# Maximal sets of mutually orthogonal frequency squares

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#### Abstract

A frequency square is a square matrix in which each row and column is a permutation of the same multiset of symbols. A frequency square is of  $type\ (n;\lambda)$  if it contains  $n/\lambda$  symbols, each of which occurs  $\lambda$  times per row and  $\lambda$  times per column. In the case when  $\lambda = n/2$  we refer to the frequency square as binary. A set of k-MOFS $(n;\lambda)$  is a set of k frequency squares of type  $(n;\lambda)$  such that when any two of the frequency squares are superimposed, each possible ordered pair occurs equally often.

A set of k-maxMOFS $(n; \lambda)$  is a set of k-MOFS $(n; \lambda)$  that is not contained in any set of (k+1)-MOFS $(n; \lambda)$ . For even n, let  $\mu(n)$  be the smallest k such that there exists a set of k-maxMOFS(n; n/2). It was shown in [1] that  $\mu(n) = 1$  if n/2 is odd and  $\mu(n) > 1$  if n/2 is even. Extending this result, we show that if n/2 is even, then  $\mu(n) > 2$ . Also, we show that whenever n is divisible by a particular function of k, there does not exist a set of k'-maxMOFS(n; n/2) for any  $k' \leq k$ . In particular, this means that  $\limsup \mu(n)$  is unbounded. Nevertheless we can construct infinite families of maximal binary MOFS of fixed cardinality. More generally, let  $q = p^u$  be a prime power and let  $p^v$  be the highest power of p that divides n. If  $0 \leq v - uh < u/2$  for  $h \geq 1$  then we show that there exists a set of  $(q^h - 1)^2/(q - 1)$ -maxMOFS(n; n/q).

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## 1 Introduction

Let  $N(n) = \{0, 1, ..., n-1\}$ . In what follows, rows and columns of an  $m \times n$  array L are indexed by N(m) and N(n), respectively, and L[i,j] denotes the entry in row i and column j of L. A frequency square F of type  $(n; \lambda_0, \lambda_1, ..., \lambda_{m-1})$  is an  $n \times n$  array such that symbol i occurs  $\lambda_i$  times in each row and  $\lambda_i$  times in each column for each  $i \in N(m)$ ; necessarily  $\sum_{i=0}^{m-1} \lambda_i = n$ . In the case where  $\lambda_0 = \lambda_1 = \cdots = \lambda_{m-1} = \lambda$  we say that F is of type  $(n; \lambda)$ . If  $\lambda = n/2$ , then we refer to the frequency square as binary. A frequency square of type (n; 1) is a Latin square of order n. Two frequency squares of type  $(n; \lambda_0, \lambda_1, \ldots, \lambda_{m-1})$  are orthogonal if each ordered pair (i, j) occurs  $\lambda_i \lambda_j$  times when the squares are superimposed. A set of mutually orthogonal frequency squares (MOFS) is a set of frequency squares in which each pair of squares is orthogonal. We use the notation k-MOFS $(n; \lambda)$  to denote k MOFS, each of type  $(n; \lambda)$ .

Research into frequency squares focuses mainly on constructions of sets of MOFS, motivated originally by problems in statistical experiment design. Hedayat, Raghavarao and Seiden [7] showed that the maximum k such that a set of k-MOFS(n; n/m) exists is  $k = (n-1)^2/(m-1)$ ; such a set is called *complete*. We give a new explanation for this result (Corollary 5). In the case when  $m = n/\lambda > 2$ , complete sets of

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MOFS of type  $(n; \lambda)$  are only known to exist when n is a prime power [10, 11, 12, 15]; a unified theory for all known constructions is given in [9].

A non-prime power result for m = 2 is given by Federer [6] (see also [15]), who showed that if there exists a Hadamard matrix of order n, then there exists a complete set of MOFS of type (n; n/2). Conversely, it is shown in [1] that there does not exist a complete set of MOFS of type (n; n/2) whenever n/2 is odd.

Two sets of frequency squares are *isomorphic* if one can be obtained from the other by some sequence of the following operations:

- Applying the same permutation to the rows of all squares in the set.
- Applying the same permutation to the columns of all squares in the set.
- Transposing all squares in the set.
- Permuting the symbols in one of the squares.
- Permuting the squares within the set (in cases where we have imposed an order on the set).

Isomorphism is an equivalence relation and the equivalence classes it induces are isomorphism classes.

A set  $\{F_1, F_2, \ldots, F_k\}$  of k-MOFS $(n; \lambda)$  is said to be maximal if there does not exist a frequency square F of type  $(n; \lambda)$  that is orthogonal to  $F_i$  for each  $1 \leq i \leq k$ . If we wish to specify that a set of k-MOFS $(n; \lambda)$  is maximal, then we may write k-maxMOFS $(n; \lambda)$ . Maintaining consistency with Latin square terminology, a 1-maxMOFS $(n; \lambda)$  is called a bachelor frequency square.

In [1] a number of existence and non-existence results are given for sets of k-maxMOFS(n; n/2), which we summarise in the next two theorems. Note that a set of  $(n-1)^2$ -MOFS(n; n/2) is complete and thus trivially maximal.

**Theorem 1.** There exists a set of k-maxMOFS(n; n/2) if:

- k = 1 and  $n \equiv 2 \pmod{4}$  (furthermore such a frequency square is unique up to isomorphism);
- n = 6 and either  $5 \le k \le 15$  or k = 17; or
- k = 5,  $n \equiv 2 \pmod{4}$  and n > 2.

**Theorem 2.** There does not exist a set of k-maxMOFS(n; n/2) if:

- k = 1 and  $n \equiv 0 \pmod{4}$ ;
- n = 4 and k < 9; or
- n = 6 and either  $k \in \{2, 3, 4, 16\}$  or  $k \ge 18$ .

The structure of this paper is as follows. In §2, we give a construction for non-complete sets of  $\max MOFS(n; n/2)$  when  $n \equiv 0 \pmod{4}$  (we are not aware of any earlier construction of this nature). For instance, we construct  $(2^v - 1)^2$ -max $MOFS(2^vc; 2^{v-1}c)$  for all  $v \geq 2$  and odd c. This is part of a more general construction for sets of  $\max MOFS(n; \lambda)$  given by Corollary 9. In §3 and §4, by exploiting the theory of integral convex polytopes, we show that for any given k there are infinitely many values of n such that a set of k'-maxMOFS(n; n/2) does not exist for any  $k' \leq k$ .

Finally, in §5, we show that a set of 2-maxMOFS(n; n/2) does not exist whenever n is divisible by 4. The case when n/2 is odd remains elusive, but we conjecture the following (which holds for n < 8 by Theorem 2):

Conjecture 3. If n is even, then there does not exist a set of 2-maxMOFS(n; n/2).

For even n, define  $\mu(n)$  to be the size of the smallest maximal set of binary MOFS of order n. Then from the above observations,  $\mu(n) = 1$  if  $n \equiv 2 \pmod{4}$ , and  $2 < \mu(n) \leqslant 9$  if  $n \equiv 4 \pmod{8}$ , but  $\limsup \mu(n)$  is unbounded. By comparison, it seems that maximal pairs of orthogonal Latin squares exist for all orders n > 6, and this has been proved for all orders that are not twice a prime [3].

### 2 A new construction for maximal MOFS

Here we give a new construction for maximal sets of MOFS via dilations of complete sets of MOFS. In the following, we consider  $n \times n$  matrices as vectors in an  $n^2$ -dimensional real vector space equipped with the inner product  $A \circ B = \sum_i \sum_j a_{ij} b_{ij}$  for matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ . Let  $J_n$  be the  $n \times n$  matrix with each entry equal to 1.

**Theorem 4.** Suppose that  $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$  is a set of k-MOFS(n; n/m) with symbols N(m). For  $r, c \in N(n)$ ,  $s \in N(m)$  and  $1 \le t \le k$  we define

$$\mathcal{R}_r[i,j] = \begin{cases} 1 & \text{if } i = r \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{C}_c[i,j] = \begin{cases} 1 & \text{if } j = c, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{S}_{s,t}[i,j] = \begin{cases} 1 & \text{if } F_t[i,j] = s, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\{J_n\} \cup \{\mathscr{R}_r : 1 \leqslant r \leqslant n-1\} \cup \{\mathscr{C}_c : 1 \leqslant c \leqslant n-1\} \cup \{\mathscr{S}_{s,t} : 1 \leqslant s \leqslant m-1, \ 1 \leqslant t \leqslant k\}$  is a linearly independent set.

Proof. Define  $\mathscr{R}'_r = n\mathscr{R}_r - J_n$ ,  $\mathscr{C}'_c = n\mathscr{C}_c - J_n$  and  $\mathscr{S}'_{s,t} = m\mathscr{S}_{s,t} - J_n$  for each r, c, s, t. It is trivial that  $\{J_n\} \cup \{\mathscr{R}_r : 1 \leqslant r \leqslant n-1\}$  and  $\{J_n\} \cup \{\mathscr{C}_c : 1 \leqslant c \leqslant n-1\}$  are both linearly independent sets, as is  $\{J_n\} \cup \{\mathscr{S}_{s,t} : 1 \leqslant s \leqslant m-1\}$  for any given t. It follows that  $\{J_n\} \cup \{\mathscr{R}'_r : 1 \leqslant r \leqslant n-1\}$ ,  $\{J_n\} \cup \{\mathscr{C}'_c : 1 \leqslant c \leqslant n-1\}$  and  $\{J_n\} \cup \{\mathscr{S}'_{s,t} : 1 \leqslant s \leqslant m-1\}$  are each linearly independent sets. It is also easy to check that

$$\mathscr{R}'_r \circ \mathscr{C}'_c = \mathscr{R}'_r \circ \mathscr{S}'_{s,t} = \mathscr{S}'_{s,t} \circ \mathscr{C}'_c = \mathscr{S}'_{s,t} \circ \mathscr{S}'_{s',t'} = 0$$

for all r, c, s, t, s', t', provided  $t \neq t'$ . Hence

$$\{J_n\} \cup \{\mathscr{R}'_r : 1 \leqslant r \leqslant n-1\} \cup \{\mathscr{C}'_c : 1 \leqslant c \leqslant n-1\} \cup \{\mathscr{S}'_{s,t} : 1 \leqslant s \leqslant m-1, \ 1 \leqslant t \leqslant k\}$$

is a linearly independent set, from which the result follows.

Corollary 5. If  $\mathcal{F}$  is a set of k-MOFS(n; n/m), then  $k \leq (n-1)^2/(m-1)$ .

*Proof.* Theorem 4 exhibited a set of 1+2(n-1)+(m-1)k independent vectors in a  $n^2$ -dimensional vector space. It follows that  $(m-1)k \leq (n-1)^2$ .

Note that previous proofs of Corollary 5 have been given in [7], [8] and [9].

Let  $\mathcal{F}$  be a set of k-MOFS(n; n/m). The d-dilation of  $\mathcal{F}$  is the set of k-MOFS(dn; dn/m) obtained by replacing every entry e in every square in  $\mathcal{F}$  by a  $d \times d$  block of entries, each equal to e. It is trivial to check that d-dilation does indeed create a new set of MOFS. We now explore the more interesting question of whether d-dilation preserves maximality. We first give some necessary conditions.

**Lemma 6.** Let  $\mathcal{F}$  be a set of k-MOFS(n; n/m). Let  $\mathcal{F}'$  be the d-dilation of  $\mathcal{F}$  and suppose that  $\mathcal{F}'$  is maximal. Then  $\mathcal{F}$  is maximal and  $d \not\equiv 0 \pmod{m}$ .

*Proof.* If  $\mathcal{F}$  was not maximal then we could extend it with a new frequency square F. But then the d-dilation of F would be orthogonal to every square in  $\mathcal{F}'$ . So we can be sure that  $\mathcal{F}$  is maximal.

If d is divisible by m then there exists a frequency square C of type (d; d/m). For example, we could create C as a circulant matrix in which its first row contains every symbol d/m times. Now build a frequency square of type (dn; dn/m) orthogonal to every square in  $\mathcal{F}'$ , by simply putting a copy of C in the position of each dilated block. This contradicts the maximality of  $\mathcal{F}'$ , and completes the proof.  $\square$ 

Next we give some sufficient conditions. We first need to explain the idea of a *relation*, which was a fundamental tool to show the existence of maximal sets of MOFS in [1] and [8]. The technique of relations was previously used in [4] and [5] (with origins in [14]) to analyse maximal sets of mutually orthogonal Latin squares.

A set  $\mathcal{F} = \{F_1, \dots, F_k\}$  of k-MOFS(n; n/m) can be written as an  $n^2 \times (k+2)$  orthogonal array  $\mathcal{O}$  in which there is a row

$$[i, j, F_1[i, j], F_2[i, j], \dots, F_k[i, j]],$$
 (1)

for each  $i \in N(n)$  and  $j \in N(n)$ . In this context it is safest to consider sets of MOFS to have an indexing that implies an ordering on the squares (and hence the order of the columns in  $\mathcal{O}$  is well-defined). Let  $Y_c$  be the set of symbols that occur in column c of  $\mathcal{O}$ . Then a relation is a (k+2)-tuple  $(X_0, \ldots, X_{k+1})$  of sets such that  $X_i \subseteq Y_i$  for  $i \in N(k+2)$ , with the property that every row (1) of  $\mathcal{O}$  has an even number of columns c for which the symbol in column c is an element of  $X_c$ . We will consider a particular type of relation from [8], for which we make the following definition. A Jedwab-Popatia relation is a relation such that  $|X_i| = 1$  for  $i \geq 2$  and at least one of  $\emptyset \subseteq X_0 \subseteq Y_0$  and  $\emptyset \subseteq X_1 \subseteq Y_1$  holds.

The following theorem is proved in [8], generalising an earlier result from [1]. It shows that under certain conditions a Jedwab-Popatia relation implies maximality. See [1, 8] for a more extensive study of the structure of relations for sets of MOFS, including restrictions on which relations can be achieved.

**Theorem 7.** Suppose  $\lambda$  is odd and let  $\mathcal{F}$  be a set of k-MOFS $(n; \lambda)$  that satisfies a Jedwab-Popatia relation. Then  $\mathcal{F}$  is maximal.

We can now present conditions which guarantee that the d-dilation of a set of MOFS is maximal.

**Theorem 8.** Suppose that  $\mathcal{F}'$  is the d-dilation of a set  $\mathcal{F}$  of k-MOFS(n; n/m). Then  $\mathcal{F}'$  is maximal if either

- $d^2 \not\equiv 0 \pmod{m}$  and  $\mathcal{F}$  is a complete set of MOFS or
- d and n/m are odd, and  $\mathcal{F}$  satisfies Jedwab-Popatia relation.

*Proof.* If both n/m and d are odd then dn/m is also odd. Also, if  $\mathcal{F}$  satisfies a Jedwab-Popatia relation then it is easy to see that  $\mathcal{F}'$  also satisfies a Jedwab-Popatia relation, and hence is maximal by Theorem 7.

Hence for the remainder of the proof we may assume that  $\mathcal{F} = \{F_1, \dots, F_k\}$  is a complete set; in other words  $k = (n-1)^2/(m-1)$ .

Aiming for a contradiction, assume that there exists a frequency square F of type (dn; dn/m) that is orthogonal to every square in the d-dilation  $\mathcal{F}' = \{F'_1, \ldots, F'_k\}$  of  $\mathcal{F}$ . Let  $X = [x_{ij}]$  be the integer matrix of order n in which  $x_{ij}$  is the number of times that symbol 0 occurs in the ij-th block of F. Since 0 occurs dn/m times in every row and column of F we have

$$X \circ \mathscr{R}_r = d^2 n/m, \tag{2}$$

$$X \circ \mathscr{C}_c = d^2 n/m, \tag{3}$$

$$X \circ J_n = d^2 n^2 / m, \tag{4}$$

for  $r, c \in N(n)$ . Also, the fact that F and  $F'_t$  are orthogonal frequency squares means that

$$X \circ \mathscr{S}_{s,t} = (dn/m)^2 \tag{5}$$

for  $s \in N(m)$  and  $1 \le t \le k$ .

However (2), (3), (4) and (5) have another simultaneous solution, namely the matrix with all entries equal to  $d^2/m$ . If  $d^2 \not\equiv 0 \pmod{m}$  then this solution is not an integer matrix, and hence is different from the solution exhibited above. Having two distinct solutions contradicts Theorem 4, so we conclude that  $\mathcal{F}'$  must be maximal.

**Corollary 9.** Let  $q = p^u$  be a prime power and let  $p^v$  be the highest power of p that divides n. If  $0 \le v - uh < u/2$  for  $h \ge 1$  then there exists a set of  $(q^h - 1)^2/(q - 1)$ -maxMOFS(n; n/q).

*Proof.* From [7] we know that a complete set  $\mathcal{F}$  of MOFS $(q^h; q^{h-1})$  exists. Let  $d = n/q^h$  and note that d is an integer because  $v \ge uh$ . Also, by assumption, the highest power of p dividing d is  $p^{v-uh}$ . It follows that  $d^2 \not\equiv 0 \pmod{q}$  since 2(v-uh) < u. Hence, applying Theorem 8 gives the result.

Applying Corollary 9 to the binary case we have q = 2, which necessitates v = h, and we find that there exists a set of  $(2^v - 1)^2$ -maxMOFS(n; n/2) whenever  $n \equiv 2^v \pmod{2^{v+1}}$ . The v = 1 case of this statement is just the existence of bachelor frequency squares, as given in Theorem 1, but for v > 1 we get something new.

Next we give an example that shows that a d-dilation of a maximal set of binary MOFS need not itself be maximal, even if d is odd. This shows that a tempting generalisation of Theorem 8 fails.

Consider the example (16) given in [1] of a set  $\mathcal{F}$  of 5-maxMOFS(6; 3) that do not satisfy a relation. Let  $\mathcal{F}'$  be the 3-dilation of  $\mathcal{F}$ . We claim that  $\mathcal{F}'$  is not maximal. Indeed, a frequency square that extends  $\mathcal{F}'$  can be obtained from the following matrix:

$$\begin{bmatrix}
0 & 3 & 2 & 1 & 3 & 0 \\
1 & 0 & 2 & 2 & 3 & 1 \\
3 & 1 & 0 & 2 & 1 & 2 \\
1 & 2 & 2 & 1 & 0 & 3 \\
1 & 1 & 2 & 2 & 0 & 3 \\
3 & 2 & 1 & 1 & 2 & 0
\end{bmatrix}.$$

We simply replace each entry c in this matrix by a frequency square of type (3; 3 - c, c); for example, a binary  $3 \times 3$  circulant block that has c positive entries in each row.

## 3 Asymptotic non-existence results

In this section we show that  $\mu(n)$ , the size of the smallest maximal set of MOFS(n, n/2), does not satisfy any bound that is uniform in n. Instead, we find that for any k there exist infinitely many n for which  $\mu(n) > k$ . This provides an interesting counterpoint to Theorem 1 and Corollary 9, both of which provide infinite families of maximal sets of binary MOFS of a fixed cardinality. For example, Corollary 9 shows that  $\mu(n) \leq 9$  for all  $n \equiv 4 \pmod{8}$ , and  $\mu(n) \leq 49$  for all  $n \equiv 8 \pmod{16}$ , and so on.

Let  $\Gamma(m)$  be the least common multiple of the integers 1, 2, ..., m. Let  $m_1 = m_2 = 1$ ,  $m_3 = 2$  and recursively define  $m_{i+1} = 2m_i(m_i - 1)\Gamma(2m_i - 1)$  for  $i \ge 3$ . We will show that:

**Theorem 10.** If  $4m_{2k}$  divides n, then there does not exist a set of k-maxMOFS(n; n/2).

Note that  $m_i$  divides  $m_{2k}$  for all i < 2k, so Theorem 10 implies that  $\mu(n) > k$  if  $4m_{2k}$  divides n. We actually prove a more general result which implies Theorem 10 but does not require the frequency squares to be orthogonal:

**Theorem 11.** If  $4m_{2k}$  divides n, then given any set  $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$  of binary frequency squares of order n, there exists a binary frequency square F which is orthogonal to every frequency square in  $\mathcal{F}$ .

Since  $\log(\Gamma(m)) \sim m$  (see, for example, [13]), the sequence  $\{m_i\}$  grows asymptotically faster than the tetration (iterated exponentiation) of k base e; so certainly  $n \gg k$ .

In the remainder of the paper, given arrays  $L_1, L_2, \ldots, L_k$  of the same dimensions, the join  $L_1 \oplus L_2 \oplus \cdots \oplus L_k$  is defined to be the array obtained by overlapping these arrays; that is, the array in which cell (r, c) contains the ordered k-tuple  $(L_1[r, c], L_2[r, c], \ldots, L_k[r, c])$ . Also, given rows  $r_1$  and  $r_2$  of any rectangular array L, we use  $L(r_1, r_2)$  to denote the two-rowed array in which the first row is equal to row  $r_1$  of L and the second row is equal to row  $r_2$  of L. We start with an elementary lemma that gives a strategy for constructing orthogonal mates for frequency squares, two rows at a time.

**Lemma 12.** Let F be a frequency square of type (n; n/2) and let  $\mathcal{R}$  be a partition of the rows of F into pairs. Suppose there exists a binary  $n \times n$  array F' such that for each  $\{r_1, r_2\} \in \mathcal{R}$ :

- each row of  $F'(r_1, r_2)$  contains n/2 zeros and n/2 ones;
- each column of  $F'(r_1, r_2)$  contains 1 zero and 1 one;
- within  $F(r_1, r_2) \oplus F'(r_1, r_2)$ , each of the pairs (0, 0), (0, 1), (1, 0) and (1, 1) occurs n/2 times.

Then F and F' are orthogonal frequency squares of type (n; n/2).

Henceforth in this section, F is the join of a set  $\{F_1, F_2, \ldots, F_k\}$  of frequency squares, each of type (n; n/2). Let  $r_1, r_2$  be two rows of F and let  $\mathcal{P}$  be an equipartition of the columns of  $F(r_1, r_2)$ . We say that  $\mathcal{P}$  is good with respect to row  $r_i$  and square  $F_j$  (where  $i \in \{1, 2\}$  and  $1 \leq j \leq k$ ) if: (a)  $|\mathcal{P}|$  is even and each element of  $\mathcal{P}$  has even cardinality; and (b) for each  $P \in \mathcal{P}$ , the number of columns c in P with  $F_j[r_i, c] = 1$  is equal to |P|/2. Note that (b) implies that for each  $P \in \mathcal{P}$ , the number of columns c in P with  $F_j[r_i, c] = 0$  is also equal to |P|/2.

**Lemma 13.** Let F be the join of a set  $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$  of frequency squares, each of type (n; n/2) and let  $\mathcal{R}$  be a partition of the rows of F into pairs. Suppose that for each  $\{r_1, r_2\} \in \mathcal{R}$ , there exists an equipartition  $\mathcal{P}$  of the columns of  $F(r_1, r_2)$  such that  $\mathcal{P}$  is good with respect to row  $r_i$  and square  $F_j$  for each  $i \in \{1, 2\}$  and  $1 \leq j \leq k$ . Then there exists a binary frequency square F' orthogonal to each frequency square in  $\mathcal{F}$ .

Proof. We construct F' two rows at a time. Let  $\{r_1, r_2\} \in \mathcal{R}$  and let  $\mathcal{P}$  be an equipartition of the columns of  $F(r_1, r_2)$  satisfying the conditions of the lemma. Let  $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$  be any partition of  $\mathcal{P}$  into two parts of equal size. For each  $P \in \mathcal{P}_i$  and  $i \in \{0, 1\}$ , place i in cell  $F'[r_1, c]$  and 1 - i in cell  $F'[r_2, c]$  for each  $c \in P$ . Repeat this process for each element of  $\mathcal{R}$ . The result follows from Lemma 12.

For the rest of this section,  $\mathcal{R}$  is a partition of the rows of F into pairs and  $\{r_1, r_2\} \in \mathcal{R}$ . Lemma 13 allows us to focus on the array  $F(r_1, r_2)$ . The next lemma is a straightforward observation.

**Lemma 14.** Let  $\mathcal{P}$  be good with respect to row  $r_i$ ,  $i \in \{1, 2\}$ , and square  $F_j$ ,  $1 \leq j \leq k$ , for the array  $F(r_1, r_2)$ . If  $\mathcal{P}'$  is an equipartition coarser than  $\mathcal{P}$  and  $|\mathcal{P}'|$  is even, then  $\mathcal{P}$  is also good with respect to row  $r_i$  and square  $F_j$ .

Let f(1) = 2 and  $f(m) = (2m-2)\Gamma(2m-1)$  for  $m \ge 2$ . Informally, the following lemma states that if  $\beta$  is divisible by f(m), then we can partition any integer partition of  $2m\beta$  with maximum part size 2m and average part size m into  $2\beta/f(m)$  integer partitions of mf(m), with average part size m in each of the smaller partitions. For the purposes of motivation, we will apply Lemma 15 before proving it in the next section, using the theory of integral convex polytopes.

**Lemma 15.** Let  $m \ge 1$  and  $\beta$  be integers such that f(m) divides  $\beta$ . Then for any non-negative integers  $x_0, x_1, \ldots, x_{2m}$  such that

$$\sum_{i=0}^{2m} ix_i = 2m\beta; \quad \sum_{i=0}^{2m} x_i = 2\beta$$
 (6)

there exist non-negative integers  $x_{i,j}$ , for  $0 \le i \le 2m$  and  $1 \le j \le 2\beta/f(m)$  such that

$$\sum_{i=0}^{2m} ix_{i,j} = mf(m); \quad \sum_{i=0}^{2m} x_{i,j} = f(m)$$
 (7)

for each  $1 \leqslant j \leqslant 2\beta/f(m)$  and

$$\sum_{j=1}^{2\beta/f(m)} x_{i,j} = x_i$$

for each  $0 \le i \le 2m$ .

Proof of Theorem 11. Consider the array  $F(r_1, r_2)$ . For  $2 \le s \le 2k$ , define  $\beta_s = n/4m_s$ . Our proof will be by induction on s. We first construct an equipartition  $\mathcal{P}$  that is good with respect to both row  $r_1$  and square  $F_1$  and row  $r_2$  and square  $F_1$ . Here each  $P \in \mathcal{P}$  is a pair of columns  $\{c, c'\}$  such that  $F_1[r_1, c] = 1 - F_1[r_1, c']$  and  $F_1[r_2, c] = 1 - F_1[r_2, c']$ . The fact that  $F_1$  is a frequency square ensures such a  $\mathcal{P}$  exists (cf. Lemma 21 later). Note also that  $|\mathcal{P}| = n/2$ , which by assumption is even.

For the inductive step, assume that there exists an equipartition  $\mathcal{P}$  which is good with respect to row  $r_i$  and square  $F_j$  for a set S of order pairs  $(i, j) \in \{1, 2\} \times \{1, 2, ..., k\}$  such that |S| = s, where  $2 \le s < 2k$  and  $|\mathcal{P}| = 2\beta_s$ . The base case s = 2 follows from the previous paragraph, since  $\beta_2 = n/4$ .

Next, let (t, u) be a fixed pair in  $(\{1, 2\} \times \{1, 2, ..., k\}) \setminus S$ . We will show that there exists an equipartition  $\mathcal{P}'$  such that: (a)  $\mathcal{P}'$  is coarser than  $\mathcal{P}$ ; (b)  $|\mathcal{P}'| = 2\beta_{s+1}$  and (c)  $\mathcal{P}'$  is good with respect to row  $r_t$  and square  $F_u$ . The result then follows by induction and Lemmas 13 and 14.

Let  $\beta = \beta_s$ ,  $m = m_s$  and  $\mathcal{Y} = \{1, 2, ..., 2\beta\}$ . Let  $\mathcal{P} = \{P_j : j \in \mathcal{Y}\}$  be the equipartition of the columns and for  $j \in \mathcal{Y}$  let  $y_j$  be the number of 1's in row  $r_t$  and square  $F_u$  within the columns of  $P_j$ . Then

$$\sum_{j \in \mathcal{V}} y_j = 2m\beta \tag{8}$$

and  $0 \leq y_j \leq 2m$  for  $j \in \mathcal{Y}$ .

Next, define  $x_i$  to be the number of indices  $j \in \mathcal{Y}$  such that  $y_j = i$ . Thus by (8):

$$\sum_{i=0}^{2m} ix_i = 2m\beta \quad \text{and} \quad \sum_{i=0}^{2m} x_i = 2\beta.$$

By definition,

$$\beta_{s+1} = \frac{n}{4m_{s+1}} = \frac{n}{4mf(m)} = \frac{\beta}{f(m)},$$

since  $m_{\ell+1} = m_{\ell} f(m_{\ell})$  for all  $\ell \geqslant 2$ . Thus by Lemma 15, there exist  $x_{i,j}$ , for  $0 \leqslant i \leqslant 2m$  and  $1 \leqslant j \leqslant 2\beta_{s+1}$  such that

$$\sum_{i=0}^{2m} i x_{i,j} = m f(m) = m_{s+1}; \quad \sum_{i=0}^{2m} x_{i,j} = f(m)$$
(9)

for  $1 \leq j \leq 2\beta_{s+1}$  and

$$\sum_{j=1}^{2\beta_{s+1}} x_{i,j} = x_i$$

for  $0 \leq i \leq 2m$ .

We now use this information to construct the coarser partition  $\mathcal{P}'$ . We do this by partitioning  $\mathcal{Y}$  into subsets  $Y_j$  for  $1 \leqslant j \leqslant 2\beta_{s+1}$ , such that  $Y_j$  contains  $x_{i,j}$  indices  $\ell$  such that  $y_{\ell} = i$  for each  $0 \leqslant i \leqslant 2m$ . Then, for each  $1 \leqslant j \leqslant 2\beta_{s+1}$ , let

$$P_j' := \bigcup_{\gamma \in Y_j} P_{\gamma}.$$

From (9), each  $P'_j$  is the union of f(m) parts of the equipartition  $\mathcal{P}$  which between them contain  $m_{s+1} = n/4\beta_{s+1}$  ones in row  $r_t$  of square  $F_u$ . Hence  $\mathcal{P}' = \{P'_j : 1 \leq j \leq 2\beta_{s+1}\}$  is an equipartition of the columns which is coarser than  $\mathcal{P}$  and is good with respect to row  $r_t$  and square  $F_u$ . This completes the proof.  $\square$ 

### 4 Proof of Lemma 15 via integral convex polytopes

In this section we prove Lemma 15. We first rephrase the problem in terms of convex polytopes and then use a handy result on the integer decomposition property of integral convex polytopes from [2].

We begin with the following definitions. A halfspace in  $\mathbb{R}^n$  is a set of the form  $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \leq b\}$  or  $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \geq b\}$ , for a fixed vector  $\mathbf{a} \in \mathbb{R}^n$  and real number b. A convex polytope  $\mathscr{P}$  is an intersection of halfspaces that is bounded. The dimension of a convex polytope  $\mathscr{P} \subseteq \mathbb{R}^n$  is the affine dimension of  $\mathscr{P}$ ; that is, the smallest dimension d such that a translation of  $\mathscr{P}$  is contained in a d-dimensional subspace of  $\mathbb{R}^n$ .

We will be interested in the following convex polytopes. Let m be a fixed positive integer. For  $b \in \mathbb{R}^+$ , let  $\mathscr{P}(b)$  be the (2m-1)-dimensional convex polytope

$$\mathscr{P}(b) = \left\{ (x_0, \dots, x_{2m}) \in \mathbb{R}^{2m+1} : \sum_{i=0}^{2m} x_i = b, \sum_{i=0}^{2m} i x_i = mb, \ x_i \geqslant 0 \text{ for } i = 0, \dots, 2m \right\}.$$

An element v of a convex polytope  $\mathscr{P}$  is a vertex if the only way to write v = ax + (1-a)y for  $a \in (0,1)$  and  $x,y \in \mathscr{P}$  is to put x = v = y. Let  $\mathbf{e}_{\ell} \in \mathbb{R}^{2m+1}$  be the  $\ell$ -th standard basis vector, where the entries are indexed from 0. Define  $V = V(b) = \{\mathbf{v}_{i,j} : 0 \le i < m < j \le 2m\} \cup \{\mathbf{v}_m\}$ , where  $\mathbf{v}_{i,j} = \frac{j-m}{j-i}b\,\mathbf{e}_i + \frac{m-i}{j-i}b\,\mathbf{e}_j$  for each  $0 \le i < m < j \le 2m$  and  $\mathbf{v}_m = b\,\mathbf{e}_m$ . It is easy to check that  $V \subseteq \mathscr{P}(b)$ . We show that in fact V is the set of vertices of  $\mathscr{P}(b)$ .

**Lemma 16.** The convex polytope  $\mathcal{P}(b)$  has vertex set V.

Proof. First we show that the elements of V are indeed vertices of  $\mathscr{P}(b)$ . All elements of  $\mathscr{P}(b)$  have non-negative entries, so v = ax + (1-a)y with  $a \in (0,1)$  and  $v, x, y \in \mathscr{P}(b)$  only if  $\operatorname{supp}(x) \subseteq \operatorname{supp}(v)$  and  $\operatorname{supp}(y) \subseteq \operatorname{supp}(v)$ . It is easy to check that  $\mathbf{v}_m$  is the only element of  $\mathscr{P}(b)$  with exactly one non-zero entry, and hence  $\mathbf{v}_m$  is a vertex. Also,  $\mathbf{v}_{i,j}$  with i < m < j is a vertex because there is no element  $v \in \mathscr{P}$  distinct from  $\mathbf{v}_{i,j}$  with  $\operatorname{supp}(v) = \{i, j\}$ .

Now we show that no element  $x = (x_0, \dots, x_{2m}) \in \mathcal{P}(b) \setminus V$  is a vertex. By the previous paragraph, x has at least two non-zero entries. If  $x_i = 0$  for all i < m, then we have the contradiction

$$\sum_{i=0}^{2m} ix_i = \sum_{i=m}^{2m} ix_i > m \sum_{i=m}^{2m} x_i = mb,$$

since  $\sum_{i=0}^{2m} x_i = b$  and  $x_j > 0$  for at least one j > m. Similarly, we cannot have that  $x_i = 0$  for all i > m. In particular, any element of  $\mathcal{P}(b)$  with exactly two non-zero entries must be one of the  $\mathbf{v}_{i,j}$ .

So, assume that x has at least 3 non-zero entries. By the previous argument, there must be i < m < j such that  $x_i$  and  $x_j$  are non-zero. Thus, we can find  $a \in (0,1)$  that is sufficiently small to ensure that  $ab\frac{j-m}{j-i} < x_i$  and  $ab\frac{m-i}{j-i} < x_j$ . Then we can write  $x = a\mathbf{v}_{i,j} + (1-a)x'$  where  $x' = (x'_0, \ldots, x'_{2m})$  with  $x'_i = \frac{1}{1-a}(x_i - ab\frac{j-m}{j-i}), x'_j = \frac{1}{1-a}(x_j - ab\frac{m-i}{j-i})$  and  $x'_\ell = \frac{1}{1-a}x_\ell$  if  $\ell \notin \{i,j\}$ . By the choice of a, all entries of x' are non-negative. Moreover,  $x' \in \mathcal{P}$ , since

$$\sum_{\ell=0}^{2m} x_{\ell}' = \frac{1}{1-a} \left( x_i - ab \frac{j-m}{j-i} \right) + \frac{1}{1-a} \left( x_j - ab \frac{m-i}{j-i} \right) + \sum_{\ell \neq i,j} \frac{x_{\ell}}{1-a}$$
$$= \frac{1}{1-a} \sum_{\ell=0}^{2m} x_{\ell} - \frac{ab}{1-a} = \frac{b}{1-a} - \frac{ab}{1-a} = b$$

and

$$\sum_{\ell=0}^{2m} \ell x_{\ell}' = \frac{i}{1-a} \left( x_i - ab \frac{j-m}{j-i} \right) + \frac{j}{1-a} \left( x_j - ab \frac{m-i}{j-i} \right) + \sum_{\ell \neq i,j} \frac{\ell x_{\ell}}{1-a}$$
$$= \frac{1}{1-a} \sum_{\ell=0}^{2m} \ell x_{\ell} - \frac{abm}{1-a} = \frac{bm}{1-a} - \frac{abm}{1-a} = bm.$$

Therefore,  $x = a\mathbf{v}_{i,j} + (1-a)x'$  with 0 < a < 1 and  $\mathbf{v}_{i,j}, x' \in \mathscr{P}$ . Moreover,  $\mathbf{v}_{i,j} \neq x'$  since x has at least 3 non-zero entries. Thus, x is not a vertex and V is the vertex set of  $\mathscr{P}(b)$ , as claimed.

For a convex polytope  $\mathscr{P}$  and an integer k, let  $k\mathscr{P} = \{k\alpha : \alpha \in \mathscr{P}\}$ . A convex polytope  $\mathscr{P}$  is integral if every vertex of  $\mathscr{P}$  has integer coordinates. We remind the reader that  $\Gamma(m) = \operatorname{lcm}(1, \ldots, m)$ . The following is immediate from Lemma 16 and the fact that  $c\mathscr{P}(b) = \mathscr{P}(bc)$  for any integers b, c.

Corollary 17. For  $m \ge 1$  if  $\Gamma(2m-1)$  divides b, then the set  $\mathcal{P}(b)$  is an integral (2m-1)-dimensional convex polytope.

A convex polytope  $\mathscr{P} \subseteq \mathbb{R}^n$  has the integer decomposition property if for all  $k \geqslant 1$ , and  $\alpha \in k\mathscr{P} \cap \mathbb{Z}^n$ , there is a way to write  $\alpha = \sum_{i=1}^k \alpha_i$  for some  $\alpha_i \in \mathscr{P} \cap \mathbb{Z}^n$  (such a convex polytope is also called integrally closed). Note that if a convex polytope  $\mathscr{P}$  has the integer decomposition property, then so does  $k\mathscr{P}$  for any integer  $k \geqslant 1$ . The following result is an immediate consequence of Theorem 1.1 in [2].

**Theorem 18.** Let  $\mathscr{P}$  be an integral convex polytope of dimension  $d \ge 2$ . Then  $(d-1)\mathscr{P}$  has the integer decomposition property.

We can now prove Lemma 15.

Proof of Lemma 15. Recall that f(1) = 2 and  $f(m) = (2m-2)\Gamma(2m-1)$  for  $m \ge 2$ . Let  $\beta$  be an integer such that  $f(m) \mid \beta$  and  $x_0, \ldots, x_{2m}$  be non-negative integers satisfying (6). By assumption,  $\mathbf{x} = (x_0, \ldots, x_{2m}) \in \mathscr{P}(2\beta) \cap \mathbb{Z}^{2m+1}$ . By Corollary 17 and Theorem 18,  $\mathscr{P}(f(m))$  is an integral convex polytope with the integer decomposition property, when m > 1. When m = 1, observe that  $\mathscr{P}(2k) = \{(a, c, a) \in \mathbb{R}^3 : 2a + c = 2k \text{ and } a, c \ge 0\}$  for any  $k \ge 1$  and that the vertex set of  $\mathscr{P}(2)$  is  $\{\mathbf{e}_0 + \mathbf{e}_2, 2\mathbf{e}_1\}$ . Therefore, for any  $\mathbf{y} \in \mathscr{P}(2k) \cap \mathbb{Z}^3$ ,  $\mathbf{y} = a(\mathbf{e}_0 + \mathbf{e}_2) + (c/2)(2\mathbf{e}_1)$  for some non-negative integers a, c such that 2a + c = 2k (which in particular means that c must be even). Thus,  $\mathscr{P}(2)$  has the integer decomposition property, when m = 1. Therefore for any  $m \ge 1$ , there exists  $\mathbf{x}_1, \ldots, \mathbf{x}_{2\beta/f(m)} \in \mathscr{P}(f(m)) \cap \mathbb{Z}^{2m+1}$  such that  $\mathbf{x} = \sum_{j=1}^{2\beta/f(m)} \mathbf{x}_j$ . Let  $x_{i,j}$  be the i-th entry of  $\mathbf{x}_j$ , where we index from 0. We show that the  $x_{i,j}$  satisfy the conclusion of Lemma 15. As  $\mathbf{x}_j \in \mathscr{P}(f(m)) \cap \mathbb{Z}^{2m+1}$  for each  $j = 1, \ldots, 2\beta/f(m), x_{0,j}, \ldots, x_{2m,j}$  are non-negative integers that satisfy (7). The final statement in the conclusion of Lemma 15 is immediate from  $\mathbf{x} = \sum_{j=1}^{2\beta/f(m)} \mathbf{x}_j$ .

## 5 A non-existence result for maximal orthogonal pairs

Theorem 10 shows that there does not exist a maximal orthogonal pair of binary frequency squares (that is, a set of 2-maxMOFS(2m; m)) if m is divisible by 48. We improve this significantly in this section by proving the following:

**Theorem 19.** If m is even, then there does not exist a set of 2-maxMOFS(2m; m).

For the remainder of the paper,  $F_1$  and  $F_2$  are binary frequency squares of order n = 2m. Initially we do not assume that  $F_1$  and  $F_2$  are orthogonal. It is plausible that in some application one might need a frequency square that is orthogonal to each member of a set of frequency squares, even though the members of that set are not themselves orthogonal. This viewpoint does materially change what is possible. For example, below are two superimposed triples of frequency squares, one of type (4; 2), and the other of type (6; 3):

$$\begin{bmatrix} 111 & 111 & 111 & 000 & 000 & 000 \\ 101 & 000 & 010 & 111 \\ 010 & 100 & 111 & 001 \\ 000 & 111 & 001 & 110 \end{bmatrix} \begin{bmatrix} 111 & 111 & 111 & 000 & 000 & 000 \\ 111 & 111 & 111 & 000 & 000 & 000 \\ 110 & 110 & 000 & 111 & 001 & 001 \\ 001 & 001 & 100 & 011 & 110 & 110 \\ 000 & 000 & 011 & 100 & 111 & 111 \\ 000 & 000 & 000 & 111 & 111 & 111 \end{bmatrix}$$

$$(10)$$

Both of these triples are *non-extendable* in the sense that there is no frequency square of the same type that is orthogonal to all squares in the triple. This contrasts with Theorem 2 which showed the non-existence

of 3-maxMOFS(n; n/2) for n < 8. Of course, any set of frequency squares of type (6; 3) that contains a bachelor square will be non-extendable. However, a computation shows that the examples in (10) are the smallest non-extendable sets of order  $n \in \{4,6\}$  that do not contain a bachelor square.

It will be convenient for us to assume that m is even from now on, although some of our statements apply also to the case when m is odd. Since Theorem 2 has completely settled the case n = 4, we will assume for the remainder of the paper that

$$8 \leqslant n \equiv 0 \pmod{4}, \quad x = \lfloor n/6 \rfloor \geqslant 1, \text{ and } y = \lfloor n/8 \rfloor \geqslant 1.$$
 (11)

For a set of rows S of a frequency square F, we define F(S) to be F restricted to the rows in S. When  $S = \{r_1, r_2\}$ , F(S) is (equivalent to)  $F(r_1, r_2)$  defined in §3. We say that two binary arrays L and L' of the same dimensions are *orthogonal* if each of the ordered pairs (1,1), (0,0), (0,1) and (1,0) occur the same number of times in  $L \oplus L'$ , where  $\oplus$  was defined in §3. A binary frequency rectangle is any matrix of 0's and 1's with the same number of 0's and 1's in each row and in each column. For an even subset S of the rows of  $F_1 \oplus F_2$ , we say that S is balanceable if there is an  $|S| \times n$  binary frequency rectangle F that is orthogonal to  $F_1(S)$  and  $F_2(S)$ . Clearly, the union of disjoint balanceable sets is balanceable, so the following generalisation of Lemma 12 is immediate.

**Lemma 20.** If there exists a partition of the rows of  $F_1 \oplus F_2$  into balanceable sets, then there is a frequency square F orthogonal to  $F_1$  and  $F_2$ .

We prove Theorem 19 by finding a suitable partition  $\mathcal{R}$  of the rows of  $F_1 \oplus F_2$  into balanceable sets and applying Lemma 20. To do this, we define tools to analyse pairs of rows in §5.3. We use these tools to describe all possible pairs of rows that do not balance and classify them into several different types. In §5.4, we show that it is not possible to have large sets of rows of a given type that pairwise do not balance. Finally in §5.5, we prove Theorem 19, using the results of the first four subsections.

#### 5.1 Preliminaries

In this subsection we define much of the notation and terminology that will be needed later in the proof of Theorem 19, as well as giving preliminary results involving those concepts. A detailed example using these definitions and results can be found in §5.2.

Define  $\psi(r)$  to be the number of cells in row r of  $F_1 \oplus F_2$  which contain (0,0). Also, given two rows  $r_1, r_2$  of a frequency square F, let  $\eta(r_1, r_2)$  be the number of columns in F containing 0 in row  $r_1$  and  $r_2$ . The following lemma is immediate from the definition of a binary frequency square.

**Lemma 21.** Let F and F' be two binary frequency squares of the same order n. Then in row r of  $F \oplus F'$ , the number of cells containing (1,1) is  $\psi(r)$ , the number of cells containing (0,1) is  $m-\psi(r)$  and the number of cells containing (1,0) is  $m-\psi(r)$ . In any rows  $r_1$  and  $r_2$  of F, the number of columns containing 1 in both row  $r_1$  and row  $r_2$  is  $\eta(r_1,r_2)$ , the number of columns containing 0 in row  $r_1$  and 1 in row  $r_2$  is  $m-\eta(r_1,r_2)$  and the number of columns containing 1 in row  $r_1$  and 0 in row  $r_2$  is  $m-\eta(r_1,r_2)$ .

For integers p and q (which may be negative), we say that a pair of rows  $\{r_1, r_2\}$  in  $F_1 \oplus F_2$  is (p, q)balanceable if there exists a  $2 \times n$  binary frequency rectangle F such that:

- $F_1(r_1, r_2) \oplus F$  has m + p occurrences of (0, 0) and
- $F_2(r_1, r_2) \oplus F$  has m + q occurrences of (0, 0).

By Lemma 21, a pair of rows  $\{r_1, r_2\}$  is balanceable, if and only if it is (0, 0)-balanceable; for such a pair of rows, F is orthogonal to both  $F_1(r_1, r_2)$  and  $F_2(r_1, r_2)$ .

In the above, if we swap the symbols 0 and 1 in F, then by Lemma 21,  $F_1(r_1, r_2) \oplus F$  and  $F_2(r_1, r_2) \oplus F$  have m - p and m - q occurrences of (0, 0), respectively. Thus a pair of rows is (p, q)-balanceable if and only if it is (-p, -q)-balanceable. The following is immediate.

**Lemma 22.** Let S be a 2(s+t)-set of rows of  $F_1 \oplus F_2$ . Let  $\mathcal{R}$  be a partition of S into pairs  $\{r_i, r_i'\}$  such that  $\{r_i, r_i'\}$  are  $(p_i, q_i)$ -balanceable for integers  $p_i, q_i$  for  $i = 1, \ldots, s+t$ . If  $\sum_{i=1}^s p_i = \sum_{i=s+1}^t p_i$  and  $\sum_{i=1}^s q_i = \sum_{i=s+1}^t q_i$ , then S is balanceable.

To analyse a pair of rows  $\{r_1, r_2\}$  in  $F_1 \oplus F_2$ , we use the following definitions. Let  $[v_i]_{i=1}^4 = [(0, 1), (1, 0), (0, 0), (1, 1)]$ . Define a  $4 \times 4$  matrix  $A' = A'(r_1, r_2) = [a'_{ij}]$  by letting  $a'_{ij}$  equal the number of columns of  $F_1 \oplus F_2$  in which  $v_i$  occurs in the first row and  $v_j$  occurs in the second row.

Lemma 21 implies that the sum of the entries in the first row of A' equals the sum of the entries in the second row of A'. Similarly, the sum of the entries in the third row of A' equals the sum of the entries in the fourth row of A'. Analogous properties hold for the columns of A'. From A' we can also determine the number of cells containing (0,0) within rows  $r_1$  and  $r_2$  of  $F_1 \oplus F_2$ . We summarise these observations in the lemma, below.

**Lemma 23.** Let  $r_1$  and  $r_2$  be two rows in  $F_1 \oplus F_2$  and  $A' = A'(r_1, r_2)$ . Then,

- the sum of the entries of A' is n = 2m;
- $a'_{11} + a'_{21} + a'_{31} + a'_{41} = a'_{12} + a'_{22} + a'_{32} + a'_{42}$ ;
- $a'_{11} + a'_{12} + a'_{13} + a'_{14} = a'_{21} + a'_{22} + a'_{23} + a'_{24}$ ;
- $a'_{13} + a'_{23} + a'_{33} + a'_{43} = a'_{14} + a'_{24} + a'_{34} + a'_{44}$ ;
- $a'_{31} + a'_{32} + a'_{33} + a'_{34} = a'_{41} + a'_{42} + a'_{43} + a'_{44}$ ;
- $\psi(r_1) = a'_{11} + a'_{13} + a'_{31} + a'_{33} = a'_{22} + a'_{24} + a'_{42} + a'_{44}$ ;
- $\psi(r_2) = a'_{22} + a'_{23} + a'_{32} + a'_{33} = a'_{11} + a'_{14} + a'_{41} + a'_{44}$

We say that a  $4 \times 4$  matrix is *admissible* if it satisfies the above equalities except possibly for the first dot point. We will sometimes write A' = B + C, where B and C are both admissible matrices.

Swapping the symbols in  $F_1$  (respectively,  $F_2$ ) corresponds to applying the permutation (12)(34) to the rows (respectively, columns) of A'. Swapping row  $r_1$  with  $r_2$  corresponds to applying the permutation (12) to both the rows and the columns of A'. Finally, swapping  $F_1$  with  $F_2$  corresponds to taking the transpose of A'. We consider two admissible matrices  $A'_1$  and  $A'_2$  to be equivalent if  $A'_2$  can be formed from  $A'_1$  by some combination of the above operations.

Given that each matrix A' may be equivalent to up to 16 matrices satisfying Lemma 23, we often consider a condensed form of  $A'(r_1, r_2)$  which we denote by  $A(r_1, r_2)$ . Given a pair of rows  $\{r_1, r_2\}$  in  $F_1 \oplus F_2$ , we define a  $3 \times 3$  matrix  $A(r_1, r_2) = [a_{ij}]$  as follows. If  $i, j \in \{1, 2\}$ , then  $a_{ij} = a'_{ij}$ . For  $1 \le i \le 2$ ,  $a_{i3} = a'_{i3} + a'_{i4}$  and for  $1 \le j \le 2$ ,  $a_{3j} = a'_{3j} + a'_{4j}$ . Finally,  $a_{33} = a'_{33} + a'_{34} + a'_{43} + a'_{44}$ . Informally, A is formed from A' be merging the last two rows and the last two columns. We have opted to give the simpler notation to this condensed format because we will use it much more often than the  $4 \times 4$  version. We consider the  $3 \times 3$  matrices  $A_1$  and  $A_2$  equivalent if  $A_2$  can be formed from  $A_1$  by some combination of swapping the first two rows, swapping the first two columns and/or taking the transpose. The next lemma is implied by Lemma 23.

**Lemma 24.** Let  $r_1$  and  $r_2$  be two rows in  $F_1 \oplus F_2$  and  $A = A(r_1, r_2)$ . Then,

- the sum of the entries of A is 2m;
- $a_{11} + a_{12} + a_{13} = a_{21} + a_{22} + a_{23} = m (a_{31} + a_{32} + a_{33})/2;$
- $a_{11} + a_{21} + a_{31} = a_{12} + a_{22} + a_{32} = m (a_{13} + a_{23} + a_{33})/2;$
- $a_{31} + a_{32} + a_{33} \equiv a_{13} + a_{23} + a_{33} \equiv 0 \pmod{2}$ .

We can determine if a pair of rows  $\{r_1, r_2\}$  is (p, q)-balanceable by considering only the condensed matrix  $A(r_1, r_2)$ , as the following lemma shows.

**Lemma 25.** Let  $r_1, r_2$  be rows in  $F_1 \oplus F_2$  and  $A = A(r_1, r_2) = [a_{ij}]$ . Suppose there exists a  $3 \times 3$  matrix  $B = [b_{ij}]$  such that

- The sum of the entries of B is m;
- $b_{11} + b_{12} + b_{13} (b_{21} + b_{22} + b_{23}) = p$ ;
- $b_{11} + b_{21} + b_{31} (b_{12} + b_{22} + b_{32}) = q$ ;
- $0 \leqslant b_{ij} \leqslant a_{ij}$  for  $1 \leqslant i \leqslant 3$  and  $1 \leqslant j \leqslant 3$ .

Then the pair of rows  $\{r_1, r_2\}$  is (p, q)-balanceable in  $F_1 \oplus F_2$ .

Proof. For  $1 \le i \le 3$  and  $1 \le j \le 3$ , partition those columns of  $F_1 \oplus F_2$  that are counted by  $a_{ij}$  into sets  $C_{ij}$  and  $C'_{ij}$  of cardinalities  $b_{ij}$  and  $a_{ij} - b_{ij}$ , respectively. Such a partition exists, since  $0 \le b_{ij} \le a_{ij}$ . It follows that  $\{C_{ij}, C'_{ij} : 1 \le i, j \le 3\}$  partitions the columns of  $F_1 \oplus F_2$ . We construct a  $2 \times 2m$  binary frequency rectangle F satisfying the properties required for  $\{r_1, r_2\}$  to be (p, q)-balanceable, as follows. For each column c, place a 0 in the first row and a 1 in the second row of F if  $c \in C_{ij}$  for some i, j and place a 1 in the first row and a 0 in the second row of F, otherwise. By Lemma 24, the total number of pairs (0,0) in  $F_1(r_1, r_2) \oplus F$  is given by:

$$b_{11} + b_{12} + b_{13} + (a_{21} - b_{21}) + (a_{22} - b_{22}) + (a_{23} - b_{23}) + (a_{31} + a_{32} + a_{33})/2 = m + p.$$

Similarly, the total number of pairs (0,0) in  $F_2(r_1,r_2) \oplus F$  is given by:

$$b_{11} + b_{21} + b_{31} + (a_{12} - b_{12}) + (a_{22} - b_{22}) + (a_{32} - b_{32}) + (a_{13} + a_{23} + a_{33})/2 = m + q.$$

We say that  $A = A(r_1, r_2)$  is (p, q)-balanceable if  $\{r_1, r_2\}$  is (p, q)-balanceable. A matrix B satisfying the conditions in Lemma 25 is said to (p, q)-balance A and  $\{r_1, r_2\}$ . When B (0, 0)-balances A, we just say that B balances A and  $\{r_1, r_2\}$ .

If B is such that it (p,q)-balances A, then taking the transpose, swapping the first two rows or swapping the first two columns of A and B, results in matrices A' and B', respectively, such that B' (q,p)-balances, (-p,q)-balances or (p,-q)-balances A', respectively. We therefore sometimes only need to consider (p,q)-balanceability up to equivalence. In particular, a matrix A can be balanced if and only if any matrix equivalent to A can be balanced.

### 5.2 A detailed example

We give four rows of  $F_1 \oplus F_2$  where  $F_1$  and  $F_2$  are each of order 8:

$r_1$	(0,0)	(0,0)	(0,0)	(1,0)	(1, 1)	(1, 1)	(0,1)	(1,1)
$r_2$	(1, 1)	(1, 1)	(1,0)	(0,0)	(0,1)	(0,1)	(0,0)	(1,0)
$r_3$	(0,0)	(0,1)	(1,0)	(1,1)	(0,0)	(1, 1)	(0,1)	(1,0)
$r_4$	(1,1)	(1,0)	(0,1)	(0,0)	(1,0)	(0,1)	(0,0)	(1,1)

Then:

Furthermore the matrix

$$B = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

(1,0)-balances both  $A(r_1,r_2)$  and  $A(r_3,r_4)$ , by Lemma 25. Note that the pair of rows  $\{r_1,r_2\}$  is not balanceable (this is an instance of exception  $E_1$  in Lemma 26, which we will prove shortly). However, by Lemma 22, the set of rows  $S = \{r_1, r_2, r_3, r_4\}$  is balanceable. Indeed we exhibit a  $4 \times 8$  binary frequency rectangle F orthogonal to both  $F_1(S)$  and  $F_2(S)$ :

	0	1	0	0	1	1	0	1	
<i>F</i> _	1	0	1	1	0	0	1	0	1
F =	1	0	0	0	1	1	1	0	1
	0	1	1	1	0	0	0	1	

#### 5.3 Pairs of rows that do not balance

We next determine all matrices that correspond to a pair of rows that is not balanceable.

**Lemma 26.** Let A be a matrix with the properties from Lemma 24. Then there exists a matrix B that balances A, unless A is equivalent to one of the follow configurations  $E_i$ ,  $1 \le i \le 6$ , where x and y are defined in (11).

			_					_			
$2x \mid 0$	)	1		2x+1	0	0			2x+1	0	0
0 0	2x	+1		0	1	2x	;		1	0	2x
0  2	x	0		0	2x	0			0	2x+2	0
$E_1:n$	$\equiv 2$ (n	nod 6)		$E_2: n \equiv$	≡ 2	(mod	6)		$E_3:n\equiv$	€ 4 (mc	od 6)
			_					_			
2x+1	0	0		2y+1		0	0		2y + 1	0	0
0	0	2x+1		2y+1		0	0		2y	0	1
0	2x+1	1		0	4y	+2	0		0	4y + 1	1
_											

*Proof.* We consider cases according to the parity of  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $a_{22}$ . For each case, we either present a matrix B that balances A or show that A must be equivalent to one of the exceptional configurations in the lemma statement. Throughout the proof, we make extensive use of Lemma 24 and, for simplicity, we omit referencing the lemma every time it is used.

Case 1:  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $a_{22}$  have the same parity.

Case 1A:  $a_{11} \equiv a_{12} \equiv a_{21} \equiv a_{22} \pmod{2}$  and  $a_{13} \equiv a_{31} \pmod{2}$ . Note that  $a_{33}$  is even. A solution for B in this case is:

$\lfloor a_{11}/2 \rfloor$	$\lceil a_{12}/2 \rceil$	$\lceil a_{13}/2 \rceil$	
$\lceil a_{21}/2 \rceil$	$\lfloor a_{22}/2 \rfloor$	$\lceil a_{23}/2 \rceil$	
$\lfloor a_{31}/2 \rfloor$	$\lfloor a_{32}/2 \rfloor$	$a_{33}/2$	

So, in all other cases we may assume that  $a_{13} \not\equiv a_{31} \pmod{2}$ . By transposing if necessary, we may assume that  $a_{13}$  is odd and  $a_{31}$  is even.

Case 1B:  $a_{11} \equiv a_{12} \equiv a_{21} \equiv a_{22} \pmod{2}$ ,  $a_{13}$  is odd and  $a_{31}$  is even and  $a_{33} > 0$ . A solution for B in this case is:

$\lfloor a_{11}/2 \rfloor$	$\lceil a_{12}/2 \rceil$	$(a_{13}-1)/2$	
$\lceil a_{21}/2 \rceil$	$\lfloor a_{22}/2 \rfloor$	$(a_{23}-1)/2$	
$a_{31}/2$	$a_{32}/2$	$a_{33}/2+1$	

Case 1C:  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$ ,  $a_{22}$  and  $a_{13}$  are odd,  $a_{31}$  is even and  $a_{33} = 0$ . As  $n \equiv 0 \pmod{4}$ , at least one of  $a_{31}$  or  $a_{32}$  is non-zero. A solution is to take B equivalent to

$(a_{11}-1)/2$	$(a_{12}-1)/2$	$(a_{13}+1)/2$	
$(a_{21}-1)/2$	$(a_{22}+1)/2$	$(a_{23}-1)/2$	
$(a_{31}+2)/2$	$a_{32}/2$	0	

Case 1D:  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$ ,  $a_{22}$  and  $a_{31}$  are even,  $a_{13}$  is odd and  $a_{33} = 0$ . If  $a_{11}$  and  $a_{31}$  are non-zero then a solution for B is

$(a_{11}-2)/2$	$a_{12}/2$	$(a_{13}+1)/2$	
$a_{21}/2$	$a_{22}/2$	$(a_{23}-1)/2$	
$(a_{31}+2)/2$	$a_{32}/2$	0	

A similar solution exists if  $a_{31} \neq 0$  and  $a_{21} \neq 0$  or  $a_{32}$  and one of  $a_{12}$  and  $a_{22}$  are non-zero. As  $n \equiv 0 \pmod{4}$ , at least one of  $a_{31}$  or  $a_{32}$  is non-zero. Therefore, without loss of generality, it suffices to consider the following configuration.

0	$a_{12}$	$a_{13}$
0	$a_{22}$	$a_{23}$
$a_{31}$	0	0

where either  $a_{12}$  or  $a_{22}$  is non-zero. If  $a_{12} \neq 0$  and  $a_{13} \neq 1$ , then a solution for B is

0	$(a_{12}+2)/2$	$(a_{13}-3)/2$	
0	$a_{22}/2$	$(a_{23}-1)/2$	
$(a_{31}+2)/2$	0	0	

A similar solution exists when  $a_{22} \neq 0$  and  $a_{23} \neq 1$ . So, without loss of generality,  $a_{12} \neq 0$  and  $a_{13} = 1$ . If  $a_{22} \neq 0$  (and  $a_{23} = 1$ ), then we have the following configuration:

0	$a_{12}$	1
0	$a_{12}$	1
$2a_{12}$	0	0

which is a contradiction as  $n \equiv 0 \pmod{4}$ . So,  $a_{22} = 0$  and we have the exceptional case  $E_1$ :

0	$a_{12}$	1	
0	0	$a_{12} + 1$	
$a_{12}$	0	0	

### Case 2: Precisely 3 of $a_{11}$ , $a_{12}$ , $a_{21}$ and $a_{22}$ have the same parity.

Without loss of generality, we can assume that  $a_{12}, a_{21}$  and  $a_{22}$  have the same parity. Necessarily  $a_{13} \not\equiv a_{23} \pmod{2}$  and  $a_{31} \not\equiv a_{32} \pmod{2}$  and  $a_{33} \equiv 1 \pmod{2}$ . In particular  $a_{33} \geqslant 1$ .

Case 2A:  $a_{11}$  even and  $a_{12}$ ,  $a_{21}$  and  $a_{22}$  are odd. Without loss of generality, we consider the three cases:  $a_{13}$  and  $a_{31}$  are both odd;  $a_{13}$  is odd and  $a_{31}$  is even;  $a_{13}$  and  $a_{31}$  are both even. Solutions for B in these respective cases are:

$a_{11}/2$	$(a_{12}+1)/2$	$(a_{13}-1)/2$	
$(a_{21}+1)/2$	$(a_{22}-1)/2$	$a_{23}/2$	,
$(a_{31}-1)/2$	$a_{32}/2$	$(a_{33}+1)/2$	

$a_{11}/2$	$(a_{12}+1)/2$	$(a_{13}-1)/2$	
$(a_{21}+1)/2$	$(a_{22}-1)/2$	$a_{23}/2$	] :
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}-1)/2$	

$a_{11}/2$	$(a_{12}+1)/2$	$a_{13}/2$
$(a_{21}+1)/2$	$(a_{22}+1)/2$	$(a_{23}-1)/2$
$a_{31}/2$	$(a_{32}-1)/2$	$(a_{33}-1)/2$

#### Case 2B: $a_{11}$ is odd, $a_{12}$ , $a_{21}$ and $a_{22}$ are even and at least one of $a_{13}$ and $a_{31}$ is odd.

Without loss of generality, we can consider the two cases when  $a_{13}$  and  $a_{31}$  are both odd, and when  $a_{13}$  is odd and  $a_{31}$  is even. Solutions in these respective cases are

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-1)/2$	
$a_{21}/2$	$a_{22}/2$	$a_{23}/2$	,
$(a_{31}-1)/2$	$a_{32}/2$	$(a_{33}+1)/2$	

$(a_{11}+1)/$	$2   a_{12}/2$	$(a_{13}-1)/2$
$a_{21}/2$	$a_{22}/2$	$a_{23}/2$
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}-1)/2$

Case 2C:  $a_{11}$  is odd and  $a_{12}$ ,  $a_{21}$ ,  $a_{22}$ ,  $a_{13}$  and  $a_{31}$  are even. For subcases (a)  $a_{33} > 1$ , (b)  $a_{22} \neq 0$ , (c)  $a_{31} \neq 0$ , (d)  $a_{12} \neq 0$ ,  $a_{21} \neq 0$ , and (e)  $a_{12} = 0$ ,  $a_{21} \neq 0$ ,  $a_{23} \geqslant 3$ , respectively, solutions for B are:

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$a_{21}/2$	$a_{22}/2$	$(a_{23}+1)/2$	,
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}-3)/2$	

$(a_{11}-1)/2$	$a_{12}/2$	$a_{13}/2$	
$a_{21}/2$	$(a_{22}-2)/2$	$(a_{23}+1)/2$	
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}+1)/2$	

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$
$a_{21}/2$	$a_{22}/2$	$(a_{23}+1)/2$
$(a_{31}-2)/2$	$(a_{32}-1)/2$	$(a_{33}+1)/2$

$(a_{11}+1)/2$	$(a_{12}-2)/2$	$a_{13}/2$	
$(a_{21}-2)/2$	$a_{22}/2$	$(a_{23}+1)/2$	] :
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}+1)/2$	Ì

$(a_{11}-1)/2$	$a_{12}/2$	$a_{13}/2$
$(a_{21}+2)/2$	$a_{22}/2$	$(a_{23}-3)/2$
$a_{31}/2$	$(a_{32}+1)/2$	$(a_{33}+1)/2$

Therefore, without loss of generality, the remaining cases are when  $a_{22} = a_{31} = a_{13} = 0$  and  $a_{33} = 1$ , and either  $a_{12} = 0 = a_{21}$  or  $a_{12} = 0$  and  $a_{23} = 1$ . If  $a_{12} = a_{21} = 0$  then we have the following exceptional case  $E_4$ :

 $\begin{array}{c|cccc}
a_{11} & 0 & 0 \\
0 & 0 & a_{11} \\
0 & a_{11} & 1
\end{array}$ 

If  $a_{12} = 0$  and  $a_{23} = 1$ , then we have the following exceptional case  $E_6$ :

$a_{11}$	0	0	
$a_{11} - 1$	0	1	
0	$2a_{11} - 1$	1	

Case 3:  $a_{11} \equiv a_{22} \pmod{2}$ ,  $a_{12} \equiv a_{21} \pmod{2}$  and  $a_{11} \not\equiv a_{12} \pmod{2}$ .

Without loss of generality, we only need to consider the case when  $a_{11}$  and  $a_{22}$  are odd and  $a_{12}$  and  $a_{21}$  are even. We necessarily have that  $a_{13} \equiv a_{23} \pmod{2}$  and  $a_{31} \equiv a_{32} \pmod{2}$  and  $a_{33}$  is even.

Case 3A:  $a_{11}$ ,  $a_{22}$  and  $a_{13}$  are odd, while  $a_{12}$  and  $a_{21}$  are even. Solutions when  $a_{31}$  is odd (respectively even) are:

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-1)/2$
$a_{21}/2$	$(a_{22}-1)/2$	
$(a_{31}-1)/2$	$(a_{32}+1)/2$	$a_{33}/2$

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-1)/2$	
$a_{21}/2$	$(a_{22}+1)/2$	$(a_{23}-1)/2$	
$a_{31}/2$	$a_{32}/2$	$a_{33}/2$	

Case 3B:  $a_{11}$  and  $a_{22}$  are odd, while  $a_{12}$ ,  $a_{21}$ ,  $a_{13}$  and  $a_{31}$  are even. If  $a_{33} \neq 0$  we have this solution

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$a_{21}/2$	$(a_{22}+1)/2$	$a_{23}/2$	
$a_{31}/2$	$a_{32}/2$	$(a_{33}-2)/2$	

So, henceforth we assume that  $a_{33} = 0$ . As  $n \equiv 0 \pmod{4}$ , at least one of  $a_{13}$  and  $a_{23}$  is non-zero and at least one of  $a_{31}$  and  $a_{32}$  are non-zero. So, without loss of generality  $a_{31} \neq 0$ . If  $a_{23} \neq 0$ , then a solution is

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$a_{21}/2$	$(a_{22}-1)/2$	$(a_{23}+2)/2$	
$(a_{31}-2)/2$	$a_{32}/2$	0	

So, without loss of generality it remains to consider the case when  $a_{23} = a_{32} = 0$  and  $a_{13} \neq 0$ . If  $a_{11} \geqslant 3$ , then a solution is

$(a_{11}-3)/2$	$a_{12}/2$	$(a_{13}+2)/2$	
$a_{21}/2$	$(a_{22}-1)/2$	0	
$(a_{31}+2)/2$	0	0	

If  $a_{11} = 1$  and  $a_{21} \neq 0$ , then a solution is

$(a_{11}-1)/2$	$a_{12}/2$	$(a_{13}+2)/2$	
$(a_{21}+2)/2$	$(a_{22}-1)/2$	0	
$(a_{31}-2)/2$	0	0	

So, without loss of generality  $a_{12} = 0 = a_{21}$  and  $a_{11} = 1$  and we get the exception  $E_2$ :

1	0	$a_{13}$	
0	$a_{13} + 1$	0	
$a_{13}$	0	0	

By equivalence, only the following case remains.

Case 4:  $a_{11} \equiv a_{21} \pmod{2}$ ,  $a_{12} \equiv a_{22} \pmod{2}$  and  $a_{11} \not\equiv a_{12} \pmod{2}$ .

Without loss of generality, we can assume that  $a_{11}$  and  $a_{21}$  are odd and we necessarily have that  $a_{13} \equiv a_{23} \pmod{2}$ ,  $a_{31} \equiv a_{32} \pmod{2}$  and  $a_{33}$  is even.

Case 4A:  $a_{11}$ ,  $a_{21}$  and  $a_{13}$  are odd, while  $a_{12}$  and  $a_{22}$  are even. Solutions when  $a_{31}$  is odd (respectively, even) are

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-1)/2$	
$(a_{21}+1)/2$	$a_{22}/2$	$(a_{23}-1)/2$	]
$(a_{31}-1)/2$	$(a_{32}+1)/2$	$a_{33}/2$	

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-1)/2$	
$(a_{21}-1)/2$	$a_{22}/2$	$(a_{23}+1)/2$	
$a_{31}/2$	$a_{32}/2$	$a_{33}/2$	

Case 4B:  $a_{11}$  and  $a_{21}$  are odd, while  $a_{12}$ ,  $a_{22}$  and  $a_{13}$  are even and  $a_{33} > 0$ . If  $a_{31}$  is odd then a solution for B is

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$(a_{21}+1)/2$	$a_{22}/2$	$a_{23}/2$	
$(a_{31}-1)/2$	$(a_{32}+1)/2$	$(a_{33}-2)/2$	

The subcase when  $a_{13}$  and  $a_{31}$  are both even requires a more thorough analysis. A solution if  $a_{13} \neq 0$  (similarly  $a_{23} \neq 0$ ), and  $a_{31} \neq 0$  are, respectively:

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-2)/2$	
$(a_{21}-1)/2$	$a_{22}/2$	$a_{23}/2$	,
$a_{31}/2$	$a_{32}/2$	$(a_{33}+2)/2$	

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$(a_{21}+1)/2$	$a_{22}/2$	$a_{23}/2$	
$(a_{31}-2)/2$	$a_{32}/2$	$a_{33}/2$	

So, it remains to consider configurations of the form

$a_{11}$	$a_{12}$	0	
$a_{21}$	$a_{22}$	0	
0	$a_{32}$	$a_{33}$	

Note that  $a_{32} \neq 0$ , since otherwise we would have the contradiction  $a_{11} = a_{22}$ . If  $a_{12} \neq 0$ , then the following is a solution for B:

$(a_{11}+1)/2$	$(a_{12}-2)/2$	0	
$(a_{21}-1)/2$	$a_{22}/2$	0	
0	$(a_{32}+2)/2$	$a_{33}/2$	

A similar solution exists if  $a_{22} \neq 0$ . The final case is when  $a_{12} = 0 = a_{22}$  (necessarily  $a_{21} = a_{11}$  and  $a_{32} = 2a_{11}$ ). As  $n \equiv 0 \pmod{4}$  and  $a_{33} \neq 0$ , we have  $a_{33} \geqslant 4$  and the following is a solution:

$(a_{11}+1)/2$	0	0	
$(a_{21}+1)/2$	0	0	] .
0	$(a_{32}+2)/2$	$(a_{33}-4)/2$	

Case 4C:  $a_{11}$  and  $a_{21}$  are odd, while  $a_{12}$ ,  $a_{22}$  and  $a_{13}$  are even and  $a_{33} = 0$ . First consider the subcase when  $a_{31}$  is odd. As  $n \equiv 0 \pmod{4}$ , at least one of  $a_{13}$  or  $a_{23}$  is non-zero. So, without loss of generality,  $a_{13} \neq 0$  and then a solution for B is:

$(a_{11}+1)/2$	$a_{12}/2$	$(a_{13}-2)/2$	
$(a_{21}-1)/2$	$a_{22}/2$	$a_{23}/2$	
$(a_{31}+1)/2$	$(a_{32}+1)/2$	0	

Now we consider the case when  $a_{13}$  and  $a_{31}$  are both even. As  $n \equiv 0 \pmod{4}$  at least one of  $a_{31}$  and  $a_{32}$  is non-zero. If  $a_{31} \neq 0$  then a solution for B is

$(a_{11}+1)/2$	$a_{12}/2$	$a_{13}/2$	
$(a_{21}+1)/2$	$a_{22}/2$	$a_{23}/2$	
$(a_{31}-2)/2$	$a_{32}/2$	0	

So, without loss of generality,  $a_{31} = 0$  and  $a_{32} \neq 0$ . If  $a_{12} \neq 0$ , then a solution is

$(a_{11}-1)/2$	$(a_{12}+2)/2$	$a_{13}/2$	
$(a_{21}+1)/2$	$a_{22}/2$	$a_{23}/2$	
0	$(a_{32}-2)/2$	0	

A similar solution exists when  $a_{22} \neq 0$ . So, without loss of generality, let  $a_{12} = a_{22} = 0$ . If  $a_{13}$  and  $a_{23}$  are both non-zero, then a solution is

$(a_{11}+1)/2$	0	$(a_{13}-2)/2$
$(a_{21}+1)/2$	0	$(a_{23}-2)/2$
0	$(a_{32}+2)/2$	0

So, without loss of generality,  $a_{23}=0$ . If  $a_{11}\geqslant 3$  and  $a_{13}\geqslant 4$ , then a solution for B is

$(a_{11}+3)/2$	0	$(a_{13}-4)/2$	
$(a_{21}-1)/2$	0	0	] .
0	$(a_{32}+2)/2$	0	

So either  $a_{11} = 1$  or  $a_{13} \leq 2$ . If  $a_{11} = 1$ , then we obtain the following exceptional case  $E_3$ :

1	0	$a_{13}$	
$a_{13} + 1$	0	0	
0	$a_{13} + 2$	0	

If  $a_{13} = 2$  or  $a_{13} = 0$  then we have following, respectively:

$a_{11}$	0	2		$a_{11}$	0	0	
$a_{11} + 2$	0	0	or	$a_{11}$	0	0	
0	$2a_{11} + 2$	0		0	$2a_{11}$	0	

The former is impossible as  $n \equiv 0 \pmod{4}$ . The latter is the exception  $E_5$ .

Before studying the exceptional configurations of Lemma 26 in more detail, we first note the following simple consequences of the lemma. We start by noting that we have shown a special case of Theorem 19.

Corollary 27. If  $n \equiv 0 \pmod{24}$ , then there is no set of 2-maxMOFS(n; n/2).

*Proof.* Lemma 26 gives a complete list of configurations which correspond to pairs of rows which are not balanceable and none can occur if  $n \equiv 0 \pmod{24}$ . The result then follows by Lemma 20.

Since we have not used the fact that  $F_1$  is orthogonal to  $F_2$  in deriving Lemma 26, we have the following, more general corollary, which improves Theorem 11 when k = 2.

Corollary 28. If  $n \equiv 0 \pmod{24}$  and  $F_1$  and  $F_2$  are any frequency squares of type (n; n/2), then there exists a frequency square F, also of type (n; n/2), that is orthogonal to both  $F_1$  and  $F_2$ .

As all exceptional configurations listed in Lemma 26 satisfy  $a_{33} \leq 1$ , we have the following.

**Corollary 29.** Let r and r' be two distinct rows in  $F_1 \oplus F_2$ . If there exists two distinct columns c and c' such that F[r,c] = F[r',c] and F[r,c'] = F[r',c'], then the pair  $\{r,r'\}$  is balanceable.

As mentioned earlier,  $A(r_1, r_2)$  is sufficient to determine the balanceability of the rows  $r_1$  and  $r_2$ . We say that  $A' = A'(r_1, r_2)$  is an  $E_i$  if  $A(r_1, r_2)$  is equivalent to  $E_i$  for  $i \in \{1, ..., 6\}$ . All such matrices characterise pairs of rows that do not balance, by Lemma 26. Unpacking that result, we find that the exceptional cases are each based on one of two underlying structures:

#### Lemma 30. Let

$$D = \begin{bmatrix} 2x & 0 & 0 & 0 \\ 0 & 0 & x & x \\ 0 & x & 0 & 0 \\ 0 & x & 0 & 0 \end{bmatrix} \quad and \quad F = \begin{bmatrix} 2y & 0 & 0 & 0 \\ 2y & 0 & 0 & 0 \\ 0 & 2y & 0 & 0 \\ 0 & 2y & 0 & 0 \end{bmatrix}.$$

Then up to equivalence  $E_i = D + B_i$  for  $i \in \{1, 2, 3, 4\}$  and  $E_i = F + B_i$  for  $i \in \{5, 6\}$ , where

$$B_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = B_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{bmatrix}$$

and

$$B_4 = B_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{bmatrix}.$$

We say that a pair of rows  $\{r, r'\}$  and the matrices A(r, r') and A'(r, r') are of type  $\alpha$ ,  $\beta$  or  $\gamma$  if, respectively, A(r, r') is equivalent to an element of  $\{E_1, E_2\}$ ,  $\{E_3, E_4\}$  or  $\{E_5, E_6\}$  from Lemma 26. We say a row r is of type  $\alpha$ ,  $\beta$  or  $\gamma$  if there is an r' such that  $\{r, r'\}$  has type  $\alpha$ ,  $\beta$  or  $\gamma$ , respectively. In the next three lemmas we further categorise rows and pairs of rows by the number  $\psi(r)$  of occurrences of (0,0) in each row r. Each lemma is a consequence of Lemma 23 and Lemma 30, by considering the exceptional configurations and equivalences.

**Lemma 31.** Let  $\{r_1, r_2\}$  be a pair of rows of type  $\alpha$ . Then the multiset  $\{\psi(r_1), \psi(r_2)\}$  is either  $\{2x, 2x+1\}$ ,  $\{2x+1, 2x+1\}$ ,  $\{x+1, x\}$  or  $\{x, x\}$ . In the first two cases we say that  $\{r_1, r_2\}$  has type  $\alpha_1$ , and we say the pair  $\{r_1, r_2\}$  has type  $\alpha_2$  otherwise.

**Lemma 32.** Let  $\{r_1, r_2\}$  be a pair of rows of type  $\beta$ . Then the multiset  $\{\psi(r_1), \psi(r_2)\}$  is one of  $\{2x + 1, 2x + 1\}$ ,  $\{2x + 1, 2x + 2\}$ ,  $\{x + 1, x + 1\}$  or  $\{x + 1, x\}$ . In the first two cases we say the pair  $\{r_1, r_2\}$  has type  $\beta_1$  and we say the pair  $\{r_1, r_2\}$  has type  $\beta_2$  otherwise.

A row r has type  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  or  $\beta_2$  if it is in a pair  $\{r, r'\}$  that is of type  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$  or  $\beta_2$ , respectively. Note that we do not claim that every row has exactly one type in the previous two lemmas; we show the contrary in Lemma 34 below. For technical reasons, we do not further classify pairs of rows of type  $\gamma$  here, only individual rows, as below.

**Lemma 33.** Let  $\{r_1, r_2\}$  be a pair of rows of type  $\gamma$ . Then the multiset  $\{\psi(r_1), \psi(r_2)\}$  is one of  $\{2y, 2y+1\}$ ,  $\{2y+1, 2y+1\}$ ,  $\{2y+1, 2y+2\}$ . For  $i \in \{1, 2\}$ , we say that row  $r_i$  has type  $\gamma_1$  if  $\psi(r_i) = 2y+1$  and type  $\gamma_2$  otherwise.

As mentioned above, it is possible for a row to have more than one of the types  $\alpha_i$ ,  $\beta_i$  or  $\gamma_i$ . We now show, however, that in most instances no such row exists.

**Lemma 34.** A row r is of two different types from the set  $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$  only if

- (i) n = 8,  $\psi(r) = 2$  and r has type  $\alpha_1$  and  $\alpha_2$ , or
- (ii) n = 20,  $\psi(r) \in \{4, 6\}$  and r has type  $\gamma_2$  and either  $\alpha_1$  or  $\alpha_2$ .

Proof. Note that by definition, r can never be of type  $\gamma_1$  and  $\gamma_2$ . By Lemmas 31 and 32, r has type  $\alpha_1$  and  $\alpha_2$  or  $\beta_1$  and  $\beta_2$  only if  $2x \leqslant x+1$  or  $2x+1 \leqslant x+1$ , respectively. As  $x \geqslant 1$ , r is never type  $\beta_1$  and  $\beta_2$  and has type  $\alpha_1$  and  $\alpha_2$  only if x=1 and  $\psi(r)=2$ . So we can assume that r has two of the three types  $\alpha$ ,  $\beta$  and  $\gamma$ . The row r can never be of type  $\alpha$  and  $\beta$ , since any type  $\alpha$  row can only occur when  $n \equiv 2 \pmod{6}$  while any row of type  $\beta$  only occurs when  $n \equiv 4 \pmod{6}$ . By Lemmas 26, 32 and 33, r has type  $\beta$  and  $\gamma$  only if  $n \equiv 4 \pmod{24}$  and at least one of  $2y \leqslant x+1$  or  $2y+2 \geqslant 2x+1$  holds. However, these inequalities imply  $n/4-1 \leqslant (n+2)/6$  and  $n/4+1 \geqslant (n-1)/3$ , respectively, both contradicting the fact that  $4 < n \equiv 4 \pmod{24}$ . Finally, by similar reasoning to the previous case, r has type  $\alpha$  and  $\gamma$  only if  $n \equiv 20 \pmod{24}$  and at least one of  $2y \leqslant x+1$  or  $2y+2 \geqslant 2x$  holds. However, this implies  $n/4-1 \leqslant (n+4)/6$  or  $n/4+1 \geqslant (n-2)/3$ ; in both cases  $n \leqslant 20$ . It follows that r has type  $\alpha$  and  $\gamma$  only if n = 20,  $\psi(r) \in \{4,6\}$  and r has type  $\gamma_2$  and one of  $\alpha_1$  or  $\alpha_2$ .

Observe the following property of types  $\alpha$  and  $\beta$  (the same does not hold for type  $\gamma$ ).

**Lemma 35.** If  $\{r_1, r_2\}$  is a pair of rows of type  $\alpha_i$  (respectively  $\beta_i$ ), where  $i \in \{1, 2\}$ , then after swapping 0 and 1 in exactly one of the frequency squares  $F_1$  and  $F_2$ , the pair  $\{r_1, r_2\}$  is of type  $\alpha_{3-i}$  (respectively,  $\beta_{3-i}$ ).

For a matrix A' = A'(r, r'), its dual is the equivalent matrix formed from A', by applying the permutation (12)(34) to its rows and columns. The following is then a corollary of Lemma 30.

Corollary 36. Let A' = A'(r, r') have type  $\alpha_1$ . Then either A' or its dual is one of the following (up to transpose):

2x	0	1	0
0	0	$\boldsymbol{x}$	x+1
0	$\boldsymbol{x}$	0	0
0	$\boldsymbol{x}$	0	0

	2x	0	0	1
O.C.	0	0	x+1	$\boldsymbol{x}$
or	0	$\boldsymbol{x}$	0	0
	0	$\boldsymbol{x}$	0	0

	2x+1	0	0	0	
c.	0	1	$\boldsymbol{x}$	$\boldsymbol{x}$	
	0	$\boldsymbol{x}$	0	0	ľ
	0	$\boldsymbol{x}$	0	0	

Let A' = A'(r, r') have type  $\beta_1$ . Then either A' or its dual is one of the following (up to transpose):

	2x + 1	0	0	0			2x	+1	(	)	(	)	0								
	1	0	x	$\boldsymbol{x}$	0	0.00		O.M.		or		O.M.		0	(	)	<i>x</i> -	+ 1	$\boldsymbol{x}$	0	m
	0	x+1	0	0	0	,		0	<i>x</i> -	+ 1	(	)	0	U	1						
	0	x+1	0	0				0	3	r	(	)	1								
г					. 1				_			0									
	2x+1	0	0	(	)			2x -	⊢ 1	(	)	0	(	)							
Ī	0	0	$\boldsymbol{x}$	x -	x+1		n <i>a</i> c	0		(	)	$\boldsymbol{x}$	$x \dashv$	- 1							
	0	x	1	(	0		r	0		$x \dashv$	- 1	0	(	)	•						
ſ	0	x+1	0	(	)			0		а	C	1	(	)							

### 5.4 Sets of rows that pairwise do not balance

We next give upper bounds on the number of rows in  $F_1 \oplus F_2$  that are pairwise not balanceable and of the same type. We begin with rows of type  $\alpha$ .

**Lemma 37.** If n > 8, then any set of four rows of type  $\alpha$  in  $F_1 \oplus F_2$  contains a balanceable pair.

Proof. Let  $r, r_1, r_2$  and  $r_3$  be 4 rows of type  $\alpha$  in  $F_1 \oplus F_2$ . We may assume that  $\{r, r_i\}$  is not balanceable for  $i \in \{1, 2, 3\}$ , since otherwise we are done. It follows from Lemmas 33 and 34 that  $\{r, r_i\}$  is of type  $\alpha$  for  $i \in \{1, 2, 3\}$ , since, even if n = 20, there can be at most one row in a non-balanceable pair which is of two different types. By Lemma 35, we can assume that  $A'(r, r_1)$  is of type  $\alpha_1$ . Lemma 31 implies that  $A'(r, r_2)$  and  $A'(r, r_3)$  both have type  $\alpha_1$ . Let  $C_{00}$  be the set of columns of  $F_1 \oplus F_2$  for which row r contains a (0, 0). By Corollary 36, by taking the dual if necessary, we can assume that  $A'(r, r_1)$  and  $A'(r, r_2)$  are each one of

2x	0	0	1		2x	0	0	0
0	0	x+1	$\boldsymbol{x}$	and	0	0	$\boldsymbol{x}$	$\boldsymbol{x}$
0	$\boldsymbol{x}$	0	0	and	0	x+1	0	0
0	x	0	0		1	x	0	0

when  $\psi(r) = 2x$  and  $A'(r, r_1)$  and  $A'(r, r_2)$  are each one of

2x+1	0	0	0		2x	0	1	0		2x	0	0	0
0	1	x	$\boldsymbol{x}$	and	0	0	$\boldsymbol{x}$	x+1	and	0	0	$\boldsymbol{x}$	$\boldsymbol{x}$
0	$\boldsymbol{x}$	0	0	and	0	$\boldsymbol{x}$	0	0	and	1	x	0	0
0	$\boldsymbol{x}$	0	0		0	$\boldsymbol{x}$	0	0		0	x+1	0	0

when  $\psi(r) = 2x + 1$ . In particular, there is at most one column in  $C_{00}$  such that row  $r_i$  of  $F_1 \oplus F_2$  does not contain (1,1), for each  $i \in \{1,2\}$ . Thus, there are at least 2x - 2 columns in  $C_{00}$  such that both rows  $r_1$  and  $r_2$  of  $F_1 \oplus F_2$  contain a (1,1). As  $x \ge 2$ , the result follows from Corollary 29.

The above lemma is also true when n = 8, using a slightly more complicated argument, but this will not be necessary. We show an analogous result for rows of type  $\beta$ .

**Lemma 38.** Any set of four rows of type  $\beta$  in  $F_1 \oplus F_2$  contains a balanceable pair.

Proof. Let  $r, r_1, r_2$  and  $r_3$  be 4 rows of type  $\beta$  in  $F_1 \oplus F_2$ . We may assume that  $\{r, r_i\}$  is not balanceable for  $i \in \{1, 2, 3\}$ , since otherwise we are done. By Lemma 34, it follows that  $\{r, r_i\}$  is of type  $\beta$  for  $i \in \{1, 2, 3\}$ . By Lemma 35, we can assume that  $A'(r, r_1)$  is of type  $\beta_1$ . Lemma 32 and Lemma 34 then imply that  $A'(r, r_2)$  and  $A'(r, r_3)$  are also of type  $\beta_1$ . Let  $C_{00}$  be the set of columns with (0, 0) in row r of  $F_1 \oplus F_2$ . By Corollary 36, by taking the dual if necessary, we can assume that  $A'(r, r_1)$  and  $A'(r, r_2)$  have 2x + 1 in cell (1, 1) and  $\psi(r) \in \{2x + 1, 2x + 2\}$ . Thus for each  $i \in \{1, 2\}$ , there are at least 2x + 1 columns of  $C_{00}$  for which  $r_i$  contains a (1, 1) and at most one column of  $C_{00}$  for which  $r_i$  does not contain (1, 1). So there are at least  $2x \geqslant 2$  columns of  $C_{00}$  that contain a (1, 1) in both row  $r_1$  and  $r_2$ . It follows from Corollary 29 that  $\{r_1, r_2\}$  is balanceable.

We now consider rows of type  $\gamma$ . We start by categorising configurations equivalent to  $E_5$ , as given below.

	2y + 1	0	0
ı	2y + 1	0	0
	0	4y + 2	0
•		$T_1$	

0	2y+1	0				
0	2y + 1	0				
4y + 2	0	0				
$T_2$						

2y + 1	2y + 1	0				
0	0	4y + 2				
0	0	0				
$T_3$						

0	0	4y+2				
2y + 1	2y + 1	0				
0	0	0				
$T_4$						

Figure 1 depicts each of the types given above as they would appear in  $F_1 \oplus F_2$ , up to permuting the columns.

	$\leftarrow$	n/4	$\longrightarrow$									
r	(0,0)	• • •	(0,0)	(1, 1)	• • •	(1,1)	(1,0)	• • •	(1,0)	(0,1)	• • •	(0,1)
$T_1$	(1, 1)	• • •	(1, 1)	(1,0)	• • •	(1,0)	(0, 1)		(0,1)	(0,0)		(0,0)
$T_2$	(0,1)		(0, 1)	(0,0)		(0,0)	(1, 1)	• • •	(1, 1)	(1,0)	• • •	(1,0)
$T_3$	(1, 1)		(1, 1)	(0, 1)		(0, 1)	(0,0)		(0,0)	(1,0)		(1,0)
												(1, 1)

Figure 1: Some type  $\gamma$  rows.

Define each  $E_6$  as type  $T_i^*$  if  $T_i$  is obtained from  $T_i^*$  by changing exactly 4 entries. We say that  $A = A(r_1, r_2)$  has type  $T_i$  and  $r_2$  has type  $T_i$  with respect to  $r_1$  if A forms an  $E_5$  of type  $T_i$ . We define type  $T_i^*$  similarly and define  $A'(r_1, r_2)$  to be the same type as  $A(r_1, r_2)$  for any pairs of rows  $r_1$  and  $r_2$ .

We find several of the pairs (p,q) for which the types defined above are (p,q)-balanceable.

**Lemma 39.** Let  $\{r,r'\}$  be a pair of rows in  $F_1 \oplus F_2$  such that A = A(r,r') is equivalent to  $E_5$  or  $E_6$ . If A is of type  $T_i^*$  for some i, then A is (p,q)-balanceable for  $(p,q) \in \{(0,1), (1,0), (1,1), (1,-1)\}$ . If A is of type  $T_1$  or  $T_2$ , then A is (1,0)-balanceable and if A is of type  $T_3$  or  $T_4$  then A is (0,1)-balanceable.

*Proof.* Consider the following array of type  $T_1^*$ ,

	2y+1	0	0
ľ	2y	0	1
ľ		4y + 1	

By Lemma 25, the following 4 matrices (p,q)-balance the array above for (p,q) = (0,1), (1,0), (1,1) and (1,-1), respectively:

y+1	0	0
y	0	1
0	2y	0

y+1	0	0
y	0	0
0	2y + 1	0

y+1	0	0
y	0	0
0	2y	1

y+1	0	0
y-1	0	1
0	2y + 1	0

By noting that an array is (p,q)-balanceable if and only if it is (-p,-q)-balanceable, it follows that the array (12) is (p,q)-balanceable for  $(p,q) \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ . As the set  $\{-1,0,1\}^2 \setminus \{(0,0)\}$  is preserved under negation of an entry or swapping the entries of the ordered pairs, it follows that any array equivalent to (12), is also (p,q)-balanceable for  $(p,q) \in \{-1,0,1\}^2 \setminus \{(0,0)\}$ . As all arrays of type  $E_6$  are equivalent, this proves the result when A = A(r,r') is of type  $T_i^*$  with  $i \in \{1,2,3,4\}$ . The matrix above that (1,0)-balances (12) also (1,0)-balances A if A is of type  $T_1$ . Swapping the first two columns of  $T_1$  results in  $T_2$ , while  $T_3$  and  $T_4$  are the transpose of  $T_1$  and  $T_2$ , respectively. It follows that A is (1,0)-balanceable if A is of type  $T_2$  and A is (0,1)-balanceable if A is of type  $T_3$  or  $T_4$ .

We also need to consider the balanceability of rows of type  $\alpha$ .

**Lemma 40.** Let  $\{r,r'\}$  be a pair of rows in  $F_1 \oplus F_2$  such that A(r,r') has type  $\alpha$ . Then A(r,r') is (0,1)-balanceable, (1,0)-balanceable and either (1,1)-balanceable or (1,-1)-balanceable.

*Proof.* Consider the arrays

of types  $E_1$  and  $E_2$ , respectively. By Lemma 25, the following matrices (p,q)-balance the arrays above for (p,q)=(0,-1), (-1,0) and (1,1), respectively:

$\boldsymbol{x}$	0	0		x	0	0		x+1	0	0	
0	0	x	and	0	0	x+1	and	0	0	x	
0	x+1	0		0	$\boldsymbol{x}$	0		0	$\boldsymbol{x}$	0	

As an array is (p,q)-balanceable if and only if it is (-p,-q)-balanceable and A=A(r,r') is equivalent to one of the arrays in (13), it follows that A is both (0,1)-balanceable and (1,0)-balanceable and either (1,1)-balanceable or (1,-1)-balanceable.

Corollary 41. Let  $r_1$ ,  $r_2$ ,  $r_3$ ,  $r_4$ ,  $r_5$  and  $r_6$  be distinct rows. If  $A(r_1, r_2)$  is of type  $\alpha$  and  $A(r_3, r_4)$  is of type  $\alpha$  or  $\gamma$ , then  $\{r_1, r_2, r_3, r_4\}$  is balanceable. Moreover, if  $A(r_1, r_2)$ ,  $A(r_3, r_4)$  and  $A(r_5, r_6)$  are each of type  $\alpha$ , then the set  $\{r_1, r_2, r_3, r_4, r_5, r_6\}$  is balanceable.

*Proof.* The first result follows directly from the previous two lemmas and Lemma 22. For the second result, by Lemma 40,  $A(r_1, r_2)$  can be (1, 1)-balanced or (1, -1)-balanced,  $A(r_3, r_4)$  can be (-1, 0)-balanced and  $A(r_5, r_6)$  can be (0, 1)-balanced. The result follows from Lemma 22.

To analyse sets of rows of type  $\gamma$  more closely, we consider tables similar to Figure 1. Let  $S_i$  be the subtable of the table in Figure 1, with rows r and  $T_i$ . Also, let  $S_i^*$  be a table formed from  $S_i$ , by replacing the row  $T_i$  with a row r' that has type  $T_i^*$  with respect to r, the first row of  $S_i$ . By Lemma 33, any pair of rows of type  $\gamma$  include one row of type  $\gamma_1$ , that is, a row with exactly n/4 occurrences of (a,b) for each  $(a,b) \in \{0,1\}^2$ . Therefore, any pair of rows of type  $\gamma$  is equivalent to the rows of  $S_i$  or an  $S_i^*$  for some  $i \in \{1,2,3,4\}$ .

Note that by swapping  $F_1$  with  $F_2$  and rearranging columns, we map  $S_1$  to  $S_3$  and  $S_2$  to  $S_4$  and vice versa. Moreover, swapping the symbols in  $F_1$  and rearranging columns maps  $S_3$  to  $S_4$  and vice versa, while fixing  $S_1$  and  $S_2$ . Thirdly, swapping the symbols in  $F_2$  and rearranging columns maps  $S_1$  to  $S_2$  and vice versa, while fixing  $S_3$  and  $S_4$ . Thus, the tables  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  are equivalent, and each  $S_i^*$  is equivalent to some  $S_1^*$ . We will exploit these facts in the following lemmas.

Next, define  $S_{i,j}$  to be the table formed by rows r,  $T_i$  and  $T_j$  from Figure 1 (if i = j then we repeat row  $T_i$ ). Let  $S_{i,j}^*$  be a table  $S_{i,j}$  with row  $T_i$  replaced by a row of type  $T_i^*$  with respect to r and/or with row  $T_j$  replaced by a row of type  $T_j^*$  with respect to r. As above, every table  $S_{i,j}$  is equivalent to either  $S_{1,1}$ ,  $S_{1,2}$  or  $S_{1,3}^*$  and each  $S_{i,j}^*$  is equivalent to  $S_{1,1}^*$ ,  $S_{1,2}^*$  or  $S_{1,3}^*$ .

In Lemma 42 below, we show that any  $S_i^*$  can be formed from  $S_i$  by swapping particular entries. The definition of such a swap is in part motivated by the following observation that will be used throughout this subsection. By inspecting Figure 1, one notices that if A(r, r') is of type  $T_3$  or  $T_4$ , then rows r and r' of  $F_1$  are complementary. Similarly, if A(r, r') is of type  $T_1$  or  $T_2$ , then rows r and r' of  $F_2$  are complementary. Let r' be the second row of  $S_1$  or  $S_2$ . Then a legitimate swap in r' is a swap which replaces two cells containing (a, b) and (c, 1 - b) with (a, 1 - b) and (c, b), or with (c, 1 - b) and (a, b), respectively. A legitimate swap in the second row of  $S_3$  or  $S_4$  is defined analogously by interchanging the roles of  $F_1$  and  $F_2$ .

**Lemma 42.** Let  $i \in \{1, 2, 3, 4\}$ . Then any  $S_i^*$  can be formed from  $S_i$  by performing one legitimate swap in the second row. In particular, every  $S_i^*$  has exactly two columns where both rows have the same first entry if  $i \in \{3, 4\}$  or the same second entry if  $i \in \{1, 2\}$ .

Proof. From the symmetries described above, it suffices to consider the case when i = 1. Let r and r' be the first and second rows of a table  $S_1^*$ , respectively. Observe that  $S_1^*$  will differ from  $S_1$  by two columns; these correspond to the 1's in the 3rd column of A(r,r'). As every  $S_i$  and  $S_i^*$  have the same first row,  $S_1^*$  in fact only differs from  $S_1$  by two cells in the second row. Let (a,b) and (c,d) be the entries in the two cells of  $S_1$  that differ from those in  $S_1^*$ . In any column of  $S_1^*$  corresponding to a 1 in the third column of A(r,r'), the ordered pairs in the two rows of  $S_1^*$  must have the same second entry. On the other hand, in every column of  $S_1$  the ordered pairs in both rows have a different second entry. So (a,b) and (c,d) in  $S_1$  are replaced with (a', 1-b) and (c', 1-d), respectively, for some a', c'. The number of 0's in row r' of each of  $F_1$  and  $F_2$  is n/2 only if we have the multiset equalities  $\{a',c'\} = \{a,c\}$  and  $\{1-b,1-d\} = \{b,d\}$ , respectively. It quickly follows that replacing (a,b) and (c,d) with (a',1-b) and (c',1-d), respectively, must be a legitimate swap.

It is an immediate corollary of Lemma 42 that any  $S_{i,j}^*$  can be formed from  $S_{i,j}$  by performing a legitimate swap in the second row and/or by performing a legitimate swap in the third row. We now consider two rows  $r_1$  and  $r_2$  that do not balance with a given row r of type  $\gamma_1$ . By the comments above, any three such rows r, r<sub>1</sub> and r<sub>2</sub> are equivalent to the rows of  $S_{1,j}$  or an  $S_{1,j}^*$  for  $j \in \{1,2,3\}$ . Next, we focus on these situations.

Here and for the remainder of the section, we use the following definition. Let C be a subset of the columns of  $F_1 \oplus F_2$ . Then let  $A_C(r_1, r_2)$  be defined analogously to  $A(r_1, r_2)$ , where each cell (i, j) of  $A_C(r_1, r_2)$  only counts columns in C.

**Lemma 43.** Let r,  $r_1$  and  $r_2$  be the three rows of  $S_{1,j}$  or an  $S_{1,j}^*$  for  $j \in \{1,2,3\}$ . Then the matrix  $A(r_1, r_2)$  is of the form A + B, where B is an admissible matrix whose entry sum is 8 and A is the following configuration,

0	0	0		0	0	4y-2		0	0	2y-1
0	0	0	,	0	0	4y - 2	,	2y - 1	0	0
0	0	8y - 4		0	0	0		0	2y - 1	2y-1

when j = 1, j = 2 and j = 3, respectively.

Proof. Let S be  $S_{1,j}$  or an  $S_{1,j}^*$  for  $j \in \{1,2,3\}$  with rows  $r, r_1, r_2$ . Choose a set C of 8 columns of S such that exactly two columns contain the pair (a,b) in row r for each (a,b) and every column involved in the legitimate swaps in  $r_1$  and  $r_2$  (if any) are in C. Note that by Lemma 42, such a set of columns exists, as a legitimate swap cannot change two cells containing the same ordered pair. Let  $C' = N(n) \setminus C$ . Then by considering Figure 1, it is easy to check that  $A = A_{C'}(r_1, r_2)$  is the configuration given in the lemma, when j = 1, j = 2 and j = 3, respectively. Finally,  $B = A_C(r_1, r_2)$  clearly has entry sum 8, and is admissible, since  $A(r_1, r_2)$  and A are admissible. As  $A(r_1, r_2) = A_C(r_1, r_2) + A_{C'}(r_1, r_2)$ , the result follows.  $\square$ 

We can now show a stronger analogue of Lemmas 37 and 38 for rows of type  $\gamma$ .

**Lemma 44.** In any set of three rows of type  $\gamma$  in  $F_1 \oplus F_2$  at least one pair of the rows is balanceable.

Proof. Let  $r, r_1, r_2$  be three rows of type  $\gamma$  in  $F_1 \oplus F_2$ . By Lemmas 33 and 34, without loss of generality, we can assume that r is of type  $\gamma_1$  and neither  $\{r, r_1\}$  nor  $\{r, r_2\}$  is balanceable. Also by equivalence, we can assume that  $r, r_1$  and  $r_2$  are the rows of  $S_{1,j}$  or an  $S_{1,j}^*$  for  $j \in \{1, 2, 3\}$ . As n > 4, a row of type  $\gamma$  exists in  $F_1 \oplus F_2$  only if  $n \ge 12$ . By comparing the configurations A in Lemma 43 with the exceptional configurations in Lemma 26,  $\{r_1, r_2\}$  is balanceable unless n = 12, j = 3 and  $A(r_1, r_2) = A + B$  where A is the last configuration in Lemma 43, so assume these conditions hold. The only exceptional configurations in Lemma 26 consistent with the last configuration of Lemma 43 are  $E_4$  and  $E_6$ . However,  $E_4$  can only occur if  $n \equiv 4 \pmod{6}$ , so we can assume that  $A(r_1, r_2)$  is equivalent to  $E_6$ .

Let S be  $S_{1,3}$  or the  $S_{1,3}^*$  with rows  $r, r_1$  and  $r_2$ . Then S can be formed from  $S_{1,3}$  by performing at most one legitimate swap in the second row and at most one legitimate swap in the third row. Thus, given that

the last two rows of  $S_{1,3}$ , that is, the rows  $T_1$  and  $T_3$  of Figure 1, correspond to the configuration

0	0	3
3	0	0
0	3	3

and  $A(r_1, r_2)$  is equivalent to  $E_6$ ,  $A(r_1, r_2)$  must be either

Now, as  $S_{1,3}$  has n/4 = 3 columns where the second and third rows both contain the pair (1,1), S must differ from  $S_{1,3}$  by exactly one legitimate swap in each of the last two rows. Moreover, the two legitimate swaps must include distinct columns that contain (1,1). The legitimate swap in row  $r_1$  replaces a (1,1) in some column c with (1,0) or (0,0). As the legitimate swap in row  $r_2$  cannot occur in column c, replacing a (1,1) with a (1,0) in row  $r_1$  is inconsistent with the configurations in (14), since the entry in cell (3,1) of  $A(r_1,r_2)$  would then be at least 1. Therefore, the legitimate swap in row  $r_1$  is one which swaps a (1,1) with a (0,0). By a similar argument, the legitimate swap in row  $r_2$  is also one which swaps a (1,1) with a (0,0). Hence,  $A(r_1,r_2)$  must be

1	0	3
3	1	0
0	3	1

which is neither of the configurations in (14). It follows that  $\{r_1, r_2\}$  is balanceable even when n = 12, completing the proof.

**Lemma 45.** If n > 8, then any set of four rows in  $F_1 \oplus F_2$  contains a balanceable pair.

Proof. By Lemma 44 we may assume that our four rows are  $r, r_1, r_2, r_3$  where r does not have type  $\gamma$ . Now, either  $n \not\equiv 2 \pmod{6}$  and none of  $r, r_1, r_2, r_3$  has type  $\alpha$ , or  $n \not\equiv 4 \pmod{6}$  and none of  $r, r_1, r_2, r_3$  has type  $\beta$ . Hence, by Lemma 37 and Lemma 38, we may assume that there is  $r' \in \{r, r_1, r_2, r_3\}$  such that r' does not have type  $\alpha$  or  $\beta$ . If  $r' \neq r$  then the pair  $\{r, r'\}$  is balanceable by Lemma 26, since it is not of type  $\alpha$ ,  $\beta$  or  $\gamma$ . For the same reason, if r' = r then  $\{r, r_i\}$  is balanceable for each  $i \in \{1, 2, 3\}$ .

We will also require a result about balancing particular sets of four rows. To prove this result, we need a refined version of Lemma 43 in very special cases, as in the lemma below. For the lemma and the remainder of the section, we use the following definition. Let  $r_1, r_2$  be rows of type  $T_{i_1}^*$  and  $T_{i_2}^*$  with respect to r, respectively. We say the legitimate swaps in  $r_1$  and  $r_2$  are disjoint if the columns involved in each swap are disjoint.

**Lemma 46.** Let r,  $r_1$  and  $r_2$  be the rows of an  $S_{1,j}^*$  for some  $j \in \{1,2\}$ . Suppose that  $r_1$  and  $r_2$  both have legitimate swaps that are disjoint. Then the entries of  $A = A(r_1, r_2)$  satisfy

- $a_{33} = 8y$  and  $a_{13} = 0 = a_{23}$  if j = 1;
- $a_{13} + a_{23} = 8y$ ,  $a_{33} = 0$  and  $a_{13}, a_{23} \ge 4y 2$  if j = 2.

*Proof.* Let S be the  $S_{1,j}^*$  with rows  $r, r_1, r_2$  and let  $r'_1$  and  $r'_2$  be second and third rows of  $S_{1,j}$ , respectively. By considering Figure 1, it is easy to check that  $A(r'_1, r'_2)$  is

0	0	0		0	0	4y+2
0	0	0	and	0	0	4y + 2
0	0	8y + 4		0	0	0

when j=1, and j=2, respectively. By Lemma 42, rows  $r_1$  and  $r_2$  differ from, respectively,  $r'_1$  and  $r'_2$  by a legitimate swap. By assumption, these legitimate swaps are disjoint, so S and  $S_{1,j}$  differ by 4 cells located in different columns; call this set of 4 columns C and let  $C'=N(n)\setminus C$ . Clearly,  $A_C(r_1,r_2)$  has entry sum 4 and  $A_{C'}(r_1,r_2)$  has entry sum 8y with all non-zero cells occurring in the last column. In each column in C the element in S in exactly one of the rows  $r_1$  and  $r_2$  has a different second entry to the element in row r, by Lemma 42. It follows that  $A_C(r_1,r_2)$  has 0's in the last column. As  $A=A(r_1,r_2)=A_C(r_1,r_2)+A_{C'}(r_1,r_2)$ , the last column of  $A(r_1,r_2)$  has entry sum 8y, which, along with the configurations above, imply that  $a_{33}=8y$  and  $a_{13}=0=a_{23}$  when j=1 and  $a_{13}+a_{23}=8y$  when j=2. Finally, as S and  $S_{1,j}$  differ in exactly 4 cells, neither  $a_{13}$  nor  $a_{23}$  can be less than 4y-2 when j=2, and the result follows.

Finally, we also require a result about three rows that do not balance with a given row of type  $\gamma_1$ .

**Lemma 47.** Let r,  $r_1$ ,  $r_2$  and  $r_3$  be rows of  $F_1 \oplus F_2$  such that r is of type  $\gamma_1$  and  $\{r, r_j\}$  has type  $T_{i_j}$  or  $T_{i_j}^*$  for  $j \in \{1, 2, 3\}$ , where  $i_1, i_2, i_3 \in \{1, 2, 3, 4\}$ . If either

- (i)  $\{i_1, i_2, i_3\} \cap \{1, 2\} \neq \emptyset$  and  $\{i_1, i_2, i_3\} \cap \{3, 4\} \neq \emptyset$ ; or
- (ii)  $\{r, r_j\}$  has type  $T_{i_j}^*$  for  $j \in \{1, 2, 3\}$  such that the legitimate swaps in two of  $r_1, r_2, r_3$  are disjoint then  $\{r, r_1, r_2, r_3\}$  is balanceable.

*Proof.* By equivalence, it suffices to consider the case when  $i_1 = 1$  and either  $i_2 = 3$  or  $i_2, i_3 \in \{1, 2\}$  and the rows satisfy (ii). First suppose that  $i_2 = 3$ . By Lemma 43,  $A(r_1, r_2)$  is

0	0	2y - 1	
2y - 1	0	0	+B
0	2y - 1	2y - 1	

where B is an admissible matrix with entry sum 8. As  $y \ge 1$ , we can write  $A(r_1, r_2)$  as

0	0	1	
1	0	0	+B'
0	1	1	

where B' is some admissible matrix with entry sum n-4=8y. If B' is not an exceptional configuration in Lemma 26, then there exists a matrix B'' which (0,0)-balances B'. The following then show that  $A(r_1, r_2)$  is (0,1)-balanceable and (1,0)-balanceable, respectively, by Lemma 25:

0	0	1			0	0	1	
1	0	0	+B''	and	0	0	0	+B''
0	0	0			0	0	1	

As  $A(r, r_3)$  is equivalent to  $E_5$  or  $E_6$ , Lemma 39 implies that  $A(r, r_3)$  is either (0, 1)-balanceable or (1, 0)-balanceable. Therefore,  $\{r, r_1, r_2, r_3\}$  is balanceable, by Lemma 22. If  $n \ge 20$ , then  $b'_{33} \ge 2y - 2 \ge 2$ , so B' cannot be an exceptional configuration in Lemma 26, by Corollary 29. Thus, if B' is an exceptional configuration in Lemma 26, then n = 12 and B' must be equivalent to  $E_1$  or  $E_2$ , as B' has entry sum 8. By Lemma 40, there are matrices B'' and B''' that (1,0)-balance and (0,-1)-balance B', respectively. Thus

	0				_	0	_	
1	0	0	+B''	and	1	0	0	+B'''
0	0	1			0	0	1	

(0,1)-balance and (-1,0)-balance  $A(r_1,r_2)$ , respectively. As before, Lemma 39 implies  $A(r,r_3)$  has to be either (0,1)-balanceable or (1,0)-balanceable, so  $\{r,r_1,r_2,r_3\}$  is balanceable by Lemma 22.

Now suppose that  $i_2, i_3 \in \{1, 2\}$  and condition (ii) is satisfied. Without loss of generality, let rows  $r_1$ and  $r_2$  have disjoint legitimate swaps. First suppose that  $i_2 = 1$ . Then by Lemma 46,  $A(r_1, r_2)$  is of the form

$$\begin{array}{c|cccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 8y \\
\end{array} + B$$

for some admissible matrix B, with entry sum 4 and 0's in the last column. In particular, B has at least one non-zero entry in the first column. Thus at least one of

1	0	0
0	0	0
0	0	4y + 1

0	0	0		0	0	0
1	0	0	or	0	0	0
0	0	4y + 1		1	0	4y + 1

0	0	0
0	0	0
1	0	4y + 1

(p,q)-balances  $A(r_1,r_2)$ , where (p,q) is (1,1), (-1,1) or (0,1), respectively. As  $A(r,r_3)$  is equivalent to  $E_6$ , Lemma 39 implies that  $A(r, r_3)$  is (1, 1)-balanceable, (-1, 1)-balanceable and (0, 1)-balanceable. Hence, Lemma 22 implies that  $\{r, r_1, r_2, r_3\}$  is balanceable.

Finally, suppose that  $i_2 = 2$ . Then, by Lemma 46,  $A = A(r_1, r_2)$  satisfies  $a_{13} + a_{23} = 8y$ ,  $a_{33} = 0$  and  $a_{13}, a_{23} \ge 4y - 2$ . In particular, the first two columns of A each have at least one non-zero entry. So if  $a_{13}$  and  $a_{23}$  are both at least 2y+1, then at least one of the following matrices (p,q)-balances A for some  $(p,q) \in \{(0,1),(1,1)\}:$ 

1	0	2y
0	0	2y + 1
0	0	0

		_							
0	2y		0	0	2y + 1		0	0	2y + 1
0	2y + 1	and	1	0	2y	and	0	0	2y
0	0		0	0	0		1	0	0
		•							

0	0	2y + 1
0	0	2y
1	0	0

Otherwise, as  $a_{13}, a_{23} \ge 4y - 2$  and  $y \ge 1$ , one of  $a_{13}$  and  $a_{23}$  is less than 2y + 1 only if y = 1 and  $a_{13}$ or  $a_{23}$  is 4y-2. If  $a_{13}=4y-2$ , then  $a_{23}=4y+2$  and so one of  $a_{11}$  and  $a_{12}$  is non-zero. It follows that at least one of the following  $(0, \pm 1)$ -balances A

1	0	2y
0	0	2y + 1
0	0	0

2y	and	0	1	2y
2y + 1		0	0	2y + 1
0		0	0	0

and so A is (0,1)-balanceable. Similarly, if  $a_{23}=4y-2$ , then A is (0,1)-balanceable. In all cases A is (0,1)balanceable or (1,1)-balanceable. As  $A(r,r_3)$  has type  $T_{i_3}^*$ , it is (0,1)-balanceable and (1,1)-balanceable, by Lemma 39. Hence,  $\{r, r_1, r_2, r_3\}$  is balanceable, by Lemma 22. 

We end this subsection with two results that limit the number of rows that  $F_1 \oplus F_2$  can have of a particular type. In both we will need to use that  $F_1$  is orthogonal to  $F_2$ , an assumption that we have not needed until now.

**Lemma 48.** Let  $F_1$  and  $F_2$  be orthogonal and n > 8. Also let  $w_{\alpha}(i)$  be the number of type  $\alpha_i$  rows and  $w_{\beta}(i)$  the number of type  $\beta$  rows in  $F_1 \oplus F_2$  for each  $i \in \{1, 2\}$ . Then

$$\max\{w_{\alpha}(1), w_{\alpha}(2), w_{\beta}(1), w_{\beta}(2)\} \leqslant \frac{3n^2}{4(n-2)} < 3n/4 + 2.$$

*Proof.* Suppose that  $w_{\alpha}(1) > \frac{3n^2}{4(n-2)}$ . Then by Lemma 31, there are more than  $\frac{3n^2}{4(n-2)} \times \frac{n-2}{3} = \frac{n^2}{4}$  occurrences of (0,0) in  $F_1 \oplus F_2$ , contradicting the assumption that  $F_1$  and  $F_2$  are orthogonal. The proofs for the other cases are similar.

**Lemma 49.** Let  $F_1$  and  $F_2$  be orthogonal and r be a row of type  $\gamma_1$  in  $F_1 \oplus F_2$ . Then there are at most  $\frac{n}{2} + 1$  rows r' such that A(r, r') has a type from the set  $\{T_1, T_1^*, T_2, T_2^*\}$ . Furthermore, if there are exactly  $\frac{n}{2} + 1$  such rows, then at least  $\frac{n}{2}$  of them must be types  $T_1^*$  or  $T_2^*$  and every column has at least one row with a legitimate swap in that column. In particular, there are at least two rows with legitimate swaps in disjoint pairs of columns. All of the above statements hold with  $T_1, T_1^*, T_2, T_2^*$  replaced respectively by  $T_3, T_3^*, T_4, T_4^*$ .

Proof. We only prove the claim about  $T_1, T_1^*, T_2, T_2^*$ , as the other case is equivalent. Let S be the submatrix of  $F_1 \oplus F_2$  with 1+t rows consisting of r and all other rows that are of types  $T_1, T_2, T_1^*$  or  $T_2^*$  with respect to r. Let S' be the  $(t+1) \times n$  matrix formed from S by replacing each row r' of type  $T_1^*$  or  $T_2^*$  with respect to r with the row r'' of type  $T_1$  or  $T_2$  with respect to r, respectively. For each  $(a,b) \in \{0,1\}^2$ , let  $C_{ab}$  be the submatrix of S with all the rows of S, and the columns for which r contains (a,b). Define  $C'_{ab}$  from S' in the analogous way, for each  $(a,b) \in \{0,1\}^2$ .

By Lemma 42, each row of S' (except the first row) can be formed from the corresponding row of S by performing a single legitimate swap. Let  $t_{ab}$  be the number of elements in  $C'_{ab}$  that differ from those in  $C_{ab}$ . We then have that  $t_{00} + t_{01} + t_{10} + t_{11} \le 2t$ . Let  $t_{\min} = \min\{t_{00}, t_{01}, t_{10}, t_{11}\}$ . By the pigeonhole principle,  $t_{\min} \le t/2$ . Recall that if A(r, r') is of type  $T_1$  or  $T_2$ , then every column has a different entry in rows r and r' of  $F_2$ . So, the columns of  $C_{a(1-b)}$  contain at least  $tn/4 - t_{a(1-b)}$  occurrences of the symbol b in  $F_2$ . As each  $C_{ab}$  is a subset of n/4 columns of  $F_1 \oplus F_2$ , we must have that  $tn/4 - t_{ab} \le n^2/8$ , for each ab. Therefore,

$$n^2/8 \geqslant \frac{tn}{4} - t_{\min} \geqslant \frac{tn}{4} - \frac{t}{2}$$

and rearranging for t implies that  $t \leq \frac{n^2}{2(n-2)} = \frac{n+2}{2} + \frac{2}{n-2}$ . Since we are assuming that  $n \geq 8$  and t is an integer, it follows that  $t \leq (n+2)/2$ , proving the first claim of the lemma.

Finally, suppose that t = (n+2)/2, and hence  $t_{ab} \ge tn/4 - n^2/8 = n/4$  for each ab. So there are at least n elements in S' that differ from the corresponding element in S. As each row of S' differs from the corresponding row in S in 0 or 2 places, there are at least n/2 rows in which S and S' differ. Also, if there were a column in which S and S' agreed, then the corresponding column of  $F_2$  would contain at least (n+2)/2 copies of some symbol, violating the fact that  $F_2$  is a frequency square. Hence there are either n/2 or n/2+1 legitimate swaps which between them cover all n columns. It follows that at least two of them must involve disjoint pairs of columns.

#### 5.5 Proof of Theorem 19

We separate the proof of Theorem 19 into cases depending on the type of rows present. We begin with the case when there is a row not of type  $\gamma$ . Here and for the remainder of the paper, we assume that  $F_1$  and  $F_2$  are orthogonal frequency square of order  $n \geq 8$ .

**Lemma 50.** Let  $n \notin \{8, 20\}$  and suppose there exists a row which is not of type  $\gamma$  in  $F_1 \oplus F_2$ . Then the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets.

Proof. By Lemma 34, no row of  $F_1 \oplus F_2$  can have two different types from the set  $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$ . In particular, any two rows not of the same type always form a balanceable pair, unless they are of types  $\gamma_1$  and  $\gamma_2$ . By Lemma 45, we can partition the rows of  $F_1 \oplus F_2$  into pairs  $\mathcal{R}$  such that at most one pair is not balanceable, by greedily selecting pairs of rows that balance. We are done unless there is a pair  $\{r, r'\} \in \mathcal{R}$  that is not balanceable. We proceed by showing there is always a way to re-pair r and r' so that all pairs in  $\mathcal{R}$  are balanceable.

Suppose first that  $\{r, r'\}$  is of type  $\alpha$  or  $\beta$ . We assume that the pair is of type  $\alpha_1$ ; the other cases are similar. Since  $n \geq 12$ , Lemma 48 implies that there exists at least two rows v and w which are not of type  $\alpha_1$ . If  $\{v, w\} \in \mathcal{R}$ , then we can replace  $\{r, r'\}$  and  $\{v, w\}$  with  $\{r, v\}$  and  $\{r', w\}$ , both of which must be balanceable. Otherwise,  $\{v, v'\}$ ,  $\{w, w'\} \in \mathcal{R}$  for some rows v' and w'. If  $\{r, v'\}$  is balanceable, we are done, as we can replace pairs  $\{r, r'\}$  and  $\{v, v'\}$  in  $\mathcal{R}$  with the balanceable pairs  $\{r, v'\}$  and  $\{r', v\}$ . We are

similarly done if any of  $\{r', v'\}$ ,  $\{r, w'\}$  or  $\{r', w'\}$  are balanceable. By Lemma 45, at least one pair from  $\{r, r', v', w'\}$  is balanceable. So we are done unless  $\{v', w'\}$  is balanceable, in which case we can replace pairs  $\{r, r'\}$ ,  $\{v, v'\}$  and  $\{w, w'\}$  in  $\mathcal{R}$  with balanceable pairs  $\{r, v\}$ ,  $\{r', w\}$  and  $\{v', w'\}$ .

Finally suppose that rows r and r' are of type  $\gamma$ . By assumption, there is a row v that is not of type  $\gamma$ . Then both  $\{r,v\}$  and  $\{r',v\}$  are balanceable. Let v' be the row such that  $\{v,v'\} \in \mathcal{R}$ . If v' is not of type  $\gamma$ , then  $\{r,v'\}$  and  $\{r',v'\}$  are both balanceable. If v' is of type  $\gamma$ , then at least one of  $\{r,v'\}$  and  $\{r',v'\}$  is balanceable, by Lemma 44. In any case, the pairs  $\{r,r'\}$  and  $\{v,v'\}$  in  $\mathcal{R}$  can be replaced with two balanceable pairs. This completes the proof.

The cases when n = 8 or n = 20 are dealt with separately in the following two lemmas.

**Lemma 51.** If n = 8, then the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets.

*Proof.* Observe that  $F_1 \oplus F_2$  has rows of neither type  $\beta$  nor  $\gamma$ , since n = 8.

Let G be the graph whose vertices are the rows of  $F_1 \oplus F_2$  with an edge between rows v and v' if and only if  $\{v, v'\}$  is balanceable. We may assume that G has no perfect matching since otherwise the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable pairs.

Let  $\mathcal{R}$  be a partition of the rows of  $F_1 \oplus F_2$  into pairs, with as few balanceable pairs as possible. Corollary 41 implies that, if  $\mathcal{R}$  contains more than one unbalanceable pair, then the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets. Thus, we may assume that  $\mathcal{R}$  contains a pair of rows  $\{r, r'\}$  of type  $\alpha$ , and the rows other than r and r' induce a clique in G. If there are two disjoint edges in G incident to r and r', then there exists a perfect matching in G. Alternatively, if there exists two disjoint pairs of rows both of which are not balanceable, then it violates our choice of  $\mathcal{R}$ . It follows that G must be the disjoint union of a  $K_7$  and  $K_1$ , where the isolated vertex is either r or r'.

Finally, we show that G cannot be the disjoint union of  $K_7$  and  $K_1$ . Assume otherwise and let r be the isolated vertex. That is, assume that  $\{r,v\}$  is not balanceable for all rows  $v \neq r$  of  $F_1 \oplus F_2$ . Without loss of generality, we can assume that r has type  $\alpha_1$ . Then  $\psi(r) \in \{2,3\}$ , by Lemma 31. If  $\psi(r) = 2$ , then the remaining rows must have 1 or 3 occurrences of (0,0) each, by Lemma 31. However, we then have the contradiction that the total number of occurrences of (0,0) in  $F_1 \oplus F_2$  is odd. Lastly, if  $\psi(r) = 3$ , then the remaining rows each have at least 2 occurrences of (0,0), by Lemma 31. However, this would mean that  $F_1 \oplus F_2$  has at least  $3 + 7 \times 2 = 17$  occurrences of (0,0), contradicting the fact that  $F_1$  and  $F_2$  are orthogonal. This completes the proof.

**Lemma 52.** Let n = 20. If there exists a row of type  $\alpha$  in  $F_1 \oplus F_2$ , then the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets.

Proof. Let  $\{r, r'\}$  be a pair of rows of type  $\alpha$ . As  $n = 20 \equiv 2 \pmod{6}$ , every row that is in a pair that is not balanceable is of type  $\alpha$  or  $\gamma$ . Within any four rows of  $F_1 \oplus F_2$ , at least one pair is balanceable by Lemma 45. So, we can partition the rows of  $F_1 \oplus F_2$  into pairs  $\mathcal{R}$ , such that  $\{r, r'\} \in \mathcal{R}$  and at most one pair other than  $\{r, r'\}$  is not balanceable. If  $\mathcal{R}$  contains a pair  $\{v, v'\}$  distinct from  $\{r, r'\}$  that is not balanceable, then  $\{v, v'\}$  must be of type  $\alpha$  or  $\gamma$ . Therefore by Corollary 41,  $\{r, r', v, v'\}$  is balanceable and it follows that the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets. So suppose that  $\{r, r'\}$  is the only pair in  $\mathcal{R}$  that is not balanceable. We show that there is always a way to re-pair the pairs in  $\mathcal{R}$  so that all pairs are balanceable. Without loss of generality, we can assume that  $\{r, r'\}$  has type  $\alpha_1$  and that  $\psi(r) \in \{6,7\}$  and  $\psi(r') = 7$  by Lemma 31.

The average value for  $\psi(\cdot)$  across all rows is 5, so by the pigeonhole principle there exists distinct pairs  $\{v,v'\}, \{w,w'\} \in \mathcal{R}$  such that  $\psi(v') \leq 4$  and  $\psi(w') \leq 5$ . It follows from Lemma 31 and Lemma 33 that  $\{r,v'\}, \{r',v'\}$  and  $\{r',w'\}$  are all balanceable pairs. If either  $\{r,v\}$  or  $\{r',v\}$  is balanceable then we can replace  $\{r,r'\}$  and  $\{v,v'\}$  by two balanceable pairs and we are done. So assume that is not the case. It then follows from Lemma 31 and Lemma 33 that  $\psi(r) = 7$  or  $\psi(v) = 7$ . By interchanging r and v if necessary, we may assume that  $\psi(r) = 7$ . It then follows that  $\{r,w'\}$  is balanceable.

Finally, we apply Lemma 45 to find that there must be a balanceable pair among  $\{r, r', v, w\}$ . This pair, together with two of the balanceable pairs  $\{r, v'\}$ ,  $\{r, w'\}$ ,  $\{r', v'\}$ ,  $\{r', w'\}$ ,  $\{v, v'\}$  and  $\{w, w'\}$ , can be used to replace the pairs  $\{r, r'\}$ ,  $\{v, v'\}$  and  $\{w, w'\}$ .

Next, we consider the case where every pair of rows is either balanceable or of type  $\gamma$ . To do so we require the following simple graph theoretical result.

**Lemma 53.** Let G be a simple graph with an even number of vertices such that each subset of three vertices induces at least one edge. Then either G has a perfect matching or G is the disjoint union of two odd cliques.

*Proof.* Suppose that G has no perfect matching. Then by Tutte's criterion there exists a set S of vertices whose removal leaves at least |S|+1 components of odd order. But the given condition means that no induced subgraph of G has more than 2 components. Given that G has an even number of vertices, the only possibility is that  $S=\emptyset$  and that G has two components, both of odd order. Considering each set of 3 vertices from 2 different components then shows that each component is a clique.

**Lemma 54.** If every pair of rows of  $F_1 \oplus F_2$  is either balanceable or of type  $\gamma$ , then the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets.

Proof. There is nothing to prove unless some rows have type  $\gamma$ , so we may assume that  $n \equiv 4 \pmod{8}$ . Let G be a graph with rows in  $F_1 \oplus F_2$  as vertices and an edge between r and r' if and only if  $\{r, r'\}$  is balanceable. A perfect matching in G corresponds to a partition of the rows of  $F_1 \oplus F_2$  into balanceable pairs. So, by Lemmas 44 and 53, we are done unless G is the union of two disjoint odd cliques. So, without loss of generality, let  $K_a$  and  $K_{n-a}$  be the connected components of G, with a < n - a and a odd. Suppose there exists 4 rows  $r, r_1, r_2, r_3$ , with r in  $K_a$  and  $r_1, r_2, r_3$  in  $K_{n-a}$ , such that  $\{r, r_1, r_2, r_3\}$  is balanceable. Then the induced subgraph of G on the remaining rows forms two disjoint even cliques and so the remaining rows can be partitioned into balanceable pairs. It then follows that the rows of  $F_1 \oplus F_2$  can be partitioned into balanceable sets.

So it suffices to show that such a set of four rows exists. Note that as G is the disjoint union of two cliques, every row is in some pair that is not balanceable. So, by assumption, every row is of type  $\gamma$ . We claim that there is at least one row in  $K_a$  that is of type  $\gamma_1$ . If there were no such row, then every row in  $K_a$  is of type  $\gamma_2$ . Inspecting Lemma 33, we see that any two  $\gamma_2$  rows form a balanceable pair, so  $K_a$  contains every  $\gamma_2$  row. Also, every  $\gamma_2$  row contains an odd number of occurrences of (0,0), while any  $\gamma_1$  row contains an even number of occurrences of (0,0). In total, there are  $n^2/4 \equiv 0 \pmod{2}$  occurrences of (0,0) in the rows of  $F_1 \oplus F_2$ . We conclude that  $K_a$  contains a row r of type  $\gamma_1$ .

If there are rows  $r_1, r_2$  and  $r_3$  in  $K_{n-a}$  such that  $A(r, r_1)$  and  $A(r, r_2)$  are of types  $T_i$  or  $T_i^*$  and  $T_j$  or  $T_j^*$  with  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$ , then by Lemma 47,  $\{r, r_1, r_2, r_3\}$  is balanceable. So we can assume that, without loss of generality, every row in  $K_{n-a}$  is of type  $T_1, T_1^*, T_2$  or  $T_2^*$ , with respect to r. By Lemma 49, if follows that n - a = n/2 + 1 and there are at least two rows  $r_1$  and  $r_2$ , such that  $r_1$  and  $r_2$  are of types  $T_1^*$  or  $T_2^*$  with respect to r and  $r_1$  and  $r_2$  have disjoint legitimate swaps. Choose any  $r_3$  in  $K_{n-a}$  distinct from  $r_1$  and  $r_2$  that is of type  $T_1^*$  or  $T_2^*$  with respect to r; such a row exists since at least  $n/2 \ge 6$  rows in  $K_{n-a}$  are type  $T_1^*$  or  $T_2^*$  with respect to r, by Lemma 49. Then by Lemma 47,  $\{r, r_1, r_2, r_3\}$  is balanceable. This completes the proof.

We can now prove Theorem 19.

Proof of Theorem 19. Let R be the set of rows of  $F_1 \oplus F_2$ . By Lemma 20, it suffices to show that R can be partitioned into balanceable sets. If all non-balanceable pairs from R are of type  $\gamma$  then we are done, by Lemma 54. So, assume that  $\{r, r'\} \subset R$  is a non-balanceable pair not of type  $\gamma$ , from which it follows, without loss of generality, that r is not of type  $\gamma$ . If  $n \notin \{8, 20\}$  then Lemma 50 implies that we can partition R into balanceable sets. Meanwhile, if n = 8, then we can partition R into balanceable sets, by Lemma 51. Finally, if n = 20, then  $n \equiv 2 \pmod{6}$ , so r must be of type  $\alpha$ , by Lemma 26. Hence, Lemma 52 completes the proof.

It may be possible to prove the analogue of Theorem 19 for  $n \equiv 2 \pmod{4}$  by similar methods. However, new configurations arise in (the analogue of) Lemma 26, making the subsequent analysis substantially more complicated. It was important for our proof that only certain rows can be in non-balanceable pairs (as shown by Lemmas 31, 32 and 33). However, if we assume that  $n \equiv 2 \pmod{4}$ , then any pair of rows in  $F_1 \oplus F_2$  that are complementary in at least one of  $F_1$  and  $F_2$  is not balanceable. Let  $F_3$  be a row in  $F_4 \oplus F_2$  and let  $F_3$  be the row which agrees with  $F_3$  and is complementary to  $F_3$  is balanceable. This means that the analogue of Lemma 45 fails for  $F_3$  (mod 4).

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