# The 2-adic complexity of Yu-Gong sequences with interleaved structure and optimal autocorrelation magnitude 

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#### Abstract

In 2008, a class of binary sequences of period $N=4\left(2^{k}-1\right)\left(2^{k}+\right.$ 1) with optimal autocorrelation magnitude has been presented by Yu and Gong based on an $m$-sequence, the perfect sequence ( $0,1,1,1$ ) of period 4 and interleaving technique. In this paper, we study the 2-adic complexity of these sequences. Our result shows that it is larger than $N-2\left\lceil\log _{2} N\right\rceil+4$ (which is far larger than $N / 2$ ) and could attain the maximum value $N$ if suitable parameters are chosen, i.e., the 2-adic complexity of this class of interleaved sequences is large enough to resist the Rational Approximation Algorithm.


Keywords $m$-sequence • interleaved sequence • optimal autocorrelation magnitude $\cdot 2$-adic complexity

## 1 Introduction

Since the interleaved structure of sequences was introduced by Gong in 3], several classes of binary sequences with this form have been constructed and were proved to have so many good pseudo-random properties, such as low autocorrelation, large linear complexity. For example, in 2010, Tang and Gong constructed three classes of sequences with optimal autocorrelation value/magnitude

[^0]using Legendre sequences, twin-prime sequences and a generalized GMW sequence, respectively [13], which were showed by Li and Tang to have large linear complexity [7]. In quick succession, Tang and Ding presented two more general constructions which include constructions in [13] as special cases and gave more sequences with optimal autocorrelation and large linear complexity [12. Later, Yan et al. also gave a generalized version for the constructions in 13] and put forward a sufficient and necessary condition for an interleaved sequence to have optimal autocorrelation 16. What's more exciting is that these sequences have also been proved to have large 2-adic complexity by Xiong et al. 14, 15 and Hu [5] using different methods respectively. Moreover, Su et al. constructed another class of sequences with optimal autocorrelation magnitude combining interleaved structure and Ding-Helleseth-Lam sequences 9 and these sequences have also been shown to have large linear complexity by Fan [2] and large 2-adic complexity by Sun et al. 10] and Yang et al. 17.

Note that each of the above mentioned sequences can be described as an interleaved form $s=I\left(s_{1}, s_{2}, s_{3}, s_{4}\right)$, i.e., the sequence $s$ is obtained by concatenating the successive rows of the matrix $I\left(s_{1}, s_{2}, s_{3}, s_{4}\right)$, in which each column is a periodic sequence $s_{i}, 1 \leq i \leq 4$. In fact, Yu and Gong also presented another description method of an interleaved structure by an indicator sequence (see Construction 1) [18]. Using this description, Yu and Gong represented an ADS (almost difference set) sequence of period $4 v$ in [1] as a $v \times 4$ interleaved structure and a product sequence of period $4 v$ in [8 as a $4 \times v$ interleaved structure respectively, which provides us a new understanding for the two sequence structures. Not only that, they also discovered another new classes of sequences with optimal autocorrelation magnitude and large linear complexity using binary $m$-sequences as the indictor sequences, which we call Yu-Gong sequences. However, the 2-adic complexity of this class of sequences has not been studied yet as far as we know.

In this paper, using the method of $\mathrm{Hu}[5$, we investigate the 2-adic complexity of a Yu-Gong sequence with an $m$-sequence as its indicator sequence, which is proved to be lower bounded by $N-2\left\lceil\log _{2} N\right\rceil+4\left(\gg \frac{N}{2}\right)$ and could attain the maximum value $N$ if suitable parameters are chosen, where $N$ is the period of the sequence.

The rest of the paper is organized as follows. Some notations and definitions are introduced in section 2 . We describe the generalized construction and the definition of a Yu-Gong sequence in Section 3. In Section 4, we point out a very interesting and useful law of the autocorrelation values of a Yu-Gong sequence. Using the method of Hu and the law of the autocorrelation distribution of a Yu-Gong sequence, we derive a lower bound on the 2-adic complexity of this sequence in Section 5.

## 2 Preliminaries

The following symbols will be used throughout the whole paper.
(1) $\mathbb{Z}_{N}$ is a ring of integers modulo $N$ and $\mathbb{Z}_{N}^{+}=\left\{t \in \mathbb{Z}_{N} \mid t \neq 0\right\}$.
(2) $\mathbb{F}_{q}$ is a finite field with $q$ elements.
(3) For positive integers $n$ and $m$ satisfying $m \mid n$, the trace function $\operatorname{Tr}_{m}^{n}(x)$ from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2^{m}}$ is defined by

$$
\operatorname{Tr}_{m}^{n}(x)=x+x^{2^{m}}+\cdots+x^{2^{m\left(\frac{n}{m}-1\right)}}, \quad x \in \mathbb{F}_{2^{n}}
$$

Let $\mathbf{s}=\left(s_{0}, s_{1}, \cdots, s_{N-1}\right)$ be a binary sequence of period $N$. Then the autocorrelation of $\mathbf{s}$ is given by

$$
\begin{equation*}
A C_{\mathbf{s}}(\tau)=\sum_{t=0}^{N-1}(-1)^{s_{t}+s_{t+\tau}}, \quad 0 \leq \tau \leq N-1 \tag{1}
\end{equation*}
$$

where $\tau$ is called a phase shift of the sequence $\mathbf{s}$ and $t+\tau$ is computed modulo $N$. The sequence $\mathbf{s}$ is called to have optimal autocorrelation if $A C_{\mathbf{s}}(\tau)$ satisfies the following:
(1) $A C_{\mathbf{s}}(\tau) \in\{N, 1,-3\}$ for $N \equiv 1(\bmod 4)$ or
(2) $A C_{\mathbf{s}}(\tau) \in\{N, 2,-2\}$ for $N \equiv 2(\bmod 4)$ or
(3) $A C_{\mathbf{s}}(\tau) \in\{N,-1\}$ for $N \equiv 3(\bmod 4)$ or
(4) $A C_{\mathbf{s}}(\tau) \in\{N, 0,-4\}$ or $\{N, 0,4\}$ for $N \equiv 0(\bmod 4)$
for all $\tau$ 's. Specially, the case (3) is called to have ideal two-level autocorrelation. Additionally, for $N \equiv 0(\bmod 4)$ and all $\tau$ 's, it is called to have perfect autocorrelation if $A C_{\mathbf{s}}(\tau) \in\{N, 0\}$ and optimal autocorrelation magnitude if $A C_{\mathbf{s}}(\tau) \in\{N, 0,4,-4\}$. So far, the sequence $(0,1,1,1)$ of period 4 is a uniquely known binary sequence with perfect autocorrelation in the sense of cyclic equivalence. Hence, it is often used to construct new sequences with good correlation and Yu-Gong sequence discussed in this paper is one of the applications.

$$
\begin{align*}
& \text { Denote } S(x)=\sum_{i=0}^{N-1} s(i) x^{i} \in \mathbb{Z}[x] \text { and suppose } \\
& \qquad \frac{S(2)}{2^{N}-1}=\frac{\sum_{i=0}^{N-1} s(i) 2^{i}}{2^{N}-1}=\frac{e}{f}, 0 \leq e \leq f, \operatorname{gcd}(e, f)=1 . \tag{2}
\end{align*}
$$

Then the integer $\left\lfloor\log _{2}(f+1)\right\rfloor$ is called the 2-adic complexity of the sequence $s$ and is denoted as $\Phi_{2}(s)$, i.e.,

$$
\begin{equation*}
\Phi_{2}(s)=\left\lfloor\log _{2}\left(\frac{2^{N}-1}{\operatorname{gcd}\left(2^{N}-1, S(2)\right)}+1\right)\right\rfloor \tag{3}
\end{equation*}
$$

where $\lfloor z\rfloor$ is the largest integer that is less than or equal to $z$.
It is well known that the 2 -adic complexity of a binary sequence $s$ with period $N$ should be larger than $\frac{N}{2}$ to resist the Rational Approximation Algorithm by Klapper et al. [6].

## 3 The interleaved structures of a binary $m$-sequence and Yu-Gong sequence

Construction 1 18]: Let each column of a $v \times w$ matrix $\mathbf{C}=\left(C_{i, j}\right)$ be given by $C(i, j)=c_{j}(i)$ and $\mathbf{c}_{j}=\left(c_{j}(0), c_{j}(1), \cdots, c_{j}(v-1)\right), 0 \leq j \leq w-1$, i.e., the matrix $\mathbf{C}$ can be expressed as

$$
\mathbf{C}=\left(\begin{array}{cccc}
c_{0}(0) & c_{1}(0) & \cdots & c_{w-1}(0)  \tag{4}\\
c_{0}(1) & c_{1}(1) & \cdots & c_{w-1}(1) \\
\vdots & \vdots & \ddots & \vdots \\
c_{0}(v-1) & c_{1}(v-1) & \cdots & c_{w-1}(v-1)
\end{array}\right)
$$

If each sequence $\mathbf{c}_{j}$ is either a cyclic shift of a binary sequence $\mathbf{a}=\left(a_{0}, a_{1}, \cdots, a_{v-1}\right)$ of period $v$ or a zero sequence and the sequence $\mathbf{u}=\left\{u_{t}\right\}$ is obtained by concatenating the successive rows of the above matrix $\mathbf{C}$, then $\mathbf{u}$ is called a $(v, w)$ interleaved sequence. By the definition, $\mathbf{c}_{j}=L^{e_{j}}(\mathbf{a}), 0 \leq j \leq w-1$, here $L^{e_{j}}$ is a cyclic $e_{j}$ left shift operation, $e_{j} \in \mathbb{Z}_{v}$ or $e_{j}=\infty$ if $\mathbf{c}_{j}$ is a zero sequence. Adding a binary sequence $\mathbf{b}=\left(b_{0}, b_{1}, \cdots, b_{w-1}\right)$ of period $w$ to the sequence $\mathbf{u}$, a new sequence $s$ will be produced, which is denoted $\mathbf{s}:=I(\mathbf{a}, \mathbf{e})+\mathbf{b}$, where $\mathbf{u}:=I(\mathbf{a}, \mathbf{e})$ and $e=\left(e_{0}, e_{1}, \cdots, e_{w-1}\right)$, and it still preserves the $(v, w)$ interleaved structure. We call $\mathbf{a}, \mathbf{e}$ and $\mathbf{b}$ the base, the shift and the indicator sequences of $\mathbf{s}$, respectively.

For a positive integer $k>1$, let $\mathbf{b}$ be a binary $m$-sequence of period $2^{2 k}-1$, i.e., $w=2^{2 k}-1$, where $b_{t}=\operatorname{Tr}_{1}^{2 k}\left(\alpha^{t}\right)$ and $\alpha$ is a primitive element of the finite field $\mathbb{F}_{2^{2 k}}, 0 \leq t \leq 2^{2 k}-2$. It is well known that $\mathbf{b}$ can be expressed as a $\left(2^{k}-1,2^{k}+1\right)$ interleaved sequence [3], i.e., $\mathbf{b}=I\left(\mathbf{a}^{\prime}, \mathbf{e}^{\prime}\right)$, where the base sequence $\mathbf{a}^{\prime}=\left(a_{0}^{\prime}, a_{1}^{\prime}, \cdots, a_{2^{k}-2}^{\prime}\right)$ is a binary $m$-sequence of period $2^{k}-1$ defined by $a_{i}^{\prime}=\operatorname{Tr}_{1}^{k}\left(\beta^{i}\right), 0 \leq i \leq 2^{k}-2, \beta=\alpha^{2^{k}+1}$ is a primitive element of $\mathbb{F}_{2^{k}}$, and the shift sequence $\mathbf{e}^{\prime}=\left(e_{0}^{\prime}, e_{1}^{\prime}, \cdots, e_{2^{k}}^{\prime}\right)$ is given by $e_{0}^{\prime}=\infty$ and $\beta^{e_{j}^{\prime}}=\operatorname{Tr}_{k}^{2 k}\left(\alpha^{j}\right)$ for $1 \leq j \leq 2^{k}$.

A Yu-Gong sequence $\mathbf{s}=I(\mathbf{a}, \mathbf{e})+\mathbf{b}$ of period $N=4\left(2^{2 k}-1\right)$, is given by a $4 \times\left(2^{2 k}-1\right)$ interleaved structure, where:
(1) $\mathbf{a}=(0,1,1,1)$ is the perfect binary sequence;
(2) $\mathbf{b}$ is the binary $m$-sequence defined as above;
(3) $\mathbf{e}$ is a sequence over $\mathbb{Z}_{4}$ represented as a $\left(2^{k}-1\right) \times\left(2^{k}+1\right)$ interleaved structure by a matrix $\mathbf{E}=\left(e_{i, j}\right)$, where

$$
e_{i, j}=\left\{\begin{array}{l}
3 i+\delta(\bmod 4), \quad \text { if } j=0  \tag{5}\\
3(i+j)(\bmod 4), \text { if } 1 \leq j \leq 2^{k}
\end{array}\right.
$$

for $0 \leq i \leq 2^{k}-2$ and $\delta=1$ or -1 .

## 4 An interesting and useful law of the autocorrelation distribution of the Yu-Gong sequence

In order to analyze the 2-adic complexity of the Yu-Gong sequence s, we need the exact order according to the value $\tau$ of the autocorrelation value $A C_{\mathbf{s}}(\tau)$ of $\mathbf{s}$, which can be given by the following two results.
Theorem 1 [18] Let $\mathbf{s}=I(\mathbf{a}, \mathbf{e})+\mathbf{b}$ be a Yu-Gong sequence of period $N=$ $4\left(2^{2 k}-1\right), k>1$. Then, it has the four-valued optimal autocorrelation of $A C_{\mathbf{s}}(\tau) \in\{N, 0, \pm 4\}$ for any $\tau$. Precisely, its complete autocorrelation is given by

$$
A C_{\mathbf{s}}(\tau)= \begin{cases}N, & \text { if } \tau=0  \tag{6}\\ 0, & \text { if }(\tau \neq 0 \text { and } x=0) \\ & \text { or }(x, y, v)=(\sigma, 0, v) \\ & \text { or }(y, v)=(\psi, 2) \\ -4, & \text { if }(x, y, v)=(\sigma, 0,0) \\ & \text { or }(y, v)=(\psi, 1) \\ & \text { or }(y, v)=(\psi, 3), \\ +4, & \text { if }(y, v)=(\psi, 0),\end{cases}
$$

where $x \equiv \tau\left(\bmod 2^{2 k}-1\right), y \equiv \tau\left(\bmod 2^{k}+1\right)$, and $v \equiv \tau(\bmod 4)$. Also, $\sigma \in \mathbb{Z}_{2^{2 k}-1}^{+}, \psi \in \mathbb{Z}_{2^{k}+1}^{+}, v \in \mathbb{Z}_{4}^{+}$, and $\mathbb{Z}_{h}^{+}=\{1,2, \cdots, h-1\}$ for a positive integer $h$.

Now, the order of the autocorrelation value of $\mathbf{s}$ can be described as follows.
Corollary 1 Let the symbols be the same as those in Theorem 1. Then the following results hold:
(1) Suppose $1 \leq \tau_{1}, \tau_{2} \leq N-1$ and $\tau_{1} \equiv \tau_{2}\left(\bmod 4\left(2^{k}+1\right)\right)$. Then the autocorrelation function of the sequence $\mathbf{s}$ satisfies $A C_{\mathbf{s}}\left(\tau_{1}\right)=A C_{\mathbf{s}}\left(\tau_{2}\right)$. Particularly, $A C_{\mathbf{s}}\left(4\left(2^{k}+1\right) i\right)=-4, i=1,2, \cdots, 2^{k}-2$;
(2) For $1 \leq \tau \leq 4\left(2^{k}+1\right)$, if we divide the set of the autocorrelation values $\left\{A C_{\mathbf{s}}(\tau)\right\}_{\tau=1}^{4\left(2^{k}+1\right)}$ of $\mathbf{s}$ into $2^{k}+1$ subsets $S_{j} ' s, j=1,2, \cdots, 2^{k}+1$, according to the order of $\tau$ and each subset contains four elements, i.e., $S_{j}=$ $\left\{A C_{\mathbf{s}}(4(j-1)+1), A C_{\mathbf{s}}(4(j-1)+2), A C_{\mathbf{s}}(4(j-1)+3), A C_{\mathbf{s}}(4(j-1)+4)\right\}$, Then

$$
\begin{array}{r}
S_{1}=S_{2}=\cdots=S_{2^{k-2}}=\{-4,0,-4,4\} ; \\
S_{2^{k-2}+1}=\{0,0,-4,4\} ; \\
S_{2^{k-2}+2}=S_{2^{k-2}+3}=\cdots=S_{3 \times 2^{k-2}}=\{-4,0,-4,4\} ; \\
S_{3 \times 2^{k-2}+1}=\{-4,0,0,4\} ; \\
S_{3 \times 2^{k-2}+2}=S_{3 \times 2^{k-2}+3}=\cdots=S_{2^{k}}=\{-4,0,-4,4\} ; \\
S_{2^{k}+1}=\{-4,0,-4,-4\} . \tag{12}
\end{array}
$$

It should be pointed out that there are $2^{k-2}$ sets in Eq. (7), $2^{k-1}-1$ sets in Eq. (9), and $2^{k-2}-1$ sets in Eq. (11).
Table 1: The autocorrelation of Yu-Gong sequence for $k=2$

| $\tau$ | $\operatorname{AC}_{\mathbf{S}}(\tau)$ |
| :---: | :---: |
| $1-20\left(S_{1}-S_{5}\right)$ | $-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,-4 ;$ |
| $21-40$ | $-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,-4 ;$ |
| $41-59$ | $-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4$ |


| $\tau$ | $\mathrm{AC}_{\mathbf{S}}(\tau)$ |
| :---: | :---: |
| $1-36\left(S_{1}-S_{9}\right)$ | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| $37-72$ | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 73-108 | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 109-144 | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| $145-180$ | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 181-216 | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 217-251 | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4 ; \end{aligned}$ |

Proof Note that $\tau_{1} \equiv \tau_{2}\left(\bmod 4\left(2^{k}+1\right)\right)$ implies $\tau_{1} \equiv \tau_{2}\left(\bmod 2^{k}+1\right)$ and $\tau_{1} \equiv \tau_{2}(\bmod 4)$. Then, from Theorem 1 , we find that $\tau_{1}$ and $\tau_{2}$ induce the same pair $(y, v)$, which leads to $A C_{\mathbf{s}}\left(\tau_{1}\right)=A C_{\mathbf{s}}\left(\tau_{2}\right)$. Other results can also be directly verified by Theorem 1 .

Example 1 By direct computation using Matlab programs, the autocorrelation distributions of Yu-Gong sequences for $k=2,3,4$ have been listed in Tables 1-3. And we have marked these three special sets $S_{2^{k-2}+1}, S_{3 \times 2^{k-2}+1}, S_{2^{k}+1}$ in red. Especially, the number of sets in Eq. (11) is 0 since $2^{k-2}-1=0$ for $k=2$ and the autocorrelation distribution is the set $\{-4,0,0,4,-4,0,-4,4,0,0,-4\}$ for $k=1$ and $\tau=1,2,3,4,5,6,7,8,9,10,11$.

## 5 The 2-adic complexity of Yu-Gong sequence

In order to derive a lower bound on the 2-adic complexity of the Yu-Gong sequence s, we need employ the method of Hu 5. It can be described as the following Lemma 1 which have also been used in several other references 4. 10, 11, 15.

Lemma 1 [5] Let $\mathbf{s}=\left(s_{0}, s_{1}, \cdots, s_{N-1}\right)$ be a binary sequence of period $N$, $S(x)=\sum_{i=0}^{N-1} s_{i} x^{i} \in \mathbb{Z}[x]$ and $T(x)=\sum_{i=0}^{N-1}(-1)^{s_{i}} x^{i} \in \mathbb{Z}[x]$. Then
$-2 S(x) T\left(x^{-1}\right) \equiv N+\sum_{\tau=1}^{N-1} A C_{\mathbf{s}}(\tau) x^{\tau}-T\left(x^{-1}\right)\left(\sum_{i=0}^{N-1} x^{i}\right) \quad\left(\bmod x^{N}-1\right)$.
The following Lemma 2 is also important to prove our main result.
Lemma 2 Let $k$ be a positive integer. Then the following results hold:

| $\tau$ | $\mathrm{AC}_{\mathbf{S}}(\tau)$ |
| :---: | :---: |
| $1-68\left(S_{1}-S_{17}\right)$ | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 69-136 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 137-204 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 205-272 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| $273-340$ | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| $341-408$ | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 409-476 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 477 - 544 | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| $545-612$ | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 613-680 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 681-748 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| $749-816$ | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| $817-884$ | $\begin{aligned} & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ; \\ & -4,0,-4,4 ;-4,0,-4,-4 ; \end{aligned}$ |
| 885-952 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,-4 ;$ |
| 953-1019 | $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ; 0,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4,4 ;-4,0,0,4 ;-4,0,-4,4 ;-4,0,-4,4 ;$ $-4,0,-4,4 ;-4,0,-4 \quad ;$ |

(1) For $k \equiv 2 \bmod 4$, we have

$$
\begin{equation*}
5 \left\lvert\, \operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right) .\right. \tag{13}
\end{equation*}
$$

But for $k \equiv 0(\bmod 4)$, we have

$$
\begin{equation*}
5 \nmid \operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right) . \tag{14}
\end{equation*}
$$

(2) $\operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right) \begin{cases}=1, & \text { if } 2^{2 k-1}-2^{k}+1 \text { is a } \\ <2^{2 k-1}, & \text { prime for } k \equiv 0(\bmod 4), \\ <\text { otherwise } .\end{cases}$
(3) $\operatorname{gcd}\left(2^{k-1}-1,2^{2^{k}+1}+1\right) \begin{cases}=1, & \text { if } k \text { is even, } \\ <2^{k-1}, & \text { otherwise. }\end{cases}$

Proof (1) Suppose $k=4 t+2$. Note that the multiplicative order of 2 modular 5 is 4 . On one hand, we have

$$
2^{2 k-1}-2^{k}+1=2^{8 t+3}-2^{4 t+2}+1 \equiv 2^{3}-2^{2}+1 \equiv 0(\bmod 5)
$$

i.e.,

$$
\begin{equation*}
5 \mid\left(2^{2 k-1}-2^{k}+1\right) \tag{15}
\end{equation*}
$$

on the other hand, since $5 \mid\left(2^{4 t}-1\right)$, we get $20 \mid\left[4\left(2^{4 t}-1\right)\right]$, i.e., $20 \mid\left(2^{k}-4\right)$, which implies $2^{k}+1 \equiv 5(\bmod 20)$ and $2\left(2^{k}+1\right) \equiv 10(\bmod 20)$. It is easy to know that the multiplicative order of 2 modular 25 is 20 . Then we have $2^{2\left(2^{k}+1\right)}+1 \equiv 2^{10}+1 \equiv 0(\bmod 25)$, i.e.,

$$
\begin{equation*}
5 \left\lvert\, \frac{2^{2\left(2^{k}+1\right)}+1}{5} .\right. \tag{16}
\end{equation*}
$$

Combining (15) and (16) we know that (13) holds. But, if $k=4 t$, we have $2^{2 k-1}-2^{k}+1 \equiv 2^{3}-1+1 \equiv 3(\bmod 5)$. Therefore, (14) holds.
(2) Suppose that $2^{2 k-1}-2^{k}+1$ is a prime. Then by little Fermat Theorem we have

$$
\begin{equation*}
\left(2^{2 k-1}-2^{k}+1\right) \mid\left(2^{2^{2 k-1}-2^{k}}-1\right) \tag{17}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\frac{2^{2\left(2^{k}+1\right)}+1}{5}\left|\left(2^{2\left(2^{k}+1\right)}+1\right),\left(2^{2\left(2^{k}+1\right)}+1\right)\right|\left(2^{4\left(2^{k}+1\right)}-1\right) . \tag{18}
\end{equation*}
$$

Furthermore, we have

$$
\begin{align*}
\operatorname{gcd}\left(2^{2^{2 k-1}-2^{k}}-1,2^{4\left(2^{k}+1\right)}-1\right) & =2^{\operatorname{gcd}\left(2^{2 k-1}-2^{k}, 4\left(2^{k}+1\right)\right)}-1 \\
& =2^{\operatorname{gcd}\left(2^{k}\left(2^{k-1}-1\right), 4\left(2^{k}+1\right)\right)}-1 \tag{19}
\end{align*}
$$

Without loss of generality, let $k \geq 2$ (for $k=1$, the conclusion is trivial). Then

$$
\begin{align*}
\operatorname{gcd}\left(2^{k}\left(2^{k-1}-1\right), 4\left(2^{k}+1\right)\right) & =4 \operatorname{gcd}\left(2^{k-1}-1,2^{k}+1\right) \\
& =4 \operatorname{gcd}\left(2^{k-1}-1,3\right)  \tag{20}\\
& = \begin{cases}4, & \text { if } k \text { is even }, \\
12, & \text { otherwise },\end{cases} \tag{21}
\end{align*}
$$

where (20) holds because $2^{k}+1=2\left(2^{k-1}-1\right)+3$. Thus, by (19), (21), we get

$$
\operatorname{gcd}\left(2^{2^{2 k-1}-2^{k}}-1,2^{4\left(2^{k}+1\right)}-1\right)=\left\{\begin{array}{l}
2^{4}-1, \text { if } k \text { is even },  \tag{22}\\
2^{12}-1, \text { otherwise },
\end{array}\right.
$$

We know that

$$
\begin{gather*}
2^{4\left(2^{k}+1\right)}-1=\left(2^{2\left(2^{k}+1\right)}-1\right)\left(2^{2\left(2^{k}+1\right)}+1\right),  \tag{23}\\
\operatorname{gcd}\left(2^{2\left(2^{k}+1\right)}-1,2^{2\left(2^{k}+1\right)}+1\right)=1,  \tag{24}\\
2^{4}-1=\left(2^{2}-1\right)\left(2^{2}+1\right)=3 \times 5,  \tag{25}\\
3 \mid\left(2^{2\left(2^{k}+1\right)}-1\right), 3 \nmid\left(2^{2\left(2^{k}+1\right)}+1\right),  \tag{26}\\
5 \mid\left(2^{2\left(2^{k}+1\right)}+1\right) \tag{27}
\end{gather*}
$$

Therefore, by (22)-(27), we have

$$
\begin{equation*}
\operatorname{gcd}\left(2^{2^{2 k-1}-2^{k}}-1,2^{2\left(2^{k}+1\right)}+1\right)=5 \text { for an even } k \tag{28}
\end{equation*}
$$

Finally, combining (13)-(14), (17)-(18) and (28), the result follows.
(3) It is easy to known that

$$
\begin{align*}
& 2^{2\left(2^{k}+1\right)}-1=\left(2^{2^{k}+1}+1\right)\left(2^{2^{k}+1}-1\right) \\
& \operatorname{gcd}\left(2^{2^{k}+1}+1,2^{2^{k}+1}-1\right)=1 \\
& \operatorname{gcd}\left(2^{k-1}-1,2^{2\left(2^{k}+1\right)}-1\right) \\
& =\operatorname{gcd}\left(2^{k-1}-1,2^{2^{k}+1}+1\right) \times \operatorname{gcd}\left(2^{k-1}-1,2^{2^{k}+1}-1\right),  \tag{29}\\
& =2^{\operatorname{gcd}\left(k-1,2\left(2^{k}+1\right)\right.}-1  \tag{30}\\
& \left.\operatorname{gcd}\left(2^{k-1}-1,2^{2^{k}+1}-1\right)=2^{\operatorname{gcd}\left(k-1,2^{k}+1\right.}\right)-1 \tag{31}
\end{align*}
$$

For an even $k, k-1$ is odd, then

$$
\operatorname{gcd}\left(k-1,2\left(2^{k}+1\right)\right)=\operatorname{gcd}\left(k-1,2^{k}+1\right),
$$

which results in

$$
\begin{equation*}
\operatorname{gcd}\left(2^{k-1}-1,2^{2\left(2^{k}+1\right)}-1\right)=\operatorname{gcd}\left(2^{k-1}-1,2^{2^{k}+1}-1\right) \tag{32}
\end{equation*}
$$

by (30)-(32). Furthermore, combining (29) and (32), we can get the desired result.

Example 2 By direct computation using Mathematica programs, we find that the smallest two positive integers $k$ 's such that $2^{2 k-1}-2^{k}+1$ are primes and $k \equiv 0(\bmod 4)$ are 4,24 .

Now, we present our main result.

Theorem 2 Let $\mathbf{s}=I(\mathbf{a}, \mathbf{e})+\mathbf{b}$ be the $Y u$-Gong sequence of period $N=$ $4\left(2^{2 k}-1\right), k>1$, introduced in Section 3. Then the 2-adic complexity $\Phi_{2}(\mathbf{s})$ of $\mathbf{s}$ satisfies the following lower bound
$\Phi_{2}(\mathbf{s}) \begin{cases}=N, & \text { if } k \equiv 0(\bmod 4) \text { and }\left(2^{2 k-1}-2^{k}+1\right) \text { is a prime, } \\ >N-\log _{2} N+1, & \text { if } k \text { is even, } \\ >N-2 \log _{2} N+4, & \text { otherwise, }\end{cases}$ i.e., the 2-adic complexity of $\mathbf{s}$ far outweight one half of the period.

Proof Above all, by Lemma 1. we know that

$$
S(2) T\left(2^{-1}\right) \equiv-\frac{1}{2} \sum_{\tau=1}^{4\left(2^{2 k}-1\right)-1} A C_{\mathbf{s}}(\tau) 2^{\tau}-2\left(2^{2 k}-1\right)\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right)
$$

From Corollary (1), if we add the value -4 to the end of the sequence $\left\{A C_{\mathbf{s}}(\tau)\right\}_{\tau=1}^{4\left(2^{2 k}-1\right)}$, we will get a sequence segment consisting $2^{k}-1$ periods of the sequence $\left\{A C_{\mathbf{s}}(\tau)\right\}_{\tau=1}^{4\left(2^{k}+1\right)}$. Therefore, we have

$$
\begin{align*}
S(2) T\left(2^{-1}\right) \equiv & -\frac{1}{2}\left\{\sum_{\tau=1}^{4\left(2^{2 k}-1\right)-1} A C_{s}(\tau) 2^{\tau}+(-4) \times 2^{4\left(2^{2 k}-1\right)}-(-4) \times 2^{4\left(2^{2 k}-1\right)}\right\} \\
& -2\left(2^{2 k}-1\right)\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right) \\
= & -\frac{1}{2}\left\{\left(\sum_{i=0}^{2^{k}-2} 2^{4\left(2^{k}+1\right) i}\right)\left(\sum_{\tau=1}^{4\left(2^{k}+1\right)} A C_{\mathbf{s}}(\tau) 2^{\tau}\right)+4 \times 2^{4\left(2^{2 k}-1\right)}\right\} \\
& -2\left(2^{2 k}-1\right)\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right) \\
\equiv & -\frac{1}{2}\left\{\left(\frac{2^{4\left(2^{2 k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1}\right)\left(\sum_{\tau=1}^{4\left(2^{k}+1\right)} A C_{s}(\tau) 2^{\tau}\right)+4\right\} \\
& -2\left(2^{2 k}-1\right)\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right) \\
= & -\frac{1}{2}\left(\frac{2^{4\left(2^{2 k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1}\right)\left(\sum_{\tau=1}^{4\left(2^{k}+1\right)} A C_{s}(\tau) 2^{\tau}\right) \\
& -2^{2 k+1}\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right) \tag{33}
\end{align*}
$$

From Corollary 1 (2), we have

$$
\begin{aligned}
& \sum_{\tau=1}^{4\left(2^{k}+1\right)} A C_{s}(\tau) 2^{\tau}=\left(\sum_{i=0}^{2^{k}} 2^{4 i}\right)\left((-4) \times 2+0 \times 2^{2}+(-4) \times 2^{3}+4 \times 2^{4}\right) \\
& -2^{4 \times 2^{k-2}}\left((-4) \times 2+0 \times 2^{2}+(-4) \times 2^{3}+4 \times 2^{4}\right)
\end{aligned}
$$

$$
\begin{align*}
& +2^{4 \times 2^{k-2}}\left(0 \times 2+0 \times 2^{2}+(-4) \times 2^{3}+4 \times 2^{4}\right) \\
& -2^{4 \times 3 \times 2^{k-2}}\left((-4) \times 2+0 \times 2^{2}+(-4) \times 2^{3}+4 \times 2^{4}\right) \\
& +2^{4 \times 3 \times 2^{k-2}}\left((-4) \times 2+0 \times 2^{2}+0 \times 2^{3}+4 \times 2^{4}\right) \\
& -2^{4 \times 2^{k}}\left((-4) \times 2+0 \times 2^{2}+(-4) \times 2^{3}+4 \times 2^{4}\right) \\
& +2^{4 \times 2^{k}}\left((-4) \times 2+0 \times 2^{2}+(-4) \times 2^{3}+(-4) \times 2^{4}\right) \\
& =4 \times 6 \times \frac{6\left(2^{4\left(2^{k}+1\right)}-1\right)}{2^{4}-1}+4 \times 2^{2^{k}+1}+4 \times 2^{3 \times\left(2^{k}+1\right)}-8 \times 2^{4 \times\left(2^{k}+1\right)} \\
& =8\left\{\frac{3\left(2^{4\left(2^{k}+1\right)}-1\right)}{2^{4}-1}+2^{2^{k}}+2^{3 \times 2^{k}+2}-2^{4 \times\left(2^{k}+1\right)}\right\} \tag{34}
\end{align*}
$$

Bringing (34) into (33) and simplifying it, we get

$$
\begin{align*}
S(2) T\left(2^{-1}\right) \equiv-4\{ & \frac{2^{4\left(2^{2 k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1}\left[\frac{2^{4\left(2^{k}+1\right)}-1}{5}+2^{2^{k}}+2^{3 \times 2^{k}+2}\right. \\
& \left.\left.-2^{4 \times\left(2^{k}+1\right)}\right]+2^{2 k-1}\right\}\left(\bmod 2^{4\left(2^{2 k}-1\right)}-1\right) . \tag{35}
\end{align*}
$$

On one hand, it is obvious that $2^{4}-1=3 \times 5$ is a factor of $2^{4\left(2^{2 k}-1\right)}-1$, on the other hand, we can derive

$$
S(2) T\left(2^{-1}\right) \equiv \begin{cases}13(\bmod 15), & \text { if } k \equiv 0(\bmod 4),  \tag{36}\\ 0(\bmod 15), & \text { if } k \equiv 1(\bmod 4), \\ 10(\bmod 15), & \text { if } k \equiv 2(\bmod 4), \\ 9(\bmod 15), & \text { if } k \equiv 3(\bmod 4)\end{cases}
$$

from (35) by direct calculation. This tells us that

$$
\begin{equation*}
\operatorname{gcd}\left(S(2) T\left(2^{-1}\right), 2^{4\left(2^{2 k}-1\right)}-1\right)>1 \quad \text { for } k \equiv 1,2,3(\bmod 4) \tag{37}
\end{equation*}
$$

In order to obtain a more exact lower bound on the 2-adic complexity of YuGong sequence, we need to give some more detailed computation. Again from (35), we get

$$
\begin{align*}
& S(2) T\left(2^{-1}\right) \equiv-2^{2 k+1}\left(\bmod \frac{2^{4\left(2^{2 k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1}\right),  \tag{38}\\
& S(2) T\left(2^{-1}\right) \equiv-4\left\{\left(2^{k}-1\right)\left[\frac{2^{4\left(2^{k}+1\right)}-1}{5}+2^{2^{k}}+2^{3 \times 2^{k}+2}-1\right]\right. \\
& \left.\quad+2^{2 k-1}\right\}\left(\bmod 2^{4\left(2^{k}+1\right)}-1\right), \tag{39}
\end{align*}
$$

where (39) comes from the following congruence

$$
\frac{2^{4\left(2^{2 k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1}=\frac{2^{4\left(2^{k}+1\right)\left(2^{k}-1\right)}-1}{2^{4\left(2^{k}+1\right)}-1} \equiv 2^{k}-1\left(\bmod 2^{4\left(2^{k}+1\right)}-1\right)
$$

Furthermore, since $2^{4\left(2^{k}+1\right)}-1=5 \times \frac{2^{2\left(2^{k}+1\right)}+1}{5} \times\left(2^{2^{k}+1}+1\right) \times\left(2^{2^{k}+1}-1\right)$, then by (39) we have

$$
\begin{align*}
& S(2) T\left(2^{-1}\right) \equiv-4\left(2^{2 k-1}-2^{k}+1\right)\left(\bmod \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right)  \tag{40}\\
& S(2) T\left(2^{-1}\right) \equiv-8\left(2^{k-1}-1\right)^{2}\left(\bmod 2^{2^{k}+1}+1\right)  \tag{41}\\
& S(2) T\left(2^{-1}\right) \equiv-2^{2 k+1}\left(\bmod 2^{2^{k}+1}-1\right) \tag{42}
\end{align*}
$$

Combining the results in Lemma 2 the proof is finished.

Example 3 To ensure the correctness of our main result, at the same time, in order to compare the actual values with the lower bounds of the 2 -adic complexity of Yu-Gong sequences obtained in this paper, we have done the following verification work by combining Matlab and Mathematica programs:
(1) For $k=1,2,3,4,5,6,7,8$, the correctness of the congruences (40)-(42) have been verified using the direct definitions of Yu-Gong sequences and the mathematical expression $S(2) T\left(2^{-1}\right)$.
(2) For $k=1,2,3,4,5,6,7,8$, the actual values of the 2-adic complexity of YuGong sequences have been determined by determining the corresponding $\operatorname{gcd}\left(S(2), 2^{N}-1\right)$ in the definition of 2-adic complexity of binary sequences. And we list a table to compare the actual values and the lower bounds of the 2 -adic complexity of Yu-Gong sequences (Please see Table 4).
(3) From the process of determining the lower bound of the 2-adic complexity of Yu-Gong sequence, the value of $\operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right)$ is a key factor, which especially affects the cases of the maximal values of the 2 -adic complexity. In Lemma 2, we proved $\operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right)=1$ only when $k \equiv 0 \bmod 4$ and $2^{2 k-1}-2^{k}+1$ is a prime. In fact, we find

$$
\begin{equation*}
\operatorname{gcd}\left(2^{2 k-1}-2^{k}+1, \frac{2^{2\left(2^{k}+1\right)}+1}{5}\right)=1 \tag{43}
\end{equation*}
$$

for each $k \in\{4,8,12,16,20,24,28,32\}$ by Mathematica programs and we can not determine it for the cases of $k \equiv 0 \bmod 4$ and $k \geq 36$ because of the limitation of computer performance. So we guess that (43) maybe hold for all $k \equiv 0(\bmod 4)$. But it is difficult for us within our capabilities to prove it now. We also sincerely invite interested readers to complete it.

Table 4: A comparison between the actual values and
the lower bounds of the 2-adic complexity of Yu-Gong sequences

| the lower bounds of the 2-adic complexity of Yu-Gong sequences |  |  |  |
| :---: | :---: | :---: | :---: |
| The value of $k$ | The period $N$ of <br> Yu-Gong sequence | The actual value of <br> the 2-adic complexity <br> of Yu-Gong sequence | The lower bound of <br> the 2-adic complexity <br> of Yu-Gong sequence <br> obtained in this paper |
| 1 | 12 | 8 | 6 |
| 2 | 60 | 60 | 55 |
| 3 | 252 | 250 | 240 |
| 4 | 1020 | 1020 | 1020 |
| 5 | 4092 | 4082 | 4072 |
| 6 | 16380 | 16380 | 16367 |
| 7 | 65532 | 65530 | 65504 |
| 8 | 262140 | 262140 | 262123 |

## References

1. Arasu, K. T., Ding, C., Helleseth, T., Kumar, P. V., Martinsen, H. M.: Almost difference sets and their sequences with optimal autocorrelation. IEEE Trans. Inform. Theory, 47(7), 2934-2943 (2001).
2. Fan, C.: The linear complexity of a class of binary sequences with optimal autocorrelation. Designs, Codes and Cryptography, 86, 2441-2450 (2018).
3. Gong, G.: Theory and applications of $q$-ary interleaved sequences. IEEE Trans. Inform. Theory, 41(2), 400-411 (1995).
4. Hofer, R., Winterhof, A.: On the 2-adic complexity of the two-prime generator. IEEE Trans. Inf. Theory, 64(8), 5957-5960 (2018).
5 . Hu, H.: Comments on "a new method to compute the 2 -adic complexity of binary sequences". IEEE Trans. Inform. Theory, 60(9), 5803-5804 (2014).
5. Klapper, A., Goresky, M.: Feedback shift registers, 2-adic span, and combiners with memory. Journal of Cryptology, 10, 111-147 (1997).
6. Li, N., Tang, X.: On the linear complexity of binary sequences of period $4 N$ with optimal autocorrelation/magnitude. IEEE Trans. Inform. Theory, 57(11), 7597-7604 (2011).
7. Lüke, H. D.: Sequences and arrays with perfect periodic correlation. IEEE Trans. Aerosp. Electron. Syst., 24(3), 287-294 (1988).
8. Su, W., Yang, Y., Fan, C.: New optimal binary sequences with period $4 p$ via interleaving Ding-Helleseth-Lam sequences. Designs, Codes and Cryptography, 86, 1329-1338 (2018).
9. Sun, Y., Wang, Q., Yan, T.: The 2-adic complexity of a class of binary sequences with optimal autocorrelation magnitude. Cryptography and Communications, 12(4), 675683(2020).
10. Sun, Y., Wang, Q., Yan, T.: The exact autocorrelation distribution and 2-adic complexity of a class of binary sequences with almost optimal autocorrelation. Cryptography and Communications, $10(3)$, 467-477 (2018).
11. Tang, X., Ding, C.: New classes of balanced quaternary and almost balanced binary sequences with optimal autocorrelation value. IEEE Trans. Inform. Theory, 56(12), 63986405 (2010).
12. Tang, X., Gong, G.: New constructions of binary sequences with optimal autocorrelation value/magnitude. IEEE Trans. Inform. Theory, 56(3), 1278-1286 (2010).
13. Xiong, H., Qu, L., Li, C.: A new method to compute the 2-adic complexity of binary sequences. IEEE Trans. Inform. Theory, 60(4), 2399-2406 (2014).
14. Xiong, H., Qu, L., Li, C.: 2-Adic complexity of binary sequences with interleaved structure. Finite Fields and Their Applications, 33, 14-28 (2015).
15. Yan, T., Chen, Z., Li, B.: A general construction of binary interleaved sequences of period $4 N$ with optimal autocorrelation. Information Sciences, 287, 26-31 (2014).
16. Yang, M., Zhang, L., Feng,K.: On the 2-adic complexity of a class of binary sequences of period $4 p$ with optimal autocorrelation magnitude. IEEE International Symposium on Information Theory (2020).
17. Yu, N. Y., Gong, G.: New binary sequences with optimal autocorrelation magnitude. IEEE Trans. Inform. Theory, 54(10), 4771-4779 (2008).

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