

# THE DUAL OF AN EVALUATION CODE

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ABSTRACT. The aim of this work is to study the dual and the algebraic dual of an evaluation code using standard monomials and indicator functions. We show that the dual of an evaluation code is the evaluation code of the algebraic dual. We develop an algorithm for computing a basis for the algebraic dual. Let  $C_1$  and  $C_2$  be linear codes spanned by standard monomials. We give a combinatorial condition for the monomial equivalence of  $C_1$  and the dual  $C_2^\perp$ . Moreover, we give an explicit description of a generator matrix of  $C_2^\perp$  in terms of that of  $C_1$  and coefficients of indicator functions. For Reed–Muller-type codes we give a duality criterion in terms of the  $v$ -number and the Hilbert function of a vanishing ideal. As an application, we provide an explicit duality for Reed–Muller-type codes corresponding to Gorenstein ideals. In addition, when the evaluation code is monomial and the set of evaluation points is a degenerate affine space, we classify when the dual is a monomial code.

## 1. INTRODUCTION

Let  $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$  be a polynomial ring over a finite field  $K = \mathbb{F}_q$  with the standard grading and let  $X = \{P_1, \dots, P_m\}$ ,  $|X| \geq 2$ , be a set of distinct points in the affine space  $\mathbb{A}^s := K^s$ . The *evaluation map*, denoted  $\text{ev}$ , is the  $K$ -linear map given by

$$\text{ev}: S \rightarrow K^m, \quad f \mapsto (f(P_1), \dots, f(P_m)).$$

The kernel of  $\text{ev}$ , denoted  $I = I(X)$ , is the *vanishing ideal* of  $X$  consisting of the polynomials of  $S$  that vanish at all points of  $X$ . This map induces an isomorphism of  $K$ -linear spaces between  $S/I$  and  $K^m$ . If  $f \in S$ , we denote the set of zeros of  $f$  in  $X$  by  $V_X(f)$ . Let  $\mathcal{L}$  be a linear subspace of  $S$  of finite dimension. The image of  $\mathcal{L}$  under the evaluation map, denoted  $\mathcal{L}_X$ , is called an *evaluation code* on  $X$  [28, 40, 42].

The basic *parameters* of the linear code  $\mathcal{L}_X$  that we consider are:

- (a) *length*:  $m = |X|$ ,
- (b) *dimension*:  $k = \dim_K(\mathcal{L}_X)$ , and
- (c) *minimum distance*:  $\delta(\mathcal{L}_X) = \min\{|X \setminus V_X(f)| : f \in \mathcal{L} \setminus I\}$ .

The dual of  $\mathcal{L}_X$ , denoted  $(\mathcal{L}_X)^\perp$ , is the set of all  $\alpha \in K^m$  such that  $\langle \alpha, \beta \rangle = 0$  for all  $\beta \in \mathcal{L}_X$ , where  $\langle \cdot, \cdot \rangle$  is the ordinary inner product in  $K^m$ . The dual of  $\mathcal{L}_X$  is an  $[m, m - k]$  linear code [27, Theorem 1.2.1]. The aim of this paper is to study  $(\mathcal{L}_X)^\perp$  by fixing a graded monomial order on  $S$  and using the information encoded in the quotient ring  $S/I$  and in the linear space  $\mathcal{L}$ .

Let  $\prec$  be a graded monomial order on  $S$ , that is, monomials are first compared by their total degrees [11, p. 54]. The monomials of  $S$  are denoted  $t^c := t_1^{c_1} \cdots t_s^{c_s}$ ,  $c = (c_1, \dots, c_s)$  in  $\mathbb{N}^s$ , where  $\mathbb{N} = \{0, 1, \dots\}$ . We denote the initial monomial of a non-zero polynomial  $f \in S$  by  $\text{in}_\prec(f)$  and

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the initial ideal of  $I$  by  $\text{in}_\prec(I)$ . A subset  $\mathcal{G} = \{g_1, \dots, g_n\}$  of  $I$  is called a *Gröbner basis* of  $I$  if  $\text{in}_\prec(I) = (\text{in}_\prec(g_1), \dots, \text{in}_\prec(g_n))$ . A monomial  $t^a$  is called a *standard monomial* of  $S/I$ , with respect to  $\prec$ , if  $t^a \notin \text{in}_\prec(I)$ . The *footprint* of  $S/I$  or *Gröbner escalier* of  $I$ , denoted  $\Delta_\prec(I)$ , is the finite set of all standard monomials of  $S/I$ . The footprint has been used in connection with many kinds of codes and their basic parameters [8, 17, 18, 19, 21, 25, 28].

If  $\mathcal{A} \subset S$ , the  $K$ -linear subspace of  $S$  spanned by  $\mathcal{A}$  is denoted by  $K\mathcal{A}$ . The linear code  $\mathcal{L}_X$  is called a *standard evaluation code* on  $X$  relative to  $\prec$  if  $\mathcal{L}$  is a linear subspace of  $K\Delta_\prec(I)$ . A polynomial  $f$  is called a *standard polynomial* of  $S/I$  if  $f \neq 0$  and  $f$  is in  $K\Delta_\prec(I)$ . As the field  $K$  and the footprint  $\Delta_\prec(I)$  are finite, there are only a finite number of standard polynomials. Any evaluation code  $\mathcal{L}_X$  on  $X$  can be regarded as a standard evaluation code on  $X$  after a suitable transformation of a generating set for  $\mathcal{L}$  [28] (Proposition 3.1). Furthermore, given  $\mathcal{L} \subset S$  and a monomial order  $\prec$ , there exists a unique linear subspace  $\tilde{\mathcal{L}}$  of  $K\Delta_\prec(I)$  such that  $\tilde{\mathcal{L}}_X = \mathcal{L}_X$  (Corollary 3.2, Example 8.1). We call  $\tilde{\mathcal{L}}$  the *standard function space* of  $\mathcal{L}_X$ . In principle, the basic parameters of  $\mathcal{L}_X$  can be computed once we determine finite generating sets for  $\tilde{\mathcal{L}}$  and  $I$  [28].

Following [4, p. 16], let  $\varphi$  be the  $K$ -linear map given by

$$\varphi: S \rightarrow K, \quad f \mapsto f(P_1) + \dots + f(P_m).$$

The kernel of  $\varphi$  is a linear subspace of  $S$  and  $S/\ker(\varphi) \simeq K$ . The linear subspace of  $S$  of all  $g \in S$  such that  $g\mathcal{L} \subset \ker(\varphi)$  is denoted by  $(\ker(\varphi): \mathcal{L})$ . The *algebraic dual* of  $\mathcal{L}_X$  relative to  $\prec$ , denoted  $\mathcal{L}^\perp$ , is the  $K$ -linear subspace of  $S$  given by

$$\mathcal{L}^\perp := (\ker(\varphi): \mathcal{L}) \cap K\Delta_\prec(I),$$

we will also call  $\mathcal{L}^\perp$  the *dual* of  $\mathcal{L}$ . The dual  $\mathcal{L}^\perp$  is isomorphic to  $(\mathcal{L}^\perp)_X$  (Lemma 3.6).

Families of linear codes that are closed under taking duals include generalized toric codes [4, Proposition 3.5], [36, Theorem 6], monomial evaluation codes over the affine space  $\mathbb{A}^s$  that are divisor closed [4, Proposition 2.4, Remark 2.5],  $q$ -ary Reed–Muller codes [12, Theorem 2.2.1], [25, Remark 4.7], projective Reed–Muller-type codes over complete intersections [22, Theorem 2], and algebraic geometry codes [40, Theorem 2.2.10]. In these cases duality formulas for the respective dual codes are given.

The next result gives a formula for the dual of  $\mathcal{L}_X$  in terms of its algebraic dual.

**Theorem 3.5.**  $(\mathcal{L}_X)^\perp$  is the standard evaluation code  $(\mathcal{L}^\perp)_X$  on  $X$  relative to  $\prec$ .

A subspace  $L$  of  $S$  is called a *monomial space* of  $S$  if  $L = K\{t^{a_1}, \dots, t^{a_k}\}$  for some  $t^{a_1}, \dots, t^{a_k}$ . We say that  $\mathcal{L}_X$  is a *monomial code* if  $\mathcal{L}$  is a monomial space of  $S$ , and we say that  $\mathcal{L}_X$  is a *standard monomial code* if the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_X$  is a monomial space of  $S$  (cf. [32, Definition 1.1]). If  $I$  is a binomial ideal, that is,  $I$  is generated by elements of the form  $t^a - t^b$ , and  $\mathcal{L}$  is a monomial space of  $S$ , then  $\mathcal{L}_X$  is a standard monomial code (Proposition 3.8). As an application of Theorem 3.5 we obtain an effective criterion for verifying whether or not the dual of an evaluation code is a monomial code (Proposition 3.10, Procedure A.1).

The formula of Theorem 3.5 can be used to compute a generating set for the algebraic dual of  $\mathcal{L}_X$ . We show an effective algorithm, based on Gaussian elimination, to compute a  $K$ -basis of any linear subspace of  $S$  of finite dimension (Theorem 3.11). This algorithm can be used to compute a  $K$ -basis for the algebraic dual  $\mathcal{L}^\perp$  of  $\mathcal{L}_X$  and also for the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_X$  (Examples 8.1, 8.3–8.6, Procedure A.1).

In Section 4 we introduce and study the  $v$ -number of  $I$  [10], and the indicator functions of  $X$  that are used in coding theory [10, 38], Cayley–Bacharach schemes [20], and interpolation

problems [29]. As is seen later in the introduction these notions are used as devices to study the duality of standard monomial codes, as well as the asymptotic behavior of the minimum distance and the duality of Reed–Muller-type codes.

Let  $\text{Ass}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$  be the set of *associated primes* of  $I$ , that is,  $\mathfrak{p}_i$  is the vanishing ideal  $I_{P_i}$  of  $P_i$  and  $I = \bigcap_{i=1}^m \mathfrak{p}_i$  is the primary decomposition of  $I$  (Lemma 2.4). The *v-number* of the ideal  $I$  at  $\mathfrak{p}_i$ , denoted  $v_{\mathfrak{p}_i}(I)$ , is given by

$$v_{\mathfrak{p}_i}(I) := \min\{d \geq 0 \mid \exists 0 \neq f \in S, \deg(f) = d, \text{ with } (I : f) = \mathfrak{p}_i\},$$

where  $(I : f) := \{g \in S \mid gf \subset I\}$  is a *colon ideal*, and the *v-number* of the ideal  $I$ , denoted  $v(I)$ , is given by  $v(I) := \min\{v_{\mathfrak{p}_i}(I)\}_{i=1}^m$ . The notion of v-number is related to indicator functions as we now explain. A polynomial  $f$  in  $S$  is called an *indicator function* for  $P_i$  if  $f(P_i) \neq 0$  and  $f(P_j) = 0$  if  $j \neq i$  [38]. Indicator functions can be computed using [29, Corollary 6.3.11]. The v-number of  $I$  at  $\mathfrak{p}_i$  is the least degree of an indicator function for  $P_i$  (Lemma 4.4).

For an ideal  $M \neq 0$  of  $S/I$ , we define  $\alpha(M)$  to be the minimum degree of the non-zero elements of  $M$ . To compute the v-number using *Macaulay2* [24] (Example 8.5), we give the following description for the v-number of  $I$  at  $\mathfrak{p}_i$  (Proposition 4.5):

$$v_{\mathfrak{p}_i}(I) = \alpha((I : \mathfrak{p}_i)/I) \text{ for all } i.$$

This computation, along with other coding theory tools, is implemented in [3].

For each point  $P_i$  there exists a unique indicator function  $f_i$  for  $P_i$  in  $K\Delta_{\prec}(I)$  satisfying  $f_i(P_i) = 1$ , furthermore the degree of  $f_i$  is  $v_{\mathfrak{p}_i}(I)$ , and  $F = \{f_1, \dots, f_m\}$  is a  $K$ -basis for  $K\Delta_{\prec}(I)$  (Proposition 4.6(a)). We call  $f_i$  the *i-th standard indicator function* for  $P_i$  and call  $F$  the set of *standard indicator functions* for  $X$ . As a byproduct we obtain an algebraic method to compute the set  $F$  (Remark 4.7, Example 8.5, Procedure A.1). We give the following formula

$$\ker(\varphi) = K\{f_i - f_m\}_{i=1}^{m-1} + I,$$

for the kernel of the map  $\varphi$  that was used earlier to define  $\mathcal{L}^\perp$  (Proposition 4.6).

Given a subset  $\Gamma \subset \Delta_{\prec}(I)$ , let  $\mathcal{L}(\Gamma)$  be the  $K$ -span of the set of all  $t^a \in \Gamma$ . Then  $\mathcal{L}(\Gamma)_X$  is called the *standard monomial code* of  $\Gamma$ . Consider two standard monomial codes  $\mathcal{L}(\Gamma_1)_X$  and  $\mathcal{L}(\Gamma_2)_X$  for some  $\Gamma_1, \Gamma_2 \subset \Delta_{\prec}(I)$ . We give a combinatorial condition for the monomial equivalence of  $\mathcal{L}(\Gamma_1)_X$  and  $\mathcal{L}(\Gamma_2)_X^\perp$ . For convenience we recall the definition of this notion. We say that two linear codes  $C_1, C_2$  in  $K^m$  are *monomially equivalent* if there is  $\beta = (\beta_1, \dots, \beta_m)$  in  $K^m$  such that  $\beta_i \neq 0$  for all  $i$  and

$$C_2 = \beta \cdot C_1 = \{\beta \cdot c \mid c \in C_1\},$$

where  $\beta \cdot c$  is the vector given by  $(\beta_1 c_1, \dots, \beta_m c_m)$  for  $c = (c_1, \dots, c_m) \in C_1$ .

To state the main result of Section 5 we will need the following definition. We say a standard monomial  $t^e \in \Delta_{\prec}(I)$  is *essential* if it appears in each standard indicator function of  $X$ .

**Theorem 5.4.** *Let  $t^e$  be essential. Then for any  $\Gamma_1, \Gamma_2 \subset \Delta_{\prec}(I)$  satisfying*

- (1)  $|\Gamma_1| + |\Gamma_2| = |X|$ ,
- (2)  $t^e$  does not appear in the reduction of  $u_1 u_2$  modulo  $I$  for every  $u_1 \in \Gamma_1$  and  $u_2 \in \Gamma_2$ ,

*we have  $\beta \cdot \mathcal{L}(\Gamma_1)_X = \mathcal{L}(\Gamma_2)_X^\perp$ , for some  $\beta = (\beta_1, \dots, \beta_m) \in K^m$  such that  $\beta_i \neq 0$  for all  $i$ . Moreover,  $\beta_i$  is the coefficient of  $t^e$  in the  $i$ -th standard indicator function  $f_i$ , for all  $i$ .*

Given an integer  $d \geq 0$ , we let  $S_{\leq d} = \bigoplus_{i=0}^d S_i$  be the  $K$ -linear subspace of  $S$  of all polynomials of degree at most  $d$  and let  $I_{\leq d} = I \cap S_{\leq d}$ . We set  $S_{\leq -1} = \{0\}$ , by convention. The function

$$H_I^a(d) := \dim_K(S_{\leq d}/I_{\leq d}), \quad d = -1, 0, 1, 2, \dots$$

is called the *affine Hilbert function* of  $S/I$ . In particular,  $H_I^a(-1) = 0$ . The *regularity index* of  $H_I^a$ , denoted  $r_0 = \text{reg}(H_I^a)$ , is the least integer  $\ell \geq 0$  such that  $H_I^a(d) = |X|$  for  $d \geq \ell$  (Proposition 2.2). Note that  $r_0 \geq 1$  because  $|X| \geq 2$ .

If  $\mathcal{L}$  is equal to  $S_{\leq d}$ , then the resulting evaluation code  $\mathcal{L}_X$  is called a *Reed-Muller-type code* of degree  $d$  on  $X$  [13, 23] and is denoted by  $C_X(d)$ .

The minimum distance of  $C_X(d)$  is simply denoted by  $\delta_X(d)$ . The  $v$ -number of  $I$  is related to the asymptotic behavior of  $\delta_X(d)$  for  $d \gg 0$ . By Proposition 2.2, there  $n$  in  $\mathbb{N}$  such that

$$|X| = \delta_X(0) > \delta_X(1) > \cdots > \delta_X(n-1) > \delta_X(n) = \delta_X(d) = 1 \quad \text{for } d \geq n.$$

The number  $n$ , denoted  $\text{reg}(\delta_X)$ , is called the *regularity index* of  $\delta_X$ . By the Singleton bound [27, p. 71], one has the inequality  $\text{reg}(\delta_X) \leq \text{reg}(H_I^a)$ . Using indicator functions we prove that  $v(I)$  is equal to  $\text{reg}(\delta_X)$  (Proposition 6.2), and consequently using Proposition 4.5 we can compute  $\text{reg}(\delta_X)$  with *Macaulay2* [24] (Example 8.1).

In Section 6 we give a duality criterion for the monomial equivalence of the linear codes  $C_X(d)$  and  $C_X(r_0 - d - 1)^\perp$  for  $-1 \leq d \leq r_0$ , where  $r_0 = \text{reg}(H_I^a)$ . As  $\dim_K(C_X(d)) = H_I^a(d)$ , a necessary condition for this equivalence is the equality

$$H_I^a(d) + H_I^a(r_0 - d - 1) = |X| \quad \text{for } -1 \leq d \leq r_0.$$

In Section 2 we characterize this equality in terms of the symmetry of  $h$ -vectors and the symmetry of the function  $\psi(d) = |\Delta_{\prec}(I) \cap S_d|$  (Proposition 2.8).

We come to one of our main results.

**Theorem 6.5.** (Duality criterion) *Let  $r_0 = \text{reg}(H_I^a)$ . The following are equivalent.*

- (a)  $C_X(d)$  is monomially equivalent to  $C_X(r_0 - d - 1)^\perp$  for  $-1 \leq d \leq r_0$ .
- (b)  $H_I^a(d) + H_I^a(r_0 - d - 1) = |X|$  for  $-1 \leq d \leq r_0$  and  $r_0 = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ .
- (c) There is  $g \in K\Delta_{\prec}(I)$  such that  $g(P_i) \neq 0$  for all  $i$  and

$$C_X(r_0 - d - 1)^\perp = (g(P_1), \dots, g(P_m)) \cdot C_X(d) \quad \text{for } -1 \leq d \leq r_0.$$

The standard polynomial  $g$  of part (c) is unique up to multiplication by a scalar from  $K^*$ , where  $K^* := K \setminus \{0\}$ . If  $F = \{f_1, \dots, f_m\}$  is the unique set of standard indicator functions for  $X$ , then  $g$  is equal to  $\sum_{i=1}^m \text{lc}(f_i)f_i$  (see Example 8.2 for an illustration). The value of  $g$  at  $P_i$  is  $\text{lc}(f_i)$ , the leading coefficient of  $f_i$ . We will use this criterion to show duality for some interesting families and recover some known results. The condition  $r_0 = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$  that appears in the duality criterion defines a Cayley–Bacharach scheme (CB-scheme) in the projective case when  $K$  is an infinite field [20, Definition 2.7], and is related to Hilbert functions.

Gorenstein and complete intersection ideals—and some of their properties—are introduced in Section 2. If  $I$  is a complete intersection generated by a Gröbner basis with  $s$  elements, then the ideal  $I$  is Gorenstein (Corollary 2.9(c)). The converse is not true (Example 8.2). If  $I$  is Gorenstein, then  $H_I^a(d) + H_I^a(r_0 - d - 1) = |X|$  for  $-1 \leq d \leq r_0$  (Corollary 2.9(a)). This result, together with the next theorem, shows that the combinatorial condition of Theorem 5.4 and the conditions of Theorem 6.5(b) are satisfied when  $I$  is a Gorenstein ideal.

**Theorem 6.11.** *Let  $F = \{f_1, \dots, f_m\}$  be the set of standard indicator functions for  $X$ . If  $I$  is Gorenstein, then  $\text{reg}(H_I^a) = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$  and  $\text{in}_{\prec}(f_i) = \text{in}_{\prec}(f_m)$  for all  $i$ .*

The following result includes the family of Reed–Muller-type codes over complete intersections and in particular—since vanishing ideals of Cartesian sets are complete intersections generated by a Gröbner basis with  $s$  elements [33, Lemma 2.3]—we recover the duality theorems for affine Cartesian codes given in [2, Theorem 5.7] and [31, Theorem 2.3].

**Corollary 6.15.** *Let  $r_0$  be the regularity index of  $H_I^0$ . If  $I$  is a Gorenstein ideal, then there is  $g \in K\Delta_{\prec}(I)$  such that  $g(P_i) \neq 0$  for all  $i$  and*

$$(g(P_1), \dots, g(P_m)) \cdot C_X(r_0 - d - 1) = C_X(d)^\perp \quad \text{for } -1 \leq d \leq r_0.$$

As an application, we produce self-dual Reed–Muller-type codes when  $I$  is Gorenstein,  $\text{char}(K) = 2$ , and  $r_0$  is odd (Corollary 6.16).

In Section 7 we give an explicit description for the algebraic dual of  $\mathcal{L}_T$  when  $\mathcal{L}$  is a monomial space of  $S$  and  $T = \{P_1, \dots, P_m\}$  is the set of points in a degenerate torus (Proposition 7.2, Example 8.7). In this case the vanishing ideal of  $T$  is a complete intersection binomial ideal and, by Proposition 3.8,  $\mathcal{L}_T$  is a standard monomial code. Let  $T = (K^*)^s$  be a torus in  $\mathbb{A}^s$  and let  $\mathcal{L}_T$  be a *generalized toric code* on  $T$  [30, 36, 37], that is,  $\mathcal{L}$  is a monomial space of  $S$ . Bras-Amorós and O’Sullivan [4, Proposition 3.5] and independently Ruano [36, Theorem 6] compute the dual of  $\mathcal{L}_T$  and show that the dual of  $\mathcal{L}_T$  is a generalized toric code. As an application we recover these results (Corollary 7.4).

The rest of this paper is devoted to study the dual of monomial codes on a degenerate affine space. Let  $K = \mathbb{F}_q$  be a finite field of characteristic  $p$ , let  $A_1, \dots, A_s$  be subgroups of the multiplicative group  $K^*$  of  $K$ , let  $B_i$  be the set  $A_i \cup \{0\}$  for  $i = 1, \dots, s$ , let

$$\mathcal{X} := B_1 \times \dots \times B_s,$$

and let  $\mathcal{L}_{\mathcal{X}}$  be a monomial code on  $\mathcal{X}$ . The set  $\mathcal{X}$  is called a *degenerate affine space*. In this case the vanishing ideal  $I = I(\mathcal{X})$  is a complete intersection binomial ideal. By Proposition 3.8, we may assume that  $\mathcal{L}$  is the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_{\mathcal{X}}$  and that  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\} \subset \Delta_{\prec}(I)$  is a  $K$ -basis for  $\mathcal{L}$  with  $a_i = (a_{i,1}, \dots, a_{i,s})$  for  $i = 1, \dots, k$ . Following [4, p. 16], we say that  $\mathcal{A}$  is *divisor-closed* if  $t^a \in \mathcal{A}$  whenever  $t^a$  divides a monomial in  $\mathcal{A}$ . To classify when the dual of  $\mathcal{L}_{\mathcal{X}}$  is a standard monomial code, we introduce a weaker notion than divisor-closed that we call *weakly divisor-closed* (Definition 7.7). The order of the multiplicative monoid  $B_i$  is denoted by  $e_i$  and the order of  $A_i$  is denoted by  $d_i$  for  $i = 1, \dots, s$ . For use below we set

$$t^{b_i} = t_1^{b_{i,1}} \dots t_s^{b_{i,s}} := \prod_{j=1}^s t_j^{d_j - a_{i,j}},$$

for  $i = 1, \dots, k$  and  $\mathcal{B} := \{t^{b_1}, \dots, t^{b_k}\}$ . Note that  $(\mathcal{L}^\perp)_{\mathcal{X}}$  is a standard monomial code if and only if  $\mathcal{L}^\perp$  is a monomial space of  $S$  because the standard function space of  $(\mathcal{L}^\perp)_{\mathcal{X}}$  is  $\mathcal{L}^\perp$ .

We come to another of our main results.

**Theorem 7.8.** *Let  $K$  be a field of characteristic  $p$  and  $\mathcal{X}$  a degenerate affine space as above. Assume that  $\gcd(p, e_i) = p$ , where  $e_i = |B_i|$ , for  $i = 1, \dots, s$ . The following are equivalent.*

- (a)  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\}$  is weakly divisor-closed.
- (b)  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ .
- (c)  $(\mathcal{L}_{\mathcal{X}})^\perp$  is a standard monomial code on  $\mathcal{X}$ .

Let  $\mathcal{L}_{\mathcal{X}}$  be a monomial standard evaluation code on  $\mathcal{X} = K^s$ . Then  $\mathcal{L}$  is generated by a subset  $\mathcal{A}$  of  $\Delta_{\prec}(I)$ . Bras-Amorós and O’Sullivan [4, Proposition 2.4, Remark 2.5] compute the dual of  $\mathcal{L}_{\mathcal{X}}$  when  $\mathcal{A}$  is divisor closed. As an application we recover this result (Corollary 7.10).

In the next result we determine the dual of  $K(S_{\leq d} \cap \Delta_{\prec}(I))$ .

**Theorem 7.11.** *Let  $K$  be a field of characteristic  $p$  and  $\mathcal{X}$  a degenerate affine space as above. Assume that  $\gcd(p, e_i) = p$ , where  $e_i = |B_i|$ , for  $i = 1, \dots, s$ . If  $-1 \leq d \leq r_0 = \sum_{i=1}^s (e_i - 1)$  and  $\mathcal{L} = K(S_{\leq d} \cap \Delta_{\prec}(I))$ , then*

$$\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \{t^{b_1}, \dots, t^{b_k}\}) = K(S_{\leq r_0 - d - 1} \cap \Delta_{\prec}(I)).$$

The codes  $C_{\mathcal{X}}(d)^\perp$  and  $C_{\mathcal{X}}(r_0 - d - 1)$  are monomially equivalent because  $I$  is a complete intersection (Corollary 6.15). We show they are equal if  $\text{char}(K)$  divides  $e_i$  for all  $i$  (Proposition 7.12). When  $\mathcal{X} = K^s$  the equality  $C_{\mathcal{X}}(d) = C_{\mathcal{X}}(r_0 - d - 1)^\perp$ ,  $r_0 = s(q - 1)$ , has long been known; see for example [12, Theorem 2.2.1] and [25, Remark 4.7].

We include one section with examples (Section 8) and an appendix with implementations of the algorithms in *Macaulay2* [24] that we used in the examples to compute bases for algebraic duals, v-numbers, and standard indicator functions (Appendix A).

For all unexplained terminology and additional information we refer the reader to [11, 29, 39, 43] (for the theory of Gröbner bases and Hilbert functions), and [27, 35, 42] (for the theory of error-correcting codes and linear codes).

## 2. PRELIMINARIES: HILBERT FUNCTIONS AND VANISHING IDEALS

In this section we introduce Hilbert functions and characterize the symmetry of the  $h$ -vector of the homogenization of a vanishing ideal.

Let  $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$  be a polynomial ring over a finite field  $K = \mathbb{F}_q$  with the standard grading and let  $I$  be an ideal of  $S$ . The Krull dimension of  $S/I$  is denoted by  $\dim(S/I)$ . We say that  $I$  has *dimension*  $k$  if  $\dim(S/I)$  is equal to  $k$ . The *height* of  $I$ , denoted  $\text{ht}(I)$ , is  $s - \dim(S/I)$ . We set  $S_{\leq d} = \bigoplus_{i=0}^d S_i$ ,  $S_{\leq -1} = \{0\}$ , and  $I_{\leq d} = I \cap S_{\leq d}$ . The function

$$H_I^a(d) := \dim_K(S_{\leq d}/I_{\leq d}), \quad d = -1, 0, 1, 2, \dots,$$

is called the *affine Hilbert function* of  $S/I$ . In particular,  $H_I^a(-1) = 0$ . For simplicity we also call  $H_I^a$  the affine Hilbert function of  $I$ . The *Hilbert function* of a graded ideal  $J$  of  $S$ , denoted  $H_J$ , is the function given by  $H_J(d) := \dim_K(S_d/J_d)$  for  $d \geq -1$ , where  $J_d = S_d \cap J$ .

Let  $u = t_{s+1}$  be a new variable. For  $f \in S$  of degree  $e$  define

$$f^h := u^e f(t_1/u, \dots, t_s/u),$$

that is,  $f^h$  is the homogenization of the polynomial  $f$  with respect to  $u$ . The *homogenization* of  $I$  is the ideal  $I^h$  of  $S[u]$  given by  $I^h := (\{f^h \mid f \in I\})$ , where  $S[u]$  is given the standard grading. One has the following two well-known facts

$$(2.1) \quad \dim(S[u]/I^h) = \dim(S/I) + 1 \text{ and } H_I^a(d) = H_{I^h}(d) \text{ for } d \geq -1,$$

where  $H_{I^h}$  is the Hilbert function of the graded ideal  $I^h$ , see for instance [43, Lemma 8.5.4]. If  $k = \dim(S/I)$ , by a Hilbert theorem [39, p. 58], there is a unique polynomial  $h_I^a(z) = \sum_{i=0}^k a_i z^i$  of degree  $k$  in  $\mathbb{Q}[z]$  such that  $h_I^a(d) = H_I^a(d)$  for  $d \gg 0$ . By convention the degree of the zero polynomial is  $-1$ . The integer  $k! a_k$ , denoted  $\deg(S/I)$ , is called the *degree* of  $S/I$ . The degree of  $S/I$  is equal to  $\deg(S[u]/I^h)$ . If  $k = 0$ , then  $H_I^a(d) = \deg(S/I) = \dim_K(S/I)$  for  $d \gg 0$ . Note that the degree of  $S/I$  is positive if  $I \subsetneq S$  and is 0 otherwise.

We say that  $I$  is a *complete intersection* if  $I$  can be generated by  $\text{ht}(I)$  elements. The ideal  $I$  and the ring  $S/I$  are called *Gorenstein* if the localization of  $S/I$  at every maximal ideal is a Gorenstein local ring in the sense of [43, Definition 2.8.3]. If  $I = I(X)$  is the vanishing ideal of a set of points in  $K^s$ , using Lemma 2.4 below, it follows that  $I$  is Gorenstein. Permitting an abuse of terminology, we say that  $I = I(X)$  is *Gorenstein* if  $S[u]/I^h$  is a Gorenstein graded ring, that is,  $S[u]/I^h$  is Cohen–Macaulay and the last Betti number in the minimal graded resolution of  $S[u]/I^h$  is equal to 1 [43, Corollary 5.3.5] (Example 8.2).

An element  $f \in S$  is called a *zero-divisor* of  $S/I$ —as an  $S$ -module—if there is  $\bar{0} \neq \bar{a} \in S/I$  such that  $f\bar{a} = \bar{0}$ , and  $f$  is called *regular* on  $S/I$  otherwise. Note that  $f$  is a zero-divisor of  $S/I$

if and only if  $(I : f) \neq I$ . An associated prime of  $I$  is a prime ideal  $\mathfrak{p}$  of  $S$  of the form  $\mathfrak{p} = (I : f)$  for some  $f$  in  $S$ . The radical of  $I$  is denoted by  $\text{rad}(I)$ . The ideal  $I$  is *radical* if  $I = \text{rad}(I)$ .

**Theorem 2.1.** [43, Lemma 2.1.19, Corollary 2.1.30] *If  $I$  is an ideal of  $S$  and  $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$  is an irredundant primary decomposition with  $\text{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$ , then the set of zero-divisors  $\mathcal{Z}_S(S/I)$  of  $S/I$  is equal to  $\bigcup_{i=1}^m \mathfrak{p}_i$ , and  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  are the associated primes of  $I$ .*

Recall  $C_X(d)$  denotes the Reed–Muller-type code of degree  $d$  on a set  $X$  of points in  $K^s$  and  $\delta(C_X(d))$  represents the minimum distance of the code.

**Proposition 2.2.** [34, Corollary 2.6] *Let  $X$  be a subset of  $K^s$  and let  $I = I(X)$  be its vanishing ideal. Then,  $H_I^a$  is increasing until it reaches the constant value  $|X|$ , and  $\delta(C_X(d))$  is decreasing, as a function of  $d$ , until it reaches the constant value 1. In particular,  $\deg(S/I) = |X|$ .*

If  $I = I(X)$  and  $X \subset K^s$ , the least integer  $r_0 \geq 0$  such that  $H_I^a(d) = |X|$  (resp.  $H_{I^h}(d) = |X|$ ) for  $d \geq r_0$ , denoted  $\text{reg}(H_I^a)$  (resp.  $\text{reg}(H_{I^h})$ ), is called the *regularity index* of  $H_I^a$  (resp.  $H_{I^h}$ ). By Eq. (2.1),  $r_0 = \text{reg}(H_I^a) = \text{reg}(H_{I^h})$ . It is known that  $\text{reg}(H_{I^h})$  equals the *Castelnuovo–Mumford regularity* of  $S[u]/I^h$  in the sense of [15, p. 55], see for instance [43, p. 346]. For this reason  $\text{reg}(H_{I^h})$  is simply called the *regularity* of  $S[u]/I^h$ .

**Lemma 2.3.** [43, Proposition 3.4.5] *Let  $X$  be a subset of  $K^s$ . Then, the ideal  $I(X)^h$  is the homogeneous vanishing ideal  $I(Y)$  of the set  $Y := \{[x, 1] \mid x \in X\}$  of projective points in  $\mathbb{P}^s$ .*

**Lemma 2.4.** [29, p. 389] *Let  $X$  be a subset of  $K^s$ , let  $P$  be a point in  $X$ ,  $P = (p_1, \dots, p_s)$ , and let  $I_P$  be the vanishing ideal of  $P$ . Then  $I_P$  is a maximal ideal of  $S$  of height  $s$ ,*

$$I_P = (t_1 - p_1, \dots, t_s - p_s), \quad \deg(S/I_P) = 1,$$

and  $I(X) = \bigcap_{P \in X} I_P$  is the primary decomposition of  $I(X)$ .

**Lemma 2.5.** *Let  $X$  be a subset of  $K^s$ . Then, the variable  $u$  is regular on  $S[u]/I(X)^h$ .*

*Proof.* We set  $I = I(X)$ . From Lemma 2.3, we get  $I^h = \bigcap_{P \in X} I_{[P,1]}$ . If  $P = (p_1, \dots, p_s)$  is a point in  $X$ , then  $I_{[P,1]}$  is generated by  $\mathcal{G} = \{t_1 - p_1u, \dots, t_s - p_su\}$ . Hence, by Theorem 2.1, it suffices to show that  $u$  is not in  $I_{[P,1]}$ . Pick a graded order with  $t_1 \succ \cdots \succ t_s \succ u$ . The set  $\mathcal{G}$  is a Gröbner basis for  $I_{[P,1]}$ . If  $u$  is in  $I_{[P,1]}$ , then  $u \in \text{in}_{\prec}(I_{[P,1]}) = (t_1, \dots, t_s)$ , a contradiction.  $\square$

Let  $I \subset S$  be an ideal, let  $\prec$  be a monomial order, and let  $\Delta_{\prec}(I)$  be the set of standard monomials of  $S/I$ . The image of  $\Delta_{\prec}(I)$ , under the canonical map  $S \mapsto S/I$ ,  $x \mapsto \bar{x}$ , is a basis of  $S/I$  as a  $K$ -vector space [1, Proposition 6.52].

**Lemma 2.6.** *Let  $I \subset S$  be an ideal and let  $\prec$  be a graded monomial order on  $S$ . Then  $H_I^a(d)$  is equal to  $H_{\text{in}_{\prec}(I)}^a(d)$  for  $d \geq 0$ ,  $H_I^a(d)$  is  $|\Delta_{\prec}(I) \cap S_{\leq d}|$ , the number of standard monomials of  $S/I$  of degree at most  $d$ , and  $\dim(S/I) = \dim(S/\text{in}_{\prec}(I))$ .*

*Proof.* By [11, Chapter 9, Section 3, Propositions 3 and 4], we have that  $H_{\text{in}_{\prec}(I)}^a(d)$  is the number of monomials of  $S$  not in the ideal  $\text{in}_{\prec}(I)$  of degree  $\leq d$ , and  $H_I^a(d)$  is equal to  $H_{\text{in}_{\prec}(I)}^a(d)$  when  $\prec$  is graded. Hence,  $H_I^a(d)$  is the number of standard monomials of  $S/I$  of degree at most  $d$ . As  $S/I$  and  $S/\text{in}_{\prec}(I)$  have the same affine Hilbert function, they have the same dimension.  $\square$

**Lemma 2.7.** *Let  $X$  be a subset of  $K^s$ ,  $I = I(X)$ ,  $r_0 = \text{reg}(H_I^a)$ , and let  $\prec$  be a graded monomial order on  $S$ . Then  $\Delta_{\prec}(I) \subset S_{\leq r_0}$ ,  $\Delta_{\prec}(I) \not\subset S_{\leq r_0-1}$ , and  $|X| = H_I^a(r_0) = |\Delta_{\prec}(I)|$ .*

*Proof.* By Proposition 2.2,  $H_I^a(r_0 - 1) < H_I^a(r_0) = |X|$ . Hence, by Lemma 2.6, it suffices to show the inclusion  $\Delta_{\prec}(I) \subset S_{\leq r_0}$ . We proceed by contradiction assuming that  $\Delta_{\prec}(I) \not\subset S_{\leq r_0}$ . Pick a monomial  $t^a$  in  $\Delta_{\prec}(I)$  with  $\deg(t^a) = d_0 > r_0$ . Then, one has the strict inclusion  $\Delta_{\prec}(I) \cap S_{\leq r_0} \subsetneq \Delta_{\prec}(I) \cap S_{\leq d_0}$  and, by Lemma 2.6, one has  $|X| = H_I^a(r_0) < H_I^a(d_0)$ , a contradiction because  $H_I^a(d) = |\bar{X}|$  for  $d \geq r_0$  (Proposition 2.2).  $\square$

Let  $J$  be a graded ideal of  $S$  and let  $F_J(z) := \sum_{i=0}^{\infty} H_J(i)z^i$  be its Hilbert series. We now introduce the notion of  $h$ -vector of  $S/J$ . By the Hilbert–Serre theorem [39, 43] there is a (unique) polynomial  $h(z) = \sum_{i=0}^r h_i z^i$ ,  $h_r \neq 0$ , with integral coefficients such that  $h(1) \neq 0$  and

$$F_J(z) = \frac{h(z)}{(1-z)^k},$$

where  $k = \dim(S/J)$ . The  $h$ -vector of  $S/J$  is defined as  $h(S/J) := (h_0, \dots, h_r)$ . We say that the  $h$ -vector of  $S/J$  is *symmetric* if  $h_i = h_{r-i}$  for  $0 \leq i \leq r$ . The  $h$ -vector of a Gorenstein graded algebra is symmetric [39]. For almost Gorenstein algebras and coordinate rings of CB-schemes their  $h$ -vectors satisfy certain interesting linear inequalities [20, 26].

**Proposition 2.8.** *Let  $I = I(X)$  be the vanishing ideal of a subset  $X$  of  $K^s$ , let  $r_0$  be the regularity index of  $H_I^a$ , let  $\prec$  be a graded monomial order on  $S[u]$  with  $t_1 \succ \dots \succ t_s \succ u$ , and let  $I^h$  be the homogenization of  $I$  with respect to  $u$ . The following are equivalent.*

- (a) *The  $h$ -vector of  $S[u]/I^h$  is symmetric.*
- (b)  *$H_I^a(d) + H_I^a(r_0 - d - 1) = |X|$  for  $-1 \leq d \leq r_0$ .*
- (c)  *$H_{\text{in}_{\prec}(I)}(d) = H_{\text{in}_{\prec}(I)}(r_0 - d)$  for  $0 \leq d \leq r_0$ .*
- (d)  *$|\Delta_{\prec}(I) \cap S_d| = |\Delta_{\prec}(I) \cap S_{r_0-d}|$  for  $0 \leq d \leq r_0$ .*

*Proof.* As  $S[u]/I^h$  is Cohen–Macaulay of dimension 1, its Hilbert series can be written as

$$(2.2) \quad F_{I^h}(z) = \frac{h_0 + h_1 z + \dots + h_{r_0} z^{r_0}}{1-z},$$

where  $h(z) = h_0 + h_1 z + \dots + h_{r_0} z^{r_0}$  is a polynomial with positive integer coefficients and the degree and regularity of  $S[u]/I^h$  are  $h(1)$  and  $r_0$ , respectively [39, 43]. The ideal  $I$  (resp.  $I^h$ ) and its initial ideal  $\text{in}_{\prec}(I)$  (resp.  $\text{in}_{\prec}(I^h)$ ) have the same affine Hilbert function (resp. Hilbert function) (Lemma 2.6). As  $u$  is not in the ideal  $\text{in}_{\prec}(I^h)$ , there is an exact sequence

$$0 \longrightarrow (S[u]/\text{in}_{\prec}(I^h))[-1] \xrightarrow{u} S[u]/\text{in}_{\prec}(I^h) \longrightarrow S[u]/(\text{in}_{\prec}(I^h), u) \longrightarrow 0.$$

Hence, noticing the equalities  $\text{in}_{\prec}(I^h) = \text{in}_{\prec}(I)S[u]$  and  $S[u]/(\text{in}_{\prec}(I^h), u) = S/\text{in}_{\prec}(I)$  and, by taking Hilbert series in this exact sequence, we obtain

$$F_{I^h}(z) = zF_{I^h}(z) + H_{\text{in}_{\prec}(I)}(0) + H_{\text{in}_{\prec}(I)}(1)z + \dots + H_{\text{in}_{\prec}(I)}(r_0)z^{r_0}.$$

Therefore, by Eq. (2.2), the  $h$ -vectors of  $S/\text{in}_{\prec}(I)$  and  $S[u]/I^h$  are equal and

$$(2.3) \quad h_i = H_{\text{in}_{\prec}(I)}(i) \text{ for } 0 \leq i \leq r_0.$$

(a)  $\Rightarrow$  (b): Note that when  $d = -1$  or  $d = r_0$  (b) holds by the definition of  $r_0$ , so we may assume that  $0 \leq d < r_0$ . Now assume that  $h(S[u]/I^h) = (h_0, \dots, h_{r_0})$  is symmetric. Hence, by Eq. (2.3), we obtain  $H_{\text{in}_{\prec}(I)}(i) = H_{\text{in}_{\prec}(I)}(r_0 - i)$  for  $0 \leq i \leq r_0$ . The affine Hilbert function of  $I$

in degree  $d$  is given by  $H_I^a(d) = H_{\text{in}_{\prec}(I)}^a(d) = \sum_{i=0}^d H_{\text{in}_{\prec}(I)}(i)$  (Lemma 2.6). Therefore

$$\begin{aligned} |X| &= \deg(S[u]/I^h) = \sum_{i=0}^{r_0} h_i = \sum_{i=0}^{r_0} H_{\text{in}_{\prec}(I)}(i) = \sum_{i=0}^d H_{\text{in}_{\prec}(I)}(i) + \sum_{i=d+1}^{r_0} H_{\text{in}_{\prec}(I)}(i) \\ &= H_I^a(d) + \sum_{i=0}^{r_0-d-1} H_{\text{in}_{\prec}(I)}(r_0-i) = H_I^a(d) + \sum_{i=0}^{r_0-d-1} H_{\text{in}_{\prec}(I)}(i) \\ &= H_I^a(d) + H_I^a(r_0-d-1). \end{aligned}$$

(b)  $\Rightarrow$  (c): As  $|X| = H_I^a(d) + H_I^a(r_0-d-1)$  and  $|X| = H_I^a(d-1) + H_I^a(r_0-d)$ , by adding the following two equalities

$$\begin{aligned} H_I^a(d) &= \sum_{i=0}^d H_{\text{in}_{\prec}(I)}(i) = H_I^a(d-1) + H_{\text{in}_{\prec}(I)}(d), \\ H_I^a(r_0-d-1) &= \sum_{i=0}^{r_0-d-1} H_{\text{in}_{\prec}(I)}(i) = H_I^a(r_0-d) - H_{\text{in}_{\prec}(I)}(r_0-d), \end{aligned}$$

we obtain the equality  $H_{\text{in}_{\prec}(I)}(d) = H_{\text{in}_{\prec}(I)}(r_0-d)$ .

(c)  $\Rightarrow$  (a): The symmetry of the  $h$ -vector of  $S[u]/I^h$  follows from Eq. (2.3).

(c)  $\Leftrightarrow$  (d): The number of standard monomials of  $I$  of degree  $d$  is  $H_{\text{in}_{\prec}(I)}(d)$  [11, p. 433], that is,  $H_{\text{in}_{\prec}(I)}(d)$  is equal to  $|\Delta_{\prec}(I) \cap S_d|$ . Hence (c) and (d) are equivalent.  $\square$

**Corollary 2.9.** *Let  $I = I(X)$  be the vanishing ideal of a subset  $X$  of  $K^s$ , let  $\prec$  be a graded monomial order on  $S$ , and let  $r_0$  be the regularity index of  $H_I^a$ . The following hold.*

- (a) *If  $I$  is Gorenstein, then  $H_I^a(d) + H_I^a(r_0-d-1) = \deg(S/I) = |X|$  for  $-1 \leq d \leq r_0$ .*
- (b) *If  $I$  is Gorenstein, then there is only one standard monomial of degree  $r_0$ .*
- (c) *If  $I$  is generated by a Gröbner basis  $\mathcal{G} = \{g_1, \dots, g_s\}$  of  $s$  elements, then  $I$  is Gorenstein.*

*Proof.* (a): As  $S[u]/I^h$  is a graded Gorenstein algebra of dimension 1, its  $h$ -vector is symmetric [39, Theorems 4.1 and 4.2]. Then, by Proposition 2.8, the equality follows.

(b): By part (a) and Proposition 2.8(d), one has  $|\Delta_{\prec}(I) \cap S_d| = |\Delta_{\prec}(I) \cap S_{r_0-d}|$  for  $0 \leq d \leq r_0$ . Setting  $d = 0$  in this equality, we get  $1 = |\Delta_{\prec}(I) \cap S_{r_0}|$ .

(c): As  $I$  is generated by the Gröbner basis  $\mathcal{G}$ , one has  $I^h = (g_1^h, \dots, g_s^h)$  [43, p. 132]. The ideals  $I$  and  $I^h$  have height  $s$ . Then,  $I^h$  is a graded ideal of height  $s$  generated by  $s$  homogeneous polynomials forming a regular sequence. Hence, by [14, Corollary 21.19],  $I^h$  is Gorenstein.  $\square$

### 3. THE DUAL OF EVALUATION CODES

To avoid repetitions, we continue to employ the notations and definitions used in Sections 1 and 2. In this section we show that the dual of an evaluation code is the evaluation code of the algebraic dual. We give an effective criterion to determine whether or not the algebraic dual is monomial and show an algorithm that can be used to compute a basis for the algebraic dual.

**Proposition 3.1.** [28] *Let  $\mathcal{L}_X$  be an evaluation code on  $X$ , let  $\prec$  be a monomial order, let  $\mathcal{G}$  be a Gröbner basis of  $I = I(X)$ , let  $\{h_1, \dots, h_k\}$  be a subset of  $\mathcal{L} \setminus \{0\}$  and for each  $i$ , let  $r_i$  be the remainder on division of  $h_i$  by  $\mathcal{G}$ . If  $\mathcal{L} = K\{h_1, \dots, h_k\}$  and*

$$\tilde{\mathcal{L}} := K\{r_1, \dots, r_k\},$$

then  $\tilde{\mathcal{L}} \subset K\Delta_{\prec}(I)$ ,  $\tilde{\mathcal{L}}_X$  is a standard evaluation code on  $X$  relative to  $\prec$  and  $\tilde{\mathcal{L}}_X = \mathcal{L}_X$ .

**Corollary 3.2.** *Let  $\mathcal{L}_X$  be an evaluation code on  $X$  and let  $\prec$  be a monomial order. Then there exists a unique linear subspace  $\tilde{\mathcal{L}}$  of  $K\Delta_{\prec}(I)$  such that  $\tilde{\mathcal{L}}_X = \mathcal{L}_X$ .*

*Proof.* The existence follows from Proposition 3.1. Assume that  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are two linear subspaces of  $K\Delta_{\prec}(I)$  such that  $(\mathcal{L}_1)_X = (\mathcal{L}_2)_X$ . Let  $P_1, \dots, P_m$  be the points of  $X$ . To show the inclusion  $\mathcal{L}_1 \subset \mathcal{L}_2$  take  $f \in \mathcal{L}_1$ . Then  $\text{ev}(f) = (f(P_1), \dots, f(P_m))$  is in  $(\mathcal{L}_1)_X$ . Thus there is  $g \in \mathcal{L}_2$  such that  $\text{ev}(f) = \text{ev}(g) = (g(P_1), \dots, g(P_m))$ . Hence  $f - g \in I(X)$  and  $f - g = 0$  because  $f$  and  $g$  are in  $K\Delta_{\prec}(I)$ . Thus  $f \in \mathcal{L}_2$ . The inclusion  $\mathcal{L}_2 \subset \mathcal{L}_1$  follows from similar reasons. Therefore  $\mathcal{L}_1 = \mathcal{L}_2$  and  $\tilde{\mathcal{L}}$  is unique.  $\square$

Recall that  $\varphi$  is the  $K$ -linear map given by

$$\varphi: S \rightarrow K, \quad f \mapsto f(P_1) + \dots + f(P_m).$$

**Definition 3.3.** Let  $\mathcal{L}_X$  be an evaluation code on  $X$  and let  $\prec$  be a monomial order on  $S$ . The unique linear subspace  $\tilde{\mathcal{L}}$  of  $K\Delta_{\prec}(I)$  such that  $\tilde{\mathcal{L}}_X = \mathcal{L}_X$  is called the *standard function space* of  $\mathcal{L}_X$ . The *dual* of  $\mathcal{L}$ , denoted  $\mathcal{L}^\perp$ , is the  $K$ -linear space given by  $\mathcal{L}^\perp := (\ker(\varphi): \mathcal{L}) \cap K\Delta_{\prec}(I)$ . We will also call  $\mathcal{L}^\perp$  the *algebraic dual* of  $\mathcal{L}_X$  relative to  $\prec$ .

**Lemma 3.4.** *Let  $\mathcal{L}_X$  be an evaluation code on  $X$  and let  $\prec$  be a monomial order on  $S$ . If  $I = I(X)$  and  $\tilde{\mathcal{L}}$  is the standard function space of  $\mathcal{L}_X$ , then*

$$\mathcal{L}^\perp = (\ker(\varphi): \mathcal{L}) \cap K\Delta_{\prec}(I) = (\ker(\varphi): \tilde{\mathcal{L}}) \cap K\Delta_{\prec}(I) = \tilde{\mathcal{L}}^\perp.$$

*Proof.* There are  $g_1, \dots, g_k$  in  $\mathcal{L} \setminus \{0\}$  such that  $\mathcal{L} = K\{g_1, \dots, g_k\}$ . By the division algorithm [11, Theorem 3, p. 63], for each  $i$ , we can write  $g_i = h_i + r_i$  for some  $h_i \in I$  and  $r_i \in K\Delta_{\prec}(I)$ . By Proposition 3.1, one has  $\tilde{\mathcal{L}} = K\{r_1, \dots, r_k\}$ . To show the inclusion “ $\subset$ ” take  $f \in \mathcal{L}^\perp$ , that is,  $f\mathcal{L} \subset \ker(\varphi)$  and  $f \in K\Delta_{\prec}(I)$ . Then  $fr_i \in \ker(\varphi)$  for all  $i$ , and consequently  $f \in \tilde{\mathcal{L}}^\perp$ . To show the inclusion “ $\supset$ ” take  $f \in \tilde{\mathcal{L}}^\perp$ , that is,  $f\tilde{\mathcal{L}} \subset \ker(\varphi)$  and  $f \in K\Delta_{\prec}(I)$ . Then  $fr_i \in \ker(\varphi)$  for all  $i$  and, since  $r_i = g_i - h_i$ , we get  $fg_i \in \ker(\varphi)$  for all  $i$ . Thus,  $f \in \mathcal{L}^\perp$ .  $\square$

**Theorem 3.5.** *Let  $\mathcal{L}_X$  be an evaluation code on  $X$  and let  $I = I(X)$  be the vanishing ideal of  $X$ . If  $\prec$  is a monomial order and  $\mathcal{L}^\perp = (\ker(\varphi): \mathcal{L}) \cap K\Delta_{\prec}(I)$ , then  $(\mathcal{L}_X)^\perp$  is the standard evaluation code  $(\mathcal{L}^\perp)_X$  on  $X$  relative to  $\prec$ .*

*Proof.* First we show the inclusion  $(\mathcal{L}_X)^\perp \subset (\mathcal{L}^\perp)_X$ . Take  $\alpha \in (\mathcal{L}_X)^\perp$ . Let  $r_0$  be the regularity index of  $H_I^a$ . The evaluation map

$$\text{ev}_{r_0}: S_{\leq r_0} \rightarrow K^m, \quad f \mapsto (f(P_1), \dots, f(P_m)),$$

is surjective since  $H_I^a(r_0) = \dim_K(S_{\leq r_0}/I_{\leq r_0}) = |X| = m$ . Then,  $\alpha = (g_1(P_1), \dots, g_1(P_m))$  for some  $g_1 \in S_{\leq r_0}$ . By the division algorithm [11, Theorem 3, p. 63], we can write  $g_1 = g_2 + g$ , where  $g_2 \in I$  and  $g \in K\Delta_{\prec}(I)$ . Thus,  $\alpha = (g(P_1), \dots, g(P_m))$ . Using that  $\alpha \in (\mathcal{L}_X)^\perp$ , we obtain

$$\langle \alpha, (f(P_1), \dots, f(P_m)) \rangle = \sum_{i=1}^m g(P_i)f(P_i) = \sum_{i=1}^m (gf)(P_i) = 0$$

for all  $f \in \mathcal{L}$ . Thus,  $g \in (\ker(\varphi): \mathcal{L}) \cap K\Delta_{\prec}(I) = \mathcal{L}^\perp$ . From the equality

$$(\mathcal{L}^\perp)_X = \{(h(P_1), \dots, h(P_m)) \mid h \in \mathcal{L}^\perp\},$$

we obtain  $\alpha \in (\mathcal{L}^\perp)_X$ . To show the inclusion  $(\mathcal{L}_X)^\perp \supset (\mathcal{L}^\perp)_X$  take  $\alpha \in (\mathcal{L}^\perp)_X$ , that is,  $\alpha = (g(P_1), \dots, g(P_m))$  for some  $g \in \mathcal{L}^\perp$ . Then,  $gf \in \ker(\varphi)$  for all  $f \in \mathcal{L}$  and

$$\langle \alpha, (f(P_1), \dots, f(P_m)) \rangle = 0$$

for all  $f \in \mathcal{L}$ . From the equality  $\mathcal{L}_X = \{(f(P_1), \dots, f(P_m)) \mid f \in \mathcal{L}\}$ , we obtain  $\alpha \in (\mathcal{L}_X)^\perp$ .  $\square$

We will need the following observation.

**Lemma 3.6.** [28, Lemma 3.1] *Let  $X$  be a subset of  $K^s$  and let  $\mathcal{L}_X$  be a standard evaluation code on  $X$  relative to a monomial order  $\prec$ . Then,  $\mathcal{L} \cap I(X) = (0)$  and  $\mathcal{L} \simeq \mathcal{L}_X$ .*

*Proof.* We set  $I = I(X)$ . Take  $f \in \mathcal{L} \cap I$  and recall that  $\mathcal{L}$  is a linear subspace of  $K\Delta_\prec(I)$ . If  $f \neq 0$ , then  $\text{in}_\prec(f) \in \text{in}_\prec(I)$ , a contradiction since all monomials of  $f$  are standard monomials of  $S/I$ . Thus,  $f = 0$ . Hence, the evaluation map gives an isomorphism between  $\mathcal{L}$  and  $\mathcal{L}_X$ .  $\square$

The next result shows that the dual of  $\mathcal{L}$  behaves well.

**Proposition 3.7.** *Let  $\mathcal{L}_X$  be a standard evaluation code on  $X$  relative to a monomial order  $\prec$  on  $S$  and let  $I = I(X)$ . The following hold.*

(a)  $\dim_K(\mathcal{L}) + \dim_K(\mathcal{L}^\perp) = |X|$ .

(b) *The conditions (b<sub>1</sub>)-(b<sub>3</sub>) are equivalent*

$$(b_1) \mathcal{L}_X \cap (\mathcal{L}_X)^\perp = (0), \quad (b_2) \mathcal{L} \cap \mathcal{L}^\perp = (0), \quad (b_3) \mathcal{L} + \mathcal{L}^\perp = K\Delta_\prec(I).$$

(c)  $\mathcal{L}_X = (\mathcal{L}_X)^\perp$  *if and only if*  $\mathcal{L} = \mathcal{L}^\perp$ .

(d)  $(\mathcal{L}^\perp)^\perp = \mathcal{L}$ .

*Proof.* (a): By [27, Theorem 1.2.1], Theorem 3.5 and Lemma 3.6, we get

$$\begin{aligned} |X| &= \dim_K(\mathcal{L}_X) + \dim_K(\mathcal{L}_X)^\perp = \dim_K(\mathcal{L}_X) + \dim_K(\mathcal{L}^\perp)_X \\ &= \dim_K(\mathcal{L}) + \dim_K(\mathcal{L}^\perp). \end{aligned}$$

(b): (b<sub>1</sub>)  $\Rightarrow$  (b<sub>2</sub>) Assume  $\mathcal{L}_X \cap (\mathcal{L}_X)^\perp = (0)$  and take  $g \in \mathcal{L} \cap \mathcal{L}^\perp$ . Then,  $\text{ev}(g) \in \mathcal{L}_X \cap (\mathcal{L}^\perp)_X$  and, because of Theorem 3.5, we get that  $\text{ev}(g)$  is in  $\mathcal{L}_X \cap (\mathcal{L}_X)^\perp = (0)$  and  $\text{ev}(g) = 0$ . Hence,  $g \in I$ , and consequently  $g = 0$  because  $g \in K\Delta_\prec(I)$ .

(b<sub>2</sub>)  $\Rightarrow$  (b<sub>3</sub>) Assume  $\mathcal{L} \cap \mathcal{L}^\perp = (0)$ . By part (a) one has

$$|X| = \dim_K(\mathcal{L}) + \dim_K(\mathcal{L}^\perp) = \dim_K(\mathcal{L} + \mathcal{L}^\perp) + \dim_K(\mathcal{L} \cap \mathcal{L}^\perp).$$

Hence,  $|X| = \dim_K(\mathcal{L} + \mathcal{L}^\perp)$ . From the inclusion  $\mathcal{L} + \mathcal{L}^\perp \subset K\Delta_\prec(I)$  and noticing that these linear spaces have dimension  $|X|$  (Lemma 2.7), we get  $\mathcal{L} + \mathcal{L}^\perp = K\Delta_\prec(I)$ .

(b<sub>3</sub>)  $\Rightarrow$  (b<sub>1</sub>) Assume  $\mathcal{L} + \mathcal{L}^\perp = K\Delta_\prec(I)$ . The evaluation map “ev” induces an isomorphism between  $K\Delta_\prec(I)$  and  $K^{|X|}$ . Then, by Theorem 3.5, we get

$$\mathcal{L}_X + (\mathcal{L}^\perp)_X = \mathcal{L}_X + (\mathcal{L}_X)^\perp = K^{|X|}$$

and the dimension of  $\mathcal{L}_X + (\mathcal{L}_X)^\perp$  is  $|X|$ . Therefore, from the equality

$$|X| = \dim_K(\mathcal{L}_X) + \dim_K(\mathcal{L}_X)^\perp = \dim_K(\mathcal{L}_X + (\mathcal{L}_X)^\perp) + \dim_K(\mathcal{L}_X \cap (\mathcal{L}_X)^\perp),$$

we obtain  $\mathcal{L}_X \cap (\mathcal{L}_X)^\perp = (0)$ .

(c):  $\Rightarrow$  Assume  $\mathcal{L}_X = (\mathcal{L}_X)^\perp$ . Let  $P_1, \dots, P_m$  be the points of  $X$ . First we show the inclusion  $\mathcal{L} \subset \mathcal{L}^\perp$ . Take  $f \in \mathcal{L}$ . Then,  $\text{ev}(f) = (f(P_1), \dots, f(P_m))$  is in  $\mathcal{L}_X$ . By Theorem 3.5,  $(\mathcal{L}_X)^\perp$  is equal to  $(\mathcal{L}^\perp)_X$ . Thus, there is  $g \in \mathcal{L}^\perp$  such that  $\text{ev}(f) = \text{ev}(g) = (g(P_1), \dots, g(P_m))$ . Then  $f - g \in I$ , and  $f = g$  because  $f, g$  are in  $K\Delta_\prec(I)$ . Thus,  $f \in \mathcal{L}^\perp$ . Now we show the inclusion

$\mathcal{L}^\perp \subset \mathcal{L}$ . Take  $f \in \mathcal{L}^\perp$ . Then,  $\text{ev}(f) = (f(P_1), \dots, f(P_m))$  is in  $(\mathcal{L}^\perp)_X$ . By Theorem 3.5,  $(\mathcal{L}^\perp)_X$  is equal to  $(\mathcal{L}_X)^\perp = \mathcal{L}_X$ . Thus, there is  $g \in \mathcal{L}$  such that  $\text{ev}(f) = \text{ev}(g) = (g(P_1), \dots, g(P_m))$ . Then  $f - g \in I$ , and  $f = g$  because  $f, g$  are standard polynomials. Thus,  $f \in \mathcal{L}$ .

$\Leftarrow$ ) Assume  $\mathcal{L} = \mathcal{L}^\perp$ . Then, by Theorem 3.5,  $\mathcal{L}_X = (\mathcal{L}^\perp)_X = (\mathcal{L}_X)^\perp$ .

(d): To show the inclusion  $(\mathcal{L}^\perp)^\perp \subset \mathcal{L}$  take  $g \in (\mathcal{L}^\perp)^\perp$ . Then  $gf \in \ker(\varphi)$  for all  $f \in \mathcal{L}^\perp$ . Hence  $\langle \text{ev}(g), \text{ev}(f) \rangle = 0$  for all  $f \in \mathcal{L}^\perp$ , that is,  $\text{ev}(g) \in ((\mathcal{L}^\perp)_X)^\perp$ . By Theorem 3.5 and the fact that the dual of  $(\mathcal{L}_X)^\perp$  is equal to  $\mathcal{L}_X$  [35, p. 26], one has  $((\mathcal{L}^\perp)_X)^\perp = ((\mathcal{L}_X)^\perp)^\perp = \mathcal{L}_X$ . Thus,  $\text{ev}(g) \in \mathcal{L}_X$  and there is  $h \in \mathcal{L}$  such that  $\text{ev}(g) = \text{ev}(h)$ . From this equality we get that  $g - h$  is in  $I$ . As  $g, h$  are in  $K\Delta_{\prec}(I)$ , it follows that  $g = h$  and  $g \in \mathcal{L}$ . To show the inclusion  $\mathcal{L} \subset (\mathcal{L}^\perp)^\perp$  take  $f \in \mathcal{L}$ . Then  $\text{ev}(f) \in \mathcal{L}_X$ . For any  $g \in \mathcal{L}^\perp$ , one has  $g\mathcal{L} \subset \ker(\varphi)$ . In particular,  $gf \in \ker(\varphi)$  for any  $g \in \mathcal{L}^\perp$ , and consequently  $f$  is in  $(\ker(\varphi): \mathcal{L}^\perp) \cap K\Delta_{\prec}(I) = (\mathcal{L}^\perp)^\perp$ .  $\square$

**Proposition 3.8.** *Let  $\mathcal{L}_X$  be an evaluation code on  $X$  and let  $\prec$  be a monomial order on  $S$ . If  $I(X)$  is a binomial ideal and  $\mathcal{L}$  is a monomial space, then  $\mathcal{L}_X$  is a standard monomial code.*

*Proof.* There exists a Gröbner basis  $\mathcal{G}$  of  $I(X)$  consisting of binomials [43, Lemma 8.2.17]. The linear space  $\mathcal{L}$  is generated by a finite set  $\{t^{a_1}, \dots, t^{a_k}\}$  of monomials. By the division algorithm [11, Theorem 3, p. 63] it follows that the remainder  $r_i$  on division of  $t^{a_i}$  by  $\mathcal{G}$  is a monomial. Hence,  $\tilde{\mathcal{L}} = K\{r_1, \dots, r_k\}$  is a monomial space.  $\square$

Let  $A = \{x^{c_1}, \dots, x^{c_s}\}$  be a finite set of monomials in a polynomial ring  $K[x_1, \dots, x_n]$ . The affine set parameterized by  $A$  is the set  $X$  of all points  $(x^{c_1}(\alpha), \dots, x^{c_s}(\alpha))$  such that  $\alpha \in K^n$ . The next result gives a wide class of standard monomial codes that includes the family of parameterized affine codes [34] and the subfamily of  $q$ -ary Reed–Muller codes [25].

**Corollary 3.9.** *If  $X$  is parameterized by monomials and  $\mathcal{L}$  is a monomial space, then  $\mathcal{L}_X$  is a standard monomial code. In particular if  $\mathcal{L} = S_{\leq d}$ , then  $\mathcal{L}_X$  is a standard monomial code.*

*Proof.* By [41, Theorem 4, p. 435],  $I(X)$  is a binomial ideal. Hence, by Proposition 3.8,  $\mathcal{L}_X$  is a standard monomial code.  $\square$

Using the equality  $(\mathcal{L}_X)^\perp = (\mathcal{L}^\perp)_X$  (Theorem 3.5) and the next result we obtain an effective criterion to verify whether or not the dual of an evaluation code is a standard monomial code.

**Proposition 3.10.** *Let  $\mathcal{L}_X$  be an evaluation code on  $X$ , let  $I$  be the vanishing ideal of  $X$ , and let  $\prec$  be a monomial order. Then,  $(\mathcal{L}^\perp)_X$  is a standard monomial code on  $X$  if and only if*

$$|(\ker(\varphi): \mathcal{L}) \cap \Delta_{\prec}(I)| = |X| - \dim_K(\mathcal{L}_X).$$

*Proof.* By Corollary 3.2, the standard function space of  $(\mathcal{L}^\perp)_X$  is equal to  $\mathcal{L}^\perp$  because  $\mathcal{L}^\perp$  is generated by standard polynomials of  $S/I$ . Then, as  $(\mathcal{L}^\perp)_X$  is a standard evaluation code, one has  $\mathcal{L}^\perp \simeq (\mathcal{L}^\perp)_X$  (Lemma 3.6). Hence, by Theorem 3.5, we get

$$(3.1) \quad \dim_K(\mathcal{L}^\perp) = \dim_K((\mathcal{L}^\perp)_X) = \dim_K(\mathcal{L}_X)^\perp = |X| - \dim_K(\mathcal{L}_X).$$

$\Rightarrow$ ) Let  $B$  be a finite monomial  $K$ -basis for  $\mathcal{L}^\perp$ . By Eq. (3.1),  $|B|$  is equal to  $|X| - \dim_K(\mathcal{L}_X)$ . Hence, the desired equality follows by noticing that  $(\ker(\varphi): \mathcal{L}) \cap \Delta_{\prec}(I) = B$ .

$\Leftarrow$ ) There are monomials  $t^{a_1}, \dots, t^{a_n}$  in  $(\ker(\varphi): \mathcal{L}) \cap \Delta_{\prec}(I)$  with  $n = |X| - \dim_K(\mathcal{L}_X)$ . By Eq. (3.1), one has  $\dim(\mathcal{L}^\perp) = n$ . Hence, as  $t^{a_i} \in \mathcal{L}^\perp$  for all  $i$ , we get  $\mathcal{L}^\perp = K\{t^{a_1}, \dots, t^{a_n}\}$ .  $\square$

**3.1. Computing a basis.** In this subsection we show an effective algorithm to compute the dimension and a  $K$ -basis for a linear subspace of  $S$  of finite dimension. Let  $(S^*)^{<\omega}$  be the set of finite subsets of  $S^* = S \setminus \{0\}$ , let  $\prec$  be the graded reverse lexicographical order (GRevLex order) on  $S$ , and let  $\sigma$  and  $\phi$  be the functions

$$\begin{aligned}\sigma, \phi: (S^*)^{<\omega} &\rightarrow (S^*)^{<\omega}, & \sigma(A) &= \{g \in A \mid \text{in}_{\prec}(g) = \text{in}_{\prec}(\max(A))\}, \\ \phi(A) &= (\{\max(A) - (\text{lc}(\max(A))/\text{lc}(g))g \mid g \in \sigma(A)\} \setminus \{0\}) \cup (A \setminus \sigma(A)),\end{aligned}$$

where  $\text{lc}(g)$  denotes the leading coefficient of  $g$  and  $\max(A)$  is any polynomial in  $A$  whose initial monomial is  $\max\{\text{in}_{\prec}(g) : g \in A\}$ . Note that  $\sigma(A)$  is the set of all polynomials in  $A$  with largest initial monomial relative to  $\prec$  and, hence, is independent of the choice of  $\max(A)$ . The following result is based on Gaussian elimination.

**Theorem 3.11.** (Basis algorithm) Let  $\mathcal{L} = KA$  be a subspace of  $S$  generated by a finite subset  $A$  of  $S^*$ . Then one can construct a  $K$ -basis for  $\mathcal{L}$  using the following algorithm:

Input:  $A$   
Output: a  $K$ -basis  $B$  for  $\mathcal{L}$   
Initialization:  $B := A$   
while  $B \neq \emptyset$  list  $\max(B)$  do  $B := \phi(B)$ .

*Proof.* As  $\sigma(B) \subset B$  we can write  $B = \{g_1, \dots, g_n\}$ , where  $\sigma(B) = \{g_1, \dots, g_r\}$ ,  $r \leq n$ , and  $\text{in}_{\prec}(g_1)$  is equal to  $\text{in}_{\prec}(g_i)$  for  $i = 1, \dots, r$ . Any  $g_i \in \sigma(B)$  can be chosen to be  $\max(B)$ . Setting  $g_1 := \max(B)$  and  $h_i := g_1 - (\text{lc}(g_1)/\text{lc}(g_i))g_i$  for  $i = 1, \dots, r$ , one has

$$\phi(B) = (\{h_i\}_{i=1}^r \setminus \{0\}) \cup \{g_i\}_{i=r+1}^n.$$

Note that  $\text{in}_{\prec}(g_1) \succ \text{in}_{\prec}(g_i)$  for  $i > r$  and  $\text{in}_{\prec}(g_1) \succ \text{in}_{\prec}(h_i)$  for  $i = 2, \dots, r$ . Thus,  $\max(B) \succ \max(\phi(B))$  and the algorithm terminates after a finite number of steps. If the algorithm terminates at  $B$ , that is,  $B \neq \emptyset$  and  $\phi(B) = \emptyset$ , then  $B = \sigma(B)$ ,  $g_1 = (\text{lc}(g_1)/\text{lc}(g_i))g_i$  for  $i = 1, \dots, r$ , and  $KB = Kg_1 = K\max(B)$ . That the output is a generating set for  $\mathcal{L} = KA$  follows by noticing that  $KB = K\max(B) + K\phi(B)$ . Finally, we show that the output is linearly independent over  $K$ . The output is the list

$$B = \{\max(A), \max(\phi(A)), \max(\phi(\phi(A))), \dots, \max(\phi^{k-1}(A))\},$$

where  $\phi^{k-1}(A) \neq \emptyset$  and  $\phi^k(A) = \emptyset$ . Since  $\max(\phi^{i-1}(A)) \succ \max(\phi^i(A))$  for  $i = 1, \dots, k-1$  it is not hard to see that  $B$  is linearly independent.  $\square$

#### 4. INDICATOR FUNCTIONS AND V-NUMBERS OF VANISHING IDEALS

Recall that  $K$  is a finite field,  $X = \{P_1, \dots, P_m\}$  is a set of points in  $K^s$ ,  $|X| \geq 2$ , and  $I = I(X)$  is its vanishing ideal. Fix a graded monomial order  $\prec$  on  $S$ . In this section we introduce and study the v-number of  $I$  and the indicator functions of  $X$ .

We begin with the notion of an indicator function of a point in  $X$  [38]. For a projective point an indicator function is called a separator [20, Definition 2.1].

**Definition 4.1.** Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$ . A polynomial  $f \in S$  is called an *indicator function* for  $P_i$  if  $f(P_i) \neq 0$  and  $f(P_j) = 0$  for all  $j \neq i$ .

An indicator function  $f$  for  $P_i$  can be normalized to have value 1 at  $P_i$  by considering  $f/f(P_i)$ . The following lemma lists basic properties of indicator functions.

**Lemma 4.2.** (a) *If  $f, g$  are indicator functions for  $P_i$  in  $K\Delta_{\prec}(I)$ , then  $g(P_i)f = f(P_i)g$ .*

(b) *The set of indicator functions for  $P_i$  is  $(I: \mathfrak{p}_i) \setminus I$ , where  $\mathfrak{p}_i$  is the vanishing ideal of  $P_i$ .*

(c) *There exists a unique, up to multiplication by a scalar from  $K^*$ , indicator function  $f$  for  $P_i$  in  $K\Delta_{\prec}(I)$ , and  $f$  is unique if  $f(P_i) = 1$ .*

(d) *If  $\mathfrak{p}_i$  is the vanishing ideal of  $P_i$ , then  $\dim_K((I: \mathfrak{p}_i)/I) = 1$  and  $(I: \mathfrak{p}_i)/I = K\bar{f}$  for any indicator function  $f$  for  $P_i$ , where  $\bar{f} = f + I$ .*

*Proof.* (a): The polynomial  $h = g(P_i)f - f(P_i)g$  vanishes at all points of  $X$ , that is,  $h \in I$ . If  $h \neq 0$ , then the initial monomial of  $h$  is in the initial ideal of  $I$ , a contradiction since all monomials of  $h$  are standard. Thus,  $h = 0$  and  $g(P_i)f = f(P_i)g$ .

(b): Note the equalities  $I = \bigcap_{j=1}^m \mathfrak{p}_j$  (Lemma 2.4) and  $(I: \mathfrak{p}_i) = \bigcap_{j \neq i} \mathfrak{p}_j$ . Let  $f$  be an indicator function for  $P_i$ , then  $f \notin \mathfrak{p}_i$  and  $f \in \mathfrak{p}_j$  for  $j \neq i$ . Thus,  $f \in (I: \mathfrak{p}_i)$  and  $f \notin I$ . Conversely, take  $f$  in  $(I: \mathfrak{p}_i) \setminus I$ . Then,  $f \in \bigcap_{j \neq i} \mathfrak{p}_j$  and  $f \notin \mathfrak{p}_i$ . Thus,  $f$  is an indicator function for  $P_i$ .

(c): The existence of  $f$  follows from the division algorithm [11, Theorem 3, p. 63] and part (b) because  $(I: \mathfrak{p}_i) \setminus I \neq \emptyset$ . The uniqueness of  $f$  follows from part (a).

(d): Let  $f$  be an indicator function for  $P_i$ . By part (b),  $f$  is in  $(I: \mathfrak{p}_i) \setminus I$ . Therefore, one has  $(I: \mathfrak{p}_i)/I \supset K\bar{f}$ . To show the other inclusion take  $\bar{0} \neq \bar{g} \in (I: \mathfrak{p}_i)/I$  and note that  $g$  is an indicator function for  $P_i$  by part (b). By the division algorithm [11, Theorem 3, p. 63], one has

$$(I: \mathfrak{p}_i)/I = \{\bar{h} \mid h \in (I: \mathfrak{p}_i) \cap K\Delta_{\prec}(I)\}.$$

Hence, using part (b), we may assume that  $f$  is an indicator function for  $P_i$  in  $K\Delta_{\prec}(I)$ , and we can write  $\bar{g} = \bar{h}$  for some  $h$  in  $(I: \mathfrak{p}_i) \cap K\Delta_{\prec}(I)$ . By part (b),  $h$  is an indicator function for  $P_i$ . Hence, by part (a), we get  $\bar{g} = \bar{h} = \lambda\bar{f}$ ,  $\lambda = h(P_i)/f(P_i)$ . Thus,  $\bar{g} \in K\bar{f}$ .  $\square$

The following numerical invariant will be used to determine the regularity index of the minimum distance of a Reed–Muller-type code (Proposition 6.2).

**Definition 4.3.** The *v-number* of  $I = I(X)$ , denoted  $v(I)$ , is given by

$$v(I) := \min\{d \geq 0 \mid \text{there is } 0 \neq f \in S, \deg(f) = d, \text{ and } \mathfrak{p} \in \text{Ass}(I) \text{ with } (I: f) = \mathfrak{p}\},$$

where  $\text{Ass}(I)$  is the set of associated primes of  $S/I$ .

The v-number is finite by the definition of associated primes and  $v(I) \geq 1$  because  $|X| \geq 2$ . Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  be the associated primes of  $I$ , that is,  $\mathfrak{p}_i$  is the vanishing ideal  $I_{P_i}$  of  $P_i$ . One can define the v-number of  $I$  at each  $\mathfrak{p}_i$  by

$$v_{\mathfrak{p}_i}(I) := \min\{d \geq 0 \mid \exists 0 \neq f \in S, \deg(f) = d, \text{ with } (I: f) = \mathfrak{p}_i\}.$$

**Lemma 4.4.** *The least degree of an indicator function for  $P_i$  is equal to  $v_{\mathfrak{p}_i}(I)$ .*

*Proof.* A polynomial  $f \in S$  is an indicator function for  $P_i$  if and only if  $(I: f) = \mathfrak{p}_i$ . This follows using that the primary decomposition of  $I$  is given by  $I = \bigcap_{j=1}^m \mathfrak{p}_j$  (Lemma 2.4) and noticing that  $(I: f) = \bigcap_{f \notin \mathfrak{p}_j} \mathfrak{p}_j$ . Hence,  $v_{\mathfrak{p}_i}(I)$  is the minimum degree of an indicator function for  $P_i$ .  $\square$

Note that the v-number of  $I$  is equal to  $\min\{v_{\mathfrak{p}_i}(I)\}_{i=1}^m$ . To compute the v-number using *Macaulay2* [24] (Example 8.5), we give a description for the v-number of  $I$  using initial degrees of certain ideals of the quotient ring  $S/I$ .

For an ideal  $M \neq 0$  of  $S/I$ , we define  $\alpha(M) := \min\{\deg(f) \mid \bar{f} \in M, f \notin I\}$ . The next result was shown in [10, Proposition 4.2] for unmixed graded ideals. For vanishing ideals we prove that the graded assumption is not needed.

**Proposition 4.5.** *Let  $I \subset S$  be the vanishing ideal of  $X$ . Then*

$$v(I) = \min\{\alpha((I: \mathfrak{p})/I) \mid \mathfrak{p} \in \text{Ass}(I)\},$$

and  $\alpha((I: \mathfrak{p})/I) = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ .

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  be the associated primes of  $I$ . If  $I$  is the vanishing ideal of a point, then  $(I: 1) = I$ ,  $(I: I) = S$ , and  $v(I) = \alpha(S/I) = 0$ . Thus, we may assume that  $X$  has at least two points. Since  $v(I) = \min\{v_{\mathfrak{p}_i}(I)\}_{i=1}^m$ , we need only show that  $\alpha((I: \mathfrak{p})/I)$  is equal to  $v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ . Fix  $1 \leq k \leq m$ . There is  $f \in S$  such that  $(I: f) = \mathfrak{p}_k$  and  $v_{\mathfrak{p}_k}(I) = \deg(f)$ . Then,  $f \in (I: \mathfrak{p}_k) \setminus I$  and

$$v_{\mathfrak{p}_k}(I) = \deg(f) \geq \alpha((I: \mathfrak{p}_k)/I).$$

Since  $I = \bigcap_{i=1}^m \mathfrak{p}_i \subsetneq \bigcap_{i \neq k} \mathfrak{p}_i = (I: \mathfrak{p}_k)$ , we can pick a polynomial  $g$  in  $(I: \mathfrak{p}_k) \setminus I$  such that  $\alpha((I: \mathfrak{p}_k)/I) = \deg(g)$ . Note that  $g \notin \mathfrak{p}_k$  since  $g \notin I$ . Therefore, from the inclusions

$$\mathfrak{p}_k \subset (I: g) = \bigcap_{i=1}^m (\mathfrak{p}_i: g) = \bigcap_{g \notin \mathfrak{p}_i} \mathfrak{p}_i \subset \mathfrak{p}_k,$$

we get  $(I: g) = \mathfrak{p}_k$ , and consequently  $v_{\mathfrak{p}_k}(I) \leq \deg(g) \leq \alpha((I: \mathfrak{p}_k)/I)$ .  $\square$

By the next result, for each  $P_i$  in  $X$  there is a unique indicator function  $f_i$  for  $P_i$  in  $K\Delta_{\prec}(I)$  of degree  $v_{\mathfrak{p}_i}(I)$  satisfying  $f_i(P_i) = 1$ . We call  $F = \{f_1, \dots, f_m\}$  the set of *standard indicator functions* for  $X$  (Example 8.5).

**Proposition 4.6.** *Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$ , let  $I = I(X)$  be its vanishing ideal, and let  $\prec$  be a graded monomial order on  $S$ . The following hold.*

- (a) *For each  $1 \leq i \leq m$  there is a unique  $f_i$  in  $K\Delta_{\prec}(I)$  such that  $f_i(P_i) = 1$  and  $f_i(P_j) = 0$  if  $j \neq i$ . The degree of  $f_i$  is  $v_{\mathfrak{p}_i}(I)$  and the set  $F = \{f_1, \dots, f_m\}$  is a  $K$ -basis for  $K\Delta_{\prec}(I)$ .*
- (b)  $\ker(\varphi) = K\{f_i - f_m\}_{i=1}^{m-1} + I$ .
- (c)  $K\{f_i - f_m\}_{i=1}^{m-1} = \ker(\varphi) \cap K\Delta_{\prec}(I) = K^{\perp}$ .
- (d) *If  $r_0 = \text{reg}(H^0)$ , then  $\deg(f_i) \leq r_0$  for all  $i$  and  $\deg(f_j) = r_0$  for some  $j$ .*

*Proof.* (a): The existence and uniqueness of  $f_i$  follows from Lemma 4.2. By Proposition 4.5, we can pick  $g \in (I: \mathfrak{p}_i) \setminus I$  such that  $v_{\mathfrak{p}_i}(I) = \alpha((I: \mathfrak{p}_i)/I) = \deg(g)$ . As  $\prec$  is graded, by the division algorithm [11, Theorem 3, p. 63], we can write  $g = h + r_g$  for some  $h \in I$  and some  $r_g \in K\Delta_{\prec}(I)$  with  $\deg(r_g) \leq \deg(g)$ . Noticing that  $r_g \in (I: \mathfrak{p}_i) \setminus I$ , we get  $\deg(r_g) = \deg(g)$ . Since  $r_g$  is an indicator function for  $P_i$  in  $K\Delta_{\prec}(I)$ , by Lemma 4.2(a), we get  $r_g = \lambda f_i$  for some  $\lambda \in K^*$ . Thus  $\deg(f_i) = v_{\mathfrak{p}_i}(I)$ . To show that  $F$  is linearly independent assume that  $\sum_{i=1}^m \lambda_i f_i = 0$  for some  $\lambda_1, \dots, \lambda_m$  in  $K$ . Hence, evaluating both sides of this equality at each  $P_j$  gives  $\lambda_j = 0$ . Now, the dimension of the linear space  $K\Delta_{\prec}(I)$  is  $m = |X|$  because  $|\Delta_{\prec}(I)| = |X|$  (Lemma 2.7). Thus,  $F$  is a  $K$ -basis for  $K\Delta_{\prec}(I)$ .

(b): First we show the inclusion “ $\supset$ ”. Clearly  $\ker(\varphi) \supset I$ . Thus, this inclusion follows by noticing that  $f_i - f_m$  is in the kernel of  $\varphi$  since  $\varphi(f_i - f_m) = f_i(P_i) - f_m(P_m) = 0$ . To show the inclusion “ $\subset$ ” take  $f \in \ker(\varphi)$ . By the division algorithm, we can write  $f = h + r_f$  for some  $h \in I$  and  $r_f \in K\Delta_{\prec}(I)$ . By part (a) we can write  $r_f = \sum_{i=1}^m \lambda_i f_i$  for some  $\lambda_i$ 's in  $K$ . Then, by noticing that  $r_f \in \ker(\varphi)$ , we get  $\varphi(r_f) = \sum_{i=1}^m \lambda_i = 0$ , and consequently

$$r_f = \left( \sum_{i=1}^{m-1} \lambda_i f_i \right) + \lambda_m f_m = \sum_{i=1}^{m-1} \lambda_i (f_i - f_m).$$

Thus,  $f = h + r_f \in I + K\{f_i - f_m\}_{i=1}^{m-1}$ .

(c): This follows from the proof of part (b).

(d): By Lemma 2.7 one has the inclusion  $K\Delta_{\prec}(I) \subset S_{\leq r_0}$ . Thus,  $\deg(f_i) \leq r_0$  for all  $i$ . Note that  $H_I^a(r_0 - 1) < H_I^a(r_0)$  by definition of  $r_0 = \text{reg}(H_I^a)$ . Then, by Lemmas 2.6 and 2.7, we obtain  $\Delta_{\prec}(I) \not\subset S_{\leq r_0 - 1}$ . Hence, we can pick  $t^a \in \Delta_{\prec}(I)$  of degree  $r_0$ . Then, by part (a),  $t^a$  is in  $K\{f_1, \dots, f_m\}$ , and consequently  $\deg(f_j) = r_0$  for some  $j$ .  $\square$

**Remark 4.7.** Using Lemma 4.2 and the ideal  $(I: \mathfrak{p}_i)/I$  of Proposition 4.5, we obtain an algebraic method to compute the standard indicator functions for  $X$  (Example 8.5, Procedure A.1).

**Remark 4.8.** It is convenient to have a matrix interpretation of the standard indicator functions for  $X$ . Recall that the evaluation map defines an isomorphism of vector spaces

$$\text{ev}: K\Delta_{\prec}(I) \rightarrow K^m.$$

The standard monomials  $\Delta_{\prec}(I)$  ordered using  $\prec$  form a basis for  $K\Delta_{\prec}(I)$ . Also an order of the points in  $X = \{P_1, \dots, P_m\}$  defines the standard basis for  $K^m$ . We let  $M_{\text{ev}}$  be the matrix of the evaluation map in these bases, i.e. the  $i$ -th column of  $M_{\text{ev}}$  consists of the values of the  $i$ -th standard monomial at  $P_1, \dots, P_m$ . Then the  $i$ -th column of the inverse  $M_{\text{ev}}^{-1}$  consists of the coefficients, relative to  $\Delta_{\prec}(I)$ , of the standard indicator function  $f_i$  for  $P_i$ .

## 5. DUALITY OF STANDARD MONOMIAL CODES

We continue with our original setup where  $K$  is a finite field,  $X$  a subset of  $K^s$  of size at least two, and  $I = I(X)$  the vanishing ideal of  $X$ . Fix a monomial order  $\prec$  and let  $\Delta_{\prec}(I)$  be the corresponding set of standard monomials. In this section we consider *standard monomial codes*, that is evaluation codes defined by subspaces which have a basis of standard monomials.

**Definition 5.1.** Given a subset  $\Gamma \subset \Delta_{\prec}(I)$ , let  $\mathcal{L}(\Gamma)$  be the  $K$ -span of the set of all monomials  $u \in \Gamma$ . Then  $\mathcal{L}(\Gamma)_X$  is called the *standard monomial code* corresponding to  $\Gamma$ .

Consider two standard monomial codes  $\mathcal{L}(\Gamma_1)_X$  and  $\mathcal{L}(\Gamma_2)_X$  for some  $\Gamma_1, \Gamma_2 \subset \Delta_{\prec}(I)$ . The main result of this section (Theorem 5.4 below) is a combinatorial condition for  $\mathcal{L}(\Gamma_1)_X$  to be monomially equivalent to the dual  $\mathcal{L}(\Gamma_2)_X^\perp$ . In what follows  $\Gamma_1\Gamma_2$  denotes the pair-wise product of the subsets, i.e.

$$\Gamma_1\Gamma_2 = \{u_1u_2 \in S \mid u_1 \in \Gamma_1, u_2 \in \Gamma_2\}.$$

Also, we write  $\bar{u} \in K\Delta_{\prec}(I)$  for the representative of  $u \in S$  modulo the ideal  $I$ .

First, recall the definition of monomial equivalence of codes and its properties.

**Definition 5.2.** We say that two linear codes  $C_1, C_2$  in  $K^m$  are *monomially equivalent* if there is  $\beta = (\beta_1, \dots, \beta_m)$  in  $K^m$  such that  $\beta_i \neq 0$  for all  $i$  and  $C_2 = \beta \cdot C_1 = \{\beta \cdot c \mid c \in C_1\}$ , where  $\beta \cdot c$  is the vector given by  $(\beta_1c_1, \dots, \beta_m c_m)$  for  $c = (c_1, \dots, c_m) \in C_1$ .

**Remark 5.3.** Monomial equivalence of codes is an equivalence relation. If  $C_2 = \beta \cdot C_1$ , then  $C_1 = \beta^{-1} \cdot C_2$ ,  $(\beta \cdot C_2)^\perp = \beta^{-1} \cdot C_2^\perp$ , and if  $C_1^\perp = \beta \cdot C_2$ , then  $C_2^\perp = \beta \cdot C_1$ .

To state the main result we will need the following definition. We say a standard monomial  $t^e \in \Delta_{\prec}(I)$  is *essential* if it appears in each standard indicator function of  $X$  (cf. Proposition 4.6).

**Theorem 5.4.** *Let  $X$  be a subset of  $K^s$  of size  $m = |X| \geq 2$ , and  $I = I(X)$  be the vanishing ideal of  $X$ . Fix a monomial order  $\prec$  and let  $t^e \in \Delta_{\prec}(I)$  be essential. Then for any  $\Gamma_1, \Gamma_2 \subset \Delta_{\prec}(I)$  satisfying*

$$(1) \quad |\Gamma_1| + |\Gamma_2| = |X|,$$

(2)  $t^e$  does not appear in  $\bar{u}$  for any  $u \in \Gamma_1\Gamma_2$ ,

we have  $\beta \cdot \mathcal{L}(\Gamma_1)_X = \mathcal{L}(\Gamma_2)_X^\perp$ , for some  $\beta = (\beta_1, \dots, \beta_m) \in (K^*)^m$ . Moreover,  $\beta_i$  is the coefficient of  $t^e$  in the  $i$ -th standard indicator function  $f_i$ , for  $i = 1, \dots, m$ .

*Proof.* Let  $\beta \in (K^*)^m$  be as in the statement of the theorem. Then, by Remark 4.8,  $\beta$  is the last row of the matrix  $M_{\text{ev}}^{-1}$ . Clearly  $\beta$  is orthogonal to all but the last column of  $M_{\text{ev}}$ . Since  $\text{ev}$  is an isomorphism, there is a unique polynomial  $g \in K\Delta_{\prec}(I)$  such that  $g(P_i) = \beta_i$  for  $P_i \in X$ ,  $i = 1, \dots, m$ . Therefore, the orthogonality property is equivalent to  $\varphi(gt^a) = 0$  for any  $t^a \in \Delta_{\prec}(I)$ ,  $t^a \neq t^e$ . By linearity, this implies

$$(5.1) \quad \varphi(gf) = 0 \quad \text{for any } f \in K(\Delta_{\prec}(I) \setminus \{t^e\}).$$

Now pick any  $h_i \in \mathcal{L}(\Gamma_i)$ , for  $i = 1, 2$ . Note that the monomials appearing in  $h_1h_2$  belong to  $\Gamma_1\Gamma_2$ . Let  $f = \overline{h_1h_2} \in K\Delta_{\prec}(I)$  be the representative of  $h_1h_2$  modulo  $I$ . Then, according to condition (2) above, the essential monomial  $t^e$  does not appear in  $f$ . Therefore, we have

$$\varphi(gh_1h_2) = \varphi(gf) = 0,$$

by Eq. (5.1). This shows that  $g\mathcal{L}(\Gamma_1) \subset (\ker(\varphi): \mathcal{L}(\Gamma_2))$ . Applying the evaluation map to both sides and using Theorem 3.5, we obtain

$$\beta \cdot \mathcal{L}(\Gamma_1)_X \subset \mathcal{L}(\Gamma_2)_X^\perp.$$

Finally, condition (1) ensures that the above inclusion is equality, as

$$\dim_K(\beta \cdot \mathcal{L}(\Gamma_1)_X) = \dim_K(\mathcal{L}(\Gamma_1)_X) = |\Gamma_1| = |X| - |\Gamma_2| = \dim_K(\mathcal{L}(\Gamma_2)_X^\perp).$$

Note that the first equality holds since  $\beta_i \neq 0$  for all  $i$ , as  $t^e$  is essential.  $\square$

**Remark 5.5.** Note that in the case when  $\Gamma_1\Gamma_2$  is contained in  $\Delta_{\prec}(I)$ , condition (2) in the statement of Theorem 5.4 can be relaxed to  $\Gamma_1\Gamma_2 \subset \Delta_{\prec}(I) \setminus \{t^e\}$ .

Duality formulas for certain toric complete intersection codes are given in [9, Theorem 3.3]. These codes are a generalization of projective evaluation codes on complete intersections.

## 6. A DUALITY CRITERION FOR REED–MULLER-TYPE CODES

In this section we concentrate on the class of evaluation codes, called Reed–Muller-type codes, defined by evaluating the subspace of polynomial of total degree up to  $d$  at a set of points  $X \subset K^s$ . As before we assume  $X = \{P_1, \dots, P_m\}$  where  $m \geq 2$ . The main result of this section is a duality criterion for Reed–Muller-type codes. It is then applied to the case when the vanishing ideal  $I = I(X)$  is Gorenstein.

**Definition 6.1.** [13, 23] Fix a degree  $d \geq 1$  and let  $S_{\leq d} = \bigoplus_{i=0}^d S_i$  be the  $K$ -linear subspace of  $S$  of all polynomials of degree at most  $d$ . If  $\mathcal{L} = S_{\leq d}$  then the resulting evaluation code  $\mathcal{L}_X$  is called a *Reed–Muller-type code* of degree  $d$  on  $X$  and is denoted by  $C_X(d)$ .

The minimum distance of  $C_X(d)$  is simply denoted by  $\delta_X(d)$ . As is seen below the  $v$ -number of  $I = I(X)$  is related to the asymptotic behavior of  $\delta_X(d)$  for  $d \gg 0$ . There is  $n \in \mathbb{N}$  such that

$$|X| = \delta_X(0) > \delta_X(1) > \dots > \delta_X(n-1) > \delta_X(n) = \delta_X(d) = 1 \quad \text{for } d \geq n,$$

see Proposition 2.2. The number  $n$ , denoted  $\text{reg}(\delta_X)$ , is called the *regularity index* of  $\delta_X$ .

**Proposition 6.2.** *If  $I = I(X)$ , then  $v(I) = \text{reg}(\delta_X) \leq \text{reg}(H_I^a)$ .*

*Proof.* By of the Singleton bound for linear codes [27, p. 71],  $\delta_X(d) = 1$  for  $d \geq \text{reg}(H_I^a)$ . Thus,  $\text{reg}(\delta_X) \leq \text{reg}(H_I^a)$ . By Lemma 4.4,  $v(I)$  is the minimum degree of the indicator functions of the points of the set  $X$ . Then, there is a point  $P_i$  and an indicator function  $f$  for  $P_i$  such that  $n_0 = v(I) = \text{deg}(f)$ . Thus,  $\delta_X(n_0) = 1$  and  $\text{reg}(\delta_X) \leq v(I)$ . If  $n = \text{reg}(\delta_X) < v(I)$ , then  $\delta_X(n) = 1$ , and consequently there is  $g \in S_{\leq n}$  and there is a point  $P_j$  such that  $g$  is an indicator function for  $P_j$ , a contradiction because  $n_0 = v(I) \leq \text{deg}(g) \leq n$ .  $\square$

**Lemma 6.3.** *Let  $I$  be the vanishing ideal of  $X$ , let  $r_0$  be the regularity index of  $H_I^a$ , and let  $\mathcal{C}$  be the set  $S_{\leq r_0-1} \cap K\Delta_{\prec}(I)$ . The following are equivalent.*

- (a)  $\mathcal{C}^\perp := (\ker(\varphi): \mathcal{C}) \cap K\Delta_{\prec}(I) = Kg$  for some  $0 \neq g \in S$ .
- (b)  $C_X(r_0 - 1)^\perp = K(g(P_1), \dots, g(P_m))$  for some  $0 \neq g \in K\Delta_{\prec}(I)$ .
- (c)  $H_I^a(r_0 - 1) + 1 = |X|$ .

*Proof.* (a)  $\Rightarrow$  (b): Under the evaluation map “ev” the left (resp. right) hand side of the equality  $\mathcal{C}^\perp = Kg$  map onto  $C_X(r_0 - 1)^\perp$  (resp.  $K(g(P_1), \dots, g(P_m))$ ).

(b)  $\Rightarrow$  (c):  $1 = \dim_K(C_X(r_0 - 1)^\perp) = |X| - \dim_K(C_X(r_0 - 1)) = |X| - H_I^a(r_0 - 1)$ .

(c)  $\Rightarrow$  (a): Using Proposition 3.7 together with the equality  $\mathcal{C} = K(S_{\leq r_0-1} \cap \Delta_{\prec}(I))$  and Lemma 2.6, we get  $\dim_K(\mathcal{C}^\perp) = |X| - \dim_K(\mathcal{C}) = |X| - H_I^a(r_0 - 1) = 1$ . Hence,  $\dim_K(\mathcal{C}^\perp) = 1$ , and  $\mathcal{C}^\perp = Kg$  for some  $0 \neq g \in S$ .  $\square$

**Proposition 6.4.** *Let  $I$  be the vanishing ideal of  $X$  and let  $r_0$  be the regularity index of  $H_I^a$ . Then, there exists  $g \in K\Delta_{\prec}(I)$  such that*

$$C_X(r_0 - 1)^\perp = K(g(P_1), \dots, g(P_m))$$

and  $g(P_i) \neq 0$  for all  $i$  if and only if  $H_I^a(r_0 - 1) + 1 = |X|$  and  $v_{\mathfrak{p}_i}(I) = r_0$  for all  $i$ .

*Proof.* By Lemma 6.3,  $C_X(r_0 - 1)^\perp = K(g(P_1), \dots, g(P_m))$  if and only if  $H_I^a(r_0 - 1) + 1 = |X|$ . Note that the latter is equivalent to having exactly one standard monomial of degree  $r_0$  in  $\Delta_{\prec}(I)$ , by Lemma 2.6. Thus we only need to prove that  $g(P_i) \neq 0$  for all  $i$  if and only if  $v_{\mathfrak{p}_i}(I) = r_0$  for all  $i$ . Recall that  $v_{\mathfrak{p}_i}(I)$  equals the degree of the  $i$ -th standard indicator function  $f_i$ , see Proposition 4.6. Let  $M_{\text{ev}}$  be the matrix defined in Remark 4.8. Since  $M_{\text{ev}}^{-1}M_{\text{ev}} = I_m$ , the  $m$ -th row  $(c_1, \dots, c_m)$  of  $M_{\text{ev}}^{-1}$  is orthogonal to the first  $m - 1$  columns of  $M_{\text{ev}}$ , i.e.  $(c_1, \dots, c_m)$  is orthogonal to  $C_X(r_0 - 1)$ . By Remark 4.8,  $c_i$  is the coefficient in  $f_i$  of the monomial of degree  $r_0$ . We obtain  $g(P_i) \neq 0$  for all  $i$  if and only if  $c_i \neq 0$  for all  $i$ , which happens if and only if  $v_{\mathfrak{p}_i}(I) = \text{deg}(f_i) = r_0$  for all  $i$ .  $\square$

We come to one of our main results.

**Theorem 6.5.** (Duality criterion) *Let  $X$  be a subset of  $K^s$ ,  $|X| \geq 2$ , let  $I = I(X)$  be its vanishing ideal, let  $r_0$  be the regularity index of  $H_I^a$ , and let  $\prec$  be a graded monomial order. The following conditions are equivalent.*

- (a)  $C_X(d)$  is monomially equivalent to  $C_X(r_0 - d - 1)^\perp$  for  $-1 \leq d \leq r_0$ .
- (b)  $H_I^a(d) + H_I^a(r_0 - d - 1) = |X|$  for  $-1 \leq d \leq r_0$  and  $r_0 = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ .
- (c) There is  $g \in K\Delta_{\prec}(I)$  such that  $g(P_i) \neq 0$  for all  $i$  and

$$C_X(r_0 - d - 1)^\perp = (g(P_1), \dots, g(P_m)) \cdot C_X(d) \quad \text{for } -1 \leq d \leq r_0.$$

Moreover, one can choose  $g = \sum_{i=1}^m \text{lc}(f_i)f_i$ , where  $f_i$  is the  $i$ -th standard indicator function and  $\text{lc}(f_i)$  is its leading coefficient.

*Proof.* (a)  $\Rightarrow$  (b): Since  $\dim_K(C_X(r_0-d-1)^\perp) = |X| - H_I^a(r_0-d-1)$  and  $\dim_K(C_X(d)) = H_I^a(d)$ , one has  $H_I^a(r_0-d-1) + H_I^a(d) = |X|$  because equivalent codes have the same dimension. As  $C_X(0)$  is equivalent to  $C_X(r_0-1)^\perp$ , there is  $\beta = (\beta_1, \dots, \beta_m)$  in  $K^m$  such that  $\beta_i \neq 0$  for all  $i$  and  $C_X(r_0-1)^\perp = \beta \cdot C_X(0) = K\beta$ . The image of  $S_{\leq r_0}$  under the evaluation map “ev” is equal to  $K^{|X|}$ . This follows from the equalities  $H_I^a(r_0) = \dim_K(S_{\leq r_0}/I_{\leq r_0}) = |X|$ . Hence, by the division algorithm, it follows that there is  $g \in K\Delta_{\prec}(I)$  such that  $\beta = (g(P_1), \dots, g(P_m))$ . Then, by Proposition 6.4, we get  $v_{\mathfrak{p}}(I) = r_0$  for  $\mathfrak{p} \in \text{Ass}(I)$ .

(b)  $\Rightarrow$  (c): This implication follows from Theorem 5.4 if we let  $\Gamma_1$  and  $\Gamma_2$  be the set of monomials in  $\Delta_{\prec}(I)$  of degree up to  $d$  and up to  $r_0-d-1$ , respectively. Then we have  $\mathcal{L}(\Gamma_1) = S_{\leq d} \cap K\Delta_{\prec}(I)$  and  $\mathcal{L}(\Gamma_2) = S_{\leq r_0-d-1} \cap K\Delta_{\prec}(I)$  and, consequently,

$$\mathcal{L}(\Gamma_1)_X = C_X(d) \quad \text{and} \quad \mathcal{L}(\Gamma_2)_X = C_X(r_0-d-1).$$

The largest standard monomial  $t^e$  with respect to  $\prec$  has total degree  $r_0$  (Lemma 2.7). By Proposition 2.8(d), there exists a unique standard monomial of degree  $r_0$  that is equal to  $t^e$ . Now, the condition  $r_0 = v_{\mathfrak{p}}(I)$  for all  $\mathfrak{p} \in \text{Ass}(I)$  means that every standard indicator function  $f_i$  has total degree  $r_0$  (Proposition 4.6(a)) and, hence, the leading monomial of every  $f_i$  is equal to  $t^e$ . Thus,  $t^e$  is an essential monomial. Furthermore, condition (1) of Theorem 5.4 translates to  $H_I^a(d) + H_I^a(r_0-d-1) = |X|$  (Lemma 2.6). Also, since  $d + (r_0-d-1) < r_0$  we see that  $\Gamma_1\Gamma_2 \subset \Delta_{\prec}(I) \setminus \{t^e\}$  and, hence, condition (2) of Theorem 5.4 is also satisfied (see Remark 5.5). Therefore, by Theorem 5.4, we get

$$\beta \cdot \mathcal{L}(\Gamma_1)_X = \mathcal{L}(\Gamma_2)_X^\perp,$$

where  $\beta_i$  is the coefficient of  $t^e$  in the  $i$ -th standard indicator function  $f_i$  for  $i = 1, \dots, m$ . Setting  $g = \sum_{i=1}^m \text{lc}(f_i) f_i$ , one has  $g(P_i) = \text{lc}(f_i) = \beta_i$  for all  $i$ .

(c)  $\Rightarrow$  (a): This follows from the definition of equivalent codes.  $\square$

The next result is known for a complete intersection homogeneous vanishing ideal  $I$  [22, Lemma 3]. As an application we show this result under weaker conditions.

**Corollary 6.6.** *Let  $X$  be a subset of  $K^s$ , let  $I$  be its vanishing ideal, and let  $r_0$  be the regularity index of  $H_I^a$ . If  $H_I^a(r_0-1) + 1 = |X|$  and  $r_0 = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ , then  $\delta(C_X(r_0-1)) = 2$ .*

*Proof.* By Theorem 6.5,  $C_X(r_0-1)^\perp$  is equivalent of  $K(1, \dots, 1) = C_X(0)$ . Hence,  $C_X(0)^\perp$  is equivalent to  $C_X(r_0-1)$ . Since  $C_X(0)$  is a repetition code, it is easy to see that  $C_X(0)^\perp$  has minimum distance 2. Therefore,  $\delta(C_X(r_0-1)) = 2$ .  $\square$

**Definition 6.7.** [20, p. 171] Let  $Y$  be a finite subset of a projective space  $\mathbb{P}^s$  over a field  $K$  and let  $I(Y)$  be its homogeneous vanishing ideal in  $R = K[t_0, \dots, t_s]$ . We say that  $Y$  is a *Cayley–Bacharach scheme* (CB-scheme) if every hypersurface of degree less than  $\text{reg}(R/I(Y))$  that contains all but one point of  $Y$  must contain all the points of  $Y$ .

To show the following proposition we need a result of Geramita, Kreuzer and Robbiano [20, Corollary 3.7] about CB-schemes. In loc. cit. the field  $K$  is assumed to be infinite [20, p. 165] but the result that we need it is seen to be valid for finite fields.

**Proposition 6.8.** *Let  $X$  be a subset of  $K^s$ , let  $I$  be its vanishing ideal, and let  $r_0$  be the regularity index of  $H_I^a$ . If  $r_0 = v_{\mathfrak{p}}(I)$  for all  $\mathfrak{p} \in \text{Ass}(I)$ , then*

- (a)  $H_I^a(d) + H_I^a(r_0-d-1) \leq |X|$  for  $0 \leq d \leq r_0$ , and
- (b)  $\dim_K(C_X(d)^\perp) \geq \dim_K(C_X(r_0-d-1))$  for  $0 \leq d \leq r_0$ .

*Proof.* (a): The ideal  $I^h$  is the homogeneous vanishing ideal  $I(Y)$  of the set  $Y = \{[P, 1] \mid P \in X\}$  of points in the projective space  $\mathbb{P}^s$  (Lemma 2.3). One has the equality  $H_I^a(d) = H_{I^h}(d)$  for  $d \geq 0$  [43, Lemma 8.5.4]. Hence, by [20, Corollary 3.7], it suffices to show that  $Y$  is a CB-scheme. Let  $f$  be a homogeneous polynomial of  $S[u]$  of degree less than  $r_0$  and let  $Q$  be a point in  $X$  such that  $f(P, 1) = 0$  for  $P \in X \setminus \{Q\}$ . We need only show that  $f(Q, 1) = 0$ . Assume that  $f(Q, 1) \neq 0$ . Consider the polynomial  $g = f(t_1, \dots, t_s, 1)$ . Then,  $g$  is an indicator function for  $Q$  of degree less than  $r_0$ , and consequently  $v_{\mathfrak{q}}(I) < r_0$ , where  $\mathfrak{q} = I_Q$ , a contradiction.

(b): This follows from part (a) and the equality  $H_I^a(d) + \dim_K(C_X(d)^\perp) = |X|$ .  $\square$

In the next theorem (Theorem 6.11 below) we show that when the vanishing ideal  $I(X)$  is Gorenstein then all standard indicator functions  $f_i$  have the same degree which equals the regularity of  $I(X)$ . The proof is based on the following lemma which relates the Castelnuovo–Mumford regularity and the socle of Artinian rings. Recall that the *socle* of an Artinian positively graded algebra  $N = \bigoplus_{d \geq 0} N_d$  is

$$\text{Soc}(N) = (0 : N_+), \quad \text{where } N_+ = \bigoplus_{d > 0} N_d.$$

**Lemma 6.9.** [43, Lemma 5.3.3] *Let  $J$  be a homogeneous ideal in a polynomial ring  $R$ . If  $R/J$  is Artinian then the Castelnuovo–Mumford regularity  $\text{reg}(R/J)$  equals the maximal degree of generators of  $\text{Soc}(R/J)$ .*

The statement can also be found in [16]. We will also need the following lemma.

**Lemma 6.10.** *Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$  and let  $I = I(X)$  be its vanishing ideal. If  $f \in S$  is an indicator function for  $P_i$  of minimum degree  $d$ , then its homogenization  $f^h$ , with respect to the variable  $u$ , is not in the ideal  $(I^h, u)$  and  $\deg(f) = v_{\mathfrak{p}_i}(I)$ .*

*Proof.* If  $f^h = g + uh$  for some  $g \in S[u]_d \cap I^h$ ,  $h \in S[u]_{d-1}$ . The ideal  $I^h$  is the homogeneous vanishing ideal of the set  $Y = \{[P_i, 1]\}_{i=1}^m$  of projective points in  $\mathbb{P}^s$  (Lemma 2.3). Hence, setting  $H = h(t_1, \dots, t_s, 1)$ , we get  $f(P_i) = f^h(P_i, 1) = H(P_i) \neq 0$  and  $f(P_j) = f^h(P_j, 1) = H(P_j) = 0$  for  $j \neq i$ , a contradiction because  $\deg(H) < \deg(f)$ . By Lemma 4.4,  $\deg(f) = v_{\mathfrak{p}_i}(I)$ .  $\square$

**Theorem 6.11.** *Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$  and let  $f_i$  be the  $i$ -th standard indicator function for  $P_i$ . If  $I = I(X)$  is a Gorenstein ideal and  $\mathfrak{p}_i$  is the vanishing ideal of  $P_i$ , then  $v_{\mathfrak{p}_i}(I) = \deg(f_i) = \text{reg}(H_I^a)$ .*

*Proof.* Let  $M = S[u]/I^h$  and consider the Artinian ring  $M/uM$ . Then

$$\text{Soc}(M/uM) = ((I^h, u) : \mathfrak{m}) / (I^h, u),$$

where  $\mathfrak{m}$  is the maximal ideal  $(t_1, \dots, t_s, u)$ . We set  $g_i = f_i^h$ . By Lemma 6.10,  $g_i \notin (I^h, u)$ . We claim that  $\overline{g_i}$ , the class of  $g_i$  modulo  $(I^h, u)$ , is in  $\text{Soc}(M/uM)$ . Indeed, let  $P_{ij}$  be the  $j$ -th coordinate of  $P_i$ . Then, for any  $1 \leq j \leq s$ , the polynomial  $(t_j - P_{ij}u)g_i$  vanishes on the set of projective points  $Y = \{[P_i, 1]\}_{i=1}^m$  and, hence,  $t_j g_i \in (I^h, u)$  (see Lemma 2.3). Also,  $u g_i \in (I^h, u)$ , trivially. Thus,  $\mathfrak{m} g_i \in (I^h, u)$  which shows the claim.

Now if  $S[u]/I^h$  is Gorenstein, then the socle of  $M/uM$  is a  $K$ -vector space of dimension 1 [43, Corollary 5.3.5] spanned by  $\overline{g_i}$ . By Lemma 6.9, the Castelnuovo–Mumford regularity of  $M/uM$  equals the degree of  $g_i$ . It remains to note that since  $u$  is regular in  $M$  (see Lemma 2.5) and  $M$  is Cohen–Macaulay, the Castelnuovo–Mumford regularity of  $M/uM$  and  $M$  are equal (cf. [5, p. 175]). Therefore,

$$v_{\mathfrak{p}_i}(I) = \deg(f_i) = \deg(g_i) = \text{reg}(M/uM) = \text{reg}(M) = \text{reg}(H_I^a),$$

where the first equality follows from Proposition 4.6 and the last equality was discussed before Lemma 2.3.  $\square$

**Remark 6.12.** Note that the above proof implies that  $\text{in}_{\prec}(f_i) = \text{in}_{\prec}(f_m)$  and the class of  $\text{in}_{\prec}(f_i)$  modulo the ideal  $(I^h, u)$  is in  $\text{Soc}(S[u]/(I^h, u))$  for all  $i$ . Indeed, by Corollary 2.9(b), there is only one standard monomial of degree  $r_0 = \text{reg}(H_I^a)$ . Hence  $\text{in}_{\prec}(f_i) = \text{in}_{\prec}(f_m)$  because  $f_i \in K\Delta_{\prec}(I)$ . We may assume that  $g_i$  is monic. Then,  $\overline{g_i} = \overline{\text{in}_{\prec}(f_i)}$  because  $f_i^h - \text{in}_{\prec}(f_i)$  is equal to  $uh_i$  for some  $h_i \in S$ , and  $\overline{\text{in}_{\prec}(f_i)}$  is in the socle of  $S[u]/(I^h, u)$ .

**Corollary 6.13.** *Let  $X$  be a set of points in  $K^s$  and let  $I = I(X)$  be its vanishing ideal. If  $I$  is Gorenstein, then  $\delta_X(d) \geq \text{reg}(H_I^a) - d + 1$  for  $1 \leq d < \text{reg}(H_I^a)$ .*

*Proof.* Let  $r_0$  be the regularity of  $H_I^a$ . If  $r_0 = 1$ , there is nothing to prove. Assume  $r_0 \geq 2$ . By Corollary 6.6,  $\delta_X(r_0 - 1) = 2$ . Hence, by Proposition 2.2, we get  $\delta_X(d) \geq (r_0 - 1 - d) + \delta_X(r_0 - 1)$ . Thus,  $\delta_X(d) \geq r_0 - d + 1$ .  $\square$

**Corollary 6.14.** *Let  $X$  be a subset of  $K^s$ , let  $I$  be its vanishing ideal, and let  $r_0$  be the regularity index of  $H_I^a$ . If  $\mathcal{C} = S_{\leq r_0 - 1} \cap K\Delta_{\prec}(I)$  and  $I$  is Gorenstein, then there is  $g \in S$  such that  $\mathcal{C}^{\perp} = Kg$  and  $g(P_i) \neq 0$  for all  $i$ .*

*Proof.* As  $I$  is Gorenstein, by Corollary 2.9, one has  $1 = H_I^a(0) = |X| - H_I^a(r_0 - 1)$  and, by Theorem 6.11, one has  $r_0 = \text{reg}(H_I^a) = \mathbf{v}_{\mathfrak{p}}(I)$  for all  $\mathfrak{p} \in \text{Ass}(I)$ . Hence, the result follows from Theorem 6.5.  $\square$

The following result can be applied to any Reed–Muller-type code  $C_X(d)$  when the vanishing ideal  $I(X)$  is a complete intersection generated by a Gröbner basis with  $s = \dim(S)$  elements (Corollary 2.9(c)). In particular, since the vanishing ideal of a Cartesian set is a complete intersection generated by a Gröbner basis with  $s$  elements [33, Lemma 2.3], we recover the duality theorems for affine Cartesian codes given in [2, Theorem 5.7] and [31, Theorem 2.3].

**Corollary 6.15.** *Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$ ,  $|X| \geq 2$ , let  $I = I(X)$  be its vanishing ideal, let  $r_0$  be the regularity index of  $H_I^a$ , and let  $\prec$  be a graded monomial order on  $S$ . If  $I$  is Gorenstein, then there is  $g \in K\Delta_{\prec}(I)$  such that  $g(P_i) \neq 0$  for all  $i$  and*

$$(g(P_1), \dots, g(P_m)) \cdot C_X(r_0 - d - 1) = C_X(d)^{\perp} \quad \text{for } -1 \leq d \leq r_0.$$

*Moreover, one can choose  $g = \sum_{i=1}^m \text{lc}(f_i)f_i$ , where  $f_i$  is the  $i$ -th standard indicator function and  $\text{lc}(f_i)$  is its leading coefficient.*

*Proof.* By Corollary 2.9(a) and Theorem 6.11, the two conditions of Theorem 6.5(b) hold, and the result follows from Theorem 6.5.  $\square$

When  $r_0$  is odd and  $\text{char}(K) = 2$  this construction produces self-dual Reed–Muller-type codes as the following Corollary shows. Recall that a linear code  $C$  is *self-dual* if  $C = C^{\perp}$ .

**Corollary 6.16.** *Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K^s$ ,  $|X| \geq 2$ , let  $I = I(X)$  be its vanishing ideal, let  $r_0$  be the regularity index of  $H_I^a$ , and let  $\prec$  be a graded monomial order. Assume  $\text{char}(K) = 2$ ,  $r_0$  is odd, and  $I$  is Gorenstein. Define  $\alpha = (\alpha_1, \dots, \alpha_m) \in (K^*)^m$  by  $\alpha_i^2 = \text{lc}(f_i)$ , where  $\text{lc}(f_i)$  is the leading coefficient of the  $i$ -th standard indicator function  $f_i$  for  $P_i$ . Then the linear code  $\alpha \cdot C_X((r_0 - 1)/2)$  is self-dual.*

*Proof.* Setting  $d = (r_0 - 1)/2$  in Corollary 6.15, we have

$$(6.1) \quad \beta \cdot C_X((r_0 - 1)/2) = C_X((r_0 - 1)/2)^{\perp},$$

for some  $\beta = (g(P_1), \dots, g(P_m)) \in (K^*)^m$ . Recall that we can choose  $g$  such that  $\beta_i = g(P_i) = \text{lc}(f_i)$ . As  $\text{char}(K) = 2$ , there exists  $\alpha_i \in K^*$  such that  $\alpha_i^2 = \beta_i$ . Then Eq. (6.1) implies that for any  $u, v \in C_X((r_0 - 1)/2)$ ,

$$\langle \alpha \cdot u, \alpha \cdot v \rangle = \langle \beta \cdot u, v \rangle = 0.$$

Therefore,  $\alpha \cdot C_X((r_0 - 1)/2)$  is contained in its dual and, by Eq. (6.1), has the same dimension as its dual, i.e. is self-dual.  $\square$

## 7. THE ALGEBRAIC DUAL OF MONOMIAL EVALUATION CODES

In this section we study the dual and the algebraic dual of two families of evaluation codes. If an evaluation code is monomial and the set of evaluation points is a degenerate torus (resp. degenerate affine space), we show that its algebraic dual is a monomial space (resp. we classify when its algebraic dual is a monomial space).

**7.1. Monomial evaluation codes on a degenerate torus.** Let  $A_1, \dots, A_s$  be subgroups of the multiplicative group  $K^*$  of the finite field  $K = \mathbb{F}_q$ , let

$$T := A_1 \times \dots \times A_s = \{P_1, \dots, P_m\}$$

be the Cartesian product of  $A_1, \dots, A_s$ , and let  $\mathcal{L}_T$  be a monomial code on  $T$ , that is,  $\mathcal{L}$  is generated by a finite set of monomials of  $S$ . The set  $T$  is called a *degenerate torus* [33]. In this subsection we determine the algebraic dual  $\mathcal{L}^\perp$  and the dual  $(\mathcal{L}_T)^\perp$  of  $\mathcal{L}_T$  in terms of the generators of  $\mathcal{L}$  and show that  $(\mathcal{L}_T)^\perp$  is a standard monomial code on  $T$ .

The order of the cyclic group  $A_i$  is denoted by  $d_i$  for  $i = 1, \dots, s$ . Let  $\prec$  be a graded monomial order on  $S$ . The vanishing ideal  $I = I(T)$  is generated by the Gröbner basis  $\mathcal{G} = \{t_i^{d_i} - 1\}_{i=1}^s$  [33, Lemma 2.3], and consequently  $\Delta_\prec(I)$  is the set of all monomials  $t^c$ ,  $c = (c_1, \dots, c_s)$ , such that  $0 \leq c_i \leq d_i - 1$  for  $i = 1, \dots, s$ . By Proposition 3.8 and Lemma 3.4, the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_T$  is a monomial space of  $S$  and  $\mathcal{L}^\perp = \tilde{\mathcal{L}}^\perp$ . Thus, we may assume that  $\mathcal{L} = \tilde{\mathcal{L}}$ . Let  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\} \subset \Delta_\prec(I)$  be the unique monomial  $K$ -basis of  $\mathcal{L}$  where

$$t^{a_i} = t_1^{a_{i,1}} \dots t_s^{a_{i,s}}, \quad a_i = (a_{i,1}, \dots, a_{i,s}) \in \mathbb{N}^s, \quad \text{for } i = 1, \dots, k.$$

The *support* of  $t^{a_i}$ , denoted  $\text{supp}(t^{a_i})$ , is the set of all  $t_j$  such that  $a_{i,j} > 0$ . To construct a monomial basis for  $\mathcal{L}^\perp$  we will need the following set of monomials. For each  $1 \leq i \leq k$ , we set  $t^{b_i} := 1$  if  $t^{a_i} = 1$  and

$$(7.1) \quad t^{b_i} = t_1^{b_{i,1}} \dots t_s^{b_{i,s}} := \prod_{t_j \in \text{supp}(t^{a_i})} t_j^{d_j - a_{i,j}} \quad \text{if } t^{a_i} \neq 1.$$

The set  $\mathcal{B} := \{t^{b_1}, \dots, t^{b_k}\}$  has cardinality  $k$ ,  $\mathcal{B} \subset \Delta_\prec(I)$ , and the support of  $t^{b_i}$  is the support of  $t^{a_i}$  for  $i = 1, \dots, k$ . The regularity index of  $H_I^a$  is  $r_0 = \sum_{i=1}^s (d_i - 1)$  [33, Proposition 2.5] and  $H_I^a(r_0) = |\Delta_\prec(I)| = |T| = d_1 \dots d_s$  (Lemma 2.7).

**Lemma 7.1.** *Let  $t^c$ ,  $c = (c_1, \dots, c_s)$ , be a monomial of  $S$ . If  $c_i \equiv 0 \pmod{d_i}$  for all  $i$ , then  $t^c \notin \ker(\varphi)$ . If  $c_i \not\equiv 0 \pmod{d_i}$  for some  $i$ , then  $t^c \in \ker(\varphi)$ .*

*Proof.* Let  $P_1, \dots, P_m$  be the points of  $T$  and let  $\beta_i$  be a generator of the multiplicative cyclic group  $A_i$  for  $i = 1, \dots, s$ . Recall that  $q = p^v$  for some prime number  $p > 0$  and  $v \in \mathbb{N}_+$ . Note that each  $d_i$  is relatively prime to  $p$  because  $d_i = |A_i|$  divides  $q - 1 = |K^*| = |(\mathbb{F}_q)^*|$ . Thus,  $\text{gcd}(m, p) = 1$  because  $m = |T| = d_1 \dots d_s$ . Assume that  $c_i \equiv 0 \pmod{d_i}$  for all  $i$ . Then,  $\varphi(t^c) = \sum_{i=1}^m t^c(P_i) = m \cdot 1$  and  $m \cdot 1 \neq 0$  because  $\text{gcd}(m, p) = 1$ . Thus,  $t^c \notin \ker(\varphi)$ . Assume

that  $c_i \not\equiv 0 \pmod{d_i}$  for some  $i$ . Then,  $c_i \geq 1$ . For simplicity of notation assume that  $i = 1$ . The cartesian set  $T$  can be partitioned as

$$T = \{P_1, \dots, P_m\} = \bigcup_{i=1}^{d_1} \{(\beta_1^i, Q) \mid Q \in A_2 \times \dots \times A_s\}.$$

Hence, setting  $T_1 = A_2 \times \dots \times A_s$ , we obtain

$$\varphi(t^c) = \sum_{i=1}^m t^c(P_i) = \left(1 + \beta_1^{c_1} + (\beta_1^2)^{c_1} + \dots + (\beta_1^{d_1-1})^{c_1}\right) \left(\sum_{Q \in T_1} t_2^{c_2} \dots t_s^{c_s}(Q)\right).$$

Hence, using the equality  $(\sum_{i=0}^{d_1-1} (\beta_1^{c_1})^i)(\beta_1^{c_1} - 1) = (\beta_1^{c_1})^{d_1} - 1 = 0$  and, noticing that  $\beta_1^{c_1} - 1 = 0$  if and only if  $c_1 \equiv 0 \pmod{d_1}$ , we get  $\sum_{i=0}^{d_1-1} (\beta_1^{c_1})^i = 0$ . Thus,  $\varphi(t^c) = 0$  and  $t^c \in \ker(\varphi)$ .  $\square$

The main result in connection with monomial evaluation codes over  $T$  is the following.

**Proposition 7.2.** (Monomial basis) *Let  $\mathcal{L}$  be a subspace with a basis of standard monomials  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\}$ . Then  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ , where  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_k}\}$  is defined in Eq. (7.1).*

*Proof.* As  $|\Delta_{\prec}(I)| = |T|$  and  $\mathcal{L}_X$  is a standard evaluation code, by Proposition 3.7, one has

$$\dim_K(\mathcal{L}^\perp) = |T| - \dim_K(\mathcal{L}) = |T| - k = |\Delta_{\prec}(I) \setminus \mathcal{B}|.$$

Thus, to show the equality  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ , we need only show that  $\Delta_{\prec}(I) \setminus \mathcal{B} \subset \mathcal{L}^\perp$ . Take  $t^\alpha \in \Delta_{\prec}(I) \setminus \mathcal{B}$ ,  $\alpha = (\alpha_1, \dots, \alpha_s)$ . Since  $\mathcal{L}$  is generated by the set  $\{t^{a_1}, \dots, t^{a_k}\}$  it suffices to show that  $t^\alpha t^{a_i}$  is in  $\ker(\varphi)$  for  $i = 1, \dots, k$ . Assume  $t^{a_i} = 1$ . Then,  $t^{b_i} = 1$  and  $t^\alpha \neq 1$ . Hence, by Lemma 7.1,  $1 \cdot t^\alpha \in \ker(\varphi)$  since  $t^\alpha$  is a standard monomial of  $S/I$  and  $\alpha \neq 0$ . Assume that  $t^{a_i} \neq 1$ . If  $\text{supp}(t^\alpha) \not\subset \text{supp}(t^{a_i})$ , then there is  $t_j \in \text{supp}(t^\alpha)$  and  $t_j \notin \text{supp}(t^{a_i})$ . Thus,  $t_j^{\alpha_j}$  divides  $t^{a_i} t^\alpha$ ,  $t_j^\ell$  does not divide  $t^{a_i} t^\alpha$  for  $\ell > \alpha_j$ , and  $\alpha_j \not\equiv 0 \pmod{d_j}$  because  $\alpha_j \leq d_j - 1$ . Hence, by Lemma 7.1, we get  $t^{a_i} t^\alpha \in \ker(\varphi)$ . The case  $\text{supp}(t^\alpha) \supset \text{supp}(t^{a_i})$  can be treated similarly. Thus, we may assume  $\text{supp}(t^\alpha) = \text{supp}(t^{a_i})$ . By definition of  $t^{b_i}$  one also has  $\text{supp}(t^{b_i}) = \text{supp}(t^{a_i})$ . As  $t^{a_i} \neq 1$ , by Eq. (7.1), we obtain

$$t^{a_i} t^\alpha = t_1^{a_i, 1 + \alpha_1} \dots t_s^{a_i, s + \alpha_s} = \prod_{t_j \in \text{supp}(h_i)} t_j^{d_j - b_{i,j} + \alpha_j}.$$

If  $c_j = d_j - b_{i,j} + \alpha_j \not\equiv 0 \pmod{d_j}$  for some  $j$ , then  $t^{a_i} t^\alpha$  is in  $\ker(\varphi)$  by Lemma 7.1. If  $c_j = d_j - b_{i,j} + \alpha_j \equiv 0 \pmod{d_j}$  for all  $j$ , then it follows readily that  $b_{i,j} = \alpha_j$  for all  $j$ , that is,  $t^{b_i} = t^\alpha$ , a contradiction since  $t^\alpha$  is not in  $\mathcal{B}$ .  $\square$

**Corollary 7.3.** *Let  $\mathcal{L}_T$  be a monomial code on  $T$  and let  $\mathcal{L}^\perp$  be its algebraic dual. Then,  $(\mathcal{L}_T)^\perp = (\mathcal{L}^\perp)_T$  and  $(\mathcal{L}_T)^\perp$  is a standard monomial code on  $T$ .*

*Proof.* The linear code  $(\mathcal{L}_T)^\perp$  is the evaluation code  $(\mathcal{L}^\perp)_T$  on  $T$  (Theorem 3.5) and  $\mathcal{L}^\perp$  is generated by the set of monomials  $\Delta_{\prec}(I) \setminus \mathcal{B}$  (Proposition 7.2). Thus,  $(\mathcal{L}_T)^\perp$  is a standard monomial code on  $T$  because its standard function space is  $\mathcal{L}^\perp$ .  $\square$

**Corollary 7.4.** [4, 36] *Let  $\mathcal{L}_T$  be a generalized toric code on  $T = (K^*)^s$ ,  $K = \mathbb{F}_q$ , and let  $\mathcal{L}^\perp$  be its algebraic dual. Then,  $(\mathcal{L}_T)^\perp = (\mathcal{L}^\perp)_T$  and  $(\mathcal{L}_T)^\perp$  is a generalized toric code.*

*Proof.* It follows at once from Corollary 7.3 by making  $A_i = K^*$  for  $i = 1, \dots, s$ .  $\square$

**7.2. Monomial evaluation codes on a degenerate affine space.** Let  $K = \mathbb{F}_q$  be a finite field of characteristic  $p$ , let  $A_1, \dots, A_s$  be subgroups of the multiplicative group  $K^*$  of the field  $K$ , let  $B_i$  be the set  $A_i \cup \{0\}$  for  $i = 1, \dots, s$ , let

$$\mathcal{X} := B_1 \times \cdots \times B_s = \{P_1, \dots, P_m\}$$

be the Cartesian product of  $B_1, \dots, B_s$ , and let  $\mathcal{L}_{\mathcal{X}}$  be a monomial code on  $\mathcal{X}$ , that is,  $\mathcal{L}$  is generated by a finite set of monomials of  $S$ . The set  $\mathcal{X}$  is called a *degenerate affine space*. In this subsection we classify when the algebraic dual  $\mathcal{L}^\perp$  is generated by monomials. We also classify when the dual  $(\mathcal{L}_{\mathcal{X}})^\perp$  of  $\mathcal{L}_{\mathcal{X}}$  is a standard monomial code, and show that in certain interesting cases  $(\mathcal{L}_{\mathcal{X}})^\perp$  is a standard monomial code.

The order of the multiplicative monoid  $B_i$  is denoted by  $e_i$  and the order of  $A_i$  is denoted by  $d_i$  for  $i = 1, \dots, s$ . Let  $\prec$  be a graded monomial order on  $S$ . By [33, Lemma 2.3], the vanishing ideal  $I = I(\mathcal{X})$  of  $\mathcal{X}$  is generated by the Gröbner basis  $\mathcal{G} = \{t_i^{e_i} - t_i\}_{i=1}^s$ , and consequently the set of standard monomials  $\Delta_{\prec}(I)$  of  $S/I$  is the set of all  $t^c$ ,  $c = (c_1, \dots, c_s)$ , such that  $0 \leq c_i \leq d_i$  for  $i = 1, \dots, s$ . By Proposition 3.8 and Lemma 3.4, the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_{\mathcal{X}}$  is a monomial space of  $S$  and  $\mathcal{L}^\perp = \tilde{\mathcal{L}}^\perp$ . Thus, we may assume that  $\mathcal{L} = \tilde{\mathcal{L}}$ . Note that  $\mathcal{L}$  has a unique basis  $\mathcal{A}$  consisting of standard monomials of  $S/I$ . We will classify when  $\mathcal{L}^\perp$  is a monomial space of  $S$  and also when  $(\mathcal{L}_{\mathcal{X}})^\perp$  is a standard monomial code in terms of this basis. Let  $\mathcal{A} = \{t^{a^1}, \dots, t^{a^k}\} \subset \Delta_{\prec}(I)$  be the unique monomial  $K$ -basis of  $\mathcal{L}$  where, as before,

$$t^{a^i} = t_1^{a_{i,1}} \cdots t_s^{a_{i,s}}, \quad a_i = (a_{i,1}, \dots, a_{i,s}) \in \mathbb{N}^s, \quad \text{for } i = 1, \dots, k.$$

To construct a candidate for a basis of  $\mathcal{L}^\perp$ , for each  $1 \leq i \leq k$ , we set

$$(7.2) \quad t^{b^i} = t_1^{b_{i,1}} \cdots t_s^{b_{i,s}} := \prod_{j=1}^s t_j^{d_j - a_{i,j}}.$$

The set  $\mathcal{B} := \{t^{b^1}, \dots, t^{b^k}\}$  has cardinality  $k$  and  $\mathcal{B} \subset \Delta_{\prec}(I)$ . The index of regularity of  $H_I^q$  is  $r_0 = \sum_{i=1}^s d_i$  [33, Proposition 2.5] and  $H_I^q(r_0) = |\Delta_{\prec}(I)| = |\mathcal{X}| = m = e_1 \cdots e_s$  (Lemma 2.7).

**Lemma 7.5.** *Let  $t^c$ ,  $c = (c_1, \dots, c_s)$ , be a monomial of  $S$ . The following hold.*

- (a) *If  $c_i \not\equiv 0 \pmod{d_i}$  for some  $i$ , then  $t^c \in \ker(\varphi)$ .*
- (b) *If  $p = \text{char}(K)$  and  $\gcd(e_i, p) = p$  for some  $i$ , then  $1 \in \ker(\varphi)$ .*
- (c) *If  $\gcd(e_i, p) = p$  and  $c_i \equiv 0 \pmod{d_i}$  for all  $i$ , and  $|\text{supp}(t^c)| < s$ , then  $t^c \in \ker(\varphi)$ .*
- (d) *If  $t^c = t_1^{\lambda_1 d_1} \cdots t_s^{\lambda_s d_s}$  and the  $\lambda_i$ 's are positive integers, then  $t^c \notin \ker(\varphi)$ .*
- (e) *If  $\gcd(e_i, p) = p$  for all  $i$ , then  $\Delta_{\prec}(I) \cap \ker(\varphi) = \Delta_{\prec}(I) \setminus \{t_1^{d_1} \cdots t_s^{d_s}\}$ .*

*Proof.* Let  $P_1, \dots, P_m$  be the points of  $\mathcal{X}$ ,  $m = |\mathcal{X}| = e_1 \cdots e_s$ , and let  $\beta_i$  be a generator of the multiplicative cyclic group  $A_i$  for  $i = 1, \dots, s$ .

(a): Assume that  $c_i \not\equiv 0 \pmod{d_i}$  for some  $i$ . Then,  $c_i \geq 1$ . For simplicity of notation assume that  $i = 1$ . The cartesian set  $\mathcal{X}$  can be partitioned as

$$\mathcal{X} = \left( \bigcup_{i=1}^{d_1} \{(\beta_1^i, Q) \mid Q \in B_2 \times \cdots \times B_s\} \right) \cup \{(0, Q) \mid Q \in B_2 \times \cdots \times B_s\}.$$

Hence, setting  $\mathcal{X}_1 = B_2 \times \cdots \times B_s$ , we obtain

$$\varphi(t^c) = \sum_{i=1}^m t^c(P_i) = \left( 1 + \beta_1^{c_1} + (\beta_1^2)^{c_1} + \cdots + (\beta_1^{d_1-1})^{c_1} \right) \left( \sum_{Q \in \mathcal{X}_1} t_2^{c_2} \cdots t_s^{c_s}(Q) \right).$$

Hence, using the equality  $(\sum_{i=0}^{d_1-1} (\beta_1^{c_1})^i)(\beta_1^{c_1} - 1) = (\beta_1^{c_1})^{d_1} - 1 = 0$  and, noticing that  $\beta_1^{c_1} - 1 = 0$  if and only if  $c_1 \equiv 0 \pmod{d_1}$ , we get  $\sum_{i=0}^{d_1-1} (\beta_1^{c_1})^i = 0$ . Thus,  $\varphi(t^c) = 0$  and  $t^c \in \ker(\varphi)$ .

(b): Assume  $\gcd(e_i, p) = p$  for some  $i$ . Then,  $\varphi(1) = (e_1 \cdots e_s) \cdot 1 = 0$ . Thus,  $1 \in \ker(\varphi)$ .

(c): By part (b), we may assume  $\text{supp}(t^c) \neq \emptyset$ , that is,  $t^c \neq 1$ . For simplicity of notation assume that  $\text{supp}(t^c) = \{t_1, \dots, t_\ell\}$ , where  $1 \leq \ell < s$ . For each  $1 \leq i \leq \ell$ , there is  $\lambda_i \in \mathbb{N}_+$  such that  $c_i = \lambda_i d_i$ . We set  $\lambda_i = 0$  for  $\ell < i \leq s$ . The set  $\mathcal{X} = \{P_1, \dots, P_m\}$  can be partitioned as

$$\mathcal{X} = A \cup (\mathcal{X} \setminus A), \quad A = A_1 \times \cdots \times A_\ell \times B_{\ell+1} \times \cdots \times B_s.$$

Note that  $t^c(P_i) = 1$  if  $P_i \in A$  and  $t^c(P_i) = 0$  if  $P_i \in \mathcal{X} \setminus A$ . Hence

$$(7.3) \quad \varphi(t^c) = \sum_{i=1}^m t^c(P_i) = \sum_{i=1}^m t_1^{\lambda_1 d_1} \cdots t_s^{\lambda_s d_s}(P_i) = ((d_1 \cdots d_\ell)(e_{\ell+1} \cdots e_s)) \cdot 1.$$

Since  $\ell < s$  and  $\gcd(e_i, p) = p$  for all  $i$ , we get  $(e_{\ell+1} \cdots e_s) \cdot 1 = 0$ . Thus,  $t^c \in \ker(\varphi)$ .

(d): As  $\gcd(d_i, p) = 1$  for all  $i$ , from Eq. (7.3), we get  $\varphi(t^c) = (d_1 \cdots d_s) \cdot 1 \neq 0$ .

(e): The inclusion “ $\subset$ ” follows from part (d). To show the inclusion “ $\supset$ ” take a monomial  $t^c = t_1^{c_1} \cdots t_s^{c_s}$  in  $\Delta_{\prec}(I) \setminus \{t_1^{d_1} \cdots t_s^{d_s}\}$ . Then,  $c_i \leq d_i$  for all  $i$  and  $c_j < d_j$  for some  $j$ . We need only show  $t^c \in \ker(\varphi)$ . By part (a), we may assume  $c_i \equiv 0 \pmod{d_i}$  for all  $i$ . Hence  $c_j = 0$ , and consequently  $|\text{supp}(t^c)| < s$ . Therefore, by part (c), we get  $t^c \in \ker(\varphi)$ .  $\square$

If  $(\mathcal{L}^\perp)_{\mathcal{X}}$  is a standard monomial code, then  $\mathcal{L}^\perp$  has a unique basis of standard monomials of  $S/I$  because  $\mathcal{L}^\perp$  is the standard function space of  $(\mathcal{L}^\perp)_{\mathcal{X}}$ . The next result identifies this basis and classifies when  $(\mathcal{L}^\perp)_{\mathcal{X}}$  is a standard monomial code.

**Proposition 7.6.** *Let  $\mathcal{L}$  be a subspace with a basis of standard monomials  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\}$ . Then  $(\mathcal{L}^\perp)_{\mathcal{X}}$  is a standard monomial code on  $\mathcal{X}$  if and only if  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ , where  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_k}\}$  is defined in Eq. (7.2).*

*Proof.*  $\Rightarrow$ ) By Proposition 3.7,  $\dim_K(\mathcal{L}^\perp) = |X| - k$ . Assume that  $\mathcal{L}^\perp = K\{t^{\gamma_1}, \dots, t^{\gamma_{m-k}}\}$ ,  $m = |X|$ ,  $k = \dim_K(\mathcal{L})$ . Take  $1 \leq \ell \leq m - k$ . Then,  $t^{\gamma_\ell}$  is in  $(\ker(\varphi): \mathcal{L}) \cap K\Delta_{\prec}(I)$ . If  $t^{\gamma_\ell}$  is in  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_k}\}$ , then  $t^{\gamma_\ell} = t^{b_i}$  for some  $i$ , and consequently  $t^{b_i} \mathcal{L} \subset \ker(\varphi)$ . Thus,  $t^{\gamma_\ell} t^{a_i} = t^{b_i} t^{a_i} = \prod_{j=1}^s t_j^{d_j}$  and, by Lemma 7.5(d),  $\prod_{j=1}^s t_j^{d_j}$  is not in  $\ker(\varphi)$ , a contradiction. Hence,  $t^{\gamma_\ell} \in \Delta_{\prec}(I) \setminus \mathcal{B}$ , and we obtain the equality  $\{t^{\gamma_1}, \dots, t^{\gamma_{m-k}}\} = \Delta_{\prec}(I) \setminus \mathcal{B}$  because one has the inclusion “ $\subset$ ” and these two sets have the same cardinality.

$\Leftarrow$ ) This part is clear since  $\Delta_{\prec}(I) \setminus \mathcal{B}$  consists of standard monomials.  $\square$

Following [4, p. 16], we say that a set of monomials  $\mathcal{A}$  of  $S$  is *divisor-closed* if  $t^a \in \mathcal{A}$  whenever  $t^a$  divides a monomial in  $\mathcal{A}$ . Families of linear codes generated by monomials that are divisor-closed are also studied in [6]. Applications of these sort of codes to polar codes are given in [7]. To classify when the algebraic dual of  $\mathcal{L}_{\mathcal{X}}$  is generated by monomials, we now introduce a weaker notion than divisor-closed (cf. [4, Remark 2.5]).

**Definition 7.7.** A set  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\} \subset \Delta_{\prec}(I)$ ,  $t^{a_i} = \prod_{j=1}^s t_j^{a_{i,j}}$ , of standard monomials of  $S/I$  is *weakly divisor-closed* if

$$\frac{t^{a_i}}{\prod_{t_j \in D} t_j^{a_{i,j}}}$$

is in  $\mathcal{A}$  for all monomials  $t^{a_i}$  in  $\mathcal{A}$  and all subsets  $D$  of  $D_i := \{t_j \mid a_{i,j} = d_j\}$ . If  $D = \emptyset$ , the product  $\prod_{t_j \in D} t_j^{a_{i,j}}$  is equal to 1 by convention.

We come to one of the main results of this section.

**Theorem 7.8.** *Let  $K$  be a field of characteristic  $p$  and let  $I$  be the vanishing ideal of  $\mathcal{X}$ . Assume that  $\gcd(p, e_i) = p$ ,  $e_i = |B_i|$ , for  $i = 1, \dots, s$ . Let  $\mathcal{L}$  have a monomial basis  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_k}\} \subset \Delta_{\prec}(I)$ . The following are equivalent.*

- (a)  $\mathcal{A}$  is weakly divisor-closed.
- (b)  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ , where  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_k}\}$  is defined in Eq. (7.2).
- (c)  $(\mathcal{L}\mathcal{X})^\perp$  is a standard monomial code on  $\mathcal{X}$ .

*Proof.* (a)  $\Rightarrow$  (b): By Lemma 2.7, one has the equality  $|\Delta_{\prec}(I)| = |\mathcal{X}|$ . Then, by Proposition 3.7, one obtains the equalities

$$\dim_K(\mathcal{L}^\perp) = |\mathcal{X}| - \dim_K(\mathcal{L}) = |\mathcal{X}| - k = |\Delta_{\prec}(I) \setminus \mathcal{B}|.$$

Thus, to show the equality  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ , we only need to show that  $\Delta_{\prec}(I) \setminus \mathcal{B} \subset \mathcal{L}^\perp$ . Take  $t^\alpha \in \Delta_{\prec}(I) \setminus \mathcal{B}$ ,  $\alpha = (\alpha_1, \dots, \alpha_s)$ . Since  $\mathcal{L}$  is generated by the set  $\{t^{a_1}, \dots, t^{a_k}\}$  it suffices to show that  $t^\alpha t^{a_i}$  is in  $\ker(\varphi)$  for  $i = 1, \dots, k$ . Fix  $1 \leq i \leq k$ . If  $\alpha_j + a_{i,j} \not\equiv 0 \pmod{d_j}$  for some  $j$ , by Lemma 7.5(a), one has  $t^\alpha t^{a_i} \in \ker(\varphi)$ . Thus, we may assume  $\alpha_j + a_{i,j} \equiv 0 \pmod{d_j}$  for  $j = 1, \dots, s$ . There are  $\lambda_1, \dots, \lambda_s$  in  $\mathbb{N}$  such that  $\alpha_j + a_{i,j} = \lambda_j d_j$  for  $j = 1, \dots, s$ . By Lemma 7.5(c), we may also assume that  $\text{supp}(t^\alpha t^{a_i}) = \{t_1, \dots, t_s\}$  and  $\lambda_j \geq 1$  for all  $j$ . If  $\lambda_j = 1$  for all  $j$ , we obtain that  $t^\alpha = t^{b_i}$ , a contradiction. If  $\lambda_j \geq 2$  for some  $j$ , since  $\alpha_j + a_{i,j} = \lambda_j d_j \leq 2d_j$ , we obtain that  $\lambda_j = 2$  and  $\alpha_j = a_{i,j} = d_j$ . Therefore, for each  $1 \leq j \leq s$  either  $\alpha_j + a_{i,j} = d_j$  or  $\alpha_j = a_{i,j} = d_j$ . Next we show that this cannot occur. We set

$$t^\delta := \frac{t^{a_i}}{\prod_{t_j \in D} t_j^{a_{i,j}}},$$

where  $D := \{t_j \mid \alpha_j = a_{i,j} = d_j\}$  is a subset of  $D_i = \{t_j \mid a_{i,j} = d_j\}$ . Since the set  $\mathcal{A}$  is weakly divisor-closed, we get  $t^\delta \in \mathcal{A}$ , and  $t^\delta = t^{a_r}$  for some  $1 \leq r \leq k$ . From the equalities  $t^\alpha t^{a_r} = t^\alpha t^\delta = \prod_{j=1}^s t_j^{d_j}$ , we obtain  $t^\alpha \in \mathcal{B}$ , a contradiction.

(b)  $\Rightarrow$  (a): Assume that  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ . Take  $t^{a_i}$  in  $\mathcal{A}$  and let  $D$  be a subset of  $D_i$ . If  $D = \emptyset$ , there is nothing to prove. For simplicity of notation we may assume that  $D = \{t_1, \dots, t_\ell\}$  for some  $1 \leq \ell \leq s$ . Then, we can write  $t^{a_i} = t_1^{d_1} \dots t_\ell^{d_\ell} t^\gamma$ , where  $t^\gamma = t_{\ell+1}^{a_{i,\ell+1}} \dots t_s^{a_{i,s}}$  and  $1 \leq a_{i,j} \leq d_j$  for  $j > \ell$ . If  $\ell = s$ , by convention  $t^\gamma = 1$ . To show that  $\mathcal{A}$  is weakly divisor-closed we need only show  $t^\gamma \in \mathcal{A}$ . We proceed by contradiction assuming  $t^\gamma \notin \mathcal{A}$ . From the equality

$$(t_1^{d_1} \dots t_\ell^{d_\ell} t_{\ell+1}^{d_{\ell+1} - a_{i,\ell+1}} \dots t_s^{d_s - a_{i,s}}) t^\gamma = t_1^{d_1} \dots t_s^{d_s},$$

we obtain that the monomial  $t^u := t_1^{d_1} \dots t_\ell^{d_\ell} t_{\ell+1}^{d_{\ell+1} - a_{i,\ell+1}} \dots t_s^{d_s - a_{i,s}}$  is not in  $\mathcal{B}$  since  $t^\gamma$  is not in  $\mathcal{A}$ . Thus,  $t^u \in \Delta_{\prec}(I) \setminus \Gamma \subset \mathcal{L}^\perp$ . Then,  $t^u \mathcal{L} \subset \ker(\varphi)$ . As  $t^{a_i}$  is in  $\mathcal{L}$ , one has

$$(7.4) \quad t_1^{2d_1} \dots t_\ell^{2d_\ell} t_{\ell+1}^{d_{\ell+1}} \dots t_s^{d_s} = t^u t^{a_i} \in \ker(\varphi),$$

a contradiction because by Lemma 7.5(d), the left hand side of Eq. (7.4) is not in  $\ker(\varphi)$ .

(b)  $\Leftrightarrow$  (c): The linear code  $(\mathcal{L}\mathcal{X})^\perp$  is the evaluation code  $(\mathcal{L}^\perp)_\mathcal{X}$  on  $\mathcal{X}$  by Theorem 3.5. Thus, that (b) and (c) are equivalent follows from Proposition 7.6.  $\square$

**Corollary 7.9.** *If  $\mathcal{X} = K$ ,  $\mathcal{L} = K\{t_1^{a_1}, \dots, t_1^{a_k}\}$ , and  $0 \leq a_1 < \dots < a_k \leq q - 1$ , then  $(\mathcal{L}\mathcal{X})^\perp$  is a standard monomial code on  $\mathcal{X}$  if and only if either  $a_k < q - 1$  or  $a_k = q - 1$  and  $1 \in \mathcal{L}$ .*

*Proof.* We set  $A_1 = K^*$ ,  $B_1 = K$ ,  $d_1 = q - 1$ , and  $e_1 = q$ . Note that  $\gcd(q, p) = p$ , where  $p = \text{char}(K)$ . By Theorem 7.8, it suffices to note that  $\mathcal{A} = \{t_1^{a_1}, \dots, t_1^{a_k}\}$  is weakly divisor-closed if and only if either  $a_k < q - 1 = d_1$  or  $a_k = q - 1 = d_1$  and  $1 \in \mathcal{L}$ .  $\square$

**Corollary 7.10.** [4, Proposition 2.4] *Let  $\mathcal{L}_X$  be a standard monomial code on  $\mathcal{X} = K^s$ , let  $\mathcal{A}$  be the monomial basis of  $\mathcal{L}$ , and let  $I$  be the vanishing ideal of  $\mathcal{X}$ . If  $\mathcal{A}$  is divisor-closed, then*

$$\mathcal{L}^\perp = \Delta_{\prec}(I) \setminus \{t_1^{q-1-c_1} \dots t_s^{q-1-c_s} : t_1^{c_1} \dots t_s^{c_s} \in \mathcal{A}\} \text{ and } (\mathcal{L}_X)^\perp = (\mathcal{L}^\perp)_X.$$

*Proof.* It follows from Theorems 3.5 and 7.8 by making  $B_i = K$  for  $i = 1, \dots, s$ .  $\square$

We now determine the algebraic dual of  $K(S_{\leq d} \cap \Delta_{\prec}(I(\mathcal{X})))$ .

**Theorem 7.11.** *Let  $K$  be a field of characteristic  $p$  and let  $I$  be the vanishing ideal of  $\mathcal{X}$ . Assume that  $\gcd(e_i, p) = p$ ,  $e_i = |B_i|$ , for all  $i$ . If  $1 \leq d < r_0 = \sum_{i=1}^s (e_i - 1)$ ,  $\mathcal{A} := S_{\leq d} \cap \Delta_{\prec}(I) = \{t^{a_1}, \dots, t^{a_k}\}$ , and  $\mathcal{L} = K\mathcal{A}$ , then*

$$\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B}) = K(S_{\leq r_0-d-1} \cap \Delta_{\prec}(I)),$$

where  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_k}\}$  is defined in Eq. (7.2).

*Proof.* We set  $\mathcal{A}^\perp := S_{\leq r_0-d-1} \cap \Delta_{\prec}(I)$ , and  $k = \dim_K(K\mathcal{A}) = H_I^a(d)$ . Note that  $|\mathcal{A}| = |\mathcal{B}|$ . We claim that  $\mathcal{A}^\perp = \Delta_{\prec}(I) \setminus \mathcal{B}$ . As  $I = I(\mathcal{X})$  is a complete intersection, by Corollary 2.9, one has  $H_I^a(d) + H_I^a(r_0 - d - 1) = |\mathcal{X}|$ . Then, using Lemmas 2.6 and 2.7, one has

$$H_I^a(r_0 - d - 1) = |\Delta_{\prec}(I) \cap S_{\leq r_0-d-1}| = |\mathcal{A}^\perp| = |\mathcal{X}| - H_I^a(d) = |\Delta_{\prec}(I) \setminus \mathcal{B}|.$$

Hence, to show that  $\mathcal{A}^\perp = \Delta_{\prec}(I) \setminus \mathcal{B}$ , we need only show the inclusion “ $\supset$ ”. Given  $c \in \mathbb{N}^n$ ,  $c = (c_1, \dots, c_s)$ , we set  $|c| := \sum_{j=1}^s c_j$ . Take  $t^c \in \Delta_{\prec}(I) \setminus \mathcal{B}$ ,  $c = (c_1, \dots, c_s)$ . We proceed by contradiction. Assume that  $t^c \notin \mathcal{A}^\perp$ , that is,  $|c| > r_0 - d - 1$ . Setting  $a = (d_1 - c_1, \dots, d_s - c_s)$ , we get  $|a| = r_0 - |c| < d + 1$ . Thus,  $|a| \leq d$  and  $t^a \in \mathcal{A}$ . As  $t^c t^a = \prod_{j=1}^s t_j^{d_j}$ , we get  $t^c \in \mathcal{B}$ , a contradiction. This proves the claim. Hence,  $K\mathcal{A}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$ . The equality  $\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B})$  follows from Theorem 7.8 because  $\mathcal{A}$  is weakly divisor-closed.  $\square$

If  $\mathcal{L} = S_{\leq d}$ , the evaluation code  $\mathcal{L}_X$  on  $\mathcal{X}$ , denoted by  $C_X(d)$ , is the Reed–Muller-type code on  $\mathcal{X}$  of degree  $d$ . The codes  $C_X(d)^\perp$  and  $C_X(r_0 - d - 1)$  are equivalent (Corollary 6.15, [2, Theorem 5.7], [31, Theorem 2.3]), the next result shows that they are equal when  $\mathcal{X}$  is a degenerate affine space and  $\text{char}(K)$  divides  $e_i$  for all  $i$ .

**Proposition 7.12.** *Let  $K$  be a field of characteristic of  $p$  such that  $\gcd(e_i, p) = p$ ,  $e_i = |B_i|$ , for all  $i$ . Then,  $C_X(d)^\perp = C_X(r_0 - d - 1)$  if  $d < r_0$  and  $C_X(d)^\perp = (0)$  if  $d = r_0$ .*

*Proof.* Let  $\prec$  be a graded monomial order. If  $\mathcal{L} = S_{\leq d}$  and  $I = I(\mathcal{X})$ , then the standard function space  $\tilde{\mathcal{L}}$  of  $\mathcal{L}_X$  is  $K(S_{\leq d} \cap \Delta_{\prec}(I))$  and  $\tilde{\mathcal{L}}_X = \mathcal{L}_X = C_X(d)$  (see Proposition 3.1, Corollary 3.2). If  $d < r_0$ , then by Theorem 7.11, we obtain the equality  $(\tilde{\mathcal{L}})^\perp = K(S_{\leq r_0-d-1} \cap \Delta_{\prec}(I))$ . Therefore, using Theorem 3.5, one has

$$\begin{aligned} C_X(d)^\perp &= (\tilde{\mathcal{L}}_X)^\perp = ((\tilde{\mathcal{L}})^\perp)_X = (K(S_{\leq r_0-d-1} \cap \Delta_{\prec}(I)))_X \\ &= (S_{\leq r_0-d-1})_X = C_X(d - r_0 - 1). \end{aligned}$$

If  $d = r_0$ , then  $C_X(d) = K^m$ ,  $m = |\mathcal{X}|$ , and  $C_X(d)^\perp = (0)$ .  $\square$

## 8. EXAMPLES

This section includes examples illustrating some of our results. In Appendix A we give the implementations in *Macaulay2* [24] that are used in some of the examples. The monomial order  $\prec$  that we use in the following examples is the graded reverse lexicographical order (GRevLex order) [43, p. 343]. This is the default order in *Macaulay2*.

**Example 8.1.** Let  $K$  be the finite field  $\mathbb{F}_3$ , let  $S = K[t_1, t_2, t_3]$  be a polynomial ring, let  $\prec$  be the GRevLex order on  $S$ , let  $I = I(X)$  be the vanishing ideal of the set of evaluation points

$$X = \{(1, 1, 1), (1, 1, -1), (0, 0, 0), (0, 0, 1), (0, 0, -1), (0, 1, 0), (0, 1, 1), (0, 1, -1)\},$$

let  $P_i$  be the point in  $X$  in the  $i$ -th position from the left, and let  $\mathfrak{p}_i$  be the vanishing ideal of  $P_i$ . The ideal  $I$  is generated by

$$\mathcal{G} = \{t_2^2 - t_2, t_1 t_2 - t_1, t_1^2 - t_1, t_3^3 - t_3, t_1 t_3^2 - t_1\}$$

and this set is a Gröbner basis for  $I$ . The Reed–Muller code  $C_X(2)$  is the standard evaluation code  $\mathcal{L}_X$  where  $\mathcal{L}$  is the standard function space of  $C_X(2)$  spanned by the set of remainders

$$\{1, t_3, t_2, t_1, t_3^2, t_2 t_3, t_1 t_3\}$$

of the monomial basis of  $S_{\leq 2}$  on division by  $\mathcal{G}$  (Proposition 3.1). The algebraic dual of  $C_X(2)$  is  $\mathcal{L}^\perp = K(t_1 + t_2 + 1)$  and the dual of  $C_X(2)$  is  $K(0, 0, 1, 1, 1, 1, 1)$ .

The homogenization  $I^h$  of the ideal  $I$  is not Gorenstein, the rings  $S/I^h$  and  $S/\text{in}_\prec(I)$  have symmetric  $h$ -vector given by  $(1, 3, 3, 1)$ , and  $r_0 = \text{reg}(H_I^a) = 3$ . The Reed–Muller code  $C_X(1)$  is the standard evaluation code  $\mathcal{L}_X$ , where  $\mathcal{L} = S_{\leq 1}$ . The algebraic dual of  $C_X(1)$  is

$$\mathcal{L}^\perp = K\{t_1 t_3 + t_2 t_3 - t_1 - t_2 - t_3 - 1, t_1 + t_2 + t_3 + 1, t_2 + t_3 - 1, t_3\}.$$

If  $d = 1$ , the linear code  $C_X(d)$  is not monomially equivalent to  $C_X(r_0 - d - 1)^\perp$  because their minimum distances are  $\delta(C_X(1)) = 2$  and  $\delta(C_X(1)^\perp) = 3$ , respectively. Hence, the duality criterion of Theorem 6.5 fails if we replace condition (b) by  $H_I^a(d) + H_I^a(r_0 - d - 1) = |X|$  for  $-1 \leq d \leq r_0$ . Condition (b) of Theorem 6.5 is not satisfied because the unique list, up to multiplication by scalars from  $K^*$ , of standard indicator function for  $X$  is:

$$\begin{aligned} f_1 &= t_1 t_3 + t_1, f_2 = t_1 t_3 - t_1, f_3 = t_2 t_3^2 - t_3^2 - t_2 + 1, f_4 = t_2 t_3^2 + t_2 t_3 - t_3^2 - t_3, \\ f_5 &= t_2 t_3^2 - t_2 t_3 - t_3^2 + t_3, f_6 = t_2 t_3^2 - t_2, f_7 = t_2 t_3^2 - t_1 t_3 + t_2 t_3 - t_1, \\ f_8 &= t_2 t_3^2 + t_1 t_3 - t_2 t_3 - t_1, \end{aligned}$$

and  $v_{\mathfrak{p}_i}(I) = \deg(f_i)$  for all  $i$  (Proposition 4.6(a)). In particular one has  $v_{\mathfrak{p}_1}(I) = 2$ ,  $v_{\mathfrak{p}_3}(I) = 3$ ,  $v(I) = 2$  and, by Proposition 6.2, the index of regularity  $\text{reg}(\delta_X)$  of  $\delta_X$  is 2. This example corresponds to Procedure A.1.

**Example 8.2.** Let  $K$  be the finite field  $\mathbb{F}_3$ , let  $S = K[t_1, t_2, t_3]$  be a polynomial ring, let  $\prec$  be the GRevLex order on  $S$ , let  $I = I(X)$  be the vanishing ideal of the set of evaluation points

$$X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0), (2, 2, 2)\},$$

let  $P_i$  be the point in  $X$  in the  $i$ -th position from the left, and let  $\mathfrak{p}_i$  be the vanishing ideal of  $P_i$ . Adapting Procedure A.1, we obtain the following data. The ideal  $I$  is generated by

$$\begin{aligned} \mathcal{G} &= \{t_2 t_3 + t_3^2 - t_3, t_1 t_3 + t_3^2 - t_3, t_2^2 - t_3^2 - t_2 + t_3, \\ & t_1 t_2 + t_3^2 - t_3, t_1^2 - t_3^2 - t_1 + t_3, t_3^3 - t_3\}, \end{aligned}$$

and this set is a Gröbner basis for  $I$ . Then,  $\mathcal{G}^h = \{g^h \mid g \in \mathcal{G}\}$  is a Gröbner basis for  $I^h$ , the homogenization of  $I$  with respect to  $u$  [43, Proposition 3.4.2]. The ideal  $I^h$  is Gorenstein

because  $I^h$  is a Cohen–Macaulay ideal of height 3 and the minimal resolution of  $S[u]/I^h$  by free  $R$ -modules,  $R = S[u]$ , is given by

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R/I^h \longrightarrow 0,$$

see [43, Corollary 5.3.5]. It is seen that  $I$  is not a complete intersection, that is,  $I$  cannot be generated by 3 elements. The graded rings  $S/I^h$  and  $S/\text{in}_{\prec}(I)$  have symmetric  $h$ -vector given by  $(1, 3, 1)$  and  $r_0 = \text{reg}(H_I^a) = 2$ . The sorted list of standard monomials of  $S/I$  is

$$\Delta_{\prec}(I) = \{1, t_3, t_2, t_1, t_3^2\},$$

and the unique set  $F = \{f_i\}_{i=1}^5$  of standard indicator functions for  $X$  with  $f_i(P_i) = 1$  for all  $i$  is

$$\begin{aligned} f_1 &= t_3 + t_1 - t_3^2, & f_2 &= t_3 + t_2 - t_3^2, & f_3 &= -t_3 - t_3^2, \\ f_4 &= 1 + t_3 - t_2 - t_1 + t_3^2, & f_5 &= t_3 - t_3^2. \end{aligned}$$

Setting  $g = -f_1 - f_2 - f_3 + f_4 - f_5$ , one has  $g(P_i) = -1$  for  $i \neq 4$  and  $g(P_4) = 1$ . By Theorem 6.5 we obtain

$$C_X(1)^\perp = (g(P_1), \dots, g(P_5)) \cdot C_X(0) = K(-1, -1, -1, 1, -1).$$

**Example 8.3.** Let  $K$  be the finite field  $\mathbb{F}_3$ , let  $S = K[t_1, t_2, t_3]$  be a polynomial ring, let  $\prec$  be the GRevLex order on  $S$ , let  $I = I(X)$  be the vanishing ideal of the set of evaluation points

$$X = \{(1, 1, -1), (0, 0, 0), (0, 0, 1), (0, 0, -1), (0, 1, 0), (0, 1, 1), (0, 1, -1)\},$$

let  $P_i$  be the point in  $X$  in the  $i$ -th position from the left, and let  $\mathfrak{p}_i$  be the vanishing ideal of  $P_i$ . Adapting Procedure A.1, we obtain the following data. The Reed–Muller code  $C_X(1)$  is the standard evaluation code  $\mathcal{L}_X$ , where  $\mathcal{L} = S_{\leq 1}$ . The algebraic dual of  $C_X(1)$  is

$$\mathcal{L}^\perp = K\{t_2t_3 - t_1 - t_2 + t_3 + 1, t_1 + t_2 + 1, t_2 - 1\},$$

the minimum distances of  $C_X(1)$  and its dual  $C_X(1)^\perp$  are  $\delta(C_X(1)) = 1$  and  $\delta(C_X(1)^\perp) = 3$ , and  $r_0 = \text{reg}(H_I^a) = 3$ . If  $d = 1$ , then

$$H_I^a(d) + H_I^a(r_0 - d - 1) = H_I^a(1) + H_I^a(1) = 8 > 7 = |X|.$$

Hence, the inequality of Proposition 6.8(a) does not hold in general. The local  $v$ -numbers are  $v_{\mathfrak{p}_1}(I) = 1$  and  $v_{\mathfrak{p}_i}(I) = 3$  for  $i \geq 2$ . In particular  $v(I) = 1$  and, by Proposition 6.2, the index of regularity  $\text{reg}(\delta_X)$  of  $\delta_X$  is 1. If  $d = 2$ , then

$$H_I^a(d) + H_I^a(r_0 - d - 1) = H_I^a(2) + H_I^a(0) = 6 + 1 = |X|,$$

the algebraic dual of  $C_X(2)$  is  $K\{t_1 + t_2 + 1\}$ , and  $\delta(C_X(2)^\perp) = 6$ . The algebraic dual of  $C_X(0)$  is equal to  $\ker(\varphi) \cap K\Delta_{\prec}(I)$  and is given by

$$\begin{aligned} &K\{t_2t_3^2 - t_2t_3 - t_3^2 - t_1 - t_2 - t_3 - 1, t_2t_3 + t_3^2 + t_1 + t_2 + t_3 + 1, \\ &t_3^2 + t_1 + t_2 + t_3, t_1 + t_2 + t_3 - 1, t_2 + t_3, t_3 + 1\}, \end{aligned}$$

$\delta(C_X(0)^\perp) = 2$  and  $\delta(C_X(2)) = 1$ . For  $d = 0$ ,  $C_X(d)^\perp$  is not equivalent to  $C_X(r_0 - d - 1)$ . This example proves that in Theorem 6.5(b) and Corollary 6.6 the assumption “ $r_0 = v_{\mathfrak{p}}(I)$  for  $\mathfrak{p} \in \text{Ass}(I)$ ” is essential.

**Example 8.4.** Let  $X = \{P_1, \dots, P_m\}$  be a subset of  $K = \mathbb{F}_q$ ,  $m \geq 2$ , let  $S = K[t_1]$  be a polynomial ring in one variable, and let  $F = \{f_1, \dots, f_m\}$  be the unique set of standard

indicator functions for  $X$  such that  $f_i(P_i) = 1$  for all  $i$ . Then the vanishing ideal  $I = I(X)$  is a principal ideal generated by  $\prod_{i=1}^m (t_1 - P_i)$ ,  $r_0 = \text{reg}(H_I^a) = m - 1$ ,

$$f_i = \prod_{j \neq i} (t_1 - P_j) \Big/ \prod_{j \neq i} (P_i - P_j),$$

and  $\text{lc}(f_i) = \left[ \prod_{j \neq i} (P_i - P_j) \right]^{-1}$  for  $i = 1, \dots, m$ . By Theorem 6.5 we obtain

$$C_X(r_0 - d - 1)^\perp = (g(P_1), \dots, g(P_m)) \cdot C_X(d) \quad \text{for } -1 \leq d \leq r_0,$$

where  $g = \text{lc}(f_1)f_1 + \dots + \text{lc}(f_m)f_m$  and  $g(P_i) = \text{lc}(f_i)$  for all  $i$ . If  $X = K$ , then

$$K^* = \{P_i - P_1, \dots, P_i - P_{i-1}, P_i - P_{i+1}, \dots, P_i - P_m\},$$

$\text{lc}(f_i) = \text{lc}(f_m)$  for  $i = 1, \dots, m$ , and  $C_X(r_0 - d - 1)^\perp = C_X(d)$  for  $-1 \leq d \leq r_0$ .

**Example 8.5.** Let  $S = K[t_1]$  be a polynomial ring in one variable over the field  $K = \mathbb{F}_7$ , let  $\beta$  be a generator of  $K^*$ , and let  $X$  be the set of points  $= \{\beta^6, \beta, \beta^4, \beta^5\} = \{1, 3, 4, 5\}$ . The vanishing ideal  $I$  of  $X$  is generated by  $(t_1 - \beta^6)(t_1 - \beta)(t_1 - \beta^4)(t_1 - \beta^5)$  and  $r_0 = \text{reg}(H_I^a) = 3$ . Let  $\prec$  be the GRevLex order. The set of standard monomials of  $S/I$  is

$$\Delta_{\prec}(I) = \{1, t_1, t_1^2, t_1^3\}.$$

If  $\mathcal{L} = K\{1, t_1, t_1^2\}$ , then  $\mathcal{L}_X = C_X(2)$ . Adapting Procedure A.1, we obtain that the algebraic dual  $\mathcal{L}^\perp$  of  $\mathcal{L}$  is  $Kg$ , where  $g$  is the polynomial  $t_1^3 - t_1^2 - 2t_1$ . Evaluating  $g$  at each point of  $X$  gives the vector  $(-2, -2, -2, -1)$  and

$$C_X(2)^\perp = K(2, 2, 2, 1).$$

The unique set, up to multiplication by scalars from  $K^*$ , of standard indicator functions for the points  $\beta^6, \beta, \beta^4, \beta^5$  are

$$\begin{aligned} f_1 &= t_1^3 + 2t_1^2 - 2t_1 + 3, & f_2 &= t_1^3 - 3t_1^2 + t_1 + 1, & f_3 &= t_1^3 - 2t_1^2 + 2t_1 - 1, \\ f_4 &= t_1^3 - t_1^2 - 2t_1 + 2, \end{aligned}$$

respectively, and they generate  $K\Delta_{\prec}(I)$  (Proposition 4.6(a)). The  $v$ -number of  $I$  at each point of  $X$  is 3,  $v(I) = 3$ , and  $C_X(r_0 - d - 1)^\perp$  is equivalent to  $C_X(d)$  for  $-1 \leq d \leq r_0$  (Corollary 6.15).

**Example 8.6.** Let  $S = K[t_1, t_2]$  be a polynomial ring over the field  $K = \mathbb{F}_3$ , let  $X$  be the set

$$X = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, -1)\},$$

let  $I = I(X)$  be the vanishing ideal of  $X$ , and let  $\prec$  be the GRevLex order on  $S$ . Adapting Procedure A.1, we obtain the following data. The ideal  $I$  is generated by

$$\mathcal{G} = \{t_1^2 - t_1, t_2^3 - t_2, t_1 t_2^2 - t_1 t_2\},$$

and this set is a Gröbner basis of  $I$ . Let  $\mathcal{L}$  be the monomial space  $K\{1, t_1, t_2\}$  of  $S$ . Then one has  $\mathcal{L} \simeq C_X(1)$ ,  $\mathcal{L}_X = C_X(1)$ ,

$$\mathcal{L}^\perp = K\{t_1 t_2 - t_1 + t_2, t_1 - 1\}, \quad C_X(1)^\perp = K\{(1, 0, 1, 0, 1), (0, 1, -1, -1, 1)\},$$

$\delta(C_X(1)) = 2$ ,  $\delta(C_X(1)^\perp) = 3$ ,  $v_{\mathfrak{p}}(I) = 2$  for  $\mathfrak{p} \in \text{Ass}(I)$ ,  $H_I^a(1) = \dim_K(C_X(1)) = 3$ ,  $H_I^a(2) = \dim_K(C_X(2)) = 5$ , and  $r_0 = \text{reg}(H_I^a) = 2$ . The unique set, up to multiplication by scalars from  $K^*$ , of standard indicator functions for the points of  $X$  of Proposition 4.6(a) are given by

$$f_1 = t_1 t_2 - t_2^2 - t_1 + 1, \quad f_2 = t_1 t_2 - t_1, \quad f_3 = t_1 t_2 + t_2^2 + t_2, \quad f_4 = t_1 t_2, \quad f_5 = t_2^2 - t_2,$$

and  $4 = H_I^a(d) + H_I^a(r_0 - d - 1) < |X| = 5$  for  $d = 1$ .

**Example 8.7.** Let  $S = K[t_1, t_2]$  be a polynomial ring over the field  $K = \mathbb{F}_7$ , let  $\beta$  be a generator of the cyclic group  $K^*$ , and let  $A_i$ ,  $i = 1, 2$ , be the cyclic groups  $A_1 = (\beta^2)$ ,  $A_2 = (\beta^3)$ . The orders of  $\beta^2$  and  $\beta^3$  are  $d_1 = 3$  and  $d_2 = 2$ , respectively. Let  $\mathcal{L}$  be the linear space generated by  $\mathcal{B} = \{1, t_1, t_2, t_1 t_2\}$  and let  $\mathcal{L}_T$  be the monomial standard evaluation code on  $T = A_1 \times A_2$  relative to the GRevLex order  $\prec$ . The vanishing ideal  $I = I(T)$  of  $T$  is generated by  $t_1^3 - 1$  and  $t_2^2 - 1$ , the index of regularity of  $H_I^a$  is 3, and the set of standard monomials of  $S/I$  is

$$\Delta_{\prec}(I) = \{1, t_1, t_2, t_1^2, t_1 t_2, t_1^2 t_2\}.$$

According to Proposition 7.2 and Corollary 7.3, the algebraic dual  $\mathcal{L}^\perp$  is given by

$$\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B}) = K\{t_1, t_1 t_2\}, \quad \text{where } \mathcal{B} = \{1, t_1^2, t_2, t_1^2 t_2\},$$

and  $(\mathcal{L}_T)^\perp = (\mathcal{L}^\perp)_T$ . The minimum distance  $\delta(\mathcal{L}_T)$  of  $\mathcal{L}_T$  is 2 and  $\delta((\mathcal{L}_T)^\perp) = 3$ .

**Example 8.8.** Let  $S = K[t_1, t_2]$  be a polynomial ring over the field  $K = \mathbb{F}_4$ . We set  $\mathcal{X} = K^2$ ,  $d_1 = d_2 = 3$ ,  $e_1 = e_2 = 4$ , and

$$t^{a_1} = 1, t^{a_2} = t_1, t^{a_3} = t_2, t^{a_4} = t_2^2, t^{a_5} = t_2^3, t^{a_6} = t_1 t_2^2.$$

Let  $\mathcal{L}$  be the linear subspace of  $S$  generated by  $\mathcal{A} = \{t^{a_1}, \dots, t^{a_6}\}$  and let  $\mathcal{L}_\mathcal{X}$  be the monomial standard evaluation code on  $\mathcal{X}$  relative to the GRevLex order  $\prec$ . The vanishing ideal  $I = I(\mathcal{X})$  of  $\mathcal{X}$  is generated by  $t_1^{e_1} - t_1$  and  $t_2^{e_2} - t_2$ , the index of regularity of  $H_I^a$  is  $r_0 = 6$ , the set of standard monomials of  $S/I$  is

$$\Delta_{\prec}(I) = \{1, t_1, t_2, t_2^2, t_1 t_2, t_1^2, t_2^3, t_1 t_2^2, t_1^2 t_2, t_1^3, t_1 t_2^3, t_1^2 t_2^2, t_1^3 t_2, t_1^3 t_2^2, t_1^2 t_2^3, t_1^3 t_2^2\},$$

and  $\mathcal{A}$  is weakly divisor-closed. Setting  $t^{b_1} = t_1^3 t_2^3$ ,  $t^{b_2} = t_1^2 t_2^3$ ,  $t^{b_3} = t_1^3 t_2^2$ ,  $t^{b_4} = t_1^3 t_2$ ,  $t^{b_5} = t_1^3$ ,  $t^{b_6} = t_1^2 t_2$ , according to Theorem 7.8, the algebraic dual  $\mathcal{L}^\perp$  is given by

$$\mathcal{L}^\perp = K(\Delta_{\prec}(I) \setminus \mathcal{B}) = K\{1, t_1, t_2, t_2^2, t_1 t_2, t_1^2, t_2^3, t_1 t_2^2, t_1 t_2^3, t_1^2 t_2^2\},$$

where  $\mathcal{B} = \{t^{b_1}, \dots, t^{b_6}\}$ , and  $(\mathcal{L}_\mathcal{X})^\perp = (\mathcal{L}^\perp)_\mathcal{X}$ . The minimum distance  $\delta(\mathcal{L}_\mathcal{X})$  of  $\mathcal{L}_\mathcal{X}$  is 4. Other examples of sets that are weakly divisor-closed are  $\mathcal{A} \cup \{t_1^3 t_2^3, t_1^3\}$  and  $\mathcal{A} \cup \{t_1^3 t_2^3, t_1^3, t_1^2 t_2^3, t_1^2\}$ .

## APPENDIX A. PROCEDURES FOR *Macaulay2*

In this appendix we give a procedure for *Macaulay2* [24] that is used in some of the examples presented in Section 8. We use the package NAGtypes, written by Anton Leykin, that defines types used by the package NumericalAlgebraicGeometry as well as other numerical algebraic geometry packages.

**Procedure A.1.** Let  $\mathcal{L}_X$  be an evaluation code and let  $\prec$  be the graded reverse lexicographical order (GRevLex order) [43, p. 343], which is the default order in *Macaulay2* [24]. This procedure computes the standard function space  $\tilde{\mathcal{L}}$  and the minimum distance of  $\mathcal{L}_X$ . It determines whether or not the algebraic dual  $\mathcal{L}^\perp$  of  $\mathcal{L}_X$  is generated by monomials. If not, it computes a generating set for  $\mathcal{L}^\perp$  and then, using the algorithm of Theorem 3.11, it computes a  $K$ -basis for  $\mathcal{L}^\perp$ . This procedure also computes the vanishing ideal  $I = I(X)$ , the regularity index  $r_0$  of the affine Hilbert function  $H_I^a$ , the unique set, up to multiplication by scalars from  $K^*$ , of the standard indicator functions for  $X$  (Remark 4.7), and the  $v$ -numbers associated to  $I$ . This procedure can be used to check the condition “ $H_I^a(r_0 - d - 1) + H_I^a(d) = |X|$  for  $0 \leq d \leq r_0$ ” using the  $h$ -vector of the homogenization of  $I$  (Proposition 2.8), and to determine whether or not the homogenization  $I^h$  of the ideal  $I$  is Gorenstein. This procedure corresponds to Example 8.1.

```

load "NAGtypes.m2"
q=3, Fq=GF(q,Variable=>a), S=Fq[t1,t2,t3]
--Evaluation points of the code:
X={{1,1,1},{1,1,-1},{0,0,0},{0,0,1},{0,0,-1}, {0,1,0},
{0,1,1},{0,1,-1}}
--Vanishing ideals of the points:
I1=ideal(t1-1,t2-1,t3-1),I2=ideal(t1-1,t2-1,t3+1),
I3=ideal(t1,t2,t3),I4=ideal(t1,t2,t3-1),I5=ideal(t1,t2,t3+1),
I6=ideal(t1,t2-1,t3),I7=ideal(t1,t2-1,t3-1),I8=ideal(t1,t2-1,t3+1)
I=intersect(I1,I2,I3,I4,I5,I6,I7,I8)--Vanishing ideal
L={I1,I2,I3,I4,I5,I6,I7,I8}--List of ideals
G=gb I, M=coker gens G
r0=regularity I-1 --Regularity of  $H_I^a$ 
--Computes the remainder of x on division by G:
div=(x)->x % G
--Monomials that define the evaluation code:
Basis=matrix{{1,t1,t2,t3}}
--The list of remainders of Basis after division by G
--gives the standard function space:
cL=toList set apply(flatten entries Basis,div)
(d,r)=(1,1)
--This is the set of all elements of the ground field Fq:
field=set(apply(1..q-1,n->a^n))+set{0}
--Var1 to Var6 are used to compute the minimum distance of  $L_X$ :
Var1=(field)^**(#cL)-(set{0})^**(#cL)
Var2=apply(toList (Var1)/deepSplice,toList)
Var3=apply(Var2,x->matrix{cL}*vector x)
Var4=set(apply(apply(Var3,entries),n->n#0))
Var5=subsets(toList set apply(toList Var4,
m->(leadCoefficient(m))^(1-m),r)
Var6=apply(apply(Var5,ideal), x-> if #(set flatten entries
leadTerm gens x)==r then degree(I+x) else 0)
md=degree M-max Var6--Minimum distance
ps=(n)->polySystem(n*cL)
--We are redefining d to be r0 to compute the
--algebraic dual
(d,r)=(r0, 1)
--These are the points of X in the right format:
B=apply(X,x->point{toList x})
--The number of elements of  $P_1$  is the length of the code
P1=apply(flatten entries basis(0,r0,M),div)
b1=apply(P1,ps)
funct1=(n)->apply(B,x->evaluate(b1#n,x))
MatA=matrix{apply(0..#P1-1,n->{a^(q-1)})}
MatB=matrix{apply(0..#cL-1,n->{a^(q-1)-1})}
funct2=(x)-> if (matrix{funct1(x)}*(MatA)==MatB) then P1#x else 0
--This is the list of standard monomials in the dual code
--If this list has  $|X|-\dim_K(L_X)$  elements, then

```

```

--the dual of L_X is monomial
dualevaluation1= set apply(0..#P1-1,funct2)-set{0}
--Now we compute the dual when the dual is not monomial
Var7=(field)^**(#flatten entries basis(0,d,M))
Var8=(set{0})^**(#flatten entries basis(0,d,M))
Var9=apply(toList (Var7-Var8)/deepSplice,toList)
Var10=apply(apply(Var9,x->basis(0,d,M)*vector x),entries)
Var11=apply(toList set(apply(Var10,n->n#0)),
m->(leadCoefficient(m))^( -1)*m)
P=rsort(toList set Var11,MonomialOrder=>GRevLex)
b=apply(P,ps)
funct3=(n)->apply(B,x->evaluate(b#n,x))
MatC=matrix{apply(0..#B-1,n->{a^(q-1)})}
MatD=matrix{apply(0..#cL-1,n->{a^(q-1)-1})}
funct4=(x)-> if (matrix{funct3(x)}*(MatC)==MatD) then P#x else 0
--This is a list of generators of the algebraic dual:
dualevaluation= set apply(0..#P-1,funct4)-set{0}
--Next we compute a K-basis for the algebraic dual
--This computes the list of polynomials of a set
--with maximum leading monomial
split=(a) -> set apply(0..#a-1, i-> if leadMonomial(a#i)==
leadMonomial(max(a)) then a#i else 0)-set{0}
--Iterating this function and taking max will give the K-basis
--for the algebraic dual
hhh=(a)->toList((set(apply(toList split(a),x->max(a)-
(leadCoefficient(max a)/leadCoefficient(x))*x))-set{0*t1}))+
(set(a)-split(a)))
--Algorithm to compute a basis for a linear subspace
--of K[t1,...,ts] of finite dimension.
--This is a K-basis for the algebraic dual:
DDD=(A=toList dualevaluation; while #A>0 list
max(A)/leadCoefficient(max(A)) do A=hhh(A))
--Next we compute the v-numbers and indicator functions for X
f=(n)->flatten flatten degrees mingens(quotient(I,L#n)/I)
p=(n)->gens gb ideal(flatten mingens(quotient(I,L#n)/I))
minA=monomialIdeal(apply(0..#L-1,p))
vnumber0=min flatten degrees minA
g=(a)->toList(set a-set{0})
N=apply(apply(0..#L-1,f),g)
--This is the list of indicator functions for X:
toList apply(0..#L-1,n->p(n))
--Checking whether or not the homogenization I^h is Gorenstein
R=Fq[t1,t2,t3,u,MonomialOrder=>GRevLex]
J=sub(I,R)
L=ideal(homogenize(gens gb J,u))
HS=hilbertSeries(L)
--This gives the h-vector of the homogenization I^h of I
reduceHilbert(HS)

```

```
--This computes the minimal graded resolution of S/I^h
--that is used to determine whether or not
--the homogenization I^h is Gorenstein
res(coker gens gb L)
```

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