)} = \{2\}, B_3^{(2)} = \{0, 4, 6\}\}$  be two families of disjoint subsets of  $\mathbb{Z}_{10}$ . It is easy to verify that

$$\bigcup_{1 \le i \le 3} D\left(B_i^{(1)}, \widetilde{B}_j^{(1)}\right) \subseteq 3 \boxtimes (\mathbb{Z}_{10} \setminus \{0\})$$

and

$$\bigcup_{1 \le i \le 3} D\left(B_i^{(2)}, \widetilde{B}_j^{(2)}\right) \subseteq 4 \boxtimes (\mathbb{Z}_{10} \setminus \{0\}).$$

According to Lemma 2,  $\mathcal{B}^{(1)}$  is an optimal (10,3,(1,2,2),5,3)-BSWEDF and  $\mathcal{B}^{(2)}$  is an optimal (10,3,(1,1,3),5,4)-BSWEDF. By Corollary 1,

$$\rho_{(10,3,5)} \ge \min_{K} \left\{ \left\lceil \frac{\widetilde{k}5(3-1)}{10-1} \right\rceil \frac{1}{3\widetilde{k}} : \sum_{1 \le i \le 3} k_i = 5 \right\} = \frac{4}{9}.$$

Thus, by Definition 11,  $\mathcal{B}^{(2)}$  is in fact not only an optimal, but a strongly optimal BSWEDF. By Corollary 3. (II), we can obtain a corresponding R-optimal weak AMD code with respect to the bound in Corollary 1 from  $\mathcal{B}^{(2)}$ .

Although the weak  $(n, m, a, \rho_{(n,m,K)} = \frac{\lambda}{km})$ -AMD code  $(S, G, \mathcal{A}, E_u)$  based on an optimal  $(n, m, K, a, \lambda)$ -BSWEDF may sometimes not correspond to an R-optimal weak AMD code with parameters  $(n, m, a, \rho_{(n,m,a)})$ , the difference  $\rho_{(n,m,K)} - \rho_{(n,m,a)}$  is not big.

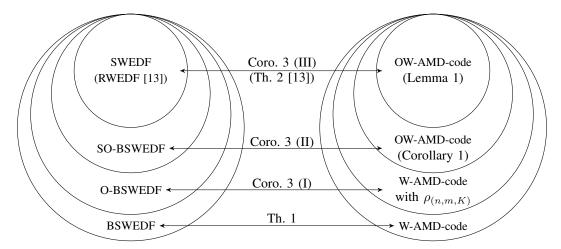


Fig. 1: The relationships between AMD codes and BSWEDFs

**Lemma 3:** Let  $a = \sum_{A \in \mathcal{A}} |A| = \sum_{1 \leq i \leq m} k_i$ . Let  $(S, G, \mathcal{A}, E_u)$  be the weak  $(n, m, a, \rho = \frac{\lambda}{\tilde{k}m})$ -AMD code based on an optimal  $(n, m, K, a, \lambda)$ -BSWEDF with  $\lambda = \lceil \frac{\tilde{k}a(m-1)}{n-1} \rceil$ , and let  $(S, G, \mathcal{A}', E_u)$  be the R-optimal weak  $(n, m, a, \rho_{(n,m,a)})$ -AMD code with respect to the bound in Corollary 1. Then we have

$$0 \le \rho_{(n,m,K)} - \rho_{(n,m,a)} \le \frac{1}{\widetilde{k}m}.$$

Proof. The lemma follows directly from the fact that

$$0 \leq \rho_{(n,m,K)} - \rho_{(n,m,a)} = \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil \frac{1}{\widetilde{k}m} - \rho_{(n,m,a)} \leq \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil \frac{1}{\widetilde{k}m} - \frac{a(m-1)}{m(n-1)} \leq \frac{1}{\widetilde{k}m}.$$

In [13], Huczynska and Paterson characterized R-optimal AMD codes  $(S, G, A, E_u)$  by reciprocally-weighted external difference families, which can be defined as follows.

**Definition 13** ([13]): Let  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  be a family of subsets of G. Let  $K = (k_1, k_2, \dots, k_m)$  with  $k_i = |B_i|$  for  $1 \leq i \leq m$  and  $\widetilde{k} = \operatorname{lcm}(k_1, k_2, \dots, k_m)$ . Then  $\mathcal{B}$  is said to be an  $(n, m, (k_1, k_2, \dots, k_m), d)$  reciprocally-weighted external difference family (RWEDF) if

$$d = \sum_{1 \le i \le m} \frac{N_i(\delta)}{k_i} \text{ for each } \delta \in G \setminus \{0\},$$

where

$$N_i(\delta) \triangleq \left| \left\{ (b_i, b_j) : b_i \in B_i, \ b_j \in \bigcup_{1 \leq t \neq i \leq m} B_t, \ \text{and} \ b_j - b_i = \delta \right\} \right|.$$

**Theorem 2** ([13]): A weak  $(n, m, a, \rho)$ -AMD code  $(S, G, \mathcal{A}, E_u)$  is R-optimal with respect to the bound in Lemma 1 if and only if there exists an (n, m, K, a, d)-RWEDF, where  $\rho = \frac{a(m-1)}{m(n-1)}$ , and  $d = \frac{a(m-1)}{n-1}$ .

Clearly,  $N_i(\delta) = \sharp \left(\delta, \bigcup_{1 \leq j \leq m \atop j \neq i} D(B_j, B_i)\right)$  for  $1 \leq i \leq m$ , and by Theorem 2 and Corollary 3 or Definitions 9 and 13, we know that an (n, m, K, a, d)-RWEDF is essentially the same as an  $(n, m, K, a, \lambda)$ -SWEDF, where  $d = \frac{\lambda}{k}$ . Therefore, Theorem 1 and Corollary 3 provide more combinatorial characterizations for various weak AMD codes  $(S, G, A, E_u)$ . These results can be viewed as a generalization of Theorem 2. As a byproduct, we have the following property for an (n, m, K, a, d)-RWEDF directly from Lemma 2 and Corollary 3. (III).

**Corollary 4:** A necessary condition for the existence of an (n, m, K, a, d)-RWEDF, or equivalently an R-optimal weak  $(n, m, a, \rho)$ -AMD code  $(S, G, A, E_u)$  with respect to Lemma 1, is  $(n-1) \mid (\widetilde{k}a(m-1))$ , where  $K = (k_1 = |A_{s_1}|, k_2 = |A_{s_2}|, \cdots, k_m = |A_{s_m}|)$  and  $\widetilde{k} = \operatorname{lcm}(k_1, k_2, \cdots, k_m)$ .

In Figure 1, we summarize the relationships between weak AMD codes and BSWEDFs, where SO-BSWEDF, O-BSWEDF, and OW-AMD-code denote strongly optimal BSWEDF, optimal BSWEDF, and R-optimal weak AMD-code, respectively.

A. Among EDFs, SEDFs, PEDFs, SWEDFs, and BSWEDFs

In general, an EDF is not necessarily an SWEDF. However, in the following cases, an EDF is always an SWEDF. First of all, we consider the regular case.

**Lemma 4:** A regular  $(n, m, k, \lambda)$ -EDF forms an  $(n, m, K = (k, k, \dots, k), a = mk, \lambda)$ -SWEDF.

The lemma follows directly from the definitions of EDF and SWEDF.

For the case of GSEDFs we have the following result.

**Lemma 5:** If  $\{B_i: 1 \leq i \leq m\}$  is an  $(n,m;k_1,k_2,\cdots,k_m;\lambda_1,\lambda_2,\cdots,\lambda_m)$ -GSEDF, then  $\{B_i: 1 \leq i \leq m\}$  is an  $(n,m,(k_1,k_2,\cdots,k_m),a,\lambda)$ -SWEDF, where  $\lambda = \sum_{1 \leq i \leq m} \frac{\lambda_i \tilde{k}}{k_i}$ .

*Proof.* Let  $\{B_i: 1 \leq i \leq m\}$  be an  $(n, m; k_1, k_2, \cdots, k_m; \lambda_1, \lambda_2, \cdots, \lambda_m)$ -GSEDF, by (1),

$$\bigcup_{1 \le j \le m, j \ne i} D(B_i, B_j) = \lambda_i \boxtimes (G \setminus \{0\}),$$

which means

$$\bigcup_{1 \le j \le m, j \ne i} D(B_j, \widetilde{B}_i) = \frac{\lambda_i \widetilde{k}}{k_i} \boxtimes (G \setminus \{0\}).$$

Thus, we have

$$\bigcup_{1 \leq i \leq m} \bigcup_{1 \leq j \leq m, \ j \neq i} D(B_j, \widetilde{B}_i) = \left( \sum_{1 \leq i \leq m} \lambda_i \frac{\widetilde{k}}{k_i} \right) \boxtimes (G \backslash \{0\}) = \lambda \boxtimes (G \backslash \{0\}),$$

i.e.,  $\{B_i: 1 \leq i \leq m\}$  is an  $(n, m, (k_1, k_2, \cdots, k_m), a, \lambda)$ -SWEDF with  $\lambda = \sum_{1 \leq i \leq m} \frac{\lambda_i \tilde{k}}{k_i}$ .

Similarly, the relationship between PEDFs and SWEDFs can be given by the following lemma.

**Lemma 6:** If  $\{B_i:1\leq i\leq m\}$  is an  $(n,m;c_1,c_2,\cdots,c_l;w_1,w_2,\cdots,w_l;\lambda_1,\lambda_2,\cdots,\lambda_l)$ -PEDF, then  $\{B_i:1\leq i\leq m\}$  is an  $(n,m,K=(|B_1|,|B_2|,\cdots,|B_m|),a,\lambda)$ -SWEDF, where  $\widetilde{k}=\operatorname{lcm}(w_1,w_2,\cdots,w_l)$  and  $\lambda=\sum_{1\leq t\leq l}\frac{\lambda_t\widetilde{k}}{w_t}$ .

*Proof.* Since  $\{B_i:1\leq i\leq m\}$  is an  $(n,m;c_1,c_2,\cdots,c_l;w_1,w_2,\cdots,w_l;\lambda_1,\lambda_2,\cdots,\lambda_l)$ -PEDF, by (3),

$$\bigcup_{\{i:|B_i|=w_t\}}\bigcup_{1\leq j\leq m,\ j\neq i}D(B_i,B_j)=\lambda_t\boxtimes (G\backslash\{0\})$$

for  $1 \le t \le l$ . By Definition 8,  $|B_i| \in \{w_j : 1 \le j \le l\}$  for  $1 \le i \le m$ . Thus, for  $K = (|B_1|, |B_2|, \cdots, |B_m|)$ , we have  $\widetilde{k} = \text{lcm}(|B_1|, |B_2|, \cdots, |B_m|) = \text{lcm}(w_1, w_2, \cdots, w_l)$ . Thus, we have

$$\bigcup_{1 \le t \le l} \bigcup_{\{i : |B_i| = w_t\}} \bigcup_{1 \le j \le m, \ j \ne i} D(B_j, \widetilde{B}_i) = \left( \sum_{1 \le t \le l} \lambda_t \frac{\widetilde{k}}{w_t} \right) \boxtimes (G \setminus \{0\}) = \lambda \boxtimes (G \setminus \{0\}),$$

i.e.,  $\{B_i: 1\leq i\leq m\}$  is an  $(n,m,K=(|B_1|,|B_2|,\cdots,|B_m|),a,\lambda)$ -SWEDF, where  $\lambda=\sum_{1\leq t\leq l}\frac{\lambda_t \tilde{k}}{w_t}$ .

In what follows, we recall an example of SWEDF which is not an EDF, or an GSEDF, or a PEDF.

**Example 3** ([20]): Let  $G = (\mathbb{Z}_{10}, +)$  and  $\mathcal{B} = \{B_1 = \{0\}, B_2 = \{5\}, B_3 = \{2, 3\}, B_4 = \{6, 4\}\}$ . Then  $\widetilde{B}_1 = \{0, 0\}, \widetilde{B}_2 = \{5, 5\}, \widetilde{B}_3 = \{2, 3\}, \widetilde{B}_4 = \{6, 4\}$ . It is easy to check

$$\bigcup_{1 \le i \le 4} \bigcup_{1 \le j \le 4, j \ne i} D(B_i, \widetilde{B}_j) = 4 \boxtimes (G \setminus \{0\}),$$

$$\bigcup_{1 \le i \le 4} \bigcup_{1 \le j \le 4, j \ne i} D(B_i, B_j) \ne \lambda \boxtimes (G \setminus \{0\}),$$

$$\bigcup_{1 \le j \le 4} D(B_1, B_j) = \{5, 8, 7, 4, 6\} \ne \lambda \boxtimes (G \setminus \{0\}),$$

and

$$\bigcup_{3 < i < 4} \bigcup_{1 < j < 4, j \neq i} D(B_i, B_j) \neq \lambda \boxtimes (G \setminus \{0\}),$$

for any positive integer  $\lambda$ . Thus,  $\mathcal{B}$  is an SWEDF which does not form an EDF, or a GSEDF, or a PEDF.

Similarly, a BEDF is not necessarily a BSWEDF in general and we have the following relationship between BEDFs and BSWEDFs.

**Lemma 7:** The regular  $(n, k, \lambda)$ -BEDF forms an  $(n, m, K = (k, k, \dots, k), a = mk, \lambda_1)$ -BSWEDF, where  $\lambda_1 \leq \lambda$ .

**Lemma 8:** If  $\mathcal{B}=\{B_i:1\leq i\leq m\}$  is an  $(n,m;k_1,k_2,\cdots,k_m;\lambda_1,\lambda_2,\cdots,\lambda_m)$ -BGSEDF, then  $\mathcal{B}$  is an  $(n,m,(k_1,k_2,\cdots,k_m),a=\sum_{1\leq i\leq m}k_i,\lambda)$ -BSWEDF, where  $\lambda\leq\sum_{1\leq i\leq m}\frac{\lambda_i\tilde{k}}{k_i}$ .

*Proof.* Since  $\mathcal{B} = \{B_i : 1 \leq i \leq m\}$  is an  $(n, m; k_1, k_2, \cdots, k_m; \lambda_1, \lambda_2, \cdots, \lambda_m)$ -BGSEDF, by (2),

$$\bigcup_{1 \le j \le m, j \ne i} D(B_i, B_j) \subseteq \lambda_i \boxtimes (G \setminus \{0\}),$$

which means

$$\bigcup_{1 \le j \le m, j \ne i} D(B_j, \widetilde{B}_i) \subseteq \lambda_i \frac{\widetilde{k}}{k_i} \boxtimes (G \setminus \{0\}).$$
(9)

Let  $\lambda$  be the smallest positive integer such that

$$\bigcup_{1 \leq i \leq m} \bigcup_{1 \leq j \leq m, \, j \neq i} D(B_j, \widetilde{B}_i) \subseteq \lambda \boxtimes (G \backslash \{0\}).$$

Thus, by (9), we have  $\lambda \leq \sum_{1 \leq i \leq m} \frac{\lambda_i \tilde{k}}{k_i}$ , i.e.,  $\mathcal{B}$  is an  $(n, m, (k_1, k_2, \cdots, k_m), a = \sum_{1 \leq i \leq m} k_i, \lambda)$ -BSWEDF.

# IV. CONSTRUCTIONS OF OPTIMAL BSWEDFS AND SWEDFS

In this section, we are going to construct BSWEDFs and SWEDFs, which are generally not EDFs, or GSEDFs, or PEDFs. We recall a well-known construction of difference families. Let q=4k+1 be a prime power. Let  $\alpha$  be a primitive element of  $\mathbb{F}_q$ ,

$$D_i^2 = \{\alpha^{i+2j} : 0 \le j \le 2k-1\}, \text{ for } i = 0,1$$
 (10)

and

$$D_i^4 = \{\alpha^{i+4j} : 0 \le j \le k-1\}, \text{ for } 0 \le i \le 3.$$
 (11)

It is well-known that  $\{D_0^2, D_1^2\}$  is a (q, 2k, 2k-1)-DF over the additive group of  $\mathbb{F}_q$ .

Construction A: Let  $S = \{S_1, S_2, S_3\}$  be the family of disjoint subsets  $\mathbb{Z}_2 \times \mathbb{F}_q$  defined as

$$S_1 = \{(0,0),(1,0)\}, \ S_2 = \{0\} \times D_0^4 \cup \{1\} \times D_2^4, \ \text{and} \ S_3 = \{0\} \times D_1^4 \cup \{0\} \times D_3^4.$$

**Theorem 3:** Let  $S = \{S_1, S_2, S_3\}$  be the family defined in Construction A. If k is odd, then S is an optimal  $(n = 2q, m = 3, (2, 2k, 2k), a = 4k + 2, \lambda = 2k + 1)$ -BSWEDF.

Before the proof we list a well-known result about  $D_0^2$  and  $D_1^2$ .

**Lemma 9:** If k is odd, then the family  $\{D_0^2, D_1^2\}$  satisfies

$$D\left(D_0^2,D_1^2\right) \cup D\left(D_1^2,D_0^2\right) = 2k \boxtimes (\mathbb{F}_q \backslash \{0\})$$

and

$$D\left(D_0^4,D_1^4\right)\cup D\left(D_0^4,D_3^4\right)\cup D\left(D_1^4,D_0^4\right)\cup D\left(D_3^4,D_0^4\right)=k\boxtimes (\mathbb{F}_q\backslash\{0\}).$$

*Proof.* By (10) and (11), we have  $D_0^2 = D_0^4 \cup D_2^4 = D_0^4 \cup (-D_0^4)$  and  $D_1^2 = D_1^4 \cup D_3^4 = D_1^4 \cup (-D_1^4)$ , where  $\alpha^{2k} = -1$ . The fact  $\{D_0^2, D_1^2\}$  is a (q, 2k, 2k - 1)-PDF means that

$$D\left(D_0^2,D_1^2\right) \cup D\left(D_1^2,D_0^2\right) = 2k \boxtimes (\mathbb{F}_q \backslash \{0\}).$$

The preceding equality can be rewritten as

$$\begin{split} 2k \boxtimes \left(\mathbb{F}_q \backslash \{0\}\right) = & D\left(D_0^2, D_1^2\right) \cup D\left(D_1^2, D_0^2\right) \\ = & D\left(D_0^4 \cup \left(-D_0^4\right), D_1^4 \cup D_3^4\right) \cup D\left(D_1^4 \cup D_3^4, D_0^4 \cup \left(-D_0^4\right)\right) \\ = & 2 \boxtimes \left(D\left(D_0^4, D_1^4\right) \cup D\left(D_0^4, D_3^4\right) \cup D\left(D_1^4, D_0^4\right) \cup D\left(D_3^4, D_0^4\right)\right), \end{split}$$

where for the last equality we use the facts  $D(-D_0^4, D_1^4 \cup D_3^4) = D(-(D_1^4 \cup D_3^4), D_0^4) = D(D_3^4 \cup D_1^4, D_0^4)$  and  $D\left(D_1^4 \cup D_3^4, -D_0^4\right) = D\left(D_0^4, -(D_1^4 \cup D_3^4)\right) = D\left(D_0^4, D_1^4 \cup D_1^4\right)$ . This completes the proof.

Proof of Theorem 3: By Definition 9, in this case,  $\widetilde{k} = \text{lcm}(2k,2) = 2k$ ,  $\widetilde{S}_1 = k \boxtimes \{(0,0),(1,0)\}$ ,  $\widetilde{S}_2 = S_2$ , and  $\widetilde{S}_3 = S_3$ . Thus,  $D(S_2,\widetilde{S}_3) = D(S_2,S_3)$  and  $D(S_3,\widetilde{S}_2) = D(S_3,S_2)$ . Recall that  $S_2 = \{0\} \times D_0^4 \cup \{1\} \times (-D_0^4)$ , which implies

$$D(S_{2}, \widetilde{S}_{3}) \cup D(S_{3}, \widetilde{S}_{2})$$

$$=D(\{0\} \times D_{0}^{4} \cup \{1\} \times (-D_{0}^{4}), \{0\} \times D_{1}^{4} \cup \{0\} \times D_{3}^{4})$$

$$\cup D(\{0\} \times D_{1}^{4} \cup \{0\} \times D_{3}^{4}, \{0\} \times D_{0}^{4} \cup \{1\} \times (-D_{0}^{4}))$$

$$= \bigcup_{i=0,1} \{i\} \times \left(D\left(D_{0}^{4}, D_{1}^{4}\right) \cup D\left(D_{0}^{4}, D_{3}^{4}\right) \cup D\left(D_{1}^{4}, D_{0}^{4}\right) \cup D\left(D_{3}^{4}, D_{0}^{4}\right)\right)$$

$$=k \boxtimes \left(\mathbb{Z}_{2} \times \left(\mathbb{F}_{q} \setminus \{0\}\right)\right),$$
(12)

where we use the fact  $D_1^4 = -D_3^4$  and the last equality holds by Lemma 9. By the fact  $\bigcup_{0 \le i \le 3} D_i^4 = \mathbb{F}_q \setminus \{0\}$ , we have

$$D(S_1, \widetilde{S}_2) \cup D(S_2, \widetilde{S}_1) = \{0\} \times D_2^4 \cup \{1\} \times D_0^4 \cup \{1\} \times D_2^4 \cup \{0\} \times D_0^4$$
$$\cup k \boxtimes (\{0\} \times D_2^4 \cup \{1\} \times D_0^4 \cup \{1\} \times D_2^4 \cup \{0\} \times D_0^4)$$
$$= (k+1) \boxtimes (\mathbb{Z}_2 \times D_0^2)$$

and

$$D(S_1, \widetilde{S}_3) \cup D(S_3, \widetilde{S}_1) = \{0\} \times D_1^2 \cup \{1\} \times D_1^2 \cup k \boxtimes (\{0\} \times D_1^2 \cup \{1\} \times D_1^2)$$
  
=  $(k+1) \boxtimes (\mathbb{Z}_2 \times D_1^2),$ 

where we use the facts  $D_i^2 = D_i^4 \cup D_{i+2}^4$  and  $D_i^4 = -D_{i+2}^4$  for i = 0, 1. The above two equalities imply that

$$\bigcup_{i=2,3} \left( D(S_1, \widetilde{S}_i) \cup D(S_i, \widetilde{S}_1) \right) = (k+1) \boxtimes \left( \mathbb{Z}_2 \times (\mathbb{F}_q \setminus \{0\}) \right). \tag{13}$$

Therefore, by (12) and (13),

$$\bigcup_{1 \le i \ne j \le 3} D(S_i, \widetilde{S}_j) = (2k+1) \boxtimes (\mathbb{Z}_2 \times (\mathbb{F}_q \setminus \{0\})) \subseteq (2k+1) \boxtimes ((\mathbb{Z}_2 \times \mathbb{F}_q) \setminus \{(0,0)\}),$$

i.e.,  $S = \{S_1, S_2, S_3\}$  is an  $(n = 2q, m = 3, (2, 2k, 2k), a = 4k + 2, \lambda = 2k + 1)$ -BSWEDF. By Lemma 2, we have

$$\lambda \ge \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil = \left\lceil \frac{2k(4k+2)2}{2q-1} \right\rceil = \left\lceil \frac{2k(8k+1)+6k}{8k+1} \right\rceil = 2k+1.$$

Thus, S is an optimal  $(n = 2q, m = 3, (2, 2k, 2k), a = 4k + 2, \lambda = 2k + 1)$ -BSWEDF.

It is easily seen from the proof of Theorem 3 that the above BSWEDFs are not EDFs, or GSEDFs, or PEDFs.

**Example 4:** Let n = 2q = 26. By Construction A, the family of sets  $S = \{S_1, S_2, S_3\}$  over  $\mathbb{Z}_{26}$  can be listed as

$$S_1 = \{0, 13\}, S_2 = \{14, 16, 22, 17, 25, 23\}, \text{ and } S_3 = \{2, 6, 18, 8, 24, 20\}.$$

It is easy to check that

$$\bigcup_{1 \le i \ne j \le 3} D(S_i, \widetilde{S}_j) = 7 \boxtimes (\mathbb{Z}_{26} \setminus \{0, 13\}),$$

which means that S is an optimal (26, 3, (2, 6, 6), 14, 7)-BSWEDF.

Let  $n_1 = 2k + 1$  and  $\{\{0\}, E_1, E_2\}$  be an  $(n_1, k, k - 1)$ -PDF over an Abelian group G of order  $n_1$ . Such kinds of PDFs exist, for example, when  $n_1$  is a prime power, and  $E_1 = D_0^2$ ,  $E_2 = D_1^2$ . Based on  $\{\{0\}, E_1, E_2\}$  we can construct a BSWEDF as follows.

**Construction B:** Let  $W = \{W_1, W_2, W_3\}$  be the family of disjoint subsets of  $\mathbb{Z}_2 \times G$ , defined as  $W_1 = \{(1,0)\}$ ,  $W_2 = \{0\} \times E_1$ , and  $W_3 = \{0\} \times E_2$ .

**Theorem 4:** The family  $W = \{W_1, W_2, W_3\}$  generated by Construction B is an optimal  $(n = 2n_1, 3, (1, k, k), 2k + 1, k + 1)$ -BSWEDF.

*Proof.* The fact that  $\{\{0\}, E_1, E_2\}$  is an  $(n_1 = 2k + 1, k, k - 1)$ -PDF means that  $D(E_1, E_2) \cup D(E_2, E_1) = k \boxtimes (G \setminus \{0\})$ . Thus, we have

$$D(W_2,\widetilde{W}_3)\cup D(W_3,\widetilde{W}_2)=D(W_2,W_3)\cup D(W_3,W_2)=k\boxtimes (\{0\}\times (G\setminus\{0\})),$$

where we apply the fact  $\widetilde{k} = \text{lcm}(1, k, k) = k = |W_2| = |W_3|$ . Note that

$$D(W_1, \widetilde{W}_2) \cup D(W_1, \widetilde{W}_3) \cup D(W_3, \widetilde{W}_1) \cup D(W_2, \widetilde{W}_1)$$

$$= \{1\} \times (-E_1) \cup \{1\} \times (-E_2) \cup D(\{0\} \times E_1, k \boxtimes \{(1,0)\}) \cup D(\{0\} \times E_2, k \boxtimes \{(1,0)\})$$

$$= (k+1) \boxtimes (\{1\} \times (G \setminus \{0\})).$$

Based on the above two equalities,

$$\bigcup_{1 \le i \ne j \le 3} D(W_i, \widetilde{W}_j) \subseteq (k+1) \boxtimes ((\mathbb{Z}_2 \times G) \setminus \{(0,0)\}),$$

i.e.,  $\mathcal{W}$  is an  $(n=2n_1, m=3, (1, k, k), a=2k+1, \lambda=k+1)$ -BSWEDF. By Lemma 2, we have

$$\lambda \ge \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil = \left\lceil \frac{k(2k+1)2}{2n_1-1} \right\rceil = \left\lceil \frac{k(4k+1)+k}{4k+1} \right\rceil = k+1.$$

Thus, W is an optimal  $(2n_1 = 4k + 2, 3, (1, k, k), 2k + 1, k + 1)$ -BSWEDF.

It is easily seen from the proof of Theorem 4 that the above BSWEDFs are not EDFs, or GSEDFs, or PEDFs.

**Example 5:** Let  $n = 2n_1 = 22$ . By Construction B, the family of sets  $\mathcal{W} = \{W_1, W_2, W_3\}$  over  $\mathbb{Z}_{22}$  can be listed as

$$W_1 = \{11\}, W_2 = \{12, 4, 16, 20, 14\}, \text{ and } W_3 = \{2, 8, 10, 18, 6\}.$$

It is easy to check that

$$\bigcup_{1 \le i \ne j \le 3} D(W_i, \widetilde{W}_j) \subseteq 6 \boxtimes (Z_{22} \setminus \{0\}),$$

which means that W is an optimal (22, 3, (1, 5, 5), 11, 6)-BSWEDF.

**Construction C:** Let q = 4k + 1 be a prime power and let  $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$  be the family of disjoint subsets of  $\mathbb{Z}_3 \times \mathbb{F}_q$ , defined as  $U_1 = \{(1,0)\}$ ,  $U_2 = \{(2,0)\}$ ,  $U_3 = \{0\} \times D_0^2$ , and  $U_4 = \{0\} \times D_1^2$ .

**Theorem 5:** The family  $U = \{U_1, U_2, U_3, U_4\}$  in Construction C is an optimal (3q = 12k + 3, 4, (1, 1, 2k, 2k), 4k + 2, 2k + 1)-BSWEDF.

*Proof.* Note that  $\widetilde{k} = \text{lcm}(1, 1, 2k, 2k) = 2k$ , which implies  $\widetilde{U}_3 = U_3$  and  $\widetilde{U}_4 = U_4$ . Lemma 9 shows that  $D(D_0^2, D_1^2) \cup D(D_1^2, D_0^2) = 2k \boxtimes (\mathbb{F}_q \setminus \{0\})$ . Thus, we have

$$D(U_3, \widetilde{U}_4) \cup D(U_4, \widetilde{U}_3) = D(U_3, U_4) \cup D(U_3, U_4) = 2k \boxtimes (\{0\} \times (\mathbb{F}_q \setminus \{0\})).$$

Recall that

$$D(U_1, \widetilde{U}_3) \cup D(U_1, \widetilde{U}_4) \cup D(U_3, \widetilde{U}_1) \cup D(U_4, \widetilde{U}_1)$$

$$= (\{1\} \times D_0^2) \cup (\{1\} \times D_1^2) \cup D(\{0\} \times D_0^2, 2k \boxtimes \{(1,0)\}) \cup D(\{0\} \times D_1^2, 2k \boxtimes \{(1,0)\})$$

$$= (\{1\} \times (\mathbb{F}_q \setminus \{0\})) \cup 2k \boxtimes (\{2\} \times (\mathbb{F}_q \setminus \{0\}))$$

and

$$\begin{split} &D(U_2,\widetilde{U}_3) \cup D(U_2,\widetilde{U}_4) \cup D(U_3,\widetilde{U}_2) \cup D(U_4,\widetilde{U}_2) \\ = &\{2\} \times D_0^2 \cup \{2\} \times D_1^2 \cup D(\{0\} \times D_0^2, 2k \boxtimes \{(2,0)\}) \cup D(\{0\} \times D_1^2, 2k \boxtimes \{(2,0)\}) \\ = &(\{2\} \times (\mathbb{F}_q \backslash \{0\})) \cup 2k \boxtimes (\{1\} \times (\mathbb{F}_q \backslash \{0\})). \end{split}$$

For the differences between  $U_1$  and  $U_2$ , we have

$$D(U_1, \widetilde{U}_2) \cup D(U_2, \widetilde{U}_1) = 2k \boxtimes \{(1,0), (2,0)\}.$$

Therefore, the above four equalities mean that

$$\begin{split} & \bigcup_{1 \leq i \neq j \leq 4} D(U_i, \widetilde{U}_j) \\ = & (2k \boxtimes \{(1,0), (2,0)\}) \cup (2k \boxtimes \{0\} \times (\mathbb{F}_q \backslash \{0\})) \cup ((2k+1) \boxtimes \{1,2\} \times (\mathbb{F}_q \backslash \{0\})) \\ \subseteq & (2k+1) \boxtimes ((\mathbb{Z}_3 \times \mathbb{F}_q) \backslash \{(0,0)\}), \end{split}$$

i.e.,  $\mathcal{U}$  is an  $(n = 3a, m = 4, (1, 1, 2k, 2k), a = 4k + 2, \lambda = 2k + 1)$ -BSWEDF.

TABLE I: Some known PDFs with parameters  $(n,\mathcal{W}=(k^{\frac{n-k+1}{k}},(k-1)^1),k-1)$ 

Parameters	Constraints	Ref.
$\left(2v, \left(3^{\frac{2v-2}{3}}, 2^1\right), 2\right),$	$v = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}, \ 2 < p_1 < p_2 < \cdots < p_r,$ and $3   (p_t - 1) \text{ for } 1 \le t \le r$	[3]
$\left(sv, ((s+1)^{\frac{sv-s}{s+1}}, s^1), s\right)$	$v = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}, \ 2 < p_1 < p_2 < \cdots < p_r,$ and $2(s+1) (p_t-1)$ for $1 \le t \le r, s = 4, 5$	[3]
$\left(6v, (7^{\frac{6v-6}{7}}, 6^1), 6\right)$	$v = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}, \ 2 < p_1 < p_2 < \cdots < p_r,$ and $28 (p_t - 1) \text{ for } 1 \le t \le r$	[3]
$\left(7v, (8^{\frac{7v-7}{8}}, 7^1), 7\right)$	$v = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}, \ 2 < p_1 < p_2 < \cdots < p_r,$ and $8   (p_t - 1) \text{ for } 1 \le t \le r, v \notin \{17, 89\}$	[3]
$(q-1,(\frac{q}{d}^{d-1},(\frac{q}{d}-1)^1),\frac{q-d}{d})$	$d q, \gcd(\frac{q}{d} - 1, (q - 1)/(\frac{q}{d} - 1)) = 1$	[8]

Herein  $p_i$ 's are primes; t, s, r and m are positive integers; q is a prime power.

By Lemma 2, we have

$$\lambda \ge \left\lceil \frac{\widetilde{k}a(m-1)}{n-1} \right\rceil = \left\lceil \frac{2k(4k+2)3}{3q-1} \right\rceil = \left\lceil \frac{2k(12k+2) + 8k}{12k+2} \right\rceil = 2k+1.$$

Thus,  $\mathcal{U}$  is an optimal (3q, 4, (1, 1, 2k, 2k), 4k + 2, 2k + 1)-BSWEDF.

It is easily seen from the proof of Theorem 5 that the above BSWEDFs are not EDFs, or GSEDFs, or PEDFs.

**Example 6:** Let n=3q=39. By Construction A, the family of sets  $\mathcal{U}=\{U_1,U_2,U_3,U_4\}$  over  $\mathbb{Z}_{39}$  can be listed as

$$U_1 = \{13\}, \ U_2 = \{26\}, \ U_3 = \{27, 30, 3, 12, 9, 36\}, \ \text{and} \ U_4 = \{15, 21, 6, 24, 18, 33\}.$$

It is easy to check that

$$\bigcup_{1 \le i \ne j \le 4} D(U_i, \widetilde{U}_j) \subseteq 7 \boxtimes (\mathbb{Z}_{39} \setminus \{0\}),$$

which means that  $\mathcal{U}$  is an optimal (39, 4, (1, 1, 6, 6), 14, 7)-BSWEDF.

## A. A construction of cyclic SWEDFs

In this subsection, we are going to construct cyclic SWEDFs, which are not regular EDFs, or GSEDFs, or PEDFs. A *cyclic* SWEDF means an SWEDF over a cyclic additive group.

A well-studied kind of PDFs  $\mathcal{R} = \{R_1, R_2, \cdots, R_l\}$  are those with parameters  $(n = (k-1)(tk+1), (k, \cdots, k, k-1), k-1)$  over  $\mathbb{Z}_n = \mathbb{Z}_{k-1} \times \mathbb{Z}_{tk+1}$  where  $\gcd(k-1, tk+1) = 1$ ,  $R_l = \mathbb{Z}_{k-1} \times \{0\}$  and l = t(k-1)+1. In Table I, we list such PDFs which can be applied in the following construction.

Construction D: Let  $V = \{V_1, V_2, \cdots, V_{t(k-1)+k-2}\}$  be the family of disjoint subsets of  $\mathbb{Z}_n$ , defined as

$$V_i = R_i$$
 for  $1 \le i \le t(k-1),$  
$$V_{t(k-1)+j} = \{(j,0)\} \text{ for } 1 \le j \le k-2.$$

**Theorem 6:** Let  $\mathcal{V}$  be the family in Construction D. Then  $\mathcal{V}$  is a cyclic  $(n, t(k-1)+k-2, K=(k, \cdots, k, 1, 1, \cdots, 1), n-1, (t+1)k^2-(t+3)k)$ -SWEDF, where the element 1 appears k-2 times and the element k appears t(k-1) times in K.

*Proof.* Since  $\mathcal{R}$  is an  $(n=(k-1)(tk+1),(k,\cdots,k,k-1),k-1)$  PDF, we can conclude that

$$\bigcup_{1 \le i \ne j \le l} D(R_i, R_j) = (n - k + 1) \boxtimes ((\mathbb{Z}_{k-1} \times \mathbb{Z}_{tk+1}) \setminus \{(0, 0)\}).$$

Recall that  $R_l = \mathbb{Z}_{k-1} \times \{0\}$ , which means

$$\bigcup_{1 \le i \le l-1} (D(R_i, R_l) \cup D(R_l, R_i)) = (2k-2) \boxtimes (\mathbb{Z}_{k-1} \times (\mathbb{Z}_{tk+1} \setminus \{0\})).$$

Thus, by Construction D, we have

$$\bigcup_{1 \leq i \neq j \leq l-1} D(V_i, \widetilde{V}_j) = \bigcup_{1 \leq i \neq j \leq l-1} D(V_i, V_j) = \bigcup_{1 \leq i \neq j \leq l-1} D(R_i, R_j)$$

$$= \left(\bigcup_{1 \leq i \neq j \leq l} D(R_i, R_j)\right) \setminus \left(\bigcup_{1 \leq i \leq l-1} (D(R_i, R_l) \cup D(R_l, R_i))\right)$$

$$= ((n-k+1) \boxtimes ((\mathbb{Z}_{k-1} \setminus \{0\}) \times \{0\})) \cup ((n-3k+3) \boxtimes (\mathbb{Z}_{k-1} \times (\mathbb{Z}_{tk+1} \setminus \{0\}))),$$
(14)

where we use the fact  $\widetilde{k} = k$ .

Note that for any  $1 \le j \le k - 2$ ,

$$\bigcup_{1 \le i \le l-1} (D(V_i, \widetilde{V}_{l-1+j}) \cup D(V_{l-1+j}, \widetilde{V}_i))$$

$$= \bigcup_{1 \le i \le l-1} (D(R_i, k \boxtimes \{(j, 0)\}) \cup D(\{(j, 0)\}, R_i)) = (k+1) \boxtimes (\mathbb{Z}_{k-1} \times (\mathbb{Z}_{tk+1} \setminus \{0\})).$$

Thus, we have

$$\bigcup_{1 \le j \le k-2} \bigcup_{1 \le i \le l-1} (D(V_i, \widetilde{V}_{l-1+j}) \cup D(V_{l-1+j}, \widetilde{V}_i)) = ((k+1)(k-2)) \boxtimes (\mathbb{Z}_{k-1} \times (\mathbb{Z}_{tk+1} \setminus \{0\})).$$
 (15)

For the last part of external differences, we have

$$\bigcup_{1 \le i \ne j \le k-2} D(V_{l-1+i}, \widetilde{V}_{l-1+j}) = \bigcup_{1 \le i \ne j \le k-2} D(\{(i,0)\}, k \boxtimes \{(j,0)\})$$

$$= k \boxtimes \left( \bigcup_{1 \le i \ne j \le k-2} D(\{(i,0)\}, \{(j,0)\}) \right)$$

$$= k(k-3) \boxtimes ((\mathbb{Z}_{k-1} \setminus \{0\}) \times \{0\}).$$
(16)

Combining (14), (15) and (16),

$$\bigcup_{1 \leq i \neq j \leq l+k-3} D(V_i, \widetilde{V}_j) \\
= \left( \bigcup_{1 \leq i \neq j \leq l-1} D(V_i, \widetilde{V}_j) \right) \cup \left( \bigcup_{1 \leq j \leq k-2} \bigcup_{1 \leq i \leq l-1} (D(V_i, \widetilde{V}_{l-1+j}) \cup D(V_{l-1+j}, \widetilde{V}_i)) \right) \cup \left( \bigcup_{1 \leq i \neq j \leq k-2} D(V_{l-1+i}, \widetilde{V}_{l-1+j}) \right) \\
= ((n-k+1+k(k-3)) \boxtimes (\mathbb{Z}_{k-1} \setminus \{0\}) \times \{0\}) \cup ((n-3k+3+(k+1)(k-2)) \boxtimes (\mathbb{Z}_{k-1} \times (\mathbb{Z}_{tk+1} \setminus \{0\}))) \\
= ((t+1)k^2 - tk - 3k) \boxtimes ((\mathbb{Z}_{k-1} \times \mathbb{Z}_{tk+1}) \setminus \{(0,0)\}),$$

where n = (k - 1)(tk + 1).

Therefore, V is a cyclic  $(n, t(k-1) + k - 2, (k, k, \dots, k, 1, 1, \dots, 1), n - 1, (t+1)k^2 - (t+3)k)$ -SWEDF, where the element 1 occurs k-2 times in K and the element k appears t(k-1) times in K. This completes the proof.

It is easily seen from the proof of Theorem 6 that the above SWEDFs are not regular EDFs, or GSEDFs, or PEDFs.

In [13], Huczynska and Paterson introduced some constructions of SWEDFs (or equivalently, RWSEDs) with the so-called bimodal property.

**Definition 14** ([13]): Let G be a finite Abelian group and  $\mathcal{B}$  be a collection  $B_1, B_2, \ldots, B_m$  of disjoint subsets of G with sizes  $k_1, k_2, \ldots, k_m$ , respectively. We say that  $\mathcal{B}$  has the *bimodal property* if for each  $\delta \in G \setminus \{0\}$  we have  $N_i(\delta) \in \{0, k_i\}$  for  $1 \le i \le m$ , where  $N_i(\delta)$  is defined in Definition 13.

The SWEDF generated by Construction D does not have the bimodal property. Let  $\mathcal V$  be the SWEDF generated by Construction D. For any  $v\in V_i$  with  $|V_i|=k$ , we have  $0\in D(V_i,\{v\})$  and  $|D(V_i,\{v\})|=|V_i|=k$ . However, by Construction D, 0 is not an element of  $V_j$  for  $1\leq j\leq l+k-3$ . Thus, the number of solutions for a-b=v for  $a\in V_i$  and  $b\in V_j$  for  $1\leq j\leq l+k-3$  and  $j\neq i$  is at most k-1, since  $\bigcup_{1\leq j\leq l+k-3}V_j=\mathbb Z_n\setminus\{0\}$ , i.e.,  $N_i(v)\leq k-1$ . Next, we show that there exists  $V_i$  with  $|V_i|=k$  satisfying  $N_i(v)\neq 0$ . If  $a-b\neq v$  for all  $a\in V_i$  and  $b\in V_j$  for  $1\leq j\leq l+k-3$  and  $j\neq i$ , then  $a\in V_i$  means that  $(a+\langle v\rangle)\setminus\{0\}\subseteq V_i$ . This is to say that  $V_i$  is the union of some cosets of  $\langle v\rangle$  besides the element 0 and  $k=\tau|\langle v\rangle|-1$  for some integer  $\tau\geq 1$ . This is impossible since there are elements v with  $|\langle v\rangle|>k+1$  in  $\mathbb Z_n\setminus\{0\}$ . Thus, the SWEDF generated by Construction D is not bimodal. For more details about SWEDFs (or equivalently, RWEDFs) with bimodal property the reader may refer to [13], [14].

Compared with the constructions in [13], Construction D can generate RWEDFs with flexible parameters without bimodal property. To the best of our knowledge, this is the first class of RWEDFs without the bimodal property, which are not regular EDFs, or GSEDFs, or PEDFs.

**Corollary 5:** Let  $\mathcal{V}$  be the family in Construction D. Then  $\mathcal{V}$  is an  $(n, t(k-1)+k-2, K=(k, \cdots, k, 1, 1, \cdots, 1), n-1, (t+1)k-t-3)$ -RWEDF without the bimodal property, where the element 1 appears k-2 times and the element k appears t(k-1) times in K.

**Example 7:** Let  $G = (\mathbb{Z}_{15}, +)$  and  $\mathcal{R} = \{R_1 = \{6, 9, 2, 8\}, R_2 = \{11, 14, 7, 13\}, R_3 = \{1, 4, 12, 3\}, R_4 = \{0, 5, 10\}\}$ . It is easy to check that  $\mathcal{R}$  is a PDF with parameters (15, (4, 4, 4, 3), 3). By Construction D, we generate a family of subsets of  $\mathbb{Z}_{15}$  as  $\mathcal{V} = \{V_1 = \{6, 9, 2, 8\}, V_2 = \{11, 14, 7, 13\}, V_3 = \{1, 4, 12, 3\}, V_4 = \{5\}, V_5 = \{10\}\}$ . It is easy to check that

$$\bigcup_{1 \le i \ne j \le 5} D(V_i, \widetilde{V}_j) = 16 \boxtimes (\mathbb{Z}_{15} \setminus \{0\}),$$

i.e., V is a (15, 5, (4, 4, 4, 1, 1), 14, 16)-SWEDF (or (15, 5, (4, 4, 4, 1, 1), 14, 4)-RWEDF). Note that  $N_3(6) = 3 \notin \{0, 4\}$ , which means the SWEDF does not have the bimodal property by Definition 14.

## V. CONCLUDING REMARKS

In this paper, we first characterized weak algebraic manipulation detection codes via bounded standard weighted external difference families (BSWEDFs). As a byproduct, we improved the known lower bound for weak algebraic manipulation detection codes. To generate optimal weak AMD codes, constructions for BSWEDFs, especially, a construction of SWEDFs without the bimodal property, were introduced.

Combinatorial structures, e.g., BSWEDFs, SWEDFs, strong external difference families (SEDFs), partitioned external difference families (PEDFs), play a key role in the constructions of weak algebraic manipulation detection (AMD) codes. There are some known results for the existence of SEDFs. However, the existence of BSWEDFs, SWEDFs, and PEDFs are generally open. Finding more explicit constructions for such combinatorial structures are not only an interesting subject for AMD codes but also an interesting problem in their own right, which is left for future research.

## ACKNOWLEDGEMENTS

The authors would like to thank Prof. Marco Buratti for the helpful discussion about difference families. This research is supported by JSPS Grant-in-Aid for Scientific Research (B) under Grant No. 18H01133.

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