Flag-transitive 4-designs and $PSL(2,q)$ groups

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Abstract This paper considers flag-transitive 4-$(q+1,k,\lambda)$ designs with $\lambda \geq 5$ and $q+1>k>4$. Let the automorphism group of a design $\mathcal{D}$ be a simple group $G=PSL(2,q)$. Depend on the fact that the setwise stabilizer G_B must be one of twelve kinds of subgroups, up to isomorphism we get the following two results. (i) If $10 \geq \lambda \geq 5$, then except $(G,G_x,G_B,k,\lambda)=(PSL(2,761),E_{761}\rtimes C_{380},S_4,24,7)$ or $(PSL(2,512),E_{512}\rtimes C_{511},D_{18},18,8)$ undecided, $\mathcal{D}$ is a 4-$(24,8,5)$, 4-$(9,8,5)$, 4-$(8,6,6)$, 4-$(10,9,6)$, 4-$(9,6,10)$, 4-$(9,7,10)$, 4-$(12,11,8)$ or 4-$(14,13,10)$ design with $G_B=D_8$, $E_8\rtimes C_7$, D_6, $E_9\rtimes C_4$, $PSL(2,2)$, D_{14}, $E_{11}\rtimes C_5$ or $E_{13}\rtimes C_6$ respectively. (ii) If $\lambda>10$, $G_B=A_4$, S_4, A_5, $PGL(2,q_0)(g>1$ even) or $PSL(2,q_0)$, where ${q_0}^g=q$, then there is no such design.

Keywords 4-Design, flag-transitive, $PSL(2,q)$

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1 Introduction

A t-(v,k,λ) design with a simple group $G=PSL(2,q)$ as an automorphism group is interesting. Many classification results about flag-transitive designs have been got. In 1986, 2-$(v,k,1)$ designs are classified by Delandtsheer[1]. In 2016, 2-(v,k,λ) symmetric designs with the sole of G equal to $PSL(2,q)$ are studyed by Alavi, Bayat and Daneshkhah[2]. In 2018, all nonsymmetric 2-designs with $(r,\lambda)=1$ are determined by Zhan and Zhou[3]. In 2019, 2-designs with $k=4$ are considered by Zhan, Ding and Bai[4]. For 3-designs, there are also many results given in[5, 6, 7, 8]. After Dai and Li[9, 10] classified flag-transitive 4-designs with $\lambda=3$ or 4, we continue this work, and get the following two theorems:

Theorem 1.1. *Let $\mathcal{D}$ be a flag-transitive* 4-$(q+1,k,\lambda)$ *design with* $10 \geq \lambda \geq 5$ *and* $q+1>k>4$, $G=PSL(2,q)$ *be an automorphism simple group.*

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Then up to isomorphism except $(G, G_x, G_B, k, \lambda) = (PSL(2,761), E_{761} \rtimes C_{380}, S_4, 24, 7)$ *or* $(PSL(2,512), E_{512} \rtimes C_{511}, D_{18}, 18, 8)$ *undecided,* $\mathcal{D}$ *is a* 4-$(24,8,5)$*,* 4-$(9,8,5)$*,* 4-$(8,6,6)$*,* 4-$(10,9,6)$*,* 4-$(9,6,10)$*,* 4-$(9,7,10)$*,* 4-$(12,11,8)$ *or* 4-$(14,13,10)$ *design with* $G_B = D_8$*,* $E_8 \rtimes C_7$*,* D_6*,* $E_9 \rtimes C_4$*,* $PSL(2,2)$*,* D_{14}*,* $E_{11} \rtimes C_5$ *or* $E_{13} \rtimes C_6$ *respectively.*

Theorem 1.2. *Let* $\mathcal{D}$ *be a flag-transitive* 4-$(q+1, k, \lambda)$ *design with* $\lambda > 10$ *and* $q+1 > k > 4$*,* $G = PSL(2,q)$ *be a simple automorphism group of* $\mathcal{D}$*,* $G_B = A_4$*,* S_4*,* A_5*,* $PGL(2,q_0)$*(*$g > 1$ *even) or* $PSL(2,q_0)$*, where* ${q_0}^g = q$*. Then there is no such design.*

2 Preliminaries

We give some useful results. The notations n, G_B and G_{xB} denote $\gcd(2, q-1)$, the setwise stabilizer and $G_x \cap G_B$ respectively, where $x \in B$.

Lemma 2.1. *Let* $\mathcal{D}$ *be a* 4-$(q+1, k, \lambda)$ *design with an automorphism group* $G = PSL(2,q)$ *flag-transitively acting on* $\mathcal{D}$*. Then*
(i) $\lambda(q-2) = \frac{k(k-1)(k-2)(k-3)}{n|G_B|}$*;*
(ii) $q = \frac{(k-1)(k-2)(k-3)}{\lambda n |G_{xB}|} + 2$*;*
(iii) $k \mid (\lambda n(q-2)|G_{xB}| + 6)$*;*
(iv) $k \mid \gcd(\frac{q(q^2-1)}{n}, \lambda n(q-2)|G_{xB}| + 6)$.

Proof. Since G is flag-transitive, then by [11] G is 2-transitive, therefore, G is also block-transitive and point-primitive. From $|G| = |G_B||B^G| = |G_{xB}||(x,B)^G|$, we get $b = \frac{|G|}{|G_B|} = \frac{|G|}{|G_{xB}|k}$. By [12], $\frac{\lambda q(q+1)(q-1)(q-2)}{k(k-1)(k-2)(k-3)} = \frac{q(q^2-1)}{n|G_B|} = \frac{q(q^2-1)}{nk|G_{xB}|}$. Obviously (i) and (ii) hold. Since $(k-1)(k-2)(k-3) = (k^2-6k+11)k - 6$, $k \mid |G_B|$ and $|G_B| \mid |G|$, (iii) and (iv) hold.

3 Proof of two theorems

First we assume that $\mathcal{D}$ is a 4-$(q+1, k, \lambda)$ design with $\lambda \geq 5$ and $q+1 > k > 4$, $G = PSL(2,q)$ is a flag-transitive automorphism group of $\mathcal{D}$, where $q = p^f \geq 4$ and p is a prime. Since $\mathcal{D}$ is flag-transitive, we get that k is one orbit length of G_B acting on $\mathcal{D}$. By [10, 13], up to conjugation G_B must be one of twelve kinds of subgroups, then several results can be got.

Lemma 3.1. $G_B \neq A_4$*,* S_4 *or* A_5 *except* $(G, G_x, G_B, k, \lambda) = (PSL(2,761), E_{761} \rtimes C_{380}, S_4, 24, 7)$ *undecided.*

Proof. Assume that $G_B = A_4$, S_4 or A_5, then $|G_B| = 12$, 24 or 60. Since $k \mid |G_B|$, we have $(|G_B|, k) = (12, 12)$, $(12, 6)$, $(24, 24)$, $(24, 12)$, $(24, 8)$, $(24, 6)$, $(60, 60)$, $(60, 30)$, $(60, 20)$, $(60, 15)$, $(60, 12)$, $(60, 10)$, $(60, 6)$ or $(60, 5)$. Note that $q > k-1$, q is a power of a prime and $\lambda \geq 5$, by Lemma 2.1(i) and (ii), we get that $(|G_B|, k, q, \lambda) = (12, 12, 32, 33)$, $(12, 12, 13, 45)$, $(12, 12, 17, 33)$, $(12, 12, 47, 11)$, $(12, 12, 101, 5)$, $(12, 6, 8, 5)$, $(24, 24, 25, 231)$, $(24, 24, 71, 77)$, $(24, 24, 79, 69)$, $(24, 24, 163, 33)$, $(24, 24, 233, 23)$, $(24, 24, 761, 7)$, $(24, 8, 16, 5)$, $(24, 8, 9, 5)$,$(60, 60, 61, 1653)$, $(60, 60, 89, 1121)$, $(60, 60, 179, 551)$, $(60, 60, 1123, 87)$, $(60, 30, 128, 87)$, $(60, 30, 31, 189)$, $(60, 30, 89, 63)$, $(60, 30, 263, 21)$, $(60, 30, 191, 29)$, $(60, 20, 53, 19)$, $(60, 20, 59, 17)$, $(60, 15, 16, 39)$, $(60, 15, 23, 13)$, $(60, 15, 41, 7)$, $(60, 12, 13, 9)$ or $(60, 10, 16, 6)$. By the lengths of orbits listed in [10, 13], we need consider $(|G_B|, k, q, \lambda) = (12, 12, 17, 33)$, $(12, 12, 47, 11)$, $(12, 12, 101, 5)$, $(24, 24, 71, 77)$, $(24, 24, 79, 69)$, $(24, 24, 233, 23)$, $(24, 24, 761, 7)$, $(60, 60, 89, 1121)$, $(60, 60, 179, 551)$ or $(60, 30, 89, 63)$. Using Magma[14] we can rule out all cases except case 7. The possible parameters are listed in Table 1, where the notation m^n means that the degree m appears n times. For example, in case 1, the group PrimitiveGroup(18,1) denotes the primitive permutation group $G = PSL(2, 17)$ acting on the set $\Omega = \{1, 2, \ldots, 18\}$. And G have two conjugacy classes of subgroups of order 12 and the possible lengths of all orbits of G_B acting on Ω are both 6, 12. There are 2 possible basic blocks B_0 of designs. However, the command Design<4,18|D> where $\mathcal{D} = B^G$ returns both structures are not 4-designs, then case 1 can be ruled out. For case 7, from $|G_x| = \frac{|G|}{v} = 289180$, we know $G_x = E_{761} \rtimes C_{380}$. There are 64 possible basic blocks B_0 of designs. However, $b = \frac{|G|}{|G_B|} = 9181465$ is too large to use Magma. This case is undecided.

Table 1: Possible parameters with $G_B = A_4$, S_4 or A_5

Case	G_B	(v, k, λ)	position	the lengths of all orbits
1	A_4	$(18, 12, 33)$	(18,1)	6,12; 6,12
2	A_4	$(48, 12, 11)$	(48,1)	12^4; 12^4; 12^4; 12^4; 12^4
3	A_4	$(102, 12, 5)$	(102,2)	6, 12^8
4	S_4	$(72, 24, 77)$	(72,1)	24^3; 24^3; 24^3
5	S_4	$(80, 24, 69)$	(80,1)	8, 24^3; 8, 24^3
6	S_4	$(234, 24, 23)$	(234,4)	6, 12, 24^9
7	S_4	$(762, 24, 7)$	(761,1)	6, 12, 24^{31}; 6, 12, 24^{31}
8	A_5	$(90, 60, 1121)$	(90,1)	30, 60; 30, 60
9	A_5	$(180, 60, 551)$	(180,1)	60^3; 60^3; 60^3
10	A_5	$(90, 60, 63)$	(90,1)	30, 60; 30, 60

Lemma 3.2. *$G_B = PGL(2, q_0)$, where ${q_0}^g = q$ and $g > 1$ even.*

Proof. Assume that $G_B = PGL(2, q_0)$, where ${q_0}^g = q$ and $g > 1$ even. Then $k = q_0 + 1$, $q_0(q_0 - 1)$ or $q_0({q_0}^2 - 1)$.

If $k = q_0 + 1$, then by Lemma 2.1(i) $n\lambda({q_0}^g - 2) = q_0 - 2$. However, there is no solution of this equation.

If $k = q_0(q_0 - 1)$, then $q_0 \geq 3$ and

$$ {q_0}^5 - 2 > n\lambda({q_0}^g - 2) = (q_0 - 2)({q_0}^2 - q_0 - 1)({q_0}^2 - q_0 - 3). $$

Thus $g = 2$ or 4. Since

$$ \begin{aligned} &(q_0 - 2)({q_0}^2 - q_0 - 1)({q_0}^2 - q_0 - 3) \\ &= ({q_0}^2 - 2)[(q_0 - 2)({q_0}^2 - 2q_0 - 1)] + (q_0 - 2) \\ &= (q_0 - 4)({q_0}^4 - 2) + ({q_0}^3 + 10{q_0}^2 - 3q_0 - 14), \end{aligned} $$

we get $({q_0}^2 - 2) \mid (q_0 - 2)$ or $({q_0}^4 - 2) \mid ({q_0}^3 + 10{q_0}^2 - 3q_0 - 14)$. However, this is impossible.

If $k = q_0({q_0}^2 - 1)$, then

$$ {q_0}^9 - 2 > n\lambda({q_0}^g - 2) = ({q_0}^3 - q_0 - 1)({q_0}^3 - q_0 - 2)({q_0}^3 - q_0 - 3), $$

Thus $g = 2$, 4, 6 or 8. Since

$$ \begin{aligned} &({q_0}^3 - q_0 - 1)({q_0}^3 - q_0 - 2)({q_0}^3 - q_0 - 3) \\ &= ({q_0}^2 - 2)({q_0}^7 - {q_0}^5 - 6{q_0}^4 + {q_0}^3 + 12q_0 - 6) + (13q_0 - 18) \\ &= ({q_0}^4 - 2)({q_0}^5 - 3{q_0}^3 - 6{q_0}^2 + 5q_0 + 12) + (4{q_0}^3 - 18{q_0}^2 - q_0 + 18) \\ &= ({q_0}^6 - 2)({q_0}^3 - 3q_0 - 6) + (3{q_0}^5 + 12{q_0}^4 + 12{q_0}^3 - 6{q_0}^2 - 17q_0 - 18) \\ &= q_0({q_0}^8 - 2) + (-3{q_0}^7 - 6{q_0}^6 + 3{q_0}^5 + 12{q_0}^4 + 10{q_0}^3 - 6{q_0}^2 - 9q_0 - 6), \end{aligned} $$

we get $({q_0}^2 - 2) \mid (13q_0 - 18)$, $({q_0}^4 - 2) \mid (4{q_0}^3 - 18{q_0}^2 - q_0 + 18)$, $({q_0}^6 - 2) \mid (3{q_0}^5 + 12{q_0}^4 + 12{q_0}^3 - 6{q_0}^2 - 17q_0 - 18)$ or $({q_0}^8 - 2) \mid (3{q_0}^7 + 6{q_0}^6 - 3{q_0}^5 - 12{q_0}^4 - 10{q_0}^3 + 6{q_0}^2 + 9q_0 + 6)$. Therefore, $(g, q_0) = (2, 2)$, $(2, 3)$. However, this is contrary with $q > k - 1$.

Lemma 3.3. *Let $G_B = PSL(2, q_0)$, where ${q_0}^g = q$. Then $\mathcal{D}$ is a 4-$(9, 6, 10)$ design with $G_B = PSL(2, 2)$, denoted by $\mathcal{D}_1$.*

Proof. Assume tht $G_B = PSL(2, q_0)$, where ${q_0}^g = q$. Then $k = q_0 + 1$, $q_0(q_0 - 1)$ if g is even or $\frac{q_0({q_0}^2 - 1)}{\gcd(2, q_0 - 1)}$.

If $k = q_0 + 1$, then $\lambda({q_0}^g - 2) = q_0 - 2$. However, there is no such $q_0 > 3$ satisfying the equation.

If $k = q_0(q_0 - 1)$, then $q_0 \geq 3$ and ${q_0}^5 - 2 > \lambda({q_0}^g - 2) = (q_0 - 2)({q_0}^2 - q_0 - 1)({q_0}^2 - q_0 - 3)$. Thus $g = 2$ or 4. The same as in Lemma 3.2, this is impossible.

If $k=\frac{q_0({q_0}^2-1)}{\gcd(2,q_0-1)}$, then ${q_0}^9-2>n^4\lambda({q_0}^g-2)=({q_0}^3-q_0-n)({q_0}^3-q_0-2n)({q_0}^3-q_0-3n)$. Note that $q+1>k>4$, we get $8\geq g\geq 2$.

Since

$$
\begin{aligned}
&({q_0}^3-q_0-n)({q_0}^3-q_0-2n)({q_0}^3-q_0-3n)\\
&=({q_0}^2-2)[{q_0}^7-{q_0}^5-6n{q_0}^4+{q_0}^3+(11n^2+1)q_0-6n]+[(11n^2+2)q_0\\
&-6n^3-12n]\\
&=({q_0}^3-2)[{q_0}^6-3{q_0}^4-(6n-2){q_0}^3+3{q_0}^2+(12n-6)q_0+(11n^2-12n\\
&+3)]+[-(6n-6){q_0}^2-(11n^2-24n+12)q_0-6n^3+22n^2-24n+6]\\
&=({q_0}^4-2)({q_0}^5-3{q_0}^3-6n{q_0}^2+5q_0+12n)+[(11n^2-7){q_0}^3-18n{q_0}^2-\\
&(11n^2-10)q_0-6n^3+24n]\\
&=({q_0}^5-2)({q_0}^4-3{q_0}^2-6nq_0+3)+[(12n+2){q_0}^4+(11n^2-1){q_0}^3-(6n\\
&+6){q_0}^2-(11n^2+12n)q_0-6n^3+6]\\
&=({q_0}^6-2)({q_0}^3-3q_0-6n)+[3{q_0}^5+12n{q_0}^4+(11n^2+1){q_0}^3-6n{q_0}^2-\\
&(11n^2+6)q_0-6n^3-12n]\\
&=({q_0}^7-2)({q_0}^2-3)+[-6n{q_0}^6+3{q_0}^5+12n{q_0}^4+(11n^2-1){q_0}^3-(6n-2)\\
&{q_0}^2-11n^2q_0-6n^3-6]\\
&=q_0({q_0}^8-2)+[-3{q_0}^7-6n{q_0}^6+3{q_0}^5+12n{q_0}^4+(11n^2-1){q_0}^3-6n{q_0}^2-\\
&(11n^2-2)q_0-6n^3],
\end{aligned}
$$

we get $(g,q_0)=(3,2)$. Thus $(q,k,\lambda)=(8,6,10)$. The generators of $G=PSL(2,8)$ acting on the set $\Omega=\{1,2,\dots,9\}$ are as follows: $g_1=(1,8)(2,4)(3,7)(5,6)$, $g_2=(2,7)(3,6)(4,5)(8,9)$, $g_3=(1,2,3,4,5,6,7)$. Then all orbits of G_B acting on Ω are $\Omega_1=\{4,6,8\}$ and $\Omega_2=\{1,2,3,5,7,9\}$. Take $B=\Omega_2$ as a possible basic block of a design, then it returns a 4-$(9,8,5)$ design $\mathcal{D}_1$. Clearly, $\mathcal{D}_1$ is also flag-transitive.

Next we consider G_B is one of the remaining four kinds of subgroups and assume that $10\geq\lambda\geq 5$.

Lemma 3.4. *Let $G_B=C_c$ or D_{2c}, where $c\mid\frac{q\pm1}{n}$, then $(\lambda,|G_B|,c,k,q)=(5,8,4,8,23)$, $(8,18,9,18,512)$, $(10,6,3,6,8)$, $(7,8,8,8,17)$, $(7,8,4,8,17)$, $(10,14,7,7,8)$ or $(6,6,3,6,7)$.*

Proof. Assume that $G_B=C_c$ or D_{2c}, where $c\mid\frac{q\pm1}{n}$. Then $k=c$ or $2c(|G_B|=2c)$.

Let $c\mid\frac{q+1}{n}$, then by Lemma 2.1(iii) $c\mid\gcd(\lambda n(q-2)|G_{xB}|+6,\frac{q+1}{n})$, that is, $c\mid\gcd(3\lambda n|G_{xB}|-6,\frac{q+1}{n})$. By Lemma 2.1(ii), $(\lambda,|G_B|,c,k,q)=(5,8,4,8,23)$, $(8,18,9,18,512)$ or $(10,6,3,6,8)$.

Let $c \mid \frac{q-1}{n}$. Assume that $(\lambda, n, |G_B|, k) = (6, 1, 2c, 2c)$, then $6(2^f-2) = (2c-1)(2c-2)(2c-3) = 2c(4c^2-12c+11)-6$, therefore, $3\frac{2^f-1}{c} = 4c^2-12c+11$. From $3 \mid (4c^2-12c+11)$, we have that $c = 3l+1$ or $3l+2$, where l is a positive integer. If $c = 3l+1$, then $2^f = l(6l+1)(6l-1)+2$, therefore, $8 \mid (l(6l+1)(6l-1)+2)$, we get $l \equiv 2 (\mod 8)$. Let $l = 8m+2$, then $2^{f-1} = (4m+1)(48m+13)(48m+11)+1$. Obviously, $m \neq 0$, therefore, $2^{f-1} \geq 5 \cdot 61 \cdot 59 + 1 > 2^{14}$. Thus $2^{14} \mid ((4m+1)(48m+13)(48m+11)+1)$, however, this is impossible. Assume that $(\lambda, n, |G_B|, k) = (6, 1, c, c)$, then $6(2^f-2) = (c-1)(c-2)(c-3) = c(c^2-6c+11)-6$, therefore, $6\frac{2^f-1}{c} = c^2-6c+11$. From $6 \mid (c^2-6c+11)$, we have that $c = 6l+1$ or $6l+5$, where l is an integer. If $c = 6l+1$, then $2^{f-1} = l(6l-1)(3l-1)+1$, therefore, $4 \mid (l(6l-1)(3l-1)+1)$, that is, $4 \mid (18l^3-9l^2+l+1)$. However, this is impossible. If $c = 6l+5$, then $2^{f-1} = (3l+2)(2l+1)(3l+1)+1$, therefore, $4 \mid ((3l+2)(2l+1)(3l+1)+1)$, that is, $4 \mid (18l^3+27l^2+13l+3)$. However, this is also impossible. Now we consider that $(\lambda, n, |G_B|, k) \neq (6, 1, 2c, 2c)$ and $(6, 1, c, c)$, then $c \mid \gcd(\lambda n(q-2)|G_{xB}|+6, \frac{q-1}{n})$, that is, $c \mid \gcd(\lambda n|G_{xB}|-6, \frac{q-1}{n})$. Thus $(\lambda, |G_B|, c, k, q) = (7, 8, 8, 8, 17)$, $(7, 8, 4, 8, 17)$, $(7, 16, 8, 16, 197)$, $(10, 14, 7, 7, 8)$, $(6, 6, 3, 6, 7)$. By employing the command Subgroups(G:OrderEqual:=b), we rule out $(\lambda, |G_B|, c, k, q) = (7, 16, 8, 16, 197)$.

Lemma 3.5. *Let $G_B = E_{q_0}$ or $E_{q_0} \rtimes C_c$, where $q_0 \mid q$, $c \mid (q_0-1)$ and $c \mid (q-1)$, then $(\lambda, i, k, q) = (5, 1, 8, 8)$, $(6, 2, 9, 9)$, $(8, 2, 11, 11)$ or $(10, 2, 13, 13)$ where $k = q_0$ and $c = \frac{k-1}{i}$.*

Proof. Let $G_B = E_{q_0}$ or $E_{q_0} \rtimes C_c$, where $q_0 \mid q$, $c \mid (q_0-1)$ and $c \mid (q-1)$. Then $k = q_0$ or cq_0.

Let $k = q_0$, then $k \mid \gcd(\lambda n(q-2)|G_{xB}|+6, q)$, that is, $k \mid \gcd(2\lambda n|G_{xB}|-6, q)$. If $|G_B| = q_0$, then $(\lambda, |G_B|, k, q) = (7, 8, 8, 32)$. If $|G_B| = cq_0$, then $|G_{xB}| = c$, therefore, $k \leq 2\lambda nc - 6$, so $\frac{k-1}{2\lambda n} < \frac{k+6}{2\lambda n} \leq c$. From $c \mid (k-1)$, we let $c = \frac{k-1}{i}$, where $i = 1, 2, \cdots, 2\lambda n - 1$. Then from $k \mid \gcd(\frac{2\lambda n(k-1)}{i} - 6, q)$, we get $k \mid \gcd(2\lambda n + 6i, q)$. All possible parameters of (λ, i, k, q) are $(5, 1, 8, 8)$, $(5, 1, 13, 13)$, $(6, 2, 9, 9)$, $(6, 3, 7, 7)$, $(7, 1, 17, 17)$, $(8, 1, 19, 19)$, $(8, 2, 11, 11)$, $(10, 1, 23, 23)$ or $(10, 2, 13, 13)$. By employing the command Subgroups(G:OrderEqual:=b), we rule out $(\lambda, i, k, q) = (5, 1, 13, 13)$, $(6, 3, 7, 7)$, $(7, 1, 17, 17)$, $(8, 1, 19, 19)$ and $(10, 1, 23, 23)$.

If $k = cq_0$, then $|G_{xB}| = 1$. From $k \mid (\lambda n(q-2)+6)$, we have that $q_0 \mid \gcd(2\lambda n - 6, q)$. Since $c \mid (q_0-1)$. We get no possible parameters.

Lemma 3.6. *Let $G_B = C_c$, D_{2c} or E_{q_0}, $E_{q_0} \rtimes C_c$, where $c \mid \frac{q \pm 1}{n}$ or $q_0 \mid q$, $c \mid (q_0-1)$ and $c \mid (q-1)$. Then except $(G, G_x, G_B, k, \lambda) = (PSL(2, 512), E_{512} \rtimes C_{511}, D_{18}, 18, 8)$ undecided, up to isomorphism $\mathcal{D}$ is a 4-$(24, 8, 5)$, 4-$(9, 8, 5)$, 4-$(8, 6, 6)$, 4-$(10, 9, 6)$, 4-$(9, 6, 10)$, 4-$(9, 7, 10)$, 4-$(12, 11, 8)$ or 4-$(14, 13, 10)$*

design with $G_B = D_8$, $E_8 \rtimes C_7$, D_6, $E_9 \rtimes C_4$, $PSL(2,2)$, D_{14}, $E_{11} \rtimes C_5$ or $E_{13} \rtimes C_6$ respectively.

Proof. We consider the cases appearing in Lemma 3.4 and 3.5 and list the results in Table 2. The lengths of all orbits and the number of the designs are listed in column 5 and column 6 respectively, where twice or five times denote such lengths of orbits appearing twice or five times.

Table 2: Possible values with $G_B = C_c$, D_{2c}, E_{q_0} or $E_{q_0} \rtimes C_c$

Case	G_B	(v,k,λ)	position	lengths	number
1	D_8	$(24,8,5)$	(24,1)	8^3	2
2	D_{18}	$(513,18,8)$	(513,10)	9, 18^{28}	?
3	D_6	$(9,6,10)$	(9,8)	3,6	1
4	D_6	$(8,6,6)$	(8,4)	2,6	1
5	D_8	$(18,8,7)$	(18,1)	2, 4^2, 8(twice)	0
6	C_8	$(18,8,7)$	(18,1)	1^2,8^2	0
7	D_{14}	$(9,7,10)$	(9,8)	2,7	1
8	E_8	$(33,8,7)$	(33,1)	1, 8^4(five times)	0
9	$E_8 \rtimes C_7$	$(9,8,5)$	(9,8)	1,8	1
10	$E_9 \rtimes C_4$	$(10,9,6)$	(10,3)	1,9	1
11	$E_{11} \rtimes C_5$	$(12,11,8)$	(12,1)	1,11	1
12	$E_{13} \rtimes C_6$	$(14,13,10)$	(14,1)	1,13	1

For case 2, from $|G_x| = \frac{|G|}{v} = 261632$, we know $G_x = E_{761} \rtimes C_{380}$. There are 28 possible basic blocks B_0 of designs. However, $b = \frac{|G|}{|G_B|} = 7456512$ is too large to use Magma.

For the remaining cases, the same as before, we can deal with them by the same method. For example, the generators of the group $G = PSL(2,23)$ acting on the set $\Omega = \{1,2,\ldots,24\}$ are listed as follows:
$g_1 = (1,8)(2,10)(3,16)(4,24)(5,15)(6,21)(7,23)(9,20)(11,22)(12,14)$
$(13,18)(17,19)$,
$g_2 = (1,19,24,6)(2,13,22,20)(3,12,23,5)(4,17,8,21)(7,14,16,15)$
$(9,11,18,10)$,
$g_3 = (1,24)(2,22)(3,23)(4,8)(5,12)(6,19)(7,16)(9,18)(10,11)(13,20)$
$(14,15)(17,21)$.

G has only one conjugacy class of subgroups of order 8 and all the orbits of G_B acting on the set Ω are listed as follows:
$\Omega_1 = \{1,4,6,8,17,19,21,24\}$,
$\Omega_2 = \{2,9,10,11,13,18,20,22\}$,
$\Omega_3 = \{3,5,7,12,14,15,16,23\}$.
First take $B_0 = \Omega_1$ as a possible basic block of a design. However, the structure is not 4-design. Now take $B_0 = \Omega_2$ or Ω_3 as a possible basic block

of a design, then we construct two flag-transitive 4-$(24,8,5)$ designs $\mathcal{D}_2$ and $\mathcal{D}_3$. By[15], it shows that both designs are isomorphic. It need to note that the design corresponding to case 3 is isomorphic to the design $\mathcal{D}_1$.

This completes the proof of Theorem 1.1 and 1.2.

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