New families of self-dual codes

Lin Sok*

Abstract

In the recent paper entitled "Explicit constructions of MDS self-dual codes" accepted in IEEE Transactions on Information Theory, doi: 10.1109/TIT.2019.2954877, the author has constructed families of MDS self-dual codes from genus zero algebraic geometry (AG) codes, where the AG codes of length n were defined using two divisors G and $D = P_1 + \cdots + P_n$. In the present correspondence, we explore more families of optimal self-dual codes from AG codes. New families of MDS self-dual codes with odd characteristics and those of almost MDS self-dual codes are constructed explicitly from genus zero and genus one curves, respectively. More families of self-dual codes are constructed from algebraic curves of higher genus.

Keywords: Self-orthogonal codes, self-dual codes, MDS codes, almost MDS codes, optimal codes, algebraic curves, algebraic geometry codes, differential algebraic geometry codes

1 Introduction

Self-dual codes are one of the most interesting classes of linear codes that find diverse applications in cryptographic protocols (secret sharing schemes) introduced in [4, 5, 18] and combinatorics [17]. It is well-known that binary self-dual codes are asymptotically good [16].

MDS codes form an optimal family of classical codes. They are closely related to combinatorial designs [17, p. 328], and finite geometries [17, p.

^{*}School of Mathematical Sciences, Anhui University, Hefei, Anhui, 230601, soklin_heng@yahoo.com

326]. Due to their largest error correcting capability for given length and dimension, MDS codes are of great interest in both theory and practice. The most well-known family of MDS linear codes is that of Reed-Solomon codes. MDS linear codes exist in a very restrict condition on their lengths as the famous MDS conjecture states: for every linear [n, k, n - k + 1] MDS code over \mathbb{F}_q , if 1 < k < q, then $n \le q + 1$, except when q is even and k = 3 or k = q - 1, in which cases $n \le q + 2$. The conjecture was proved by Ball [1] for q a prime. However, for self-dual case, the conjecture may not be true.

Due to the reasons mentioned above, MDS self-dual codes have been of much interest to many researchers. As we have already known that determining the parameters of a given linear code is a challenging problem in coding theory. However, the parameters of an MDS self-dual code are completely determined by its length. Constructions of MDS self-dual codes are valuable. For classical constructions of MDS self-dual codes, we refer to [2, 8, 14, 11]. Existing families of MDS self-dual codes can be described as follows. Grassl et al. [10] constructed MDS codes of all lengths over \mathbb{F}_{2^m} and of all highest possible length over finite fields of odd characteristics. Jin et al. [13] proved the existence of MDS self-dual codes over \mathbb{F}_q in odd characteristic for $q \equiv 1 \pmod{4}$ and for q a square of a prime for some restricted lengths. Using the same technique developed in [13], more families of MDS self-dual codes have been constructed in [24, 7]. Tong et al. [23] gave constructions of MDS Euclidean self-dual codes through cyclic duadic codes. The families of known MDS self-dual codes are summarized in Table 1.

The discovery of algebraic geometry codes in 1981 was due to Goppa [9], where they were also called geometric Goppa codes. Goppa showed in his paper [9] how to construct linear codes from algebraic curves over a finite field. Despite a strongly theoritical construction, algebraic geometry (AG) codes have asymptotically good parameters, and it was the first time that linear codes improved the so-called Gilbert-Vasharmov bound. Self-dual AG codes were studied by Stichtenoth [20] and Driencourt et al. [6], where they first characterized such codes. However, the construction of MDS self-dual AG codes with odd characteristics or almost MDS self-dual AG codes was not considered there.

On the contrary to the MDS case, almost MDS codes exist more frequently, and it is thus worth exploring families of self-dual codes in such a case and those of optimal self-dual codes. In [20], Stichtenoth gave constructions of self-orthogonal AG codes (and self-dual AG codes for some special cases) but did not consider an embedding the self-orthogonal codes into the

Table 1: Existing families of MDS self-dual codes, $\eta\colon$ the quadratic character of \mathbb{F}_q

q	n	References
$q = 2^m$	$n \le q$	
$\frac{q-2}{q=p^m, p \text{ odd prime}}$ $q=r^2$	= -	[10]
$q = p$, p odd prime $q = r^2$	$n = q + 1$ $n \le r$	[10]
$q = r^2, r \equiv 3 \pmod{4}$	$n = 2tr, t \le (r-2)/2$ $n \equiv 3 \pmod{4}, (n-1) (q-1)$	[13]
$q \equiv 3 \pmod{4}$	$n \equiv 3 \pmod{4}, (n-1) (q-1)$	
$q \equiv 1 \pmod{4}$	$\frac{(n-1) (q-1)}{n (q-1), n < q-1}$	[23]
$q \equiv 1 \pmod{4}$	1(1)/ 1	
q odd	$(n-1) (q-1), \eta(1-n) = 1$	
q odd	$(n-2) (q-1), \eta(2-n) = 1$	
$q = r^2, r \text{ odd}$	$n = tr, t \text{ even }, 1 \le t \le r$	
$q = r^2, r \text{ odd}$	$n = tr + 1, t \text{ odd } 1 \le t \le r$	
$q = r^s, r \text{ odd }, s \ge 2$	n = lr, l even , 2l (r-1)	[24]
$q = r^s, r \text{ odd }, s \ge 2$	$n = lr, l \text{ even }, (l-1) (r-1), \eta(1-l) = 1$	
$q = r^s, r \text{ odd }, s \ge 2$	$n = lr + 1, l \text{ odd }, l (r - 1), \eta(l) = 1$	
$q = r^s, r \text{ odd }, s \ge 2$	$n = lr + 1, l \text{ odd }, (l-1) (r-1), \eta(l-1) = \eta(-1) = 1$	
$q = p^m, p \text{ odd prime}$	n = pr + 1, r m	
$\frac{q = p^m, p \text{ odd prime}}{q = p^m}$	$n = 2p^e, 1 \le e < m, \eta(-1) = 1$ n (q-1), (q-1)/n even	
$q = p^m$		
$q=p^m, m \text{ even }, r=p^s, s \frac{m}{2}$	$n = 2tr^{\ell}, 0 \le \ell < m/s, 1 \le t \le (r-1)/2$	
$q = p^m, q \equiv 1 \pmod{4}$	$n = 2p^{\ell}, 0 \le \ell < m$	
$q = p^m, m \text{ even }, r = p^s, s \frac{m}{2}$	$n = (2t+1)r^{\ell} + 1, 0 \le \ell < m/s, 0 \le t \le (r-1)/2 \text{ or } (\ell,t) = (m/s,0)$	[7]
$q = p^m, q \equiv 1 \pmod{4}$	$n = p^{\ell} + 1, 0 < \ell < m$	
$q = p^m$	$(n-2) (q-1), \eta(2-n) = 1$	
$q = p^m$	$\frac{(n-2) (q-1),\eta(2-n)=1}{n=n_0,p n_0,(n_0-1) (q-1)}$	
$q = p^m, q \equiv 1 \pmod{4}$	$n = n_0 + 1, p n_0, (n_0 - 1) (q - 1)$	
$q = p^m, q \equiv 1 \pmod{4}$	$n = p^r + 1, 1 < r < m, r m$	
$q = p^m, q$ a square	$n = n_0, (n_0 - 1) (q - 1) $	
$q = p^m, q$ a square	$n = n_0 + 1, (n_0 - 1) (q - 1)$	
$q = p^m, q$ a square	$n = 2n_0, n_0 \text{ odd }, (n_0 - 1) (q - 1)$	
$q = p^m$	$n = n_0, n_0 \frac{(q-1)}{2}$	
$q = p^m, q \equiv 1 \pmod{4}$	$n = n_0 + 1, n_0 \frac{(q-1)}{2}$	
$q = p^m, q \equiv 1 \pmod{4}$	$n = 2p^r, 1 \le r < m, r m$	[19]
$q = p^m, m = 2m_0$	$n = (t+1)n_0 + 2, n_0 = p^{m_0} - 1, n_0 \equiv 0 \pmod{4}, t \text{ odd } 0 < t \le \frac{n_0}{2} + 1$	
$q = p^m, m = 2m_0$	$n = (t+1)n_0 + 2, n_0 = p^{m_0} - 1, n_0 \equiv 2 \pmod{4}, 0 < t \le \frac{n_0}{2}$	
$q = p^m, q$ a square	$n = (t+1)n_0 + 2, n_0 = \frac{q-1}{p^r+1}$ even, $1 \le r < m, \frac{n_0(p^r+1)}{2(p^r-1)}$ odd $t \ge t \le p^r$	
$q = p^m, q$ a square	$n = (t+1)n_0 + 2, n_0 = \frac{q-1}{p^r+1} \text{ even, } 1 \le r < m, \frac{n_0(p^r+1)}{(p^r-1)} \text{ even , } 1 \le t \le p^r$	

self-dual ones.

In this paper, we will discover more families of optimal self-dual codes from algbraic curves over finite fields. Optimal self-dual codes are constructed from rational points on the curves and embedding their orthogonal subcodes. We improve the construction [19] and other known constructions over \mathbb{F}_q with q a prime (see Theorem 2) and also give explict constructions of the cosets of \mathbb{F}_q with desired properties (see Lemma 10 and Lemma 11). Due to Lemma 6 and Lemma 7, new classes of self-dual codes with prescribed minimum distance are constructed. Additionally, we construct MDS self-dual codes with new parameters [24, 12, 13]₃₇, [32, 16, 17]₄₁, [26, 13, 14]₆₁, [42, 21, 22]₆₁, [50, 25, 26]₇₃, [24, 12, 13]₈₁, almost MDS self-dual codes with new parameters [16, 8, 8]₉, [16, 8, 8]₁₆, [18, 9, 9]₁₆, [20, 10, 10]₁₆, [22, 11, 11]₁₆, [24, 12, 12]₁₆ and optimal self-dual codes with new parameters [28, 14, 12]₉, [26, 13, 12]₁₆, [28, 14, 13]₁₆, [30, 15, 14]₁₆, [32, 16, 15]₁₆.

The paper is organized as follows: Section 2 gives preliminaries and background on algebraic geometry codes. Section 3 provides explicit constructions of self-dual codes from various algebraic curves. New families of self-dual codes are presented as well some numerical examples are also given. We end up with concluding remark in Section 4.

2 Preliminaries

Let \mathbb{F}_q be the finite field with q elements. A linear code of length n and dimension k over \mathbb{F}_q , denoted as q-ary [n,k] code, is a k-dimensional subspace of \mathbb{F}_q^n . The (Hamming) weight $\operatorname{wt}(\mathbf{x})$ of a vector $\mathbf{x} = (x_1, \ldots, x_n)$ is the number of nonzero coordinates in it. The minimum distance (or minimum weight) d(C) of C is $d(C) := \min\{\operatorname{wt}(\mathbf{x}) | \mathbf{x} \in C, \mathbf{x} \neq \mathbf{0}\}$. The parameters of an [n,k] code with minimum distance d are written [n,k,d]. If C is an [n,k,d] code, then from the Singleton bound, its minimum distance is bounded above by

$$d \le n - k + 1$$
.

A code meeting the above bound is called *Maximum Distance Separable* (MDS). A code is called *almost* MDS if its minimum distance is one unit less than the MDS case. A code is called *optimal* if it has the highest possible minimum distance for its length and dimension. The *Euclidean inner product* of $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in \mathbb{F}_q^n is $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$. The *dual* of C, denoted by C^{\perp} , is the set of vectors orthogonal to every

codeword of C under the Euclidean inner product. A linear code C is called self-orthogonal if $C \subset C^{\perp}$ and self-dual if $C = C^{\perp}$. It is well-known that a self-dual code can only exist for even lengths.

We refer to Stichtenoth [21] for undefined terms related to algebraic function fields.

Let \mathcal{X} be a smooth projective curve of genus g over \mathbb{F}_q . The field of rational functions of \mathcal{X} is denoted by $\mathbb{F}_q(\mathcal{X})$. Function fields of algebraic curves over a finite field can be characterized as finite separable extensions of $\mathbb{F}_q(x)$. We identify points on the curve \mathcal{X} with places of the function field $\mathbb{F}_q(\mathcal{X})$. A point on \mathcal{X} is called rational if all of its coordinates belong to \mathbb{F}_q . Rational points can be identified with places of degree one. We denote the set of \mathbb{F}_q -rational points of \mathcal{X} by $\mathcal{X}(\mathbb{F}_q)$.

A divisor G on the curve \mathcal{X} is a formal sum $\sum_{P \in \mathcal{X}} n_P P$ with only finitely many nonzeros $n_P \in \mathbb{Z}$. The support of G is defined as $supp(G) := \{P | n_P \neq 0\}$. The degree of G is defined by $deg(G) := \sum_{P \in \mathcal{X}} n_P deg(P)$. For two divisors $G = \sum_{P \in \mathcal{X}} n_P P$ and $H = \sum_{P \in \mathcal{X}} m_P P$, we say that $G \geq H$ if $n_P \geq m_P$ for all places $P \in \mathcal{X}$.

It is well-known that a nonzero polynomial $f(x) \in \mathbb{F}_q(x)$ can be factorized into irreducible factors as $f(x) = \alpha \prod_{i=1}^{s} p_i(x)^{e_i}$, with $\alpha \in \mathbb{F}_q^*$. Moreover, any irreducible polynomial $p_i(x)$ corresponds to a place, say P_i . We define the valuation of f at P_i as $v_{P_i}(f) := t$ if $p_i(x)^t | f(x)$ but $p_i(x)^{(t+1)} \not| f(x)$.

For a nonzero rational function f on the curve \mathcal{X} , we define the "principal" divisor of f as

$$(f) := \sum_{P \in \mathcal{X}} v_P(f) P.$$

If Z(f) and N(f) denotes the set of zeros and poles of f respectively, we define the zero divisor and pole divisor of f, respectively by

$$(f)_0 := \sum_{P \in Z(f)} v_P(f)P,$$

$$(f)_{\infty} := \sum_{P \in N(f)} -v_P(f)P.$$

Then $(f) = (f)_0 - (f)_\infty$, and it is well-known that the principal divisor f has degree 0.

We say that two divisors G and H on the curve \mathcal{X} are equivalent if G = H + (z) for some rational function $z \in \mathbb{F}_q(\mathcal{X})$.

For a divisor G on the curve \mathcal{X} , we define

$$\mathcal{L}(G) := \{ f \in \mathbb{F}_q(\mathcal{X}) \setminus \{0\} | (f) + G \ge 0 \} \cup \{0\},\$$

and

$$\Omega(G) := \{ \omega \in \Omega \setminus \{0\} | (\omega) - G > 0 \} \cup \{0\},\$$

where $\Omega := \{fdx | f \in \mathbb{F}_q(\mathcal{X})\}$, the set of differential forms on \mathcal{X} . It is well-known that, for a differential form ω on \mathcal{X} , there exists a unique a rational function f on \mathcal{X} such that

$$\omega = f dt$$

where t is a local uniformizing parameters. In this case, we define the divisor associated to ω by

$$(\omega) = \sum_{P \in \mathcal{X}} v_P(\omega) P,$$

where $v_P(\omega) := v_P(f)$.

Through out the paper, we let $D = P_1 + \cdots + P_n$, called the rational divisor, where $(P_i)_{1 \leq i \leq n}$ are places of degree one, and G a divisor with $supp(D) \cap supp(G) = \emptyset$. Define the algebraic geometry code by

$$C_{\mathcal{L}}(D,G) := \{ (f(P_1),\ldots,f(P_n)) | f \in \mathcal{L}(G) \},$$

and the differential algebraic geometry code as

$$C_{\Omega}(D,G) := \{ (\operatorname{Res}_{P_1}(\omega), \dots, \operatorname{Res}_{P_n}(\omega)) | \omega \in \Omega(G-D) \},$$

where $\operatorname{Res}_{P}(\omega)$ denotes the residue of ω at point P.

The parameters of an algebraic geometry code $C_{\mathcal{L}}(D,G)$ is given as follows.

Theorem 1. [21, Corollary 2.2.3] Assume that 2g - 2 < deg(G) < n. Then the code $C_{\mathcal{L}}(D,G)$ has parameters [n,k,d] satisfying

$$k = \deg(G) - g + 1 \text{ and } d \ge n - \deg(G). \tag{1}$$

The dual of the algebraic geometry code $C_{\mathcal{L}}(D,G)$ can be described as follows.

Lemma 1. [21, Theorem 2.2.8] With above notation, the two codes $C_{\mathcal{L}}(D,G)$ and $C_{\Omega}(D,G)$ are dual to each other.

Moreover, the differential code $C_{\Omega}(D,G)$ is determined as follows.

Lemma 2. [21, Proposition 2.2.10] With the above notation, assume that there exists a differential form ω satisfying

- 1. $v_{P_i}(\omega) = -1, 1 \le i \le n \text{ and }$
- 2. $Res_{P_i}(\omega) = Res_{P_j}(\omega)$ for $1 \le i \le n$.

Then $C_{\Omega}(D,G) = a \cdot C_{\mathcal{L}}(D,D-G+(\omega))$ for some $a \in (\mathbb{F}_a^*)^n$.

3 Self-dual algebraic geometry codes

In this section, we will construct self-dual codes from algebraic geometry codes. Self-dual codes can be constructed directly from Lemma 3 or from their self-orthogonal subcodes by extending the basis of the existing codes.

Lemma 3. [20, Corollary 3.4] With the above notation, assume that there exists a differential form ω satisfying

- 1. $v_{P_i}(\omega) = -1, 1 \le i \le n \text{ and }$
- 2. $Res_{P_i}(\omega) = Res_{P_j}(\omega) = a_i^2, \ 1 \le i \le n, \ for \ some \ a_i \in \mathbb{F}_q^*.$

Then the following statements hold.

- 1. If $2G \leq D + (\omega)$, then there exists a divisor G' such that $C_{\mathcal{L}}(D,G) \sim C_{\mathcal{L}}(D,G')$, and $C_{\mathcal{L}}(D,G')$ is self-orthogonal.
- 2. If $2G = D + (\omega)$, then there exists a divisor G' such that $C_{\mathcal{L}}(D, G) \sim C_{\mathcal{L}}(D, G')$, and $C_{\mathcal{L}}(D, G')$ is self-dual.

The existence of self-dual algebraic geometry codes can be given as follows.

Proposition 1. [22, Corollary 3.1.49, p.292] With the above notation, assume that $N = |\mathcal{X}(\mathbb{F}_q)| > 2g$. Then there exists a self-dual code with parameters $[n, \frac{n}{2}, \geq \frac{n}{2} - g + 1]$ over \mathbb{F}_q for some n even such that $n \geq N - 2g - 1$.

The following lemma will be applied many times for constructing a q-ary self-dual code of length n (if it exists for such a length).

Lemma 4. Let n be an odd positive integer and C a q-ary self-orthogonal code with parameters $[n, \frac{n-1}{2}]$. Then there exists a self-orthogonal code C_0 with parameters $[n+1, \frac{n-1}{2}]$ and a self-dual code C_0' with parameters $[n+1, \frac{n+1}{2}]$ such that $C_0 \subset C_0' \subset C_0^{\perp}$.

Proof. Let G be the generator matrix of C and C_0 be a self-orthogonal code obtained from C by lengthening one zero coordinate. Clearly, the code C_0 has parameters $[n+1, \frac{n-1}{2}]$, and C_0^{\perp} has parameters $[n+1, \frac{n+1}{2}+1]$. Denote G_0 the generator matrix of C_0 , that is,

$$G_0 = \left(\begin{array}{cc} 0 \\ G & \vdots \\ 0 \end{array}\right).$$

Let \mathbf{x} be a nonzero element in the quotient space C_0^{\perp}/C_0 such that $\mathbf{x} \cdot \mathbf{x} = 0$. Then the code C_0' with its following generator matrix G_0' is self-dual with parameters $[n+1, \frac{n+1}{2}]$:

$$G_0' = \begin{pmatrix} & & 0 \\ G & \vdots \\ & & 0 \\ \hline & \mathbf{x} \end{pmatrix}.$$

Moreover, we have the following inclusion

$$C_0 \subset C_0' \subset C_0^{\perp}$$
.

Similarly, we have the following embedding.

Lemma 5. Let n be an even positive integer and C a q-ary self-orthogonal code with parameters $[n, \frac{n}{2} - 1]$. Then there exists a self-dual code C' (if it exists for such a length) with parameters $[n, \frac{n}{2}]$ such that $C \subset C' \subset C^{\perp}$.

Lemma 6. Let \mathcal{X} be a smooth projective curve having genus g. Let n be an odd positive integer and $D = P_1 + \cdots + P_n$ be a divisor on \mathcal{X} . Assume that there exists a differential form ω satisfying

1.
$$v_{P_i}(\omega) = -1$$
, for $i = 1, ..., n$ and

2. $Res_{P_i}(\omega) = Res_{P_i}(\omega) = a_i^2$ with $a_i \in \mathbb{F}_q^*$ for $1 \le i, j \le n$.

If $G = \frac{(2g-3+n)}{2}P_{\infty}$ with $supp(G) \cap supp(D) = \emptyset$, then there exists a self-orthogonal code $C_{\mathcal{L}}(D,G)$ with parameters $[n,\frac{n-1}{2},\frac{n+3}{2}-g]$. Moreover, the code $C_{\mathcal{L}}(D,G)$ can be embedded into a self-dual $[n+1,\frac{n+1}{2},\geq \frac{n+1}{2}-g]$ code C' (if a self-dual code exists for such a length n).

Proof. Choose U as a subset of \mathbb{F}_q with its size |U|=n so that $\omega=\frac{dx}{h}$, where $h(x)=\prod_{\alpha\in U}(x-\alpha)$, satisfying the above two conditions. Then the divisor $(\omega)=(2g-2+n)P_{\infty}-D$, and thus $2G\leq (\omega)+D$. From Lemma 3 and Theorem 1, there exists a self-orthogonal code $C_{\mathcal{L}}(D,G)$ with parameters $[n,\frac{n-1}{2},\geq\frac{n+3}{2}-g]$. The second assertion follows from Lemma 4. First note that

$$C_{\mathcal{L}}(D,G)^{\perp} = a \cdot C_{\mathcal{L}}(D,D-G+(\omega)) \text{ (from Lemma 2)}$$

= $a \cdot C_{\mathcal{L}}\left(D,\frac{(2g-1+n)}{2}P_{\infty}\right)$ (2)

We now calculate the lower bound on the minimum distance of the dual code.

$$d(C_{\mathcal{L}}(D,G)^{\perp}) \geq n - \frac{(2g-1+n)}{2}$$
 (due to (2) and Theorem 1)
= $\frac{n+1}{2} - g$.

The minumum distance of C' follows from the fact that $C' \subset C_{\mathcal{L}}^{\perp}(D,G)$, and this completes the proof.

Lemma 7. Let \mathcal{X} be a smooth projective curve having genus g. Let n be an even positive integer and $D = P_1 + \cdots + P_n$ be a divisor on \mathcal{X} . Assume that there exists a differential form ω satisfying

1.
$$v_{P_i}(\omega) = -1$$
, for $i = 1, ..., n$ and

2.
$$Res_{P_i}(\omega) = Res_{P_j}(\omega) = a_i^2$$
 with $a_i \in \mathbb{F}_q^*$ for $1 \le i, j \le n$.

If $G = \frac{(2g-2+n)}{2}P_{\infty}$ with $supp(G) \cap supp(D) = \emptyset$, then there exists a self-orthogonal code $C_{\mathcal{L}}(D,G)$ with parameters $[n,\frac{n}{2},\frac{n}{2}+1-g]$.

Proof. The result follows from the same reasoning as that in Lemma 6. \Box

3.1 Self-dual codes from projective lines

In this subsection, we will discover new families of MDS self-dual codes based on the work from [19]. In what follows, we let for $a \in \mathbb{F}_q$, $\eta(a) := 1$ if a is a square in \mathbb{F}_q , and $\eta(a) := -1$ if a is not a square in \mathbb{F}_q .

The following two lemmas [19] will be used to construct self-dual codes of genus zero.

Lemma 8. [19, Lemma 6] For $G = sP_{\infty}$ with $s \leq \lfloor \frac{n-2}{2} \rfloor$, if $(h'(P_i))_{1 \leq i \leq n}$ are squares in \mathbb{F}_a^* , then $C_{\mathcal{L}}(D, G - (1/\sqrt{h'}))$ is an MDS self-orthogonal code.

The following lemma is useful for constructing a self-dual code from its self-orthogonal subcode.

Lemma 9. [19, Lemma 7] Let $q \equiv 1 \pmod{4}$. Assume that $G = (k-1)P_{\infty}$, n = 2k + 1, and $(h'(P_i))_{1 \leq i \leq n}$ are squares in \mathbb{F}_q^* . Then the q-ary self-orthogonal code $C_{\mathcal{L}}(D, G - (1/\sqrt{h'}))$ with parameters [n, k] can be embedded into a q-ary MDS self-dual [n + 1, k + 1] code.

Now, we construct MDS self-dual codes from Lemma 8 and Lemma 9.

Theorem 2. Let $q = p^m$ be an odd prime power. If $\eta(-1) = \eta(n) = 1$, n|(q-1) and n even, then there exists an [2n+2, n+1, n+2] self-dual code over \mathbb{F}_q .

Proof. Let $U_n = \{\alpha \in \mathbb{F}_q^* | \alpha^n = 1\}$. Let $\beta_1 \in \mathbb{F}_q^*$ such that $\beta_1^n - 1$ is a nonzero square in \mathbb{F}_q . Put $U = U_n \cup \beta_1 U_n \cup \{0\}$, and write

$$h(x) = \prod_{\beta \in U} (x - \beta).$$

Then we have that

$$h'(x) = ((n+1)x^n - 1)(x^n - \beta_1^n) + nx^n(x^n - 1).$$

Consider the following quadratic equation

$$a^2 + b^2 = 1. (3)$$

For any q, (3) has T=(q-1)-4 solutions, say $(a_1,\pm b_1),\ldots,(a_{\frac{T}{2}},\pm b_{\frac{T}{2}})$, with $(a_i,b_i)\neq (0,\pm 1),(\pm 1,0)$. Take $\beta_1=\sqrt[n]{a_i^2}$ for some $1\leq i\leq t,(t<\frac{T}{2})$. Then we get $1-\beta_1^n=1-a_i^2=b_i^2$ which are squares in \mathbb{F}_q^* .

We have that -1, n are squares in \mathbb{F}_q . Moreover, since β_1^n and $(\beta_1^n - 1)$ are squares in \mathbb{F}_q^* , it implies that $h'(\beta)$ is a square in \mathbb{F}_q^* for any $\beta \in U$. Now, the fact that all the roots of h(x) are simple gives rise to a self-orthogonal code with parameters [2n+1,n,n+2]. Thus, by Lemma 9, it can be embedded into a q-ary self-dual code with parameters [2n+2,n+1,n+2].

Example 1. We construct MDS self-dual codes with new parameters as follows.

- 1. Taking q = 37, n = 12, we obtain a self-dual code over \mathbb{F}_{37} with parameters [26, 13, 14].
- 2. Taking q = 61, n = 12, 20 we obtain self-dual code over \mathbb{F}_{61} with parameters [26, 13, 14], [42, 21, 22], respectively.
- 3. Taking q = 73, n = 24, we obtain a self-dual code over \mathbb{F}_{73} with parameters [50, 25, 26].

Remark 1. In the proof of Theorem 2, we have found many values of β_i such that $1 - \beta_i^n$ is a square. Furthermore, if there exist β_1 and β_2 such that $\beta_1^n - \beta_2^n$ is again a square, then we can construct an MDS self-dual code of length 3n + 2 over \mathbb{F}_q . For example, taking q = 41, n = 10 and considering two non-zero multiplicative cosets of U_n yields a self-dual code over \mathbb{F}_{41} with parameters [32, 16, 17]. The generator matrix of the self-dual code over \mathbb{F}_{41} is given as follows.

```
 \begin{pmatrix} 20 & 15 & 20 & 11 & 6 & 11 & 22 & 10 & 39 & 15 & 5 & 9 & 20 & 31 & 33 & 40 \\ 37 & 5 & 11 & 12 & 3 & 15 & 4 & 10 & 37 & 18 & 35 & 8 & 3 & 26 & 32 & 8 \\ 12 & 29 & 9 & 14 & 13 & 11 & 8 & 30 & 16 & 20 & 17 & 22 & 3 & 9 & 13 & 10 \\ 2 & 17 & 11 & 25 & 14 & 3 & 1 & 27 & 38 & 5 & 1 & 15 & 36 & 2 & 1 & 21 \\ 27 & 21 & 21 & 13 & 20 & 7 & 36 & 15 & 29 & 30 & 25 & 20 & 1 & 11 & 2 & 32 \\ 39 & 7 & 2 & 1 & 26 & 25 & 5 & 38 & 38 & 13 & 33 & 20 & 17 & 15 & 7 & 36 \\ 40 & 39 & 34 & 15 & 18 & 12 & 6 & 28 & 25 & 10 & 21 & 23 & 8 & 35 & 26 & 26 \\ 40 & 39 & 34 & 15 & 18 & 12 & 6 & 28 & 25 & 10 & 21 & 23 & 8 & 35 & 26 & 26 \\ I_{16} & 30 & 36 & 28 & 2 & 1 & 11 & 12 & 28 & 2 & 27 & 34 & 35 & 4 & 4 & 20 & 2 \\ 22 & 18 & 5 & 24 & 5 & 40 & 23 & 9 & 34 & 40 & 12 & 34 & 9 & 34 & 33 & 31 \\ 20 & 13 & 9 & 12 & 31 & 35 & 37 & 33 & 26 & 37 & 23 & 39 & 29 & 18 & 25 & 19 \\ 11 & 19 & 18 & 16 & 38 & 40 & 2 & 29 & 8 & 30 & 30 & 10 & 12 & 2 & 20 & 30 \\ 34 & 13 & 10 & 13 & 18 & 28 & 19 & 14 & 28 & 31 & 4 & 34 & 24 & 9 & 31 & 35 \\ 24 & 31 & 21 & 40 & 12 & 23 & 25 & 4 & 17 & 27 & 13 & 4 & 31 & 40 & 23 & 30 \\ 40 & 31 & 36 & 35 & 28 & 38 & 21 & 31 & 14 & 20 & 16 & 36 & 20 & 37 & 34 & 21 \\ 9 & 10 & 23 & 11 & 36 & 23 & 30 & 9 & 16 & 22 & 27 & 32 & 37 & 26 & 39 & 26 \\ 36 & 4 & 32 & 32 & 4 & 4 & 10 & 14 & 12 & 14 & 20 & 30 & 29 & 34 & 8 & 21 \end{pmatrix}
```

The two following lemmas play the key role in determining whether the difference of two special elements in \mathbb{F}_q is a square or not and also in determining the number of cosets of a multiplicate subgroup of \mathbb{F}_q^* .

Lemma 10. Let $q = p^m$ with p an odd prime, $n = \frac{q-1}{p^r+1}$ and for $\alpha_i, \alpha_j \in \mathbb{F}_q$ with $\alpha_i \neq \alpha_j$, denote $\alpha_{ij} = \alpha_i^n - \alpha_j^n$. Then for ω a primitive element of \mathbb{F}_q , we have the following equality:

$$\alpha_{ij} = \frac{\omega^{\frac{n(p^r+1)}{2(p^r-1)}}}{\alpha_i \alpha_j}.$$
 (4)

Proof. Raising α_{ij} to the power $p^r - 1$, we get

$$\begin{array}{ll} \alpha_{ij}^{p^r-1} & = \frac{\left(\alpha_i^n - \alpha_j^n\right)^{p^r}}{\alpha_i^n - \alpha_j^n} = \frac{\left(\alpha_i^{np^r} - \alpha_j^{np^r}\right)}{\alpha_i^n - \alpha_j^n} \\ & = \frac{\left(\alpha_i^{q-1-n} - \alpha_j^{q-1-n}\right)}{\alpha_i^n - \alpha_j^n} = \frac{\frac{1}{\alpha_i^n} - \frac{1}{\alpha_j^n}}{\alpha_i^n - \alpha_j^n} \\ & = \frac{\omega \frac{n(p^r + 1)}{2}}{\alpha_i^n \alpha_j^n}, \end{array}$$

where the last equality come from the fact that $\omega^{\frac{q-1}{2}}=-1$.

By taking the $(p^r - 1)$ -th root, the result follows.

Lemma 11. Let $q = p^m$ with p an odd prime, r|m, $n = \frac{q-1}{p^r-1}$ and for $\alpha_i, \alpha_j \in \mathbb{F}_q$ with $\alpha_i \neq \alpha_j$, denote $\alpha_{ij} = \alpha_i^n - \alpha_j^n$. Then for ω a primitive element of \mathbb{F}_q , we have the following:

$$\alpha_{ij} \in \mathbb{F}_{p^r}.$$
 (5)

Proof. Raising α_{ij} to the power p^r , we get

$$\alpha_{ii}^{p^r} = (\alpha_i^{np^r} - \alpha_i^{np^r}) = (\alpha_i^{q-1+n} - \alpha_i^{q-1+n}) = \alpha_i^n - \alpha_i^n = \alpha_{ii}.$$

Thus the result follows.

Theorem 3. Let $q = p^m$ be an odd square and n even. Put s = (t+1)n.

- 1. If $\frac{n(p^r+1)}{2(p^r-1)}$ is even, then there exists a self-dual code over \mathbb{F}_q with parameters $[s, \frac{s}{2}, \frac{s}{2} + 1]$, with $n = \frac{q-1}{p^r+1}$, for $1 \le t \le p^r$.
- 2. If $\frac{n(p^r+1)}{2(p^r-1)}$ is odd, then there exists a self-dual code over \mathbb{F}_q with parameters $[s, \frac{s}{2}, \frac{s}{2} + 1]$, with $n = \frac{q-1}{p^r+1}$, for t odd and $1 \le t \le p^r$.
- 3. There exists a self-dual code over \mathbb{F}_q with parameters $[s, \frac{s}{2}, \frac{s}{2} + 1]$, with $n = \frac{q-1}{p^r-1}, r|\frac{m}{2}$, for $1 \le t \le p^r 2$.

Proof. Let U_n be a multiplicative subgroup of \mathbb{F}_q^* of order n, say $U_n = \{u_1, \ldots, u_n\}$. Let $\alpha_1 U_n, \ldots, \alpha_t U_n$ be t nonzero cosets of U_n , where $(\alpha_i)_{1 \leq i \leq t}$ will be determined later. Put $U = U_n \cup \left(\bigcup_{i=1}^t \alpha_i U_n\right)$, and write

$$h(x) = \prod_{\alpha \in U} (x - \alpha).$$

Clearly, all the roots of h(x) are simple. The derivative of h(x) is given by

$$h'(x) = nx^{n-1} \prod_{i=1}^{t} (x^n - \alpha_i^n) + nx^{n-1} (x^n - 1) \left(\sum_{i=1}^{t} \prod_{j=1, j \neq i}^{t} (x^n - \alpha_j^n) \right).$$

For $1 \le j \le t, 1 \le s \le n$, we have

$$h'(u_s) = nu_s^{n-1}(\alpha_j^n - 1) \prod_{i=1, i \neq j}^t (1 - \alpha_i^n),$$

$$h'(\alpha_j u_s) = n(\alpha_j u_s)^{n-1}(\alpha_j^n - 1) \prod_{i=1, i \neq j}^t (\alpha_j^n - \alpha_i^n).$$

For $1 \le i, j \le t$ and $n = \frac{q-1}{p^r+1}$, we have from (4)

$$\alpha_{ij} = \alpha_i^n - \alpha_j^n = \frac{\omega^{\frac{n(p^r+1)}{2(p^r-1)}}}{\alpha_i \alpha_j},$$

where ω is a primitive element of \mathbb{F}_q .

Fixing j and taking all the product of α_{ij} for $1 \leq i \leq t, i \neq j$, we get that

$$\prod_{i=1, i\neq j}^{t} (\alpha_j^n - \alpha_i^n) = \prod_{i=1, i\neq j}^{n} \frac{\omega^{\frac{n(p^r+1)}{2(p^r-1)}}}{\alpha_i \alpha_j}.$$

Obviously, n and $(u_s)_{1 \leq s \leq n}$ are squares in \mathbb{F}_q for q a square. Now, the squareness of $h'(u_s)$ and $h'(\alpha_j u_s)$ depend on the parity of $T = \frac{n(p^r + 1)}{2(p^r - 1)}$.

If T is even, then α_i is chosen to be a square element in \mathbb{F}_q , and thus $(1 - \alpha_i^n)$ and $(\alpha_i^n - \alpha_i^n)$ are square elements in \mathbb{F}_q due to (4) of Lemma 10.

If T is odd, then α_i is chosen to be a non-square element in \mathbb{F}_q , and thus $(1 - \alpha_i^n)$ and $(\alpha_i^n - \alpha_i^n)$ are again square elements in \mathbb{F}_q due to (4).

In conclusion, we have

- 1. If $\frac{n(p^r+1)}{2(p^r-1)}$ is even, then $h'(u_s)$ and $h'(\alpha_j u_s)$ are squares in \mathbb{F}_q^* for $1 \le s \le n$ and $1 \le j \le t$ with $t \in \{1, \ldots, p^r\}$.
- 2. If $\frac{n(p^r+1)}{2(p^r-1)}$ is odd, then $h'(u_s)$ and $h'(\alpha_j u_s)$ are squares in \mathbb{F}_q^* for $1 \leq s \leq n$ and $1 \leq j \leq t$ with t odd and $t \in \{1, \ldots, p^r\}$.

For $1 \leq i, j \leq t$ and $n = \frac{q-1}{p^r-1}$, from (5) in Lemma 11, we get that $\alpha_{ij} = \alpha_i^n - \alpha_j^n \in \mathbb{F}_{p^r}$, and hence it is a square if $r|\frac{m}{2}$. We have shown that $h'(\alpha)$ is a nonzero square in \mathbb{F}_q for any $\alpha \in U$, and thus the constructed code is self-dual by Lemma 8.

Example 2. Taking $q = 9^2$, $n = \frac{9^2-1}{9+1} = 8$, t = 2, we get an MDS self-dual with parameters [24, 12, 13]. These parameters are new. The generator matrix of the code is given as follows.

$$\begin{pmatrix} & w^{48} & w^{20} & w^{72} & w^{77} & w^{19} & w^{16} & w^{44} & w^{10} & w^{37} & w^{76} & w^{47} & w^{56} \\ & w^{20} & w^{26} & w^{73} & w^{56} & 2 & w^{26} & w^{60} & w^{55} & w^{65} & w^{47} & w^{41} & w^{47} \\ & w^{32} & w^{33} & w^{34} & w^{14} & w^{54} & w^2 & w^7 & w^{26} & w^{65} & w^{75} & w^{47} & w^{76} \\ & w^{37} & w^{16} & w^{14} & w^{16} & w^{53} & w^{99} & w^{53} & w^{43} & w^{8} & w^{25} & w^{25} & w^{77} \\ & w^{19} & 2 & w^{14} & w^{13} & w^{4} & w^{5} & w^{57} & w^{9} & w^{13} & w^{66} & w^{15} & w^{50} \\ & I_{12} & w^{16} & w^{26} & w^{42} & w^{19} & w^{5} & 2 & w^{9} & w^{17} & w^{13} & w^{7} & w^{60} & w^{44} \\ & & w^{4} & w^{20} & w^{7} & w^{53} & w^{17} & w^{49} & 2 & w^{45} & w^{19} & w^{2} & w^{26} & w^{16} \\ & & w^{10} & w^{55} & w^{66} & w^{3} & w^{9} & w^{17} & w^{5} & w^{44} & w^{13} & w^{54} & 2 & w^{19} \\ & & w^{37} & w^{65} & w^{25} & w^{48} & w^{43} & w^{13} & w^{59} & w^{14} & w^{14} & w^{74} & w^{33} & w^{32} \\ & & w^{36} & w^{7} & w^{75} & w^{25} & w^{26} & w^{47} & w^{2} & w^{14} & w^{14} & w^{74} & w^{33} & w^{32} \\ & & w^{47} & w^{41} & w^{7} & w^{65} & w^{15} & w^{60} & w^{66} & w^{19} & w^{37} & v^{72} & w^{60} & w^{8} \end{pmatrix}$$

Theorem 4. Let q be an odd prime power with $q \equiv 1 \pmod{4}, 1 \leq r < m, r | \frac{m}{2}$ and n even with $\eta(n) = 1$. Put s = (t+1)n.

- 1. If $\frac{n(p^r+1)}{2(p^r-1)}$ is odd, then there exists a self-dual code over \mathbb{F}_q with parameters $[s, \frac{s}{2}, \frac{s}{2} + 1]$, with $n = \frac{q-1}{p^r+1}$, for t odd and $1 \le t \le p^r$.
- 2. If $\frac{n(p^r+1)}{2(p^r-1)}$ is even, then there exists a self-dual code over \mathbb{F}_q with parameters $[s,\frac{s}{2},\frac{s}{2}+1]$, with $n=\frac{q-1}{p^r+1}$, for $1\leq t\leq p^r$.
- 3. There exists a self-dual code over \mathbb{F}_q with parameters $[s, \frac{s}{2}, \frac{s}{2} + 1]$, with $n = \frac{q-1}{p^r-1}, r|\frac{m}{2}$, for $1 \le t \le p^r 2$.

Proof. The proof follows from that of Theorem 3.

Table 2: Numbers of rational points of elliptic curves

Elliptic curve $\mathcal{E}_{1,b,c}$	m	$\#\mathcal{E}_{1,b,c}(\mathbb{F}_{2^m})$
	m odd	$q+1-2\sqrt{q}$
$y^2 + y = x^3$	$m \equiv 0 \pmod{4}$	$q+1-2\sqrt{q}$
	$m \equiv 2 \pmod{4}$	$q+1+2\sqrt{q}$
$y^2 + y = x^3 + x$	$m \equiv 1,7 \pmod{8}$	$q+1+2\sqrt{q}$
y + y = x + x	$m \equiv 3, 5 \pmod{8}$	$q+1-2\sqrt{q}$
$y^2 + y = x^3 + x + 1$	$m \equiv 1,7 \pmod{8}$	$q+1+2\sqrt{q}$
y + y = x + x + 1	$m \equiv 3, 5 \pmod{8}$	$q+1-2\sqrt{q}$
$y^2 + y = x^3 + bx(Tr_1^m(b) = 1)$	m even	q+1
$y^2 + y = x^3 + c \ (Tr_1^m(c) = 1)$	$m \equiv 0 \pmod{4}$	$q+1+2\sqrt{q}$
$y + y - x + c \left(I r_1 \left(c \right) - 1 \right)$	$m \equiv 2 \pmod{4}$	$q+1-2\sqrt{q}$

3.2 Self-dual codes from elliptic curves and hyper-elliptic curves

In this subsection, we will consider elliptic curves and hyper-elliptic curves over \mathbb{F}_q , q even.

First, we will consider elliptic curves in Weierstrass form to construct self-dual codes. Let $q = p^m$ and an elliptic curve defined by the equation

$$\mathcal{E}_{a,b,c}: y^2 + ay = x^3 + bx + c,$$
 (6)

where $a, b, c \in \mathbb{F}_q$. Let S be the set of x-components of the affine points of $\mathcal{E}_{a,b,c}$ over \mathbb{F}_q , that is,

$$S_{a,b,c} := \{ \alpha \in \mathbb{F}_q | \exists \beta \in \mathbb{F}_q \text{ such that } \beta^2 + a\beta = \alpha^2 + b\alpha + c \}.$$
 (7)

For $q=2^m$, any $\alpha \in S_{1,b,c}$ gives exactly two points with x-component α , and we denote these two points corresponding to α by $P_{\alpha}^{(1)}$ and $P_{\alpha}^{(2)}$. Then the set of all rational points of $\mathcal{E}_{1,b,c}$ over \mathbb{F}_q is $\{P_{\alpha}^{(1)} | \alpha \in S_{1,b,c}\} \cup \{P_{\alpha}^{(2)} | \alpha \in S_{1,b,c}\} \cup \{P_{\infty}\}$. The numbers of rational points of elliptic curves \mathcal{E} over \mathbb{F}_q are given in Table 2.

Lemma 12 (Hilbert's Theorem 90). Let $q = p^m$. The equation $y^p - y = k$ has solutions over \mathbb{F}_q if and only if $Tr_{\mathbb{F}_q/\mathbb{F}_p}(k) = 0$.

Lemma 13. Let $q_0 = 2^m$, $m \ge 2$ and $q = q_0^2$. If α is an element in \mathbb{F}_{q_0} , then $Tr_{\mathbb{F}_q/\mathbb{F}_2}(\alpha) = 0$, and $Tr_{\mathbb{F}_q/\mathbb{F}_2}(\alpha + \alpha^3) = 0$.

Proof. For any $\alpha \in \mathbb{F}_{q_0}$, we have

$$\begin{aligned} \operatorname{Tr}_{\mathbb{F}_{q}/\mathbb{F}_{2}}(\alpha) &= \operatorname{Tr}_{\mathbb{F}_{q_{0}}/\mathbb{F}_{2}} \left(\operatorname{Tr}_{\mathbb{F}_{q}/\mathbb{F}_{q_{0}}}(\alpha) \right) \\ &= \operatorname{Tr}_{\mathbb{F}_{q_{0}}/\mathbb{F}_{2}} \left(\alpha + \alpha^{q_{0}} \right) \\ &= \operatorname{Tr}_{\mathbb{F}_{q_{0}}/\mathbb{F}_{2}} \left(\alpha \right) + \operatorname{Tr}_{\mathbb{F}_{q_{0}}/\mathbb{F}_{2}} \left(\alpha^{q_{0}} \right) \\ &= 0. \end{aligned}$$

where the first and second equality come from the properties of the trace function and the last one from the fact that $\alpha \in \mathbb{F}_{q_0}$. Since $\mathbb{F}_{q_0}^*$ is a multiplicative group, the second part follows.

Proposition 2. Let $q_0 = 2^m$ and $q = q_0^2$. Then there exists a $[2q_0, q_0, d \ge q_0]$ self-dual code over \mathbb{F}_q .

Proof. Consider the elliptic curve defined by

$$\mathcal{E}_{1,1,0}: y^2 + y = x^3 + x.$$

From Lemma 13, we get that \mathbb{F}_{q_0} is a subset of $S_{1,1,0}$. Put $U = \mathbb{F}_{q_0}$ and $h(x) = \prod_{\alpha \in U} (x - \alpha)$. Then the residue $\operatorname{Res}_{P_{\alpha}}(\omega) = \frac{1}{h'(P_{\alpha})}$ is a square for any $\alpha \in U$, and by Lemma 3, the constructed code $a \cdot C_{\mathcal{L}}(D, G)$ is self-dual, where $a_i^2 = \operatorname{Res}_{P_i}(\omega)$.

Theorem 5. Let $q = 2^m$ and $U = \{\alpha \in \mathbb{F}_q | Tr(\alpha^3 + \alpha) = 0\}$. Then there exists a self-dual code over \mathbb{F}_q with parameters $[2n, n, d \geq n]$ for $1 \leq n \leq |U|$.

Proof. Let U be defined as in the theorem. Put

$$h(x) = \prod_{\alpha \in U} (x - \alpha).$$

Since any element in \mathbb{F}_q (q even) is a square in \mathbb{F}_q , we conclude that $h'(\alpha)$ is a nonzero square in \mathbb{F}_q for any $\alpha \in U$.

Consider the elliptic curve defined by

$$\mathcal{E}_{1,1,0}: y^2 + y = x^3 + x.$$

From Lemma 12, we get that U is a subset of $S_{1,1,0}$. Put

$$D = \sum_{\alpha \in U_0 \subset U} \left(P_{\alpha}^{(1)} + P_{\alpha}^{(2)} \right) = P_1 + \dots + P_s, s = 2|U_0|, G = \frac{s}{2} P_{\infty}, \omega = \frac{dx}{h}.$$

Then the residue $\operatorname{Res}_{P_{\alpha}}(\omega) = \frac{1}{h'(P_{\alpha})}$ is a square for any $\alpha \in U_0$, and by Lemma 3, the constructed code $a \cdot C_{\mathcal{L}}(D, G)$ is self-dual, where $a_i^2 = \operatorname{Res}_{P_i}(\omega)$.

Example 3. The elliptic curve

$$\mathcal{E}_{1,1,0}: y^2 + y = x^3 + x,$$

has rational points in the set $\{P_{\infty} = (1:0:0), (1:0:1), (1:1:1), (w^3:w^7:1), (w^3:w^9:1), (w^6:w^3:1), (w^6:w^{14}:1), (w^{12}:w^6:1), (w^{12}:w^{13}:1), (w^{10}:w:1), (w^{10}:w^4:1), (w^{11}:w:1), (w^{11}:w^4:1), (w^5:w^2:1), (w^5:w^8:1), (w^7:w^2:1), (w^7:w^8:1), (w^{13}:w^2:1), (w^{13}:w^8:1)\}.$ Put $D = P_1 + \dots + P_{18}, G = 9P_{\infty}$. The code $C_{\mathcal{L}}(D,G)$ is self-dual. The set $\{\frac{x^iy^j}{z^{i+j}}|(i,j)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1)\}\}$ is a basis for $C_{\mathcal{L}}(D,G)$, and thus its generator matrix is given by

$$\mathcal{G} = \begin{pmatrix} 11111111111111111\\ 01w^7w^9w^3w^{14}w^6w^{13}ww^4w^4w^2w^8w^2w^8w^2w^8\\ 01w^{14}w^3w^6w^{13}w^{12}w^{11}w^2w^8w^2w^8w^4ww^4w^4w\\ 01w^6w^{12}w^9w^{12}w^3w^9w^3w^{12}w^3w^{12}w^6w^9w^6w^9w^6w^9\\ 11w^3w^3w^6w^6w^{12}w^{12}w^{10}w^{10}w^{11}w^{11}w^5w^5w^7w^7w^{13}w^{13}\\ 01w^{10}w^{12}w^9w^5w^3w^{10}w^{11}w^{14}w^{12}1w^7v^{13}w^911w^6\\ 01w^2w^6w^{12}w^4w^9w^8w^{12}w^3w^{13}w^4w^9w^6w^{11}w^8w^2w^{14}\\ 11w^6w^6w^{12}w^{12}w^9w^9w^5w^5w^7w^7w^{10}w^{10}w^{14}w^{14}w^{11}w^{11}\\ 01w^{13}11w^{11}1w^7w^6w^9w^8w^{11}w^{12}w^3ww^7w^{13}w^4 \end{pmatrix}$$

By Magma [3], the code with generator matrix $a \cdot \mathcal{G}$ is self-dual, and it has parameters [18,9,9] over \mathbb{F}_{16} , where $a = (w^5, w^5, w^{12}, w^{12}, w^7, w^7, 1, 1, w^2, w^2, w^{14}, w^{14}, w^4, w^4, w^9, w^9, w^{10}, w^{10})$.

This code is an almost MDS code. We also find almost MDS self-dual codes over \mathbb{F}_{16} with parameters [20, 10, 10], [22, 11, 11], [24, 12, 12].

Corollary 1. Let $q = 2^m$ and $U = \{\alpha \in \mathbb{F}_q | Tr(\alpha^3) = 0\}$. Then there exists a self-dual code over \mathbb{F}_q with parameters $[2n, n, d \ge n]$ for $1 \le n \le |U|$.

Theorem 6. Let $q = 2^m, m \ge 3$ and $U = \{\alpha \in \mathbb{F}_q | Tr(\alpha^5) = 0\}$. Then there exists a self-dual code over \mathbb{F}_q with parameters $[2n, n, d \ge n - 1]$ for $1 \le n \le |U|$.

Proof. Let U be defined as in the theorem. Put

$$h(x) = \prod_{\alpha \in U} (x - \alpha).$$

Since any element in \mathbb{F}_q (q even) is a square in \mathbb{F}_q , we conclude that $h'(\alpha)$ is a nonzero square in \mathbb{F}_q for any $\alpha \in U$.

Consider the hyper-elliptic curve defined by

$$\mathcal{X}: \ y^2 + y = x^5. \tag{8}$$

From Lemma 12, we get that U is a subset of the solution to (8). Put $D = \sum_{\alpha \in U_0 \subset U} \left(P_{\alpha}^{(1)} + P_{\alpha}^{(2)}\right) = P_1 + \dots + P_s, s = 2|U_0|, G = (\frac{s}{2} + 1)P_{\infty}$ and $\omega = \frac{dx}{h}$. Then the residue $\operatorname{Res}_{P_{\alpha}}(\omega) = \frac{1}{h'(P_{\alpha})}$ is a square for any $\alpha \in U_0$, and by Lemma 3, the constructed code $a \cdot C_{\mathcal{L}}(D, G)$ is a $[s, \frac{s}{2}, d \geq \frac{s}{2} - 1]$ self-dual code, where $a_i^2 = \operatorname{Res}_{P_i}(\omega)$.

Example 4. The hyper-elliptic curve defined by

$$y^2 + y = x^5,$$

$\mathcal{G} = \begin{pmatrix} 111111111111111111111111\\ 01010101w^4w^4w^4w^4w^4w^4w^2w^8w^2w^8w^2w^8w^2w^8w^2w^8\\ 0101010101w^2w^8w^2w^8w^2w^8w^2w^8w^2w^8w^2w^8w^2w^8w^2w^8\\ 0101010101w^2w^8w^2w^8w^2w^8w^2w^8w^4ww^4ww^4w^4w^4w^4w\\ w^{12}w^{12}w^3w^3w^6w^6w^9w^9w^{11}w^{11}w^2w^2w^5w^5w^8w^8www^{13}w^{13}w^4w^4w^7w^7w^{10}w^{10}\\ 0w^{12}0w^30w^60w^9w^{12}1w^3w^6w^6w^9w^9w^{12}w^3w^91w^6w^6w^{12}w^91w^{12}w^3\\ 0w^{12}0w^30w^60w^9w^{13}w^4w^4w^{10}w^7w^{13}w^{10}ww^5w^2w^2w^{14}w^8w^5w^{11}w^8w^{14}w^{11}\\ w^9w^9w^6w^6w^{12}u^{12}w^3w^3w^7w^7w^4w^4w^{10}w^{10}ww^2w^2w^{211}w^{11}w^8w^8w^{14}w^{14}w^5w^5\\ 0w^90w^60w^{12}0w^3w^8w^{11}w^5w^8w^{11}w^{14}w^2w^5w^4w^{10}w^{13}w^4w^{10}ww^7w^7w^{13}\\ 0w^90w^60w^{12}0w^3w^91w^6w^{12}w^{12}w^3w^3w^9w^6w^31w^{12}w^{12}w^9w^31w^9w^6\\ w^6w^6w^9w^9w^3w^3w^{12}w^{12}w^3w^3w^6w^611w^9w^9w^3w^3w^9w^9w^{12}w^{12}w^6w^611\\ 0w^60w^90w^30w^{12}w^4w^7w^7w^{10}ww^4w^{10}w^{13}w^5w^{11}w^{11}w^2w^{14}w^5w^8w^{14}w^{2}w^8\\ w^3w^3w^{12}w^{12}w^9w^9w^6w^6w^{14}w^{14}w^8w^8w^5w^5w^2w^2w^4w^4w^7w^7www^{13}w^{13}w^{10}w^{10}\\ 0w^30w^{12}0w^90w^61w^3w^9w^{12}w^6w^9w^3w^6w^6w^{12}w^{12}w^{12}w^3\\ 11111111w^{10}w^{10}w^{10}w^{10}w^{10}w^{10}w^{10}w^{5}w^5w^5w^5w^5w^5w^5w^5w^5w^5\\ w^{12}w^{12}w^3w^3w^6w^6w^9w^9w^6w^6w^{12}w^{12}11w^3w^3w^6w^6w^3w^9w^9w^{12}w^{12}111\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{13}w^4w^4w^{10}w^{10}\\ w^9w^9w^6w^6w^{12}w^{12}w^3w^3w^2w^2w^{14}w^{14}w^5w^5w^{11}w^{11}w^7w^7www^{13}w^{14}w^4w^{10}\\ w^9w^9w^6w^6w^{12}w^{12$

By Magma [3], the code with generator matrix $a \cdot \mathcal{G}$ is self-dual, and it has parameters [26, 13, 12] over \mathbb{F}_{16} , where $a = (w^{14}, w^{14}, w, w, w^6, w^6, w^{10}, w^{10}, w^9, w^9, w^4, w^4, w^6, w^6, w^8, w^8, w^6, w^6, w^3, w^3, w^7, w^7, w, w, w^{13}, w^{13})$. We also find self-dual codes over \mathbb{F}_{16} with parameters [28, 14, 13], [30, 15, 14], [32, 16, 15].

Next, we will consider hyper-elliptic curves over \mathbb{F}_q , $q=p^m$ with p an odd prime.

Theorem 7. Let $q = p^m$ and t be a positive odd integer such that gcd(t, q - 1) = 1. If $\eta(n) = 1$ and 4n|(q-1), then there exists a self-dual code with parameters $[2n, n, d \ge n + \frac{t-3}{2}]$.

Proof. Denote $C_j = \{j \times i \pmod{q-1} | i = 0, 1, \ldots\}$. For θ a primitive element of \mathbb{F}_q , let $U_n = \{\theta^i | i \in C_{\frac{q-1}{n}}\}$, and label the elements of U_n as $\alpha_1, \ldots, \alpha_n$. Under the condition 4n|(q-1), the set U_n is a multiplicative subgroup of \mathbb{F}_q^* of order n. Put

$$h(x) = \prod_{\alpha \in U_n} (x - \alpha).$$

Clearly, all the roots of h(x) are simple, and the derivative $h'(x) = nx^{n-1}$, and thus for any $\alpha \in U_n$, we have that $h'(\alpha)$ is a square. Consider the elliptic curve defined by

$$\mathcal{X}: \ y^2 = x^t.$$

Since $\gcd(t, q - 1) = 1$, the set $\{x^t | x \in \mathbb{F}_q\}$ is in bijection with \mathbb{F}_q . For any $\alpha \in U_n$, there are two places, say $P_{\alpha}^{(1)}$ and $P_{\alpha}^{(2)}$, arising from x-component α . Put $D = \sum_{\alpha \in U_n} P_{\alpha}^{(1)} + P_{\alpha}^{(2)} = P_1 + \cdots + P_s, s = 2n, G = nP_{\infty}$ and $\omega = \frac{dx}{h}$.

With the choice of $\alpha_i \in U_n$ and $\beta_i^2 = \alpha_i^t$, the residue $\operatorname{Res}_{P_{\alpha_i}}(\omega) = \frac{1}{\beta_i h'(\alpha_i)}$ is a square for any $\alpha_i \in U_n$, and by Lemma 3, the constructed code $a \cdot C_{\mathcal{L}}(D, G)$ is self-dual, where $a_i^2 = \operatorname{Res}_{P_i}(\omega)$.

Corollary 2. Let $q = p^m$. Then we have the following:

- 1. if gcd(3, q-1) = 1, $\eta(n) = 1$ and 4n|(q-1), then there exists a self-dual code with parameters $[2n, n, d \ge n]$;
- 2. If gcd(5, q-1) = 1, $\eta(n) = 1$ and 4n|(q-1), then there exists a self-dual code with parameters $[2n, n, d \ge n-1]$.

Example 5. The hyper-elliptic curve over \mathbb{F}_{25} defined by

$$y^2 = x^5$$
,

has rational points in the set $\{P_{\infty} = (1:0:0), (1:1:1), (1:4:1), (w^{20}:w^2:1), (w^{20}:w^{14}:1), (w^{16}:w^4:1), (w^{16}:w^{16}:1), (4:2:1), (4:3:1), (w^8:w^8:1), (w^8:w^{20}:1), (w^4:w^{10}:1), (w^4:w^{22})\}$. Put $D = P_1 + \cdots + P_{12}$ and $G = 7P_{\infty}$. The set $\{\frac{x^iy^j}{z^{i+j}}|(i,j) \in \{(0,0), (0,1), (1,0), (1,1), (2,0), (3,0)\}\}$ is a basis for $C_{\mathcal{L}}(D,G)$, and thus its generator matrix is given by

By Magma [3], the code with generator matrix $a \cdot \mathcal{G}$ is self-dual, and it has parameters [12, 6, 5] over \mathbb{F}_{25} , where $a = (1, 3, w^{21}, w^{15}, 3, 1, w^{15}, w^{21}, 1, 3, w^{21}, w^{15})$.

By considering curves in higher genus, we can release the gcd condition in Theorem 7.

Theorem 8. Let $q = p^m$ with p an odd prime.

- 1. If n is odd, $\eta(n) = 1$ and 4n|(q-1), then there exists a self-dual code with parameters $[2n, n, d \ge \frac{n}{2} + 2]$.
- 2. If n is even, $\eta(n) = 1$ and 2n|(q-1), then there exists a self-dual code with parameters $[2n, n, d \ge \frac{n}{2} + 2]$.

Proof. Assume that n is odd. Let U_n and h(x) be defined as in Theorem 7. Consider an algebraic curve given by

$$\mathcal{X}: \ y^2 = x^n.$$

Take $\omega = \frac{dx}{h}$ and $G = \frac{3n-4}{2}P_{\infty}$. Then by Lemma 3, the code $a \cdot C_{\mathcal{L}}(D,G)$, where $a_i = \operatorname{Res}_{P_i}(\omega)$, is self-dual with parameters $[2n, n, \frac{n}{2} + 2]$, and this proves point 1).

For point 2), we put $U_n = \{a^2 | a \in \mathbb{F}_q, a^n = 1\}$. The rest follows from the same reasoning as the first part.

3.3 Self-dual codes from other curves

In this subsection, we will construct self-dual codes over \mathbb{F}_q from algebraic curves of high genus.

Let $q_0 = p^m, q = q_0^2$ and \mathcal{X} be the Hermitian curve over \mathbb{F}_q defined by

$$\mathcal{X}: \ y^{q_0} + y = x^{q_0 + 1}.$$

The Hermitian curve \mathcal{X} has genus $g = \frac{q_0(q_0-1)}{2}$, and for any $\alpha \in \mathbb{F}_q$, $x - \alpha$ has q zeros of degree one in \mathcal{X} . All rational points of the curve \mathcal{X} different from the point at infinity are obtained in this way. Self-orthogonal AG codes from Hermitian curves were already considered in [20]. In what follows, we embed those codes into the self-dual ones and provide the parameters of the latter codes. We also construct new families of self-dual codes from this curve.

Theorem 9. Let p be an odd prime, $q_0 = p^m$, $q = q_0^2$, $g = \frac{q_0(q_0-1)}{2}$. Put $d_0 = \frac{s}{2} + 1 - g$, s' = s + 1, $d'_0 = \frac{s'}{2} - g$.

1. If p|n,(n-1)|(q-1), then there exists a q-ary self-dual code with parameters $[s,\frac{s}{2},d\geq d_0]$ (resp. $[s',\frac{s'}{2},d\geq d'_0]$), where $s=q_0n$ with n even (resp. n odd).

- 2. If r|m, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \ge d'_0]$, where $s = q_0 p^r$.
- 3. If (n-1)|(q-1), then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \geq d'_0]$, where $s = q_0(2n-1)$.
- 4. If $n|\frac{q-1}{2}$, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$ (resp. $[s', \frac{s'}{2}, d \ge d'_0]$), where $s = q_0 n$ with n even (resp. n odd).
- 5. If $1 \le r < m$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \ge d'_0]$, where $s = q_0(2p^r 1)$.
- 6. If $n = q_0 1$, $n \equiv 0 \pmod{4}$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \geq d'_0]$, where $s = q_0(n(t+1)+1)$, for t odd, $0 \leq t \leq \frac{n}{2} + 1$.
- 7. If $n = q_0 1$, $n \equiv 2 \pmod{4}$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \geq d'_0]$, where $s = q_0(n(t+1)+1)$, $0 \leq t \leq \frac{n}{2}$.
- 8. If $1 \le r < m$ and $\frac{n(p^r+1)}{2(p^r-1)}$ is odd, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$ (resp. $[s', \frac{s'}{2}, d \ge d'_0]$), where $s = q_0(t+1)n$ (resp. $s = q_0((t+1)n+1)+1$), $n = \frac{q-1}{p^r+1}$, for t odd, $1 \le t \le p^r$.
- 10. If $1 \le r < m$, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$ (resp. $[s', \frac{s'}{2}, d \ge d'_0]$), where $s = q_0(t+1)n$ (resp. $s = q_0((t+1)n+1) + 1$), $n = \frac{q-1}{p^r-1}, r|\frac{m}{2}$, for $1 \le t \le p^r 2$.
- 11. If t is even such that $1 \le t \le q_0$, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$, where $s = q_0(q_0t)$.
- 12. If t is odd such that $1 \le t \le q_0$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \ge d'_0]$, where $s = q_0(q_0 t)$.
- 13. If $r = p^k$, $k|m, 0 \le \ell < m/k$, $1 \le t \le (r-1)/2$, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$, where $s = q_0(2tr^{\ell})$.

- 14. If $0 \le \ell < 2m$, then there exists a q-ary self-dual code with parameters $[s, \frac{s}{2}, d \ge d_0]$, where $s = q_0(2p^{\ell})$.
- 15. If $r = p^k, k|m, 0 \le \ell < m/k, 0 \le t \le (r-1)/2$ or $(\ell, t) = (m/k, 0)$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \ge d'_0]$, where $s = q_0(2t+1)r^{\ell}$.
- 16. If $0 \le \ell < 2m$, then there exists a q-ary self-dual code with parameters $[s', \frac{s'}{2}, d \ge d'_0]$, where $s = q_0 p^{\ell}$.

Proof. It should be noted that each x-component $\alpha \in \mathbb{F}_q$ gives q places of degree one. Let U be a subset of $\{\alpha \in \mathbb{F}_q | \beta^{q_0} + \beta = \alpha^{q_0+1}\}$ such that q|U| = s. Put

$$h(x) = \prod_{\alpha \in U} (x - \alpha)$$
 and $\omega = \frac{dx}{h}$.

For each case, it is enough to prove that the residue $\operatorname{Res}_{P_{\alpha}}(\omega)$ of ω at place P_{α} is a nonzero square for any $\alpha \in U$, that is, $h'(\alpha)$ is a nonzero square in \mathbb{F}_q . Take U as follows.

- for 1), $U = \{ \alpha \in \mathbb{F}_q | \alpha^n = \alpha \},$
- for 2), $U = \{ \alpha \in \mathbb{F}_q | \alpha^{p^r} = \alpha \},$
- for 3), $U = U_{n-1} \cup \alpha_1 U_{n-1}$, where $U_{n-1} = \{\alpha \in \mathbb{F}_q | \alpha^{n-1} = 1\}$ and $\alpha_1 \in \mathbb{F}_q \setminus U_{n-1}$ such that $1 \alpha_1^{n-1}$ is a square,
- for 4), $U = \{ \alpha \in \mathbb{F}_q | \alpha^n = 1 \}$,
- for 5), $U = U_n \cup \alpha_1 U_n \cup \{0\}$, where $n = p^r 1$, $U_n = \{\alpha \in \mathbb{F}_q | \alpha^n = 1\}$ and $\alpha_1 \in \mathbb{F}_q \setminus U_n$ such that $1 \alpha^n$ is a square,
- for 6)-10), take $U = \{0\} \cup V$ or U = V, where $V = U_n \cup \alpha_1 U_n \cup \cdots \cup \alpha_t U_n$, $U_n = \{\alpha \in \mathbb{F}_q | \alpha^n = 1\}$ and $\alpha_1, \ldots, \alpha_t \in \mathbb{F}_q \setminus U_n$ as in Theorem 3,
- for 10)-12), label the elements of \mathbb{F}_{q_0} as a_1, \ldots, a_{q_0} . For some fixed element $\beta \in \mathbb{F}_q \backslash \mathbb{F}_{q_0}$, take $U = \{a_k \beta + a_j | 1 \leq k, j \leq q_0\}$,
- for 13)-16), label the element of \mathbb{F}_r as a_0, \ldots, a_{r-1} , take H as an \mathbb{F}_r -subspace and set $H_i = H + a_i \beta$ for some fixed element $\beta \in \mathbb{F}_q \backslash \mathbb{F}_r$. Put $U = H_0 \cup \cdots \cup H_{2t-1}$ or $U = H_0 \cup \cdots \cup H_{2t}$.

For 1)-5), it can be easily checked that $h'(\alpha)$ is a square for any $\alpha \in U$.

For 6)–10), it has been already checked, in Theorem 3, that $h'(\alpha)$ is a square for any $\alpha \in U$.

For 11)–12), it was proved in [24, Theorem 2] that $h'(\alpha)$ is a square for any $\alpha \in U$.

For 13)–16), it was proved in [7, Theorem 4] that $h'(\alpha)$ is a square for any $\alpha \in U$.

Example 6. The Hermitian curve defined over \mathbb{F}_9 has all rational points in the set $\{P_{\infty} = (1:0:0), (0:0:1), (0:w^2:1), (0:w^6:1), (1:w:1), (1:w^3:1), (1:2:1), (w^2:w:1), (w^2:w^3:1), (w^2:2:1), (2:w:1), (2:w^3:1), (2:2:1), (w^6:w:1), (w^6:w^3:1), (w^6:2:1), (w:1:1), (w:w^5:1), (w:w^7:1), (w^3:1:1), (w^3:w^5:1), (w^3:w^7:1), (w^5:1:1), (w^5:w^5:1), (w^5:w^7:1), (w^7:1:1), (w^7:w^5:1), (w^7:w^7:1)\}. Put <math>D = P_1 + \cdots + P_{27}, G = 15P_{\infty}$. The code $C_{\mathcal{L}}(D,G)$ has parameters [27, 13, 12]. The set $\{\frac{x^iy^j}{z^{i+j}}|(i,j)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(3,0),(3,1)\}\}$ is a basis for the code $C_{\mathcal{L}}(D,G)$, and thus its generator matrix is given by

$$\mathcal{G} = \begin{pmatrix} 11111111111111111111111111\\ 0w^2w^6w^32ww^32ww^32w^321w^5w^71w^5w^71w^5w^71w^5w^7\\ 022w^2w^61w^2w^61w^2w^61w^2w^61w^2w^61w^2w^61w^2w^6\\ 0w^6w^2w^3w2w^3w2w^3w2w^3w21w^7w^51w^7w^51w^7w^5\\ 000111w^2w^2w^2222w^6w^6w^6www^3w^3w^3w^5w^5w^5w^7w^7w^7\\ 0000w^22w^3w^5w^6w^5w^71w^7w^2w^2w^61w^31w^2w^5w^22w^72w^6\\ 000w^2w^6121w^2w^6w^2212w^6w^3w^7w^3w^7w^3w^7w^5\\ 0000w^3w^2w^5w^3w^6w^7w^51ww^7w^2u1w^6w^3w^21y^52w^2v^7w^6\\ 0000w^32w^5w^3w^6w^7w^51w^7w^7w^3w^7w^3w^7w^5\\ 000111222111222w^2w^2w^6w^6w^6w^2w^2w^2w^6w^6w^6\\ 000w^32w^5v^71w^33w^5w^71w^7w^6w^5w^5w^7w^7w^6w^5w^7w^7w^6w^5w^5\\ 000w^2w^61w^6w^22w^2w^61w^6w^22w^221w^612w^221w^612\\ 000111w^6w^6w^6222w^2w^2w^3w^3w^3www^7w^7w^7w^5w^5w^5\\ 000w^32w^7ww^2w^5v^71w^3w^5w^6w^31w^2w^61w^72w^6w^5w^22 \end{pmatrix}$$

Take $\mathcal{G}' = \begin{pmatrix} \mathcal{G} & 0 \\ g_{14} \end{pmatrix}$, where $g_{14} = (0, 1, 1, 2, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2$

Theorem 10. Let $q_0 = p^m, q = q_0^2$ be an odd prime power, $g = \frac{(q_0 - 1)^2}{4}$ and $s = q_0 \frac{q_0^2 + 1}{2}$. Then,

- 1. there exists a $[s, \frac{s}{2}, d \geq \frac{s}{2} g + 1]$ self-dual code if s is even, and
- 2. there exists a $[s+1, \frac{s+1}{2}, d \ge \frac{s+1}{2} g]$ self-dual code if s is odd.

Table 3: Self-dual codes of length s from Hermitian curves defined over \mathbb{F}_9

Theorem 9	Length s	Distance	Lower bound	Extended length	Distance	Lower bound
1)	3.3	3	3	10	3	2
1)	3.9	12	12	28	12	11
2)	3.3	3	3	10	3	2
3)	3(2.3-1)	6	6	16	6	5
4)	3.2	3	1	_	_	_
4)	3.4	4	4	_	_	_
5)	3(1+1)2	4	4	_	_	_
6)	3((1+1)2+1)	6	6	16	6	5
11)	3 (3.2)	7	7	_	_	

Proof. Consider an algebraic curve defined by

$$\mathcal{X}: y^{q_0} + y = x^{\frac{q_0+1}{2}}.$$

The curve has genus $g = \frac{(q_0-1)^2}{4}$. Put

$$U = \{ \alpha \in \mathbb{F}_q | \exists \beta \in \mathbb{F}_q \text{ such that } \beta^{q_0} + \beta = \alpha^{\frac{q_0 + 1}{2}} \}.$$

The set U is the set of x-component solutions to the Hermitian curve whose elements are squares in \mathbb{F}_q . There are $\frac{q_0^2+1}{2}$ square elements in \mathbb{F}_q , and this gives rise to $q_0 \frac{q_0^2+1}{2}$ rational places. Write

$$h(x) = \prod_{\alpha \in U} (x - \alpha)$$
 and $\omega = \frac{dx}{h}$.

Then $h(x) = x^n - x$, where $n = \frac{q_0^2 + 1}{2}$, and thus $h'(x) = nx^{n-1} - 1$. Since q is a square, we have that $h'(\alpha) = n - 1$ is a square for any $\alpha \in U \setminus \{0\}$. Put $D = \sum_{\alpha \in U} \left(P_{\alpha}^{(1)} + \dots + P_{\alpha}^{(q_0)} \right) = P_1 + \dots + P_s, s = q_0 \frac{q_0^2 + 1}{2}$. Set

$$G = \begin{cases} (g - 1 + \frac{s}{2})P_{\infty} & \text{if } s \text{ is even,} \\ (g - 1 + \frac{s - 1}{2})P_{\infty} & \text{if } s \text{ is odd.} \end{cases}$$

Then the residue $\operatorname{Res}_{P_{\alpha}}(\omega) = \frac{1}{h'(P_{\alpha})}$ is a square for any $\alpha \in U$, and by Lemma 3, the constructed code $a \cdot C_{\mathcal{L}}(D,G)$ is self-orthogonal, where $a_i^2 = \operatorname{Res}_{P_i}(\omega)$. If s is even, then point 1) follows, otherwise the self-orthogonal code can be embedded into a self-dual code using Lemma 6, and thus point 2) follows. \square

Example 7. There exist self-dual codes with parameters $[16, 8, \geq 7]_{3^2}$, $[66, 33, \geq 29]_{5^2}$, $[176, 88, \geq 79]_{7^2}$, $[370, 185, \geq 169]_{9^2}$, $[672, 336, \geq 311]_{11^2}$, $[1106, 553, \geq 517]_{13^2}$, $[2466, 1233, \geq 1169]_{17^2}$, $[3440, 1720, \geq 1639]_{19^2}$, $[7826, 3913, \geq 3769]_{25^2}$. We now calculate the exact distance of the self-dual code over \mathbb{F}_{3^2} . The algebraic curve over \mathbb{F}_9 defined by

$$y^3 + y = x^2$$

has all rational points in the set $\{P_{\infty} = (1:0:0), (0:0:1), (0:w^2:1), (0:w^6:1), (1:w:1), (1:w^3:1), (1:2:1), (2:w:1), (2:w^3:1), (2:2:1), (w^2:1:1), (w^2:w^5:1), (w^2:w^7:1), (w^6:1:1), (w^6:w^5:1), (w^6:w^7:1)\}$. Put $D = P_1 + \cdots + P_{15}$, $G = 7P_{\infty}$. The code $C_{\mathcal{L}}(D,G)$ has parameters [15, 7, 8].

The set $\{\frac{x^iy^j}{z^{i+j}}|(i,j)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2)\}\}$ is a basis for the code $C_{\mathcal{L}}(D,G)$, and thus its generator matrix is given by

Take

$$\mathcal{G}' = \left(\begin{array}{c} a \cdot \mathcal{G} & 0 \\ g_8 \end{array}\right),$$

Theorem 11. Let $q_0 = p^m, q = q_0^2$ be an odd prime power, $g = \frac{(q_0 - 1)^2}{4}$ and $s = q_0 \frac{q_0^2 - 1}{2}$. Then,

- 1. there exists a $[s, \frac{s}{2}, d \ge \frac{s}{2} g + 1]$ self-dual code if s is even, and
- 2. there exists a $[s+1, \frac{s+1}{2}, d \ge \frac{s+1}{2} g]$ self-dual code if s is odd.

Proof. Consider the same setting as the proof of Theorem 10. Take $U' = U \setminus \{0\}$, and write

$$h(x) = \prod_{\alpha \in U'} (x - \alpha)$$
 and $\omega = \frac{dx}{h}$.

The rest follows with the same reasoning as that in Theorem 10.

Example 8. There exist self-dual codes with parameters $[12, 6, 6]_{3^2}$, $[60, 30, \ge 27]_{5^2}$, $[168, 84, \ge 76]_{7^2}$, $[360, 180, \ge 165]_{9^2}$, $[660, 330, \ge 306]_{11^2}$, $[1092, 546, \ge 511]_{13^2}$, $[2448, 1224, \ge 1161]_{17^2}$, $[3420, 1710, \ge 1630]_{19^2}$, $[7800, 3900, \ge 3757]_{25^2}$.

We update parameters of MDS self-dual codes from the previous constructions in Table 4.

4 Conclusion

In this correspondence, we have constructed new families of optimal q-ary Euclidean self-dual codes from algebraic curves. With the same spirit, constructing more families of Euclidean self-dual codes from genus zero and genus one curves (over \mathbb{F}_q with q a prime) is worth considering. Characterization and constructions of Hermitian self-dual codes from algebraic geometry codes are also valuable.

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Table 4: MDS self-dual codes of length n over \mathbb{F}_q , -: no self-dual code exists with such a pair (n,q), +: known parameters, ?: unknown parameters, *: new parameters

n/q	11	13	16	17	19	23	25	27	29	31	32	37	41	43	47	49	53	61	73	81
2	_	+	+	+	_	_	+	_	+	_	+	+	+	_	_	+	+	+	+	+
4	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6	_	+	+	+	_	_	+	_	+	_	+	+	+	_	_	+	+	+	+	+
8	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
10	_	+	+	+	_	_	+	_	+	_	+	+	+	_	_	+	+	+	+	+
12	+	6	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
14	_	+	+	?	_	_	+	_	+	_	+	+	+	_	_	+	+	+	+	+
16			+	?	?	+	+	+	+	+	+	+	+	+	+	+	?	+	?	+
18				+	_	_	+	_	+	_	+	+	+	_	_	+	?	?	+	+
20					+	?	+	?	?	+	+	?	+	+	+	+	?	+	+	+
22	_				_	_	?	_	?	_	+	?	+	?	?	+	?	+	?	+
24						+	?	?	?	+	+	?	?	+	+	+	?	?	+	*
26	_				_	_	+	_	?	_	+	*	?	_	_	+	+	*	+	+
28								+	?	?	+	?	?	?	?	+	?	?	?	+
30	_				_	_			+		+	?	?	_	_	?	?	+	?	+
32										+	+	?	*	?	?	?	?	?	?	+
34	_				_	_		_		_		?	?	_	_	+	?	?	?	+
36												?	?	?	?	+	?	?	+	+
38	_				_	_		_		_		+	?	_	_	+	?	?	+	?
40													?	?	?	?	?	?	?	+
42	_				_	_		_		_			+	_	_	+	?	*	?	+
44														+	?	?	?	?	?	?
46	_				_	_		_		_				_	_	?	?	?	?	+
48															+	?	?	?	?	?
50	_				_	_		_		_				_	_	+	?	?	*	+
52																	?	?	?	+
62	_				_	_		_		_				_	_			+	?	+
72																		?	?	+

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