MDS Symbol-Pair Codes from Repeated-Root Cyclic Codes

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Abstract: Symbol-pair codes are proposed to combat pair-errors in symbol-pair read channels. The minimum symbol-pair distance is of significance in determining the error-correcting capability of a symbol-pair code. Maximum distance separable (MDS) symbol-pair codes are optimal in the sense that such codes can achieve the Singleton bound. In this paper, two new classes of MDS symbol-pair codes are proposed utilizing repeated-root cyclic codes over finite fields with odd characteristic. Precisely, these codes poss minimum symbol-pair distance ten or twelve, which is bigger than all the known MDS symbol-pair codes from constacyclic codes.

Keywords: MDS symbol-pair code, AMDS symbol-pair code, minimum symbol-pair distance, constacyclic codes, repeated-root cyclic codes

1 Introduction

In information theory, noisy channels are analyzed generally by dividing the message into independent information units. With the development of modern high-density data storage systems, the reading process may be lower than that of the process used to store the data. Motivated by this situation, a new coding framework named symbol-pair code was proposed by Cassuto and Blaum (2010) to guard against pair-errors over symbol-pair channels in [1]. Cassuto and Blaum firstly studied symbol-pair codes on pair-error correctability conditions, code construction, decoding methods and asymptopic bounds in [1,2]. Shortly afterwards, Cassuto and Litsyn [3] established that codes for correcting pair-errors exist with strictly higher rates compared to codes for the Hamming metric with the same relative distance. Later, researchers

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further investigated symbol-pair codes, including the construction of symbol-pair codes [5, 6, 8, 9, 12, 14, 17–19, 21], some decoding algorithms of symbol-pair codes [15, 20, 25, 27, 28] and the symbol-pair weight distribution of some linear codes [10, 11, 13, 22, 26].

The minimum symbol-pair distance plays an important role in determining the error-correcting capability of a symbol-pair code. Cassuto and Blaum [1] determined that a code C with minimum symbol-pair distance d_p can correct up to $\lfloor \frac{d_p-1}{2} \rfloor$ symbol-pair errors. In 2012, Chee et al. [6] derived a Singleton-type bound on symbol-pair codes. Similar to classical error-correcting codes, the symbol-pair codes achieving the Singleton-type bound are called MDS symbol-pair codes. Recently, the construction of MDS symbol-pair codes has attracted the attention of many researchers. In general, there are two methods to construct MDS symbol-pair codes. The first one is based on linear codes with certain properties, such as MDS codes [5,6] and constacyclic (cyclic) codes [8,17–19,21]. The second method is to construct MDS symbol-pair codes by utilizing interleaving techniques [5,6], Eulerian graphs [5,6], projective geometry [9] and algebraic geometry codes over elliptic curves [9].

In Table 1, we summarize all currently known MDS symbol-pair codes from constacyclic codes. As we can see, most known codes in Table 1 poss a fairly small symbol-pair distance.

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$(n, a_p)_q$	Condition	Reference
$(n, 5)_q$	$n \left \left(q^2 + q + 1 ight) ight.$	[17], [19]
$(n, 6)_q$	$n ig(q^2+1ig)$	[17], [19]
$(n, 6)_q$	$n \mid (q^2 - 1), n \text{ odd or } n \text{ even and } v_2(n) < v_2(q^2 - 1)$	[19]
$(n, 6)_q$	$q \ge 3, n \ge q+4, n \left(q^2 - 1\right)$	[8]
$(lp, 5)_p$	$p \ge 5, l > 2, \gcd(l, p) = 1, l (p - 1)$	[8]
$(p^2 + p, 6)_p$	$p \geq 3$	[18]
$(2p^2 - 2p, 6)_p$	$p \ge 3$	[18]
$(3p,6)_p$	$p \ge 5$	[8]
$(3p, 7)_p$	$p \ge 5$	[8]
$(4p, 7)_p$	$p \equiv 3 (\mathrm{mod}4)$	[18]
$(3p,8)_p$	3 (p - 1)	[8]
$(3p, 10)_p$	3 (p - 1)	Theorem 1
$(3p, 12)_p$	3 (p-1)	Theorem 2

Table 1: Known MDS symbol-pair codes from constacyclic codes

where q is a power of prime p.

The construction of symbol-pair codes with comparatively large minimum symbol-pair distance

is a very interesting problem. It is shown in [9] that there exist q-ary MDS symbol-pair codes from algebraic geometry codes over elliptic curves with larger minimum symbol-pair distance. But their lengths are bounded by $q + 2\sqrt{q}$. Inspired by the aforementioned works, in this paper, we propose two new classes of p-ary MDS symbol-pair codes with length n = 3p by employing repeated-root cyclic codes. Notably, these codes poss minimum symbol-pair distance 10 or 12, which is bigger than all the known MDS symbol-pair codes from constacyclic codes.

The rest of this paper is organized as follows. In Section 2, we introduce some basic notations and results on symbol-pair codes and constacyclic codes. By means of repeated-root cyclic codes, we investigate MDS symbol-pair codes in Section 3. In Section 4, we make some conclusions.

2 Preliminaries

In this section, we review some basic notations and results on symbol-pair codes and constacyclic codes, which will be used to prove our main results in the sequel.

2.1 Symbol-pair Codes

Let $q = p^m$ and \mathbb{F}_q denote the finite field with q elements, where p is a prime and m is a positive integer. Throughout this paper, let \star be an element in \mathbb{F}_q^* and **0** denotes the all-zero vector. Let n be a positive integer. From now on, we always take the subscripts modulo n. For any vector $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ in \mathbb{F}_q^n , the symbol-pair read vector of \mathbf{x} is

$$\pi(\mathbf{x}) = ((x_0, x_1), (x_1, x_2), \cdots, (x_{n-2}, x_{n-1}), (x_{n-1}, x_0)).$$

Observe that every vector $\mathbf{x} \in \mathbb{F}_q^n$ has a unique pair representation $\pi(\mathbf{x})$. Denote by \mathbb{Z}_n the residue class ring $\mathbb{Z}/n\mathbb{Z}$. Recall that the *Hamming weigh* of \mathbf{x} is

$$w_H(\mathbf{x}) = \left| \left\{ i \in \mathbb{Z}_n \, \middle| \, x_i \neq 0 \right\} \right|.$$

Accordingly, the symbol-pair weight of \mathbf{x} is defined by

$$w_{p}(\mathbf{x}) = \left| \left\{ i \in \mathbb{Z}_{n} \mid (x_{i}, x_{i+1}) \neq (0, 0) \right\} \right|.$$

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$, the *symbol-pair distance* between \mathbf{x} and \mathbf{y} is

$$d_p(\mathbf{x}, \mathbf{y}) = \left| \left\{ i \in \mathbb{Z}_n \mid (x_i, x_{i+1}) \neq (y_i, y_{i+1}) \right\} \right|.$$

A code C is said to have minimum symbol-pair distance d_p if

$$d_{p} = \min \left\{ d_{p} \left(\mathbf{x}, \, \mathbf{y} \right) \, \middle| \, \mathbf{x}, \, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y} \right\}.$$

Elements of C are called *codewords* in C. It is shown in [1,2] that for any $0 < d_H(C) < n$,

$$d_H(\mathcal{C}) + 1 \le d_p(\mathcal{C}) \le 2 \cdot d_H(\mathcal{C}). \tag{1}$$

Similar to classical error-correcting codes, the size of a symbol-pair code satisfies the following Singleton bound.

Lemma 1. ([5]) Let $q \ge 2$ and $2 \le d_p \le n$. If C is a symbol-pair code with length n and minimum symbol-pair distance d_p , then $|C| \le q^{n-d_p+2}$.

The symbol-pair code achieving the Singleton bound is called a maximum distance separable (MDS) symbol-pair code. For a linear code of length n, dimension k and minimum symbol-pair distance d_p , if $d_p = n - k + 1$, then it is called an almost maximum distance separable (AMDS) symbol-pair code.

2.2 Constacyclic Codes

In this subsection, we review some basic concepts of constacyclic codes. For any $\eta \in \mathbb{F}_q^*$, the η -constacyclic shift τ_η on \mathbb{F}_q^n is defined as

$$\tau_{\eta}(x_0, x_1, \cdots, x_{n-1}) = (\eta x_{n-1}, x_0, \cdots, x_{n-2}).$$

A linear code C is an η -constacyclic code if $\tau_{\eta}(\mathbf{c}) \in C$ for any codeword $\mathbf{c} \in C$. An η -constacyclic code is called a *cyclic code* if $\eta = 1$ and a *negacyclic code* if $\eta = -1$. Note that each codeword $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$ can be identified with a polynomial

$$c(x) = c_0 + c_1 x + \dots + c_{n-1} x^{n-1}.$$

In this paper, we always regard the codeword **c** in C as the corresponding polynomial c(x). Indeed, a linear code C is an η -constacyclic code if and only if it is an ideal of the principle ideal ring $\mathbb{F}_q[x]/\langle x^n - \eta \rangle$. Consequently, there is a unique monic polynomial $g(x) \in \mathbb{F}_q[x]$ with $g(x) \mid (x^n - \eta)$ and

$$\mathcal{C} = \langle g(x) \rangle = \left\{ f(x) g(x) \pmod{x^n - \eta} \mid f(x) \in \mathbb{F}_q[x] \right\}.$$

We refer g(x) as the generator polynomial of C and the dimension of C is $n - \deg(g(x))$.

An η -constacyclic code of length n over \mathbb{F}_q is called a *simple-root* constacyclic code if n and p are relatively co-prime and a *repeated-root* constacyclic code if $p \mid n$. Note that simple-root constacyclic codes can be characterized by their defining sets. Furthermore, the BCH bound and the Hartmann-Tzeng bound for simple-root cyclic codes can be obtained by calculating

the consecutive roots of the generator polynomial [16,23]. However, repeated-root cyclic codes cannot be directly characterized by sets of zeros.

Let $\mathcal{C} = \langle g(x) \rangle$ be a repeated-root cyclic code of length lp^e over \mathbb{F}_q with gcd(l, p) = 1 and

$$g(x) = \prod_{i=1}^{r} m_i(x)^{e_i}$$

the factorization of g(x) into distinct monic irreducible polynomials $m_i(x) \in \mathbb{F}_q[x]$ of multiplicity e_i . For any $0 \leq t \leq p^e - 1$, we denote $\overline{\mathcal{C}}_t$ the simple-root cyclic code of length l over \mathbb{F}_q with generator polynomial

$$\overline{g}_t(x) = \prod_{1 \leq i \leq r, \, e_i > t} m_i(x)$$

If this product turns out to be $x^{l} - 1$, then \overline{C}_{t} contains only the all-zero codeword and we set $d_{H}(\overline{C}_{t}) = \infty$. If all $e_{i}(1 \leq i \leq s)$ satisfy $e_{i} \leq t$, then we set $\overline{g}_{t}(x) = 1$ and $d_{H}(\overline{C}_{t}) = 1$.

The following lemma obtained from [4] indicates that the minimum Hamming distance of C can be derived from $d_H(\overline{C}_t)$, which will be used to determine the minimum Hamming distance of codes in Section 3.

Lemma 2. ([4]) Let C be a repeated-root cyclic code of length lp^e over \mathbb{F}_q , where l and e are positive integers with gcd (l, p) = 1. Then

$$d_H(\mathcal{C}) = \min\left\{P_t \cdot d_H\left(\overline{\mathcal{C}}_t\right) \mid 0 \le t \le p^e - 1\right\}$$
(2)

where

$$P_t = w_H \left((x-1)^t \right) = \prod_i (t_i + 1)$$
(3)

with t_i 's being the coefficients of the radix-p expansion of t.

In the sequel, we recall the result of Lemma 3 in [21], which will be used in Theorem 1.

Lemma 3. ([21]) Let C be a repeated-root cyclic code of length lp^e over \mathbb{F}_q and $c(x) = (x^l - 1)^t v(x)$ a codeword in C with Hamming weight $d_H(C)$, where l and e are positive integers with $gcd(l, p) = 1, 0 \le t \le p^e - 1$ and $(x^l - 1) \nmid v(x)$. Then

$$w_H\left(c(x)\right) = P_t \cdot N_v$$

where P_t is defined as (3) in Lemma 2 and $N_v = w_H (v(x) \mod (x^l - 1))$.

In this paper, we will employ repeated-root cyclic codes to construct new MDS symbol-pair codes. The following two lemmas will be applied in our later proof.

Lemma 4. ([8]) Let C be an $[n, k, d_H]$ constacyclic code over \mathbb{F}_q with $2 \leq d_H < n$. Then we have $d_p(C) \geq d_H + 2$ if and only if C is not an MDS code, i.e., $k < n - d_H + 1$.

Lemma 5. ([8]) Let C be an $[lp^e, k, d_H]$ repeated-root cyclic code over \mathbb{F}_q and g(x) the generator polynomial of C, where gcd (l, p) = 1 and l, e > 1. If $d_H(C)$ is prime and one of the following two conditions is satisfied

(1)
$$l < d_H(\mathcal{C}) < lp^e - k;$$

(2) $x^l - 1$ is a divisor of g(x) and $2 < d_H(\mathcal{C}) < lp^e - k$,

then $d_p(\mathcal{C}) \geq d_H(\mathcal{C}) + 3$.

3 Constructions of MDS Symbol-Pair Codes

In this section, for n = 3p, we propose two new classes of MDS symbol-pair codes from repeated-root cyclic codes by analyzing the system of certain linear equations over \mathbb{F}_p . Interestingly, the minimum symbol-pair distance of these codes ranges in $\{10, 12\}$, which is bigger than all the known codes in Table 1. For preparation, we define the following notations.

Let n and A_i be positive integers for any $1 \le i \le n$ and

$$\mathcal{V}(A_1, \cdots, A_n) = (A_1 \mod 3, \cdots, A_n \mod 3).$$

Denote by $\mathcal{CS}(A_1, \dots, A_n) = (a_1, \dots, a_n)$ the rearrangement of $\mathcal{V}(A_1, \dots, A_n)$ with $a_i \leq a_j$ for any i < j. For instance, $\mathcal{CS}(5, 10, 4) = (1, 1, 2)$.

Now we present a class of MDS symbol-pair codes with length 3p and minimum symbol-pair distance 10.

Theorem 1. Let p be an odd prime with 3 | (p-1). Then there exists an MDS $(3p, 10)_p$ symbolpair code.

Proof. Let \mathcal{C} be a repeated-root cyclic code of length 3p over \mathbb{F}_p with generator polynomial

$$g(x) = (x - 1)^4 (x - \omega)^2 (x - \omega^2)^2$$

where ω is a primitive third root of unity in \mathbb{F}_p .

Note that Lemma 2 yields that C is a [3p, 3p - 8, 5] cyclic code. Precisely, recall that $\overline{g}_t(x)$ is the generator polynomial of \overline{C}_t . If $t \in \{0, 1\}$, then $\overline{g}_t(x) = x^3 - 1$ and

$$P_t \cdot d_H\left(\overline{\mathcal{C}}_t\right) = \infty.$$

If t = 2, then $\overline{g}_2(x) = x - 1$ and

$$P_2 \cdot d_H\left(\overline{\mathcal{C}}_2\right) = 3 \cdot 2 = 6.$$

If t = 3, then $\overline{g}_3(x) = x - 1$ and

$$P_3 \cdot d_H\left(\overline{\mathcal{C}}_3\right) = 4 \cdot 2 = 8.$$

If $4 \le t \le p-1$, then $\overline{g}_t(x) = 1$ and

$$P_t \cdot d_H\left(\overline{\mathcal{C}}_t\right) = t + 1 \ge 5.$$

Due to the equality (2), one immediately has $d_H(\mathcal{C}) = 5$.

Since $(x^3 - 1) | g(x)$ and $2 < 5 = d_H(\mathcal{C}) < 3p - (3p - 8) = 8$, by Lemma 5, one gets $d_p(\mathcal{C}) \ge 8$. Suppose that there exists a codeword f(x) in \mathcal{C} with Hamming weight 7 such that f(x) has 7 consecutive nonzero entries. Denote

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6$$

where $f_i \in \mathbb{F}_p^*$ for any $0 \le i \le 6$. It follows that the degree of g(x) is less than or equal to the degree of f(x), i.e., $8 \le 6$, which is impossible. Hence there does not exist a codeword in \mathcal{C} with Hamming weight 7 and symbol-pair weight 8. Similarly, it can be verified that there does not exist a codeword in \mathcal{C} with Hamming weight 8 and symbol-pair weight 9.

In the sequel, we claim that there does not exist a codeword in C with Hamming weight 5 and symbol-pair weight 8 (or 9). Let c(x) be a codeword in C with Hamming weight 5. Assume that c(x) has factorization $c(x) = (x^3 - 1)^t v(x)$, where $0 \le t \le p - 1$, $(x^3 - 1) \nmid v(x)$ and $v(x) = v_0(x^3) + x v_1(x^3) + x^2 v_2(x^3)$. Then by Lemma 3, one can conclude that

$$5 = w_H\left(\left(x^3 - 1\right)^t\right) \cdot w_H\left(v(x) \mod \left(x^3 - 1\right)\right) = (1+t) N_v$$

where $N_v = w_H (v(x) \mod (x^3 - 1))$. It follows that $(N_v, t) = (1, 4)$, which implies that the symbol-pair weight of c(x) cannot be 8 (or 9).

In order to derive that C is an MDS $(3p, 10)_p$ symbol-pair code, we need to prove that there does not exist a codeword c(x) in C with $(w_H(c(x)), w_p(c(x))) = (6, 8), (6, 9)$ or (7, 9).

Firstly, on the contrary, suppose that c(x) is a codeword in C with Hamming weight 6 and symbol-pair weight 8. Then its certain cyclic shift must have the form

$$(\star, \star, \star, \star, \star, 0, \star, 0),$$

 $(\star, \star, \star, \star, 0, \star, \star, 0)$

$$(\star, \star, \star, \mathbf{0}, \star, \star, \star, \mathbf{0})$$
.

Without loss of generality, in this paper, we always suppose that the first coordinate of a codeword is 1.

- For the subcase of $(\star, \star, \star, \star, \star, \mathbf{0}, \star, \mathbf{0})$. Let $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^l$ with $6 \le l \le 3p - 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{V}(l) \in \{0, 1\}$, then by $c(1) = c(\omega) = c(\omega^2) = 0$, one can immediately obtain

$$\begin{cases} 1 + a_1 + a_2 + a_3 + a_4 + a_5 = 0, \\ 1 + a_1 \omega + a_2 \omega^2 + a_3 + a_4 \omega + a_5 \omega^l = 0, \\ 1 + a_1 \omega^2 + a_2 \omega + a_3 + a_4 \omega^2 + a_5 \omega^{2l} = 0 \end{cases}$$

This leads to $a_2 = 0$, a contradiction.

- If $\mathcal{V}(l) = 2$, then $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ indicates $a_3 = 0$, a contradiction.
- For the subcase of $(\star, \star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0})$. Let $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^l + a_5 x^{l+1}$ with $5 \le l \le 3p - 3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{V}(l) \in \{0, 2\}$, then it follows from $c(1) = c(\omega) = c(\omega^2) = 0$ that $a_1 = 0$ or $a_2 = 0$, a contradiction.
 - If $\mathcal{V}(l) = 1$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one immediately has $a_3 = 0$, a contradiction.
- For the subcase of $(\star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0})$. Let $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^l + a_4 x^{l+1} + a_5 x^{l+2}$ with $4 \le l \le 3p - 4$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$. Then for any $4 \le l \le 3p - 4$, by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can derive $a_i = 0$ for some $3 \le i \le 5$, a contradiction.

Secondly, assume that there exists a codeword c(x) in C with Hamming weight 6 and symbolpair weight 9. There are three subcases to be considered:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^{l_1} + a_5 x^{l_2}$ with $5 \le l_1 < l_2 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $CS(l_1, l_2) = (1, 2)$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can obtain that $a_3 = 0$, a contradiction.
 - If $\mathcal{CS}(l_1, l_2) \neq (1, 2)$, then $c(1) = c(\omega) = c(\omega^2) = 0$ indicates that $a_1 = 0$ or $a_2 = 0$, a contradiction.

or

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_1+1} + a_5 x^{l_2}$ with $4 \le l_1 < l_2 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{V}(l_1, l_2) \in \{(0, 2), (1, 0), (2, 1)\}$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can immediately derive $a_i = 0$ for some $3 \le i \le 5$, a contradiction.
 - If $\mathcal{V}(l_1, l_2) \notin \{(0, 2), (1, 0), (2, 1)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ implies that 1 = 0 or $a_i = 0$ for some $i \in \{1, 2\}$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^{l_1} + a_3 x^{l_1+1} + a_4 x^{l_2} + a_5 x^{l_2+1}$ with $3 \le l_1 < l_2 \le 3p 3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{CS}(l_1, l_2) = (0, 0)$, then one can deduce that $p \nmid l_1 l_2 (l_2 l_1)$. By $c(1) = c(\omega) = c(\omega^2) = 0$, one immediately gets

$$1 + a_2 + a_4 = a_1 + a_3 + a_5 = 0$$

follows from $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ that

 It

$$l_1 a_2 + l_2 a_4 = l_1 a_3 + l_2 a_5 = 0$$

Then one can conclude that $a_2 + a_3 = a_4 + a_5 = 0$ due to $c^{(2)}(1) = 0$. Hence the fact $c^{(3)}(1) = 0$ indicates that $l_2(l_2 - l_1)a_5 = 0$, a contradiction.

- If $CS(l_1, l_2) = (1, 2)$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can immediately deduce $a_3 = 0$ or $a_5 = 0$, a contradiction.
- If $CS(l_1, l_2) \notin \{(0, 0), (1, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ yields that 1 = 0 or $a_i = 0$ for some $1 \le i \le 5$, a contradiction.

Thirdly, suppose that c(x) is a codeword in C with Hamming weight 7 and symbol-pair weight 9. Then we ought to discuss the following three subcases:

- For the subscase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^l$ with $7 \le l \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $\mathcal{V}(l) = 0$, then by $c(1) = c(\omega) = c(\omega^2) = 0$, one can immediately obtain $a_1 + a_4 = 0$. It follows from $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ that $a_1 + 4a_4 = 0$. Hence $3a_4 = 0$, a contradiction.
 - If $\mathcal{V}(l) \in \{1, 2\}$, then $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ implies that $a_3 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^l + a_6 x^{l+1}$ with $6 \le l \le 3p-3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.

- If $\mathcal{V}(l) \in \{0, 1\}$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one has $a_2 = 0$ or $a_3 = 0$, a contradiction.
- If $\mathcal{V}(l) = 2$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ indicates that $a_4 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^l + a_5 x^{l+1} + a_6 x^{l+2}$ with $5 \le l \le 3p-4$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$. For any $5 \le l \le 3p-4$, $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ yields that $a_i = 0$ for some $4 \le i \le 6$, a contradiction.

Consequently, C is an MDS $(3p, 10)_p$ symbol-pair code. This completes the proof.

In what follows, we construct a class of MDS symbol-pair codes with length 3p and minimum symbol-pair distance 12, which is the maximum minimum symbol-pair distance for all known MDS symbol-pair codes from constacyclic codes.

Theorem 2. Let p be an odd prime with 3 | (p-1). Then there exists an MDS $(3p, 12)_p$ symbolpair code.

Proof. Let \mathcal{C} be a repeated-root cyclic code of length 3p over \mathbb{F}_p with generator polynomial

$$g(x) = (x - 1)^5 (x - \omega)^3 (x - \omega^2)^2$$

where ω is a primitive third root of unity in \mathbb{F}_p . It follows from Lemma 2 that \mathcal{C} is a [3p, 3p-10, 6] code. Since \mathcal{C} is not MDS, by Lemma 4, one can get $d_p(\mathcal{C}) \geq 8$. With a similar manner to the proof of Theorem 1, it can be verified that there does not exist a codeword c(x) in \mathcal{C} with $(w_H(c(x)), w_p(c(x))) = (9, 10)$ or (10, 11). Besides, the proof of Theorem 1 yields that there does not exist a codeword c(x) in \mathcal{C} with $(w_H(c(x)), w_p(c(x))) = (6, 8), (6, 9), (7, 8), (7, 9)$ or (8, 9).

To determine that C is an MDS $(3p, 12)_p$ symbol-pair code, it is sufficient to derive that there does not exist a codeword c(x) in C with $(w_H(c(x)), w_p(c(x))) = (6, 10), (6, 11), (7, 10),$ (7, 11), (8, 10), (8, 11) or (9, 11).

Firstly, we suppose that c(x) is a codeword in C with Hamming weight 6 and symbol-pair weight 10. Without loss of generality, we just consider the following two subcases:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_2} + a_5 x^{l_3}$ with $4 \le l_1 < l_2 < l_3 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{CS}(l_1, l_2, l_3) = (0, 1, 2)$, then it can be checked that by $c(1) = c(\omega) = c(\omega^2) = 0$, one can immediately get 1 = 0 or $a_i = 0$ for some $1 \le i \le 5$, a contradiction.

- If $\mathcal{CS}(l_1, l_2, l_3) \neq (0, 1, 2)$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ yields that $a_i = 0$ for some $1 \le i \le 5$, a contradiction.

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^{l_1} + a_3 x^{l_1+1} + a_4 x^{l_2} + a_5 x^{l_3}$ with $3 \le l_1 < l_2 < l_3 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.
 - If $\mathcal{V}(l_1, l_2, l_3) \in \{(0, 0, 0), (0, 1, 1), (0, 2, 2), (1, 0, 2), (1, 2, 0), (2, 1, 2), (2, 2, 1)\},$ then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, it follows that $a_i = 0$ for some $1 \le i \le 5$, a contradiction.
 - If $\mathcal{V}(l_1, l_2, l_3) \in \{(0, 1, 0), (0, 0, 1)\}$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = c^{(2)}(1) = c^{(2)}(\omega) = 0$, one can immediately obtain $a_4 = 0$ or $a_5 = 0$ since $p \nmid l_3(l_3 l_1)$ or $p \nmid l_2(l_2 l_1)$. This leads to a contradiction.
 - For other conditions, it can be verified that $c(1) = c(\omega) = c(\omega^2) = 0$ indicates 1 = 0 or $a_i = 0$ for some $1 \le i \le 5$, which is impossible.

Secondly, we assume that c(x) is a codeword in C with Hamming weight 6 and symbol-pair weight 11. Let $c(x) = 1 + a_1 x + a_2 x^{l_1} + a_3 x^{l_2} + a_4 x^{l_3} + a_5 x^{l_4}$ with $3 \le l_1 < l_2 < l_3 < l_4 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 5$.

- If $\mathcal{CS}(l_1, l_2, l_3, l_4) \in \{(0, 0, 0, 1), (0, 1, 1, 1), (0, 1, 2, 2)\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ indicates that $a_i = 0$ for some $2 \le i \le 5$, a contradiction.
- If $\mathcal{CS}(l_1, l_2, l_3, l_4) = (0, 0, 1, 1)$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = c^{(2)}(1) = c^{(2)}(\omega) = 0$, one has $a_i = 0$ for some $2 \le i \le 5$, a contradiction.
- If $\mathcal{CS}(l_1, l_2, l_3, l_4) \notin \{(0, 1, 1, 1), (0, 1, 2, 2), (0, 0, 1, 1), (0, 0, 0, 1)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ yields that 1 = 0 or $a_i = 0$ for some $1 \le i \le 5$, a contradiction.

Thirdly, we suppose that c(x) is a codeword in C with Hamming weight 7 and symbol-pair weight 10. There are five subcases to be discussed:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^{l_1} + a_6 x^{l_2}$ with $6 \le l_1 < l_2 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $CS(l_1, l_2) \in \{(0, 0), (0, 1), (1, 1)\}$, then by $c(1) = c(\omega) = c(\omega^2) = 0$, one can immediately deduce $a_2 = 0$, a contradiction.
 - If $\mathcal{CS}(l_1, l_2) \in \{(1, 2), (2, 2)\}$, then $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ yields that $a_3 = 0$, a contradiction.

- If $\mathcal{CS}(l_1, l_2) = (0, 2)$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one has $a_4 = 0$, a contradiction.

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^{l_1} + a_5 x^{l_1+1} + a_6 x^{l_2}$ with $6 \le (l_1+1) < l_2 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $\mathcal{V}(l_1, l_2) \in \{(0, 0), (0, 1), (2, 0), (2, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ implies that $a_1 = 0$ or $a_2 = 0$, a contradiction.
 - If $\mathcal{V}(l_1, l_2) \in \{(0, 2), (1, 0), (1, 1), (1, 2), (2, 1)\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can immediately get $a_i = 0$ for some $4 \le i \le 6$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_1+1} + a_5 x^{l_1+2} + a_6 x^{l_2}$ with $6 \le (l_1+2) < l_2 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$. For any $6 \le (l_1+2) < l_2 \le 3p-2$, it follows from $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ that $a_4 = 0$ or $a_5 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_1+1} + a_5 x^{l_2} + a_6 x^{l_2+1}$ with $5 \le (l_1+1) < l_2 \le 3p-3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $CS(l_1, l_2) \in \{(0, 0), (1, 1), (2, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ indicates that 1 = 0 or $a_i = 0$ for some $1 \le i \le 2$, a contradiction.
 - If $\mathcal{CS}(l_1, l_2) \in \{(0, 1), (0, 2), (1, 2)\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can conclude that $a_5 = 0$ or $a_6 = 0$, a contradiction.

Fourthly, we assume that c(x) is a codeword in C with Hamming weight 7 and symbol-pair weight 11. There are three subcases to be considered:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^{l_1} + a_5 x^{l_2} + a_6 x^{l_3}$ with $5 \le l_1 < l_2 < l_3 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $\mathcal{CS}(l_1, l_2, l_3) \in \{(0, 1, 2), (1, 1, 2), (1, 2, 2)\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can immediately have $a_i = 0$ for some $1 \le i \le 6$, a contradiction.
 - If $\mathcal{CS}(l_1, l_2, l_3) \notin \{(0, 1, 2), (1, 1, 2), (1, 2, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ implies that $a_1 = 0$ or $a_2 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_1+1} + a_5 x^{l_2} + a_6 x^{l_3}$ with $5 \le (l_1+1) < l_2 < l_3 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.

- If $\mathcal{V}(l_1, l_2, l_3) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)\}$, then by $c(1) = c(\omega) = c(\omega^2) = 0$, one can derive that 1 = 0, a contradiction.
- If $\mathcal{V}(l_1, l_2, l_3) \notin \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ induces that $a_i = 0$ for some $1 \le i \le 6$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^{l_1} + a_3 x^{l_1+1} + a_4 x^{l_2} + a_5 x^{l_2+1} + a_6 x^{l_3}$ with $4 \le (l_1 + 1) < l_2 < l_3 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 6$.
 - If $\mathcal{V}(l_1, l_2, l_3) \in \{(1, 1, 1), (1, 1, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = 0$ yields 1 = 0, a contradiction.
 - If $\mathcal{V}(l_1, l_2, l_3) \in \{(0, 0, 0), (0, 0, 1)\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = c^{(2)}(1) = c^{(2)}(\omega) = 0$, one can deduce that $a_4 = 0$ or $a_5 = 0$, a contradiction.
 - If $\mathcal{V}(l_1, l_2, l_3) \notin \{(0, 0, 0), (0, 0, 1), (1, 1, 1), (1, 1, 2)\}$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ leads to 1 = 0 or $a_i = 0$ for some $1 \le i \le 6$, a contradiction.

Fifthly, we suppose that c(x) is a codeword in C with Hamming weight 8 and symbol-pair weight 10. There are four subcases to be considered:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^l$ with $8 \le l \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $8 \le l \le 3p-2$, it can be verified that $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ induces that $a_4 = 0$ or $a_5 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^l + a_7 x^{l+1}$ with $7 \le l \le 3p 3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $7 \le l \le 3p 3$, $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ implies that $a_i = 0$ for some $3 \le i \le 5$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^l + a_6 x^{l+1} + a_7 x^{l+2}$ with $6 \le l \le 3p 4$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $6 \le l \le 3p 4$, it follows from $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ that $a_i = 0$ for some $5 \le i \le 7$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^l + a_5 x^{l+1} + a_6 x^{l+2} + a_7 x^{l+3}$ with $5 \le l \le 3p - 5$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $5 \le l \le 3p - 5$, by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can derive that $a_5 = 0$ or $a_6 = 0$, a contradiction.

Sixthly, we suppose that c(x) is a codeword in C with Hamming weight 8 and symbol-pair weight 11. Without loss of generality, it suffices to consider the following five subcases:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^{l_1} + a_7 x^{l_2}$ with $7 \le l_1 < l_2 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$.
 - If $\mathcal{CS}(l_1, l_2) = (1, 2)$, then it can be verified that by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can obtain $a_3 = 0$, a contradiction.
 - If $CS(l_1, l_2) \neq (1, 2)$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ indicates $a_4 = 0$ or $a_5 = 0$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^{l_1} + a_6 x^{l_1+1} + a_7 x^{l_2}$ with $7 \le (l_1 + 1) < l_2 \le 3p 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$.
 - If $\mathcal{V}(l_1, l_2) = (1, 2)$, then by $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can get $a_3 = 0$, a contradiction.
 - If $\mathcal{V}(l_1, l_2) \neq (1, 2)$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ implies $a_i = 0$ for some $1 \le i \le 7$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^{l_1} + a_5 x^{l_1+1} + a_6 x^{l_1+2} + a_7 x^{l_2}$ with $7 \le (l_1+2) < l_2 \le 3p-2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $7 \le (l_1+2) < l_2 \le 3p-2$, it can be verified that by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can deduce $a_i = 0$ for some $4 \le i \le 6$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^{l_1} + a_5 x^{l_1+1} + a_6 x^{l_2} + a_7 x^{l_2+1}$ with $6 \le (l_1 + 1) < l_2 \le 3p 3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$.
 - If $\mathcal{V}(l_1, l_2) = (1, 1)$, then $c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ yields $a_3 = 0$, a contradiction.
 - If $\mathcal{V}(l_1, l_2) \neq (1, 1)$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can derive $a_i = 0$ for some $1 \le i \le 7$, a contradiction.
- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^{l_1} + a_4 x^{l_1+1} + a_5 x^{l_1+2} + a_6 x^{l_2} + a_7 x^{l_2+1}$ with $6 \le (l_1+2) < l_2 \le 3p-3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 7$. For any $6 \le (l_1+2) < l_2 \le 3p-3$, it follows from $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ that $a_i = 0$ for some $1 \le i \le 5$, a contradiction.

Finally, we assume that c(x) is a codeword in C with Hamming weight 9 and symbol-pair weight 11. There are four subcases to be discussed:

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^l$ with $9 \le l \le 3p - 2$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 8$.
 - If $\mathcal{V}(l) \in \{0, 1\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can obtain $a_5 = 0$, a contradiction.
 - If $\mathcal{V}(l) = 2$, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ induces

$$\begin{cases} 1 + a_3 + a_6 = a_1 + a_4 + a_7 = a_2 + a_5 + a_8 = 0, \\ 3a_3 + 6a_6 = a_1 + 4a_4 + 7a_7 = 2a_2 + 5a_5 + la_8 = 0 \end{cases}$$

which indicates that

$$a_2 = \frac{l-5}{3}a_8, \quad a_3 = -2, \quad a_4 = -2a_7, \quad a_5 = -\frac{l-2}{3}a_8, \quad a_6 = 1.$$
 (4)

The fact $c^{(2)}(1) = c^{(2)}(\omega) = 0$ indicates

$$\begin{cases} 2a_2 + 20a_5 + l(l-1)a_8 + 6a_3 + 30a_6 + 12a_4 + 42a_7 = 0, \\ 2a_2 + 20a_5 + l(l-1)a_8 + (6a_3 + 30a_6)\omega + (12a_4 + 42a_7)\omega^2 = 0 \end{cases}$$

which yields

$$a_7 = \omega$$
 and $a_8 = \frac{18\,\omega^2}{(l-2)\,(l-5)}$

due to (4). It follows from $c^{(3)}(1) = 0$ that

$$6 a_3 + 24 a_4 + 60 a_5 + 120 a_6 + 210 a_7 + l (l-1) (l-2) a_8 = 0$$

which yields $l = 2 - 3 \omega^2$. By $c^{(4)}(1) = 0$, one can get

$$24 a_4 + 120 a_5 + 360 a_6 + 840 a_7 + l (l-1) (l-2) (l-3) a_8 = 0.$$

This implies $(l^2 + l - 12) \omega + 24 = 0$, which leads to

$$\left(\left(2 - 3\,\omega^2 \right)^2 + 2 - 3\,\omega^2 - 12 \right) \omega + 24 = 0.$$

Thus $-5\omega = 0$, a contradiction.

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^l + a_8 x^{l+1}$ with $8 \le l \le 3p - 3$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 8$.
 - If $\mathcal{V}(l) \in \{0, 2\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can conclude $a_4 = 0$ or $a_5 = 0$, a contradiction.

 $- \text{ If } \mathcal{V}(l) = 1, \text{ then } c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0 \text{ implies}$ $a_2 = \frac{l-4}{3}a_8, \quad a_3 = -2, \quad a_4 = -\frac{l-1}{3}a_7, \quad a_5 = -\frac{l-1}{3}a_8, \quad a_6 = 1.$ (5)

It can be verified by (5) and $c^{(2)}(1) = c^{(2)}(\omega) = 0$ that

$$a_7 = \frac{18\,\omega}{(l-1)\,(l-4)}$$
 and $a_8 = \frac{18\,\omega^2}{(l-1)\,(l-4)}$

According to $c^{(3)}(1) = 0$, one can obtain $l = -\omega^2 - 4\omega$. Hence $c^{(4)}(1) = 0$ yields $3\omega^2 = 0$, a contradiction.

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^l + a_7 x^{l+1} + a_8 x^{l+2}$ with $7 \le l \le 3p - 4$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 8$.
 - If $\mathcal{V}(l) = 0$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one can derive

$$a_2 = \frac{l-3}{3}a_8, \quad a_3 = -\frac{l}{l-3}, \quad a_4 = -\frac{l}{3}a_7, \quad a_5 = -\frac{l}{3}a_8, \quad a_6 = \frac{3}{l-3}.$$
 (6)

By (6) and $c^{(2)}(1) = c^{(2)}(\omega) = 0$, one has

$$a_7 = \frac{3\omega}{l-3}$$
 and $a_8 = \frac{3\omega^2}{l-3}$

Then $c^{(3)}(1) = 0$ implies

$$6a_3 + 24a_4 + 60a_5 + l(l-1)(l-2)a_6 + l(l-1)(l+1)a_7 + l(l+1)(l+2)a_8 = 0$$

which yields $\omega^2 = \omega$, a contradiction.

- If
$$\mathcal{V}(l) = 1$$
, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ indicates
 $a_2 = \frac{l-4}{3}a_7, \quad a_3 = -\frac{l+2}{l-1}, \quad a_4 = -\frac{l-1}{3}a_6, \quad a_5 = -\frac{l-1}{3}a_7, \quad a_8 = \frac{3}{l-1}.$ (7)

It follows from (7) and $c^{(2)}(1) = c^{(2)}(\omega) = 0$ that

$$a_6 = \frac{3(l+2)\omega}{(l-1)(l-4)}$$
 and $a_7 = \frac{3(l+2)\omega^2}{(l-1)(l-4)}$.

By $c^{(3)}(1) = 0$, one can deduce that $3\omega^2 = 0$, a contradiction.

- If $\mathcal{V}(l) = 2$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one has

$$a_2 = \frac{l-5}{3}a_6, a_3 = -\frac{l+1}{3}a_7, a_4 = -\frac{l+1}{3}a_8, a_5 = -\frac{l-2}{3}a_6, a_7 = \frac{3}{l-2}.$$
 (8)

Due to (8) and $c^{(2)}(1) = c^{(2)}(\omega) = 0$, one can immediately obtain that

$$a_6 = \frac{3(l+1)\omega^2}{(l-2)(l-5)}$$
 and $a_8 = \frac{3\omega}{l-2}$

Thus $c^{(3)}(1) = 0$ leads to 3 = 0, a contradiction.

- For the subcase of $c(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^l + a_6 x^{l+1} + a_7 x^{l+2} + a_8 x^{l+3}$ with $6 \le l \le 3p - 5$ and $a_i \in \mathbb{F}_p^*$ for any $1 \le i \le 8$.
 - If $\mathcal{V}(l) \in \{0, 1\}$, then by $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$, one gets $a_6 = 0$ or $a_7 = 0$, a contradiction.

If
$$\mathcal{V}(l) = 2$$
, then $c(1) = c(\omega) = c(\omega^2) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(1)}(\omega^2) = 0$ induces
 $a_2 = \frac{3}{l-2}a_8, a_3 = -\frac{l+1}{3}a_6, a_4 = -\frac{l+1}{3}a_7, a_5 = -\frac{l+1}{l-2}a_8, a_6 = \frac{3}{l-2}.$ (9)

By (9) and $c^{(2)}(1) = c^{(2)}(\omega) = 0$, one can derive

$$a_7 = \frac{3\omega}{l-2}$$
 and $a_8 = \omega^2$.

It follows from $c^{(3)}(1) = 0$ that $l = -3\omega^2 - 1$. Then $c^{(4)}(1) = 0$ yields $l\omega + 2 = -\omega - 1 = 0$, a contradiction.

As a result, C is an MDS $(3p, 12)_p$ symbol-pair code. The desired result follows.

Remark 1. Note that if C is a repeated-root cyclic code of length 3p over \mathbb{F}_p with generator polynomial

$$g(x) = (x - 1)^4 (x - \omega)^3 (x - \omega^2)^2$$

where ω is a primitive third root of unity in \mathbb{F}_p . Due to Theorem 1, we can conclude that \mathcal{C} is an AMDS $(3p, 10)_p$ symbol-pair code. Indeed, by Lemma 2, one can immediately get $d_H(\mathcal{C}) =$ 5. Since $(x^3 - 1) | g(x)$ and $2 < 5 = d_H(\mathcal{C}) < 3p - (3p - 9) = 9$, Lemma 5 indicates that $d_p(\mathcal{C}) \geq 8$. It follows from the proof of Theorem 1 that there does not exist a codeword c(x) in \mathcal{C} with $(w_H(c(x)), w_p(c(x))) = (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)$ or (8, 9). Therefore, the inequality (1) shows that \mathcal{C} is an AMDS $(3p, 10)_p$ symbol-pair code.

In what follows, we present two examples to illustrate the result in Theorems 1 and 2.

Example 1. (1) Let C be a repeated-root cyclic code of length 21 over \mathbb{F}_7 with generator polynomial

$$g(x) = (x-1)^4 (x-2)^2 (x-2^2)^2.$$

By the computation software MAGMA, it can be verified that C is a [21, 13, 5] code and the minimum symbol-pair distance of C is 10, which coincides with our result in Theorem 1.

(2) Let C be a repeated-root cyclic code of length 21 over \mathbb{F}_7 with generator polynomial

$$g(x) = (x-1)^5 (x-2)^3 (x-2^2)^2$$

MAGMA experiments yield that C is a [21, 11, 6] code and the minimum symbol-pair distance of C is 12, which is consistent with our result in Theorem 2.

4 Conclusions

In this paper, for n = 3p, we construct two new classes of MDS symbol-pair codes over \mathbb{F}_p with p an odd prime by employing repeated-root cyclic codes:

- [3p, 3p 8, 5] code with $d_p = 10;$
- [3p, 3p 10, 6] code with $d_p = 12$.

As mentioned in Table 1, these codes poss minimum symbol-pair distance bigger than all the known MDS symbol-pair codes from constacyclic codes. Note that alongside with larger minimum symbol-pair distance, much more cases need to be considered, which has not been explored.

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