# AUTOMORPHISM SUBGROUPS FOR DESIGNS WITH $\lambda=1$ 

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#### Abstract

Given an integer $k \geq 3$ and a group $G$ of odd order, if there exists a $2-(v, k, 1)$-design and if $v$ is sufficiently large then there is such a design whose automorphism group has a subgroup isomorphic to $G$. Weaker results are obtained when $|G|$ is even.


## 1. Introduction

About 40 years ago Babai [Ba, p. 8] proposed the following "subgroup problem": PROBLEM 2.7. Prove for every $k \geq 3$, that, given a finite group $G$, there is a BIBD of block size $k$ (a $2-(v, k, 1)$-design) $X$ such that $G \widetilde{\leq}$ Aut $X$.
R.M. Wilson proved this when $k$ is a multiple of $|G|$ [Ba, p. 8]; [LW, Theorem 12.1] contains this when $k-1$ is a multiple of $|G|$. (These results are also in [Wi3, p. 311].)

In this note we will prove other special cases of Babai's problem:
Theorem 1.1. Given an integer $k \geq 3$ and a group $G$ of odd order, if $v$ satisfies the divisibility conditions for a $2-(v, k, 1)$-design and is sufficiently large then there is a 2-( $v, k, 1)$-design whose automorphism group has a subgroup isomorphic to $G$.

When $k=3$ stronger results appear in [Ca and [DK].
Theorem 1.2. Given an odd integer $k \geq 3$ and a group $G$ of even order such that $(k,|G|)=1$, there are infinitely many $v$ for which there is a $2-(v, k, 1)$-design whose automorphism group has a subgroup isomorphic to $G$.

Theorem 1.3. Consider an even integer $k \geq 4$ and a group $G$ of even order. Assume that every prime power dividing $|G|$ either divides $k$ or is relatively prime to $k(k-1)$. Then there are infinitely many $v$ for which there is a $2-(v, k, 1)$-design whose automorphism group has a subgroup isomorphic to $G$.

When $k$ or $k-1$ is a prime power, see [Ba, p. 8] or [Ka2 for a stronger type of result: there are infinitely many $2-(v, k, 1)$-designs $D$ for which $G \cong$ Aut $D$. Babai [Ba, Conjecture 2.8] asked for such a stronger result for arbitrary $k \geq 3$, but this presently seems out of reach: there appears to be no method for recovering the classical geometry underlying one of our designs as was done when $k$ or $k-1$ is a prime power. See Remark 2.5 and Section 8 for further comments about proving this stronger result.

The single idea behind the above three theorems is to place copies of a $2-(p, k, 1)-$ design in the lines of an affine space $A G(d, p)$ in a $G$-invariant manner, for large $d$ and a suitable prime $p$; this occurred in [Ka1, Sec. III.C] for a very different purpose. The 2 - $(p, k, 1)$-designs we use admit suitable automorphism groups (which are cyclic for Theorems 1.2 and (1.3), and are special cases of lovely results in Wi1, Wi3, LW.

Theorem 1.2 is proved in Section 3, while Theorem 1.3 is in Section 5 The remainder of this paper is devoted to Theorem 1.1. Section 2 contains a proof that there are infinitely many designs behaving as in Babai's problem when $|G|$ is odd, while Propositions 6.2 and 7.1 (based on Theorem 4.1) contain the background needed for the proof of Theorem 1.1 at the end of Section 7 .

All of our proofs are the same for abelian and nonabelian groups. In all of the results mentioned above $|G|$ is tiny relative to $v$. Our theorems do not deal with the case $|G| \equiv 0(\bmod 4)$ and $k \equiv 2(\bmod 4)$. The case $(|G|, k) \neq 1 \neq(|G|, k-1)$ seems especially difficult when $|G|$ is even.

Preliminaries: If $G$ is a group of permutations $x \mapsto x^{g}$ of a set $X$, and $L \subseteq X$, then $G_{L}:=\left\{g \in G \mid L^{g}=L\right\}$ is the set-stabilizer of $L$ in $G$, which induces the subgroup $G_{L}^{L}$ of the symmetric group $\operatorname{Sym}(L)$.

A permutation group $C$ on a set $X$ is semiregular if $x^{c} \neq x$ whenever $x \in X$ and $1 \neq c \in C$; and $C$ is regular if it is transitive and semiregular. If $\langle c\rangle$ and $\left\langle c^{\prime}\right\rangle$ are semiregular cyclic groups of permutations of $X$ having the same order then $c$ and $c^{\prime}$ are conjugate in $\operatorname{Sym}(X)$.

We will use the same symbol to denote a design and its set of points.

## 2. OdD ORDER

Theorem 2.1 (Wilson Wi1, pp. 22-26]). Given $k \geq 3$, for all sufficiently large primes $p \equiv 1(\bmod k(k-1))$ there is a $2-(p, k, 1)$-design $E$ whose set of points is $F:=\mathbb{F}_{p}$ and whose automorphism group contains $\{x \mapsto x+b \mid b \in F\}$.

Moreover, if $p=1+k(k-1) t$ with $t$ odd then $E$ can be chosen so that $\{x \mapsto s x \mid$ $\left.s \in F, s^{t}=1\right\}$ is also a group of automorphisms of $E$.

If $t=(p-1) / k(k-1)$ is odd, the subgroup of $F^{*}$ of order $2 t$ factors as $S \times\langle-1\rangle$ for a subgroup $S$ of order $t$. Then Wi1 obtains $A \subset F$ such that $\{s A+b \mid s \in S, b \in F\}$ is the set of blocks of $E$.

The preceding theorem lets us handle Babai's problem when $|G|$ is odd:
Theorem 2.2. Given an integer $k \geq 3$ and a group $G$ of odd order, there are infinitely many $v$ for which there is a $2-(v, k, 1)$-design whose automorphism group has a subgroup isomorphic to $G$.
Proof. By Dirichlet's Theorem there is a prime $p \equiv 1+k(k-1)|G|(\bmod 2 k(k-1)|G|)$. If we write $p-1=k(k-1) t$, it follows that $(p-1) /\{k(k-1)\}=t$ is odd and divisible by $|G|$. As above, let $F=\mathbb{F}_{p}$ and let $S$ be the subgroup of $F^{*}$ of order $t$.

We will prove the theorem by using suitable powers $v=p^{d}$. Let $V=F^{d}$, where $d$ is chosen so that $G$ is (isomorphic to) a group of permutations of a basis of $V$ and hence is in GL $(V)$. (For example, any integer $d \geq|G|$ can be chosen.)

We will use the affine space $\mathbf{A}:=\mathrm{AG}(d, p)$ whose set of points is $V$. Clearly $G<\operatorname{GL}(V)<\operatorname{AGL}(V)$. (Here $\operatorname{AGL}(V)=\{v \mapsto v M+c \mid M \in \operatorname{GL}(V), c \in V\}$ is Aut $\mathbf{A}$ if $d>1$.) Let $\mathcal{L}$ be a set of representatives of the orbits of $G$ on the lines of A.

Let $L \in \mathcal{L}$. View $L$ as $F$, so the group $\operatorname{AGL}(1, p)$ of $p(p-1)$ affine transformations $x \mapsto a x+b$ for $a \in F^{*}, b \in F$, corresponds to the affine group AGL $(L)$ on $L$ obtained from $\operatorname{AGL}(V)$. Then $\{x \mapsto s x+b \mid s \in S, b \in F\}$ corresponds to a subgroup $S(L)$ of AGL $(L)$ of order $p t$. Each subgroup of $\operatorname{AGL}(L)$ of order dividing $|S|=t$ lies in $S(L)$ (since the quotient group AGL $(1, p) /\{x \mapsto x+b \mid b \in F\}$ is isomorphic to the cyclic group $F^{*}$ ).

The set-stabilizer $G_{L}$ induces on $L$ a subgroup $G_{L}^{L}$ of AGL $(L)$. Since $|G|$ divides $t=|S|$ so does $\left|G_{L}^{L}\right|$. Then $G_{L}^{L} \leq S(L)$ by the preceding paragraph. (In fact, $G_{L}^{L}$ is even more restricted since $p>\left|G_{L}^{L}\right|$, but we will not need this fact.)

Use each $L \in \mathcal{L}$ as the set of points of a $2-(p, k, 1)$-design $D_{L}$ behaving as $E$ does at the end of Theorem [2.1, so $G_{L}^{L} \leq S(L) \leq$ Aut $D_{L}$. (The end of Theorem 2.1 required that that $t$ and $|G|$ are odd.)

For each $L \in \mathcal{L}$ let $\mathcal{B}_{L}$ be the set of blocks of $D_{L}$. If $g \in G$ let $D_{L^{g}}$ denote the design $\left(D_{L}\right)^{g}$ whose set of points is $L^{g}$ and whose set of blocks is $\left(\mathcal{B}_{L}\right)^{g}$.

This well-defined: if $L^{g}=L^{g^{\prime}}$ for $g, g^{\prime} \in G$ then $\left(D_{L}\right)^{g}=\left(D_{L}\right)^{g^{\prime}}$. For, if $h=g^{\prime} g^{-1}$ then $h \in G$ and $L^{h}=L$, so the permutation $h^{L}$ induced by $h$ on $L$ lies in $G_{L}^{L} \leq$ Aut $D_{L}$. Then $h^{L}$ sends $D_{L}$ to itself, so $\left(D_{L}\right)^{g}=\left(D_{L}\right)^{g^{\prime}}$, as required.

Define a design $D$ as follows:
points are the points of $\mathbf{A}$
blocks are the elements of $\bigcup_{L \in \mathcal{L}, g \in G}\left(\mathcal{B}_{L}\right)^{g}$.
It is elementary that $D$ is a $2-\left(p^{d}, k, 1\right)$-design: any two points lie in a unique line $L^{g}$ for $L \in \mathcal{L}$ and $g \in G$, and then in a unique member of $\left(\mathcal{B}_{L}\right)^{g}$. Since $G$ is in $\operatorname{AGL}(V)$ and permutes the sets $\left(D_{L}\right)^{g}$ it is a subgroup of Aut $D$.

Remark 2.3. By the last sentence of Theorem[2.1, the first paragraph of the above proof contains a solution to Babai's problem for the cyclic group of order $|G|$. The proofs of Theorems 1.2 and 1.3 involve something similar: a cyclic group case of Babai's problem is used to deal with much more general groups.
Remark 2.4. Placing designs on the blocks of another design is standard Wi1, p. 28]. Preserving the automorphism group is less standard. The above simple method was used in Ka1, Sec. III.C] to construct flag-transitive designs; preserving a group of automorphisms of the larger design was as essential there as it is here.

Remark 2.5. We used $\mathbf{A}$ with an arbitrary group $G$ of odd order. Given the action of $G$ on $V$, the groups $G_{L}$ and $G_{L}^{L}$ are known; since $p>|G|$, the group $G_{L}^{L}$ is cyclic.

However, there is flexibility with the designs $D_{L}$. We only needed to have $G_{L}^{L} \leq$ Aut $D_{L}$ (for each $L \in \mathcal{L}$ ) in order for the proof to work. Thus, each of the original designs $D_{L}(L \in \mathcal{L})$ can be replaced by $\left(D_{L}\right)^{h(L)}$ for any permutation $h(L)$ of the points of $L$ that normalizes $G_{L}^{L}$.

Suitable changes of this sort might provide a way to obtain a 2 - $\left(p^{d}, k, 1\right)$-design $D^{\prime}$ such that $G \cong$ Aut $D^{\prime}$. For this purpose it appears to be necessary to recover the affine space $\mathbf{A}$ from some such design $D^{\prime}$. However, we have been unable to do this (cf. Section 8).

Remark 2.6. On the other hand, each design $D_{L}$ admits the group $S(L)<$ $\operatorname{AGL}(L)=\operatorname{AGL}(V)_{L}^{L}$ as a group of automorphisms that is regular on blocks: $\{s A+b \mid s \in S, b \in F\}$ is the set of $|S| p=p(p-1) / k(k-1)$ blocks of each design constructed in Wi1, p. 22] starting from a suitable initial block $A \subset F$. Once again this uses the fact that $t$ and $|G|$ are odd.
Remark 2.7. If $B$ is a block of the design $D$ constructed in the proof of Theorem 2.2 then $G_{B}^{B}=1$. For, $B$ is in a unique line $L$ of $\mathbf{A}$, so $L$ is fixed by $G_{B}$. Then $G_{B}^{L} \leq S(L)$ as in the above proof. However, as already noted in the preceding remark, $S(L)$ is regular on the blocks of $D_{L}$, so $G_{B}^{L} \leq S(L)_{B}=1$ and hence $G_{B}^{B}=1$.

This will be crucial in Section 7
Note that $G$ has many fixed points, so there are many lines of $\mathbf{A}$ fixed pointwise by many elements of $G$.

## 3. Theorem 1.2

When $|G|$ is even we use a consequence of a theorem of Lamken and Wilson LW]; but first we need a prime:

Lemma 3.1. Let $k \geq 3$, and let $h$ be a multiple of 4 such that $(k, h)=1$. Then there are infinitely many primes $p>h$ satisfying the following conditions for some integer $n$ :
(i) $p=1+(k-1) n$,
(ii) $n(n-1) \equiv 0(\bmod k)$,
(iii) $n(n-1) \equiv 0(\bmod 4 k)$ if $k \equiv 3(\bmod 4)$, and
(iv) $(p-1, h)=(k-1, h)$.

Proof. Let $w$ be a positive integer such that $k w \equiv 1(\bmod h)$. Then $(1+k(k-$ 1) $w, h k(k-1))=(1+k(k-1) w, h)=(1+(k-1), h)=1$. By Dirichlet's Theorem there are infinitely many integers $y$ such that $p:=1+k(k-1) w+\{h k(k-1)\} y=$ $1+(k-1) n$ is prime, where $n:=k w+h k y \equiv 0(\bmod k)$. Then (ii) is clear, and (iii) holds: $n-1=(k w-1)+h k y$ is a multiple of $h$ and hence of 4. Finally, (iv) holds: $(p-1, h)=(k(k-1) w+\{h k(k-1)\} y, h)=(k w(k-1), h)=(k-1, h)$.

Theorem 3.2 (Lamken and Wilson [LW, Theorem 12.1]). Given $k$, for all sufficiently large $p$ satisfying the first three conditions of Lemma3.1 there is a $2-(p, k, 1)$ design $E$ such that Aut $E$ has a cyclic subgroup of order $k-1$ having one fixed point and semiregular on the remaining points.

Proof of Theorem 1.2, We imitate the proof of Theorem 2.2. In Lemma 3.1 let $h:=|G|$, where we increase $G$ if necessary in order to have $h$ divisible by 4 . (Admittedly this is annoying.) Choose a sufficiently large $p>|G|$ so that the lemma applies. Choose $d$ sufficiently large so that $G$ is (isomorphic to) a subgroup of the symmetric group $S_{d}$ and hence also of $\operatorname{AGL}(d, p)$. The points of our design $D$ are the points of $\mathbf{A}=\mathrm{AG}(d, p)$.

Let $L \in \mathcal{L}$, where $\mathcal{L}$ is a set of representatives of the orbits of $G$ on the lines of $\mathbf{A}$. Then $G_{L}^{L} \leq \operatorname{AGL}(L) \cong \operatorname{AGL}(1, p)$ and $p>|G| \geq\left|G_{L}^{L}\right|$, so $G_{L}^{L}$ is a cyclic group of order dividing $(p-1,|G|)=(k-1,|G|)$ by Lemma3.1(iv). This cyclic group fixes a point, and all remaining orbits have length $\left|G_{L}^{L}\right|$; all permutations of $L$ having this cycle structure are conjugate in $\operatorname{Sym}(L)$. After identifying $L$ with the set of points of the design in Theorem 3.2 and conjugating by an element of $\operatorname{Sym}(L)$, we may assume that $G_{L}^{L}$ is contained in the cyclic group of order $k-1$ provided by Theorem 3.2. Thus, $L$ is the set of points of a design $D_{L}$, isomorphic to the design $E$ in that theorem, such that $G_{L}^{L} \leq$ Aut $D_{L}$.

Now repeat the last three paragraphs of the proof of Theorem 2.2,

## 4. Moore and Ray-Chaudhuri

Wilson Wi1, p. 29] credits Ray-Chaudhuri for the following generalization of a standard, fundamental result due to Moore [Mo, p. 276]:

Theorem 4.1. A 2-( $w, k, 1)$-design $W$, a transversal design $T D(k, y-x)$ and $a$ $2-(y, k, 1)$-design $Y$ with an $x$-point subdesign $X$ produce a $2-(w(y-x)+x, k, 1)$ design.

Here a transversal design $T D(k, n)$ consists of $k n$ points, $n^{2}$ subsets of size $k$ called "blocks", and a partition of the points into $k$ "groups" of size $n$, such that each block meets each group in a single point and any two points in different groups are in a unique block.

The following proof is based on Wi1, pp. 29-30], and is included since we need properties of the constructed design.

Proof. If $Z:=Y-X$ as a set of points, then $X \cup(W \times Z)$ will be the set of points of our new design. Let $A$ be a block of $W$, hence of size $k$. There is a transversal design $T D(k, y-x)$ on $A \times Z$ whose set of groups is $\{a \times Z \mid a \in A\}$ and whose set of blocks will be denoted $\mathcal{B}_{A \times Z}$; this transversal design, denoted $T_{A \times Z}$, has nothing to do with the design on $Y$.

Imitating Moore [M0, p. 276] produces a new design as follows:
points: elements of $X \cup(W \times Z)$;
blocks are of four sorts:

- the blocks of $X$,
- for each $a \in W$ and each block $B$ of $Y$ not inside $X$,
$-a \times B$ if $B \cap X=\emptyset$, or
$-x \cup(a \times(B-x))$ if $B \cap X=x$, and
- $\bigcup\left\{\mathcal{B}_{A \times Z} \mid A\right.$ is a block of $\left.W\right\}$.

There is no conflict between the blocks in $T_{A \times Z}$ for different choices of $A$ : distinct intersecting sets $A \times Z$ and $A^{\prime} \times Z$ intersect in a group $a \times Z$ that meets each block of $T_{A \times Z}$ or $T_{A^{\prime} \times Z}$ once.

The only other part the proof worth a comment concerns a pair $\left(a_{1}, z_{1}\right),\left(a_{2}, z_{2}\right) \in$ $W \times Z$ with $a_{1} \neq a_{2}$. Since $a_{1} \neq a_{2}$ there is a unique block $A$ of $W$ containing them, and $\left(a_{1}, z_{1}\right)$ and $\left(a_{2}, z_{2}\right)$ belong to different groups $a_{1} \times Z$ and $a_{2} \times Z$ of $T_{A \times Z}$. Then there is a unique block in $\mathcal{B}_{A \times Z}$ containing them.

Remark 4.2. The existence of a $T D(k, n)$ is equivalent to the existence of a set of $k-2$ mutually orthogonal Latin squares of order $n$ Wi2, Lemma 2.1]. If $N(n)$ denotes the maximum number of mutually orthogonal Latin squares of order $n$, then CES proves that there is an integer $n_{0}$ such that $N(n) \geq \frac{1}{3} n^{1 / 91}$ if $n>n_{0}$ (and there are better bounds known [Wi2]). Thus, if $n(k):=\max \left(n_{0},(3 k)^{91}\right)$ then

$$
\begin{equation*}
\text { If } n>n(k) \text { then there is a } T D(k, n) \tag{4.3}
\end{equation*}
$$

## 5. Nets and Even $k$

As in Sections 2 and 3 the proof of Theorem 1.3 requires a suitable design on a prime number of points. Whereas Theorem 1.2 used a $2-(p, k, 1)$-design having a cyclic automorphism group of order $k-1$ fixing one point and semiregular on the remaining points (Theorem 3.2), this time we need a $2-(p, k, 1)$-design having a cyclic automorphism group of order $k$ fixing one point and semiregular on the remaining points (Theorem(5.4). For this purpose we use Theorem4.1 and transversal designs. However, it will be easier to start with nets.
5.1. Nets. The dual of a transversal design $T D(k, n)$ is a $(k, n)$-net: a set of $n^{2}$ points and $k n$ subsets of size $n$ called "lines" such that distinct lines meet at most once and the points are partitioned into $k$ "parallel classes" each consisting of $n$ lines. (Parallel classes correspond to groups.) The examples we need arise from unions of $k$ parallel classes of lines of a desarguesian affine plane $\operatorname{AG}(2, n)$; the translation group of the plane acts as a group of automorphisms of the net. Clearly these examples exist whenever $n$ is a prime power and $k \leq n+1$.

Lemma 5.1. Let $q$ and $m$ be powers $>1$ of a prime $p$, and $E=\mathbb{F}_{q^{m}} \supset F=\mathbb{F}_{q}$. Let $\sigma: x \mapsto x^{q}$ and let $T: E \rightarrow F$ be the trace map. If $a \in E-\operatorname{Ker} T$ and $h: x \mapsto x \sigma+a$, then $\langle h\rangle$ has order pm and is semiregular on $E$.

Proof. By induction, $h^{i}: x \mapsto x \sigma^{i}+\sum_{j=0}^{i-1} a \sigma^{j}$ for all $i \geq 1$, so $h^{m}: x \mapsto x \sigma^{m}+T(a)=$ $x+T(a)$ and $h$ has order $p m$.

If $x \in E$ then $T(x):=\sum_{j=0}^{m-1} x \sigma^{j}$ and $T(x(\sigma-1))=T(x) \sigma-T(x)=0$, so $\operatorname{Im}(\sigma-$ $1) \subseteq \operatorname{Ker} T$. If $i \geq 1, d:=(m, i)<m$ and $x \sigma^{i}=x$, then $T(x)=(m / d) \sum_{j=0}^{d-1} x \sigma^{j}=$ 0 since $m / d$ is a multiple of $p$, so $\operatorname{Im}(\sigma-1)+\operatorname{Ker}\left(\sigma^{i}-1\right) \subseteq \operatorname{Ker} T$.

For semiregularity, let $0<i<p m$ and suppose that $h^{i}$ fixes $x$. Then $x\left(1-\sigma^{i}\right)=$ $\sum_{j=0}^{i-1} a \sigma^{i}$, so $x\left(1-\sigma^{i}\right)(\sigma-1)=a\left(\sigma^{i}-1\right)$ and $x(\sigma-1)+a \in \operatorname{Ker}\left(\sigma^{i}-1\right)$. If $i \neq m$ then $(m, i)<m$; we have seen that this implies that $\operatorname{Ker} T$ contains $\operatorname{Im}(\sigma-1)+$ $\operatorname{Ker}\left(\sigma^{i}-1\right)$ and hence also $a$, which is not the case. If $i=m$ then we obtain the same contradiction: $0=x\left(1-\sigma^{m}\right)=\sum_{j=0}^{m-1} a \sigma^{j}=T(a)$.

Lemma 5.2. Let $q$ and $m$ be powers $>1$ of a prime $p$. Let $3 \leq k<q$. Then there is a $\left(k, q^{m}\right)$-net having a cyclic automorphism group of order $m$ that is semiregular on both the points and lines and leaves invariant each parallel class.

Proof. We use the notation of the preceding lemma. Consider the affine plane $\mathrm{AG}\left(2, q^{m}\right)$ defined using $E$. Our net will consist of the points of this plane and any union of $k$ parallel classes of lines of the form $y=t x+b$ with $1 \neq t \in F$ (so $t \sigma=t$ ).

Let $g:(x, y) \mapsto\left(x^{h}, y^{h}\right)$. By the preceding lemma, $\langle g\rangle$ has order $p m$ and is semiregular on points. Moreover, if $i \geq 1$ then $g^{i}$ sends the line $\{(x, t x+b) \mid x \in F\}$ to the parallel line $\left\{\left(x \sigma^{i}+a_{i}, t\left(x \sigma^{i}\right)+b \sigma^{i}+a_{i}\right) \mid x \in F\right\}$, where $a_{i}:=\sum_{j=0}^{i-1} a \sigma^{j}$. As above, $a_{i}(\sigma-1)=a\left(\sigma^{i}-1\right)$.

We still need semiregularity on lines. If $0<i<p m$ and $g^{i}$ fixes a line $y=t x+b$ of the net, then $t\left(x \sigma^{i}\right)+b \sigma^{i}+a_{i}=t\left(x \sigma^{i}+a_{i}\right)+b$, so $b\left(\sigma^{i}-1\right)=a_{i}(t-1)$. Then $b(\sigma-1)\left(\sigma^{i}-1\right)=a\left(\sigma^{i}-1\right)(t-1)$, so $b(\sigma-1)-a(t-1) \in \operatorname{Ker}\left(\sigma^{i}-1\right)$. If $i \neq m$ then $a(t-1) \in \operatorname{Im}(\sigma-1)+\operatorname{Ker}\left(\sigma^{i}-1\right) \subseteq \operatorname{Ker} T$ (as seen above), which is impossible since $0 \neq t-1 \in F$ and $a$ is not in the $F$-space $\operatorname{Ker} T$. Thus, $i=m$ and $0=b\left(\sigma^{m}-1\right)=a_{m}(t-1)=T(a)(t-1)$, which is again impossible since $t \neq 1$.

This proves that $\left\langle g^{p}\right\rangle$ behaves as required.
Lemma 5.3. Let $k \geq 3$ be an integer and let $p_{1}, \ldots, p_{r}$ be its distinct prime factors. For each $i$ let $m_{i}>k$ be a power of $p_{i}$, so $k \mid \pi:=\prod_{i} m_{i}$.

Then for each integer $s>n(k)$ there is a $\left(k, s \prod_{i} m_{i}^{m_{i}}\right)$-net having a cyclic automorphism group of order $\pi$ that is semiregular on both points and lines while leaving invariant each parallel class.

Remark 4.2 contains the definition of $n(k)$. We emphasize that $s$ and the $m_{i}$ are not related.

Proof. For each $i$, by using Lemma 5.2 with $q=m=m_{i}$ we obtain a $\left(k, m_{i}^{m_{i}}\right)$-net $N_{i}$ having a cyclic automorphism group $C_{i}$ of order $m_{i}$ that is semiregular on both points and lines and leaves invariant each parallel class.

By Remark 4.2 if $s>n(k)$ then there is a $(k, s)$-net $N_{\infty}$. The net required in the lemma is a product $N=N_{1} \cdots N_{r} N_{\infty}$, which we now define.

Let $X_{i}$ be the set of points of $N_{i}$ and $\mathcal{L}_{i 1}, \ldots, \mathcal{L}_{i k}$ the parallel classes of $N_{i}$, so $\bigcup_{j} \mathcal{L}_{i j}$ is the set of lines of $N_{i}$. Then $N$ is defined as follows: $X:=X_{1} \times \cdots \times X_{r} \times X_{\infty}$ is its set of points, while its parallel classes are $\mathcal{L}_{j}:=\mathcal{L}_{1 j} \times \cdots \times \mathcal{L}_{r j} \times \mathcal{L}_{\infty j}, 1 \leq j \leq k$, and $\bigcup_{j} \mathcal{L}_{j}$ is its set of lines.

In general, the groups Aut $N_{i}$ are not involved in Aut $N$ since we used an arbitrary ordering of the parallel classes of each $N_{i}$. However, for our purposes this is not a problem since $C_{i}$ leaves invariant each parallel class of $N_{i}$ and we will use the identity on $N_{\infty}$ : there is a cyclic automorphism group $C \cong \prod_{i} C_{i}$ of $N$ of order $\pi=\prod_{i} m_{i}$, consisting of all $\left(x_{1}, \ldots, x_{r}, x_{\infty}\right) \mapsto\left(x_{1}^{c_{1}}, \ldots, x_{r}^{c_{r}}, x_{\infty}\right)$ for $c_{i} \in C_{i}$. (This is an automorphism of $N: c_{i}$ permutes the lines in each $\mathcal{L}_{i j}$, so if $\left(x_{1}, \ldots, x_{r}, x_{\infty}\right) \in$ $\left(L_{1 j}, \ldots, L_{r j}, L_{\infty j}\right) \in \mathcal{L}_{j}$ then $\left.\left(x_{1}^{c_{1}}, \ldots, x_{r}^{c_{r}}, x_{\infty}\right) \in\left(L_{1 j}^{c_{1}}, \ldots, L_{r j}^{c_{r}}, L_{\infty j}\right) \in \mathcal{L}_{j}.\right)$

The only way a point (or line) of $N$ can be fixed by the preceding element of $C$ is for a point (or line) of every component to be fixed, and then each $c_{i}=1$ by the semiregularity of each $C_{i}$.
5.2. Theorem 1.3. Following the models in Sections 2 and 3 we need a prime $p$ and a $2-(p, k, 1)$-design admitting a suitable automorphism group.

Proposition 5.4. Given integers $k \geq 3$ and $h \geq 1$ such that $(k-1, h)=1$, there are infinitely many primes $p$ such that $(p-1, h)$ divides some power of $k$ and there is a 2-( $p, k, 1)$-design having a cyclic automorphism group of order $k$ fixing one point and semiregular on the remaining points.

Proof. We will use a design in Theorem4.1] whose set of points is $U:=X \cup(W \times Z)$, $Z:=Y-X$, where $X$ is a subset of size 1 of the design $Y$ BS Corollary 2C.1]. For this we need three ingredients involving one choice of a suitable prime $q>h$, a suitable choice in Lemma 5.3 of the $m_{i}$ such that $\pi=\prod_{i} m_{i}$ is divisible by $k$, and infinitely many $s$ :
(1) a 2-( $q k, k, 1)$-design $W$ having a cyclic automorphism group $C$ of order $k$ that is semiregular on points and whose $q$ point-orbits are blocks of $W$ [Wi3, Theorem 1.2 and p. 308],
(2) a transversal design $T=T D(k,(k-1) h \pi s)$ having a cyclic automorphism group of order $k$ that is semiregular on both points and lines ( $T$ exists for all sufficiently large $s$ by Lemma 5.3), and
(3) a 2- $(y, k, 1)$-design $Y$ with $y:=1+k(k-1)(\pi / k) s$, and an arbitrary point $X$ of $Y$ ( $Y$ exists for all sufficiently large $s$ Wi1, Theorem 1.1]). (Note that we do not have any information concerning automorphisms of $Y$.)
Moreover, we require that
(4) $p=1+q k \cdot(k-1) \pi s=1+|W|(y-1)$ is prime, and
(5) $(p-1, h)$ divides some power of $k$ (a condition in the proposition).

We will proceed in four steps.
(I) Number Theory: $\pi, p$ and $s$. Write $h=h_{0} h^{\prime}$, where $\left(k, h^{\prime}\right)=1$ and all primes dividing $h_{0}$ also divide $k$. Then $\left(k(k-1), h^{\prime}\right)=1$ since $(k-1, h)=1$, so that $h^{\prime}$ is odd. We may assume that $h_{0}$ divides the product $\pi$ of the $m_{i}$ used in Lemma 5.3

We have a prime $q>h$ in (1), so $\left(q k(k-1) \pi, h^{\prime}\right)=1$ since $\left(k, h^{\prime}\right)=1=(k-1, h)$. Let $t$ be a positive integer such that $q k(k-1) \pi t \equiv 1\left(\bmod h^{\prime}\right)$. Then $\left(t, h^{\prime}\right)=1$ and $\left(1+q k(k-1) \pi t, q k(k-1) \pi h^{\prime}\right)=\left(1+q k(k-1) \pi t, h^{\prime}\right)=\left(1+1, h^{\prime}\right)=1$ since $h^{\prime}$ is odd. By Dirichlet's Theorem there are infinitely many integers $f$ such that

$$
p:=1+q k(k-1) \pi t+q k(k-1) \pi h^{\prime} f
$$

is a prime.
Choose $s:=t+h^{\prime} f$ with $f$ so large that the designs in (2) and (3) exist. Clearly (4) holds.

Moreover, $(p-1, h)$ divides $\left(q(k-1)\left(t+h^{\prime} f\right), h^{\prime}\right) k \pi h_{0}=\left((k-1) t, h^{\prime}\right) k \pi h_{0}=$ $k \pi h_{0}$, which divides some power of $k$, as required in (5).

Now that we have $W, T$ and $Y$ we need to turn the set $U=X \cup(W \times Z)$ of size $p$ into a design.
(II) The cyclic group $\bar{C}$. We need a group $\bar{C} \cong C$ of permutations of $U$. Extend each $c \in C$ (cf. (1)) to a permutation $\bar{c}$ of $U$ that fixes the point in $X$ and sends $(a, z) \mapsto\left(a^{c}, z\right)$ for $a \in W, z \in Z$. Then $\bar{C}=\{\bar{c} \mid c \in C\}$ is a group of $k$ permutations of $U$ fixing $X$ and semiregular on the remaining points. This is not yet a group of automorphisms of anything.

We will construct the design in Theorem4.1 by placing (in (III)) copies of the transversal design $T$ in the sets $B \times Z$ of size $k(y-1)$ arising from blocks $B$ of $W$, and (in (IV)) copies of $Y$ in the sets $Y_{a}:=X \cup(a \times Z)$ of size $y$ for $a \in W$.
(III) Copies of $T$. We will use copies of the transversal design $T$ in (2) as the transversal designs occurring in the proof of Theorem 4.1. (Recall that $y-1=$ $k(k-1) h \pi s$.)

In view of the point-orbits in (1), the stabilizer in $C$ of a block of $W$ is either 1 or $C$.

Let $\mathcal{B}$ be a set of orbit representatives of $C$ on the blocks of $W$. If $B \in \mathcal{B}$ let $T_{B \times Z} \cong T$ have $B \times Z$ as its set of points and $\{b \times Z \mid b \in B\}$ as its set of groups. If the stabilizer of $B$ in $C$ is $1, T_{B \times Z}$ is placed in $B \times Z$ arbitrarily. If $B$ is a $C$-orbit we have to be more careful. Initially, place $T_{B \times Z}$ arbitrarily. We then have two semiregular cyclic permutation groups of order $k$ on $B \times Z$ : one is the restriction $\bar{C}^{B \times Z}$ of $\bar{C}$ to $B \times Z$, and the other is the cyclic automorphism group of $T_{B \times Z}$ provided by (2). These cyclic groups of order $k$ are conjugate by an element of $\operatorname{Sym}(B \times Z)$; conjugate by such an element in order to assume that $T_{B \times Z}$ has been placed in $B \times Z$ so that the cyclic groups coincide, and hence so that $\bar{C}^{B \times Z} \leq$ Aut $T_{B \times Z}$.

For $B \in \mathcal{B}$ and $c \in C$ let $T_{B^{c} \times Z}$ denote the transversal design $\left(T_{B \times Z}\right)^{\bar{c}}$ having $(B \times Z)^{\bar{c}}=B^{c} \times Z$ as its set of points. This is well-defined: if $B^{c}=B^{c^{\prime}}$ then $\bar{c}^{\prime} \bar{c}^{-1}$ induces the permutation $\left(\bar{c}^{\prime} \bar{c}^{-1}\right)^{B \times Z}$ of $B \times Z$, which is an automorphism of $T_{B \times Z}$ by the preceding paragraph, so that $\left(T_{B \times Z}\right)^{\bar{c}}=\left(T_{B \times Z}\right)^{\bar{c}^{\prime}}$.
(IV) Copies of $Y$. Next we place copies of the design $Y$ into the sets $Y_{a}=$ $X \cup(a \times Z), a \in W$, in the same manner. Namely, let $\mathcal{W}$ be a set of orbit representatives of $C$ on the points of $W$. For $a \in \mathcal{W}$ place a copy of the design $Y$ in $Y_{a}$ using the bijection $X \mapsto X, z \mapsto(a, z)$ with $z \in Z$; then let $Y_{a^{c}}:=\left(Y_{a}\right)^{\bar{c}}$ for $c \in C$. As usual, this is well-defined since $a^{c}=a^{c^{\prime}}$ implies that $c=c^{\prime}$ by semiregularity (cf. (2)).

Using the construction in the proof of Theorem4.1 we obtain a 2 - $(p, k, 1)$-design $U$ having $\bar{C}$ as a group of $k$ automorphisms that fixes the point $X$ and is semiregular on the remaining points.
Proof of Theorem 1.3. Let $h=|G|$ and $p>h$ be as in the preceding proposition. We imitate the proof of Theorem 2.2, regarding $G$ as a group of automorphisms of $\mathbf{A}=$ $\mathrm{AG}(d, p)$ for any sufficiently large $d$. Let $L \in \mathcal{L}$, where $\mathcal{L}$ is a set of representatives of the orbits of $G$ on the lines of $\mathbf{A}$. Then $G_{L}^{L}$ has order dividing $(p(p-1),|G|)=$ ( $p-1,|G|$ ); by the proposition, this divides some power of $k$, and hence divides $k$ by an hypothesis of the theorem.

The cyclic group $G_{L}^{L}$ fixes a point and is semiregular on the remaining points. After identifying $L$ with the set of points of the design $U$ in Proposition 5.4 and conjugating by an element of $\operatorname{Sym}(L)$, we may assume that $G_{L}^{L}$ is contained in the cyclic group of order $k$ provided by the proposition. Thus, $L$ is the set of points of a design $D_{L}$, isomorphic to $U$, such that $G_{L}^{L} \leq$ Aut $D_{L}$.

Now complete the proof by repeating the last three paragraphs of the proof of Theorem 2.2.

## 6. LARGE DESIGNS

The Doyen-Wilson Theorem [DW states that, whenever $y \geq 2 x+1$ and there are Steiner triple systems on $y$ and $x$ points, there is a Steiner triple system on $y$ points having a subsystem on $x$ points. The following is a significant generalization of that result DLL:
Theorem 6.1. If $k \geq 3$ then there is an integer $x_{0}(k)>k$ such that, if $x>x_{0}(k)$, $y>x k, x-1 \equiv y-1 \equiv 0(\bmod k-1)$ and $x(x-1) \equiv y(y-1) \equiv 0(\bmod k(k-1))$, then there is a 2-(y,k,1)-design having an x-point subdesign.

We use this for a result concerning large designs ( $n(k)$ appears in Remark 4.2):
Proposition 6.2. Let $\mathcal{S}$ be a set of 2-( $u, k, 1)$-designs, let $\overline{\mathcal{S}}$ be the set of all such $u$ that occur for $\mathcal{S}$, and let $w \in \overline{\mathcal{S}}$. Assume that, if $x$ and $y$ are as in Theorem 6.1 with $y-x>n(k)$, then $x+w(y-x) \in \overline{\mathcal{S}}$.

Then $\overline{\mathcal{S}}$ contains all sufficiently large $u$ satisfying the divisibility conditions for a 2-(u, $k, 1)$-design.

Our argument imitates Ca . Note that the hypothesis involves only the initial existence of one $w \in \overline{\mathcal{S}}$.

Proof. For $x_{0}(k)$ in Theorem [6.1] let $x_{1}>x_{0}(k)>k$ be any representative for a congruence class $(\bmod k(k-1))$ of integers such that there exists a $2-\left(x_{1}, k, 1\right)$ design. Consider any integer $a \geq w x_{1} n(k) \geq w x_{1}$. Choose $y-x=x_{1} k(k-1) a>$ $a>n(k)$, so $y-x>n(k)$, and then choose $x=x_{1}+k(k-1) t$ with $0 \leq t<a$, so $x \geq x_{1}>x_{0}(k)$. Then $x$ and $y=x_{1}+x_{1} k(k-1) a+k(k-1) t$ satisfy $y>k x$. (For, since $t<a$ and $x_{1}>k$, we have $y-k x=(k-1)\left(-x_{1}+x_{1} k a-k(k-1) t\right)$, where $x_{1}(k a-1)-k(k-1) t>k(k a-1)-k(k-1) a>0$. $)$

Since $x$ and $y$ satisfy the divisibility conditions and the requirements $x>x_{0}(k)$ and $y>k x$ in Theorem6.1, there is a $2-(y, k, 1)$-design having an $x$-point subdesign. Theorem 4.1 also needs a $T D(k, y-x)$, which exists since $y-x>n(k)$ (cf. (4.3)). By hypothesis, Theorem 4.1 produces a $2-(u, k, 1)$-design such that

$$
\begin{equation*}
u:=x+w(y-x) \in \overline{\mathcal{S}}, \text { with } u=x_{1}+w x_{1} k(k-1) a+k(k-1) t . \tag{6.3}
\end{equation*}
$$

Here $u-1 \equiv 0(\bmod k-1)$ and $u(u-1) \equiv 0(\bmod k(k-1))$. We will show that the set of all $u$ obtained in (6.3) contains the set of all sufficiently large $u \equiv x_{1}$ $(\bmod k(k-1))$ satisfying these divisibility conditions.

Given $a$, we have $y-x=x_{1} k(k-1) a$ and $x=x_{1}+k(k-1) t$. By choosing $t=0, \ldots, a-1$, we realize

$$
u=x_{1}+w x_{1} k(k-1) a, \ldots, x_{1}+w x_{1} k(k-1) a+k(k-1)(a-1) .
$$

For $y-x=x_{1} k(k-1)(a+1)$, we realize

$$
u=x_{1}+w x_{1} k(k-1)(a+1), \ldots, x_{1}+w x_{1} k(k-1)(a+1)+k(k-1) a
$$

In order not to leave any gaps, we require that these intervals abut or overlap. This occurs as long as $x_{1}+w x_{1} k(k-1) a+k(k-1) a \geq x_{1}+w x_{1} k(k-1)(a+1)$, that is, $a \geq w x_{1}$, which is a condition already satisfied by $a$. So we can achieve all sufficiently large $x \equiv x_{1}(\bmod k(k-1))$.

Now let $x_{1}>x_{0}(k)$ run through a set of representatives for the congruence classes $\bmod k(k-1)$ that satisfy the divisibility conditions for a $2-\left(x_{1}, k, 1\right)$-design.

## 7. Theorem 1.1

We call an automorphism group of a design 1-blocked if the set-stabilizer of any block is the identity on the block; our basic example was in Remark 2.7. This notion is preserved by the construction in Section 4 .

Proposition 7.1. Let $k \geq 3$ and let $G$ be a 1-blocked automorphism group of $a$ $2-(w, k, 1)$-design $W$. Then a 2-( $y, k, 1)$-design $Y$ with a subdesign $X$ on $x$ points, together with a transversal design $T D(k, y-x)$, produce a $2-(w(y-x)+x, k, 1)$ design $U$ such that $G$ is isomorphic to a 1-blocked subgroup of Aut $U$.

Proof. We use the construction and notation in the proof of Theorem 4.1 Each $g \in G$ induces on $U$ the permutation $\bar{g}$ sending $b \mapsto b$ and $(a, z) \mapsto\left(a^{g}, z\right)$ for $b \in X$, $a \in W, z \in Z$. Clearly $G \cong \bar{G}:=\{\bar{g} \mid g \in G\}$.

For each orbit-representative $A$ of $G$ on the blocks of $W$ we have a transversal design $T D_{A \times Z}$ whose set of points is $A \times Z$ and whose set of groups is $\{a \times Z \mid$ $a \in A\}$. If $g \in G$ then $(A \times Z)^{\bar{g}}=A^{g} \times Z$ for a block $A^{g}$ of $W$; let $T D_{A^{g} \times Z}:=$ $\left(T D_{A \times Z}\right)^{\bar{g}}$. As in the proof of Theorem 2.2 this is well-defined: if $A^{g} \times Z=A^{g^{\prime}} \times Z$ with $g, g^{\prime} \in G$ then $A^{g}=A^{g^{\prime}}$, so $g^{\prime} g^{-1}=1$ on $A$ since $G$ is 1-blocked, and hence $\left(T D_{A \times Z}\right)^{\bar{g}}=\left(T D_{A \times Z}\right)^{\bar{g}^{\prime}}$ since $\bar{g}^{\prime} \bar{g}^{-1}=1$ on $A \times Z$.

By the construction in Section 4 each $\bar{g}$ permutes the designs $T D_{A^{\prime} \times Z}$ with $A^{\prime}$ a block of $W$, and is the identity on any other block of $U$ (i. e., a block of $X$, or else $a \times B$ or $b \cup(a \times(B-b))$ if $a \in W$ and if $B \cap X=b)$. Thus, $\bar{G} \leq$ Aut $U$.

We need to verify that $\bar{G}$ is 1-blocked. Consider a block $E$ of $U$ fixed by $\bar{g} \in \bar{G}$. By Section 4, either $E$ is contained in $X \cup(a \times B)$ for $a \in W$ and a block $B$ of $Y$, or $E$ is a block of some $T D_{A \times Z}$. In the former case it is clear that $\bar{g}=1$ on $E$, so we are left with $E$ in $T D_{A \times Z}$. In view of the construction in Section 4, $A$ is uniquely determined by $E$ and hence is fixed by $g$. Since $G$ is 1-blocked on $W$, it follows that $g=1$ on $A$. Then $\bar{g}=1$ on $A \times Z$ and hence on $E$. Thus, $\bar{G}$ is a 1-blocked subgroup of Aut $U$.

Remark 7.2. An automorphism group of even order cannot be 1-blocked. For, an involution interchanges two points, hence fixes the block containing them and acts nontrivially on that block.

Proof of Theorem 1.1. Apply Proposition 6.2 to the set $\mathcal{S}$ of 2- $(v, k, 1)$-designs whose automorphism group has a 1-blocked subgroup isomorphic to $G$. By Theorem 2.2 and Remark 2.7, $\mathcal{S}$ contains some $2-(v, k, 1)$-design.

We defined $n(k)$ in Remark 4.2 and $x_{0}(k)$ in Theorem6.1. Let $x>\max \left(n(k), x_{0}(k)\right)$ and $y>k x$ be integers such that there are $2-(x, k, 1)$ - and $2-(y, k, 1)$-designs. By Theorem 6.1 there is a $2-(y, k, 1)$-design having an $x$-point subdesign. Since $y-x>k x-x>n(k)$ there is a $T D(k, y-x)$ by (4.3). Then $x+v(y-x) \in \overline{\mathcal{S}}$ by Proposition 7.1 Now use Proposition 6.2.

## 8. Conjectures

Our theorems are significantly weaker than the corresponding results in Ba , Ka2, DK, where $G$ is isomorphic to the full automorphism group of the constructed design. We conclude with a conjecture concerning affine spaces that would produce designs with this stronger property.

Conjecture 8.1. Given an integer $s \geq 14$, a prime $p \equiv 1(\bmod s)$, and an affine space $\mathbf{A}^{\prime}$ having the same set of points as the original affine space $\mathbf{A}=\mathrm{AG}(d, p)$, such that
for any subspaces $X$ of $\mathbf{A}$ and $Y^{\prime}$ of $\mathbf{A}^{\prime}$,
either $X \cap Y^{\prime}=\emptyset$ or $\left|X \cap Y^{\prime}\right| \equiv 1(\bmod s)$.
Conjecture: $\mathbf{A}=\mathbf{A}^{\prime}$.
Note that it is essential here that $p$ is prime. For suppose that $p=p_{0}^{e}>p_{0}$ for a prime $p_{0} \equiv 1(\bmod s)$. Let $\mathbf{A}_{0}=\mathrm{AG}\left(e d, p_{0}\right)$, let $\mathbf{A}$ be the set of affine $\mathbb{F}_{p}$-subspaces of $\mathbf{A}_{0}$. If $g \in \operatorname{AGL}\left(d n, p_{0}\right)-\operatorname{AGL}(d, p)$ then $\mathbf{A}^{\prime}:=\mathbf{A}^{g}$ provides a counterexample to the conclusion in the preceding conjecture.

The condition $s \geq 14$ reflects the fact that 14 is the smallest integer $s=k-1 \geq 2$ such that neither $s$ nor $s+1$ is a prime power: when one of these is a prime power the desired result is already known $\mathrm{Ba}, \mathrm{Ka}$.

Theorem 8.2. Assume that the preceding conjecture is correct. Under the hypotheses in any of Theorems 1.1-1.3, for infinitely many $v$ there is a $2-(v, k, 1)$-design $D$ such that $G \cong$ Aut $D$.

Proof. By Ba, Ka2] we may assume that neither $k$ nor $k-1$ is a prime power, so that $p>s:=k-1 \geq 14$. Each theorem in Section 1 uses 2 - $(p, k, 1)$-designs constructed in Theorems 2.1 or 3.2, or Proposition 5.4. In view of those constructions, in the situation of any of the theorems in Section 1, there are 2-( $p, k, 1$ )-designs $E_{1}, E_{2}, E_{3}$ with $\mathrm{AG}(1, p)$ as their set of points such that there is no isomorphism between any two of these designs that lies in AGL $(1, p)$. (Namely, start with a design $E_{1}$, and for $i=2,3$ apply an $i$-cycle of points to the blocks of $E_{1}$ in order to obtain the blocks of $E_{i}$.)

Let $d>4$ be as in the proofs, so we are using $\mathbf{A}=\mathrm{AG}(d, p)$ based on a $d$-space $V$. Let $\left\{v_{1}, \ldots, v_{d}\right\}$ be a basis of $V$. There is a connected graph $\Gamma$ with vertex set $\left\{v_{1}, \ldots, v_{d}\right\}$ such that $G \cong$ Aut $\Gamma$.

Let $c:=\sum_{1}^{d} v_{t}$. Place $E_{1}$ in each affine 1 -space $\left\langle v_{i}\right\rangle$, place $E_{2}$ in each affine 1 -space $\left\langle v_{i}+v_{j}\right\rangle+c$ such that $\left\{v_{i}, v_{j}\right\}$ is an edge of $\Gamma$, and place $E_{3}$ in every other affine 1 -space of $\mathbf{A}$. (Note that, since $d>4$, if $\left\langle v_{i}+v_{j}\right\rangle+c=\left\langle v_{i^{\prime}}+v_{j^{\prime}}\right\rangle+c$ then $\{i, j\}=\left\{i^{\prime}, j^{\prime}\right\}$.)

This produces a 2 - $\left(p^{d}, k, 1\right)$-design $D$ with $G$ (isomorphic to) a subgroup of Aut D. (Compare Ka2.)

Let $h \in$ Aut $D$ and consider the affine space $\mathbf{A}^{\prime}=\mathbf{A}^{h}$. If $X$ and $Y^{\prime}$ are subspaces of $\mathbf{A}$ and $\mathbf{A}^{\prime}$, respectively, and if $X \cap Y^{\prime} \neq \emptyset$, then $X \cap Y^{\prime}$ is the intersection of subdesigns and so is a subdesign. Then $\left|X \cap Y^{\prime}\right| \equiv 1(\bmod s)$. Thus, $\mathbf{A}^{h}=\mathbf{A}$ by our hypothesis concerning Conjecture 8.1, so $h$ is an automorphism of $\mathbf{A}$, and hence permutes the lines of $\mathbf{A}$. Then $h$ also permutes the designs we have placed inside these lines, so $h$ permutes the lines $\left\langle v_{i}\right\rangle$. The intersection of these lines is 0 , so $h$ is a linear transformation. By construction, $h$ also permutes the lines $\left\langle v_{i}+v_{j}\right\rangle+c$, so it induces an automorphism of $\Gamma$, and hence agrees with some $g \in G=$ Aut $\Gamma$ in its action on the lines $\left\langle v_{i}\right\rangle$. Now $h^{\prime}:=h g^{-1}$ fixes each line $\left\langle v_{i}\right\rangle$, and hence is a diagonal transformation: $v_{t}^{h^{\prime}}=a_{t} v_{t}$ for some $a_{i} \in K^{*}$ and all $t$. Since $h^{\prime}$ fixes each vertex of $\Gamma$ it fixes each edge: $\left\langle v_{i}+v_{j}\right\rangle+c=\left(\left\langle v_{i}+v_{j}\right\rangle+c\right)^{h^{\prime}}=\left\langle a_{i} v_{i}+a_{j} v_{j}\right\rangle+\sum_{1}^{d} a_{t} v_{t}$. It follows that all $a_{t}=1$ for $t \neq i, j$, so $h^{\prime}=1$ since $\Gamma$ is connected, and then $h \in G$.

Remark 8.3. The fact that $X \cap Y^{\prime}$ is a subdesign imposes arithmetic and structural conditions that can be included in the hypotheses of the above conjecture.
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