# MDS Symbol-Pair Codes from Repeated-Root Cyclic Codes 

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#### Abstract

Symbol-pair codes are proposed to combat pair-errors in symbol-pair read channels. The minimum symbol-pair distance is of significance in determining the errorcorrecting capability of a symbol-pair code. Maximum distance separable (MDS) symbolpair codes are optimal in the sense that such codes can achieve the Singleton bound. In this paper, two new classes of MDS symbol-pair codes are proposed utilizing repeated-root cyclic codes over finite fields with odd characteristic. Precisely, these codes poss minimum symbol-pair distance ten or twelve, which is bigger than all the known MDS symbol-pair codes from constacyclic codes.


Keywords: MDS symbol-pair code, AMDS symbol-pair code, minimum symbol-pair distance, constacyclic codes, repeated-root cyclic codes

## 1 Introduction

In information theory, noisy channels are analyzed generally by dividing the message into independent information units. With the development of modern high-density data storage systems, the reading process may be lower than that of the process used to store the data. Motivated by this situation, a new coding framework named symbol-pair code was proposed by Cassuto and Blaum (2010) to guard against pair-errors over symbol-pair channels in [1]. Cassuto and Blaum firstly studied symbol-pair codes on pair-error correctability conditions, code construction, decoding methods and asymptopic bounds in [1,2]. Shortly afterwards, Cassuto and Litsyn [3] established that codes for correcting pair-errors exist with strictly higher rates compared to codes for the Hamming metric with the same relative distance. Later, researchers

[^0]further investigated symbol-pair codes, including the construction of symbol-pair codes [5, 6, [8, 9, 12, [14, 17, 19, [21], some decoding algorithms of symbol-pair codes [15, [20, [25, 27, [28] and the symbol-pair weight distribution of some linear codes [10, 11, 13, 22, 26].

The minimum symbol-pair distance plays an important role in determining the error-correcting capability of a symbol-pair code. Cassuto and Blaum [1] determined that a code $\mathcal{C}$ with minimum symbol-pair distance $d_{p}$ can correct up to $\left\lfloor\frac{d_{p}-1}{2}\right\rfloor$ symbol-pair errors. In 2012, Chee et al. [6] derived a Singleton-type bound on symbol-pair codes. Similar to classical error-correcting codes, the symbol-pair codes achieving the Singleton-type bound are called MDS symbol-pair codes. Recently, the construction of MDS symbol-pair codes has attracted the attention of many researchers. In general, there are two methods to construct MDS symbol-pair codes. The first one is based on linear codes with certain properties, such as MDS codes [5,6] and constacyclic (cyclic) codes [8, 17-19, 21]. The second method is to construct MDS symbol-pair codes by utilizing interleaving techniques [5, [6], Eulerian graphs [5, [6], projective geometry [9] and algebraic geometry codes over elliptic curves 9 .

In Table [1, we summarize all currently known MDS symbol-pair codes from constacyclic codes. As we can see, most known codes in Table $\mathbb{1}$ poss a fairly small symbol-pair distance.

Table 1: Known MDS symbol-pair codes from constacyclic codes

| $\left(n, d_{p}\right)_{q}$ | Condition | Reference |
| :---: | :---: | :---: |
| $(n, 5)_{q}$ | $n \mid\left(q^{2}+q+1\right)$ | $[17],[19]$ |
| $(n, 6)_{q}$ | $n \mid\left(q^{2}+1\right)$ | $[17],[19]$ |
| $(n, 6)_{q}$ | $n \mid\left(q^{2}-1\right), n$ odd or $n$ even and $v_{2}(n)<v_{2}\left(q^{2}-1\right)$ | $[19]$ |
| $(n, 6)_{q}$ | $q \geq 3, n \geq q+4, n \mid\left(q^{2}-1\right)$ | $[8]$ |
| $(l p, 5)_{p}$ | $p \geq 5, l>2, \operatorname{gcd}(l, p)=1, l \mid(p-1)$ | $[8]$ |
| $\left(p^{2}+p, 6\right)_{p}$ | $p \geq 3$ | $[18]$ |
| $\left(2 p^{2}-2 p, 6\right)_{p}$ | $p \geq 3$ | $[18]$ |
| $(3 p, 6)_{p}$ | $p \geq 5$ | $[8]$ |
| $(3 p, 7)_{p}$ | $p \geq 5$ | $[8]$ |
| $(4 p, 7)_{p}$ | $p \equiv 3(\bmod 4)$ | $[18]$ |
| $(3 p, 8)_{p}$ | $3 \mid(p-1)$ | $[8]$ |
| $(3 p, 10)_{p}$ | $3 \mid(p-1)$ | Theorem[1] |
| $(3 p, 12)_{p}$ | $3 \mid(p-1)$ | Theorem[2] |

where $q$ is a power of prime $p$.
The construction of symbol-pair codes with comparatively large minimum symbol-pair distance
is a very interesting problem. It is shown in $[9$ that there exist $q$-ary MDS symbol-pair codes from algebraic geometry codes over elliptic curves with larger minimum symbol-pair distance. But their lengths are bounded by $q+2 \sqrt{q}$. Inspired by the aforementioned works, in this paper, we propose two new classes of $p$-ary MDS symbol-pair codes with length $n=3 p$ by employing repeated-root cyclic codes. Notably, these codes poss minimum symbol-pair distance 10 or 12 , which is bigger than all the known MDS symbol-pair codes from constacyclic codes.

The rest of this paper is organized as follows. In Section 2, we introduce some basic notations and results on symbol-pair codes and constacyclic codes. By means of repeated-root cyclic codes, we investigate MDS symbol-pair codes in Section 3. In Section 4, we make some conclusions.

## 2 Preliminaries

In this section, we review some basic notations and results on symbol-pair codes and constacyclic codes, which will be used to prove our main results in the sequel.

### 2.1 Symbol-pair Codes

Let $q=p^{m}$ and $\mathbb{F}_{q}$ denote the finite field with $q$ elements, where $p$ is a prime and $m$ is a positive integer. Throughout this paper, let $\star$ be an element in $\mathbb{F}_{q}^{*}$ and $\mathbf{0}$ denotes the all-zero vector. Let $n$ be a positive integer. From now on, we always take the subscripts modulo $n$. For any vector $\mathbf{x}=\left(x_{0}, x_{1}, \cdots, x_{n-1}\right)$ in $\mathbb{F}_{q}^{n}$, the symbol-pair read vector of $\mathbf{x}$ is

$$
\pi(\mathbf{x})=\left(\left(x_{0}, x_{1}\right),\left(x_{1}, x_{2}\right), \cdots,\left(x_{n-2}, x_{n-1}\right),\left(x_{n-1}, x_{0}\right)\right)
$$

Observe that every vector $\mathbf{x} \in \mathbb{F}_{q}^{n}$ has a unique pair representation $\pi(\mathbf{x})$. Denote by $\mathbb{Z}_{n}$ the residue class ring $\mathbb{Z} / n \mathbb{Z}$. Recall that the Hamming weigh of $\mathbf{x}$ is

$$
w_{H}(\mathbf{x})=\left|\left\{i \in \mathbb{Z}_{n} \mid x_{i} \neq 0\right\}\right| .
$$

Accordingly, the symbol-pair weight of $\mathbf{x}$ is defined by

$$
w_{p}(\mathbf{x})=\left|\left\{i \in \mathbb{Z}_{n} \mid\left(x_{i}, x_{i+1}\right) \neq(0,0)\right\}\right| .
$$

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_{q}^{n}$, the symbol-pair distance between $\mathbf{x}$ and $\mathbf{y}$ is

$$
d_{p}(\mathbf{x}, \mathbf{y})=\left|\left\{i \in \mathbb{Z}_{n} \mid\left(x_{i}, x_{i+1}\right) \neq\left(y_{i}, y_{i+1}\right)\right\}\right| .
$$

A code $\mathcal{C}$ is said to have minimum symbol-pair distance $d_{p}$ if

$$
d_{p}=\min \left\{d_{p}(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}\right\} .
$$

Elements of $\mathcal{C}$ are called codewords in $\mathcal{C}$. It is shown in [1, 2] that for any $0<d_{H}(\mathcal{C})<n$,

$$
\begin{equation*}
d_{H}(\mathcal{C})+1 \leq d_{p}(\mathcal{C}) \leq 2 \cdot d_{H}(\mathcal{C}) . \tag{1}
\end{equation*}
$$

Similar to classical error-correcting codes, the size of a symbol-pair code satisfies the following Singleton bound.

Lemma 1. (5) Let $q \geq 2$ and $2 \leq d_{p} \leq n$. If $\mathcal{C}$ is a symbol-pair code with length $n$ and minimum symbol-pair distance $d_{p}$, then $|\mathcal{C}| \leq q^{n-d_{p}+2}$.

The symbol-pair code achieving the Singleton bound is called a maximum distance separable (MDS) symbol-pair code. For a linear code of length $n$, dimension $k$ and minimum symbol-pair distance $d_{p}$, if $d_{p}=n-k+1$, then it is called an almost maximum distance separable (AMDS) symbol-pair code.

### 2.2 Constacyclic Codes

In this subsection, we review some basic concepts of constacyclic codes. For any $\eta \in \mathbb{F}_{q}^{*}$, the $\eta$-constacyclic shift $\tau_{\eta}$ on $\mathbb{F}_{q}^{n}$ is defined as

$$
\tau_{\eta}\left(x_{0}, x_{1}, \cdots, x_{n-1}\right)=\left(\eta x_{n-1}, x_{0}, \cdots, x_{n-2}\right) .
$$

A linear code $\mathcal{C}$ is an $\eta$-constacyclic code if $\tau_{\eta}(\mathbf{c}) \in \mathcal{C}$ for any codeword $\mathbf{c} \in \mathcal{C}$. An $\eta$-constacyclic code is called a cyclic code if $\eta=1$ and a negacyclic code if $\eta=-1$. Note that each codeword $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in \mathcal{C}$ can be identified with a polynomial

$$
c(x)=c_{0}+c_{1} x+\cdots+c_{n-1} x^{n-1} .
$$

In this paper, we always regard the codeword $\mathbf{c}$ in $\mathcal{C}$ as the corresponding polynomial $c(x)$. Indeed, a linear code $\mathcal{C}$ is an $\eta$-constacyclic code if and only if it is an ideal of the principle ideal ring $\mathbb{F}_{q}[x] /\left\langle x^{n}-\eta\right\rangle$. Consequently, there is a unique monic polynomial $g(x) \in \mathbb{F}_{q}[x]$ with $g(x) \mid\left(x^{n}-\eta\right)$ and

$$
\mathcal{C}=\langle g(x)\rangle=\left\{f(x) g(x)\left(\bmod x^{n}-\eta\right) \mid f(x) \in \mathbb{F}_{q}[x]\right\} .
$$

We refer $g(x)$ as the generator polynomial of $\mathcal{C}$ and the dimension of $\mathcal{C}$ is $n-\operatorname{deg}(g(x))$.
An $\eta$-constacyclic code of length $n$ over $\mathbb{F}_{q}$ is called a simple-root constacyclic code if $n$ and $p$ are relatively co-prime and a repeated-root constacyclic code if $p \mid n$. Note that simple-root constacyclic codes can be characterized by their defining sets. Furthermore, the BCH bound and the Hartmann-Tzeng bound for simple-root cyclic codes can be obtained by calculating
the consecutive roots of the generator polynomial [16, 23]. However, repeated-root cyclic codes cannot be directly characterized by sets of zeros.

Let $\mathcal{C}=\langle g(x)\rangle$ be a repeated-root cyclic code of length $l p^{e}$ over $\mathbb{F}_{q}$ with $\operatorname{gcd}(l, p)=1$ and

$$
g(x)=\prod_{i=1}^{r} m_{i}(x)^{e_{i}}
$$

the factorization of $g(x)$ into distinct monic irreducible polynomials $m_{i}(x) \in \mathbb{F}_{q}[x]$ of multiplicity $e_{i}$. For any $0 \leq t \leq p^{e}-1$, we denote $\overline{\mathcal{C}}_{t}$ the simple-root cyclic code of length $l$ over $\mathbb{F}_{q}$ with generator polynomial

$$
\bar{g}_{t}(x)=\prod_{1 \leq i \leq r, e_{i}>t} m_{i}(x)
$$

If this product turns out to be $x^{l}-1$, then $\overline{\mathcal{C}}_{t}$ contains only the all-zero codeword and we set $d_{H}\left(\overline{\mathcal{C}}_{t}\right)=\infty$. If all $e_{i}(1 \leq i \leq s)$ satisfy $e_{i} \leq t$, then we set $\bar{g}_{t}(x)=1$ and $d_{H}\left(\overline{\mathcal{C}}_{t}\right)=1$.

The following lemma obtained from [4] indicates that the minimum Hamming distance of $\mathcal{C}$ can be derived from $d_{H}\left(\overline{\mathcal{C}}_{t}\right)$, which will be used to determine the minimum Hamming distance of codes in Section 3.

Lemma 2. ([4]) Let $\mathcal{C}$ be a repeated-root cyclic code of length lpe over $\mathbb{F}_{q}$, where $l$ and $e$ are positive integers with $\operatorname{gcd}(l, p)=1$. Then

$$
\begin{equation*}
d_{H}(\mathcal{C})=\min \left\{P_{t} \cdot d_{H}\left(\overline{\mathcal{C}}_{t}\right) \mid 0 \leq t \leq p^{e}-1\right\} \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{t}=w_{H}\left((x-1)^{t}\right)=\prod_{i}\left(t_{i}+1\right) \tag{3}
\end{equation*}
$$

with $t_{i}$ 's being the coefficients of the radix-p expansion of $t$.

In the sequel, we recall the result of Lemma 3 in [21], which will be used in Theorem (1)
Lemma 3. ([21]) Let $\mathcal{C}$ be a repeated-root cyclic code of length lpe over $\mathbb{F}_{q}$ and $c(x)=$ $\left(x^{l}-1\right)^{t} v(x)$ a codeword in $\mathcal{C}$ with Hamming weight $d_{H}(\mathcal{C})$, where $l$ and $e$ are positive integers with $\operatorname{gcd}(l, p)=1,0 \leq t \leq p^{e}-1$ and $\left(x^{l}-1\right) \nmid v(x)$. Then

$$
w_{H}(c(x))=P_{t} \cdot N_{v}
$$

where $P_{t}$ is defined as (3) in Lemma 圆 and $N_{v}=w_{H}\left(v(x) \bmod \left(x^{l}-1\right)\right)$.

In this paper, we will employ repeated-root cyclic codes to construct new MDS symbol-pair codes. The following two lemmas will be applied in our later proof.

Lemma 4. ([8]) Let $\mathcal{C}$ be an $\left[n, k, d_{H}\right]$ constacyclic code over $\mathbb{F}_{q}$ with $2 \leq d_{H}<n$. Then we have $d_{p}(\mathcal{C}) \geq d_{H}+2$ if and only if $\mathcal{C}$ is not an MDS code, i.e., $k<n-d_{H}+1$.

Lemma 5. ([8]) Let $\mathcal{C}$ be an $\left[l p^{e}, k, d_{H}\right]$ repeated-root cyclic code over $\mathbb{F}_{q}$ and $g(x)$ the generator polynomial of $\mathcal{C}$, where $\operatorname{gcd}(l, p)=1$ and $l, e>1$. If $d_{H}(\mathcal{C})$ is prime and one of the following two conditions is satisfied
(1) $l<d_{H}(\mathcal{C})<l p^{e}-k$;
(2) $x^{l}-1$ is a divisor of $g(x)$ and $2<d_{H}(\mathcal{C})<l p^{e}-k$,
then $d_{p}(\mathcal{C}) \geq d_{H}(\mathcal{C})+3$.

## 3 Constructions of MDS Symbol-Pair Codes

In this section, for $n=3 p$, we propose two new classes of MDS symbol-pair codes from repeated-root cyclic codes by analyzing the system of certain linear equations over $\mathbb{F}_{p}$. Interestingly, the minimum symbol-pair distance of these codes ranges in $\{10,12\}$, which is bigger than all the known codes in Table [1. For preparation, we define the following notations.

Let $n$ and $A_{i}$ be positive integers for any $1 \leq i \leq n$ and

$$
\mathcal{V}\left(A_{1}, \cdots, A_{n}\right)=\left(A_{1} \bmod 3, \cdots, A_{n} \bmod 3\right)
$$

Denote by $\mathcal{C S}\left(A_{1}, \cdots, A_{n}\right)=\left(a_{1}, \cdots, a_{n}\right)$ the rearrangement of $\mathcal{V}\left(A_{1}, \cdots, A_{n}\right)$ with $a_{i} \leq a_{j}$ for any $i<j$. For instance, $\mathcal{C S}(5,10,4)=(1,1,2)$.

Now we present a class of MDS symbol-pair codes with length $3 p$ and minimum symbol-pair distance 10 .

Theorem 1. Let $p$ be an odd prime with $3 \mid(p-1)$. Then there exists an $M D S(3 p, 10)_{p}$ symbolpair code.

Proof. Let $\mathcal{C}$ be a repeated-root cyclic code of length $3 p$ over $\mathbb{F}_{p}$ with generator polynomial

$$
g(x)=(x-1)^{4}(x-\omega)^{2}\left(x-\omega^{2}\right)^{2}
$$

where $\omega$ is a primitive third root of unity in $\mathbb{F}_{p}$.
Note that Lemma 2 yields that $\mathcal{C}$ is a $[3 p, 3 p-8,5]$ cyclic code. Precisely, recall that $\bar{g}_{t}(x)$ is the generator polynomial of $\overline{\mathcal{C}}_{t}$. If $t \in\{0,1\}$, then $\bar{g}_{t}(x)=x^{3}-1$ and

$$
P_{t} \cdot d_{H}\left(\overline{\mathcal{C}}_{t}\right)=\infty
$$

If $t=2$, then $\bar{g}_{2}(x)=x-1$ and

$$
P_{2} \cdot d_{H}\left(\overline{\mathcal{C}}_{2}\right)=3 \cdot 2=6
$$

If $t=3$, then $\bar{g}_{3}(x)=x-1$ and

$$
P_{3} \cdot d_{H}\left(\overline{\mathcal{C}}_{3}\right)=4 \cdot 2=8
$$

If $4 \leq t \leq p-1$, then $\bar{g}_{t}(x)=1$ and

$$
P_{t} \cdot d_{H}\left(\overline{\mathcal{C}}_{t}\right)=t+1 \geq 5
$$

Due to the equality (2), one immediately has $d_{H}(\mathcal{C})=5$.
Since $\left(x^{3}-1\right) \mid g(x)$ and $2<5=d_{H}(\mathcal{C})<3 p-(3 p-8)=8$, by Lemma 5, one gets $d_{p}(\mathcal{C}) \geq 8$. Suppose that there exists a codeword $f(x)$ in $\mathcal{C}$ with Hamming weight 7 such that $f(x)$ has 7 consecutive nonzero entries. Denote

$$
f(x)=f_{0}+f_{1} x+f_{2} x^{2}+f_{3} x^{3}+f_{4} x^{4}+f_{5} x^{5}+f_{6} x^{6}
$$

where $f_{i} \in \mathbb{F}_{p}^{*}$ for any $0 \leq i \leq 6$. It follows that the degree of $g(x)$ is less than or equal to the degree of $f(x)$, i.e., $8 \leq 6$, which is impossible. Hence there does not exist a codeword in $\mathcal{C}$ with Hamming weight 7 and symbol-pair weight 8. Similarly, it can be verified that there does not exist a codeword in $\mathcal{C}$ with Hamming weight 8 and symbol-pair weight 9 .

In the sequel, we claim that there does not exist a codeword in $\mathcal{C}$ with Hamming weight 5 and symbol-pair weight 8 (or 9 ). Let $c(x)$ be a codeword in $\mathcal{C}$ with Hamming weight 5 . Assume that $c(x)$ has factorization $c(x)=\left(x^{3}-1\right)^{t} v(x)$, where $0 \leq t \leq p-1,\left(x^{3}-1\right) \nmid v(x)$ and $v(x)=v_{0}\left(x^{3}\right)+x v_{1}\left(x^{3}\right)+x^{2} v_{2}\left(x^{3}\right)$. Then by Lemma 3, one can conclude that

$$
5=w_{H}\left(\left(x^{3}-1\right)^{t}\right) \cdot w_{H}\left(v(x) \bmod \left(x^{3}-1\right)\right)=(1+t) N_{v}
$$

where $N_{v}=w_{H}\left(v(x) \bmod \left(x^{3}-1\right)\right)$. It follows that $\left(N_{v}, t\right)=(1,4)$, which implies that the symbol-pair weight of $c(x)$ cannot be 8 (or 9 ).

In order to derive that $\mathcal{C}$ is an $\operatorname{MDS}(3 p, 10)_{p}$ symbol-pair code, we need to prove that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(6,8),(6,9)$ or $(7,9)$.

Firstly, on the contrary, suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 6 and symbol-pair weight 8 . Then its certain cyclic shift must have the form

$$
\begin{aligned}
& (\star, \star, \star, \star, \star, \mathbf{0}, \star, \mathbf{0}), \\
& (\star, \star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0})
\end{aligned}
$$

or

$$
(\star, \star, \star, \mathbf{0}, \star, \star, \star, \mathbf{0}) .
$$

Without loss of generality, in this paper, we always suppose that the first coordinate of a codeword is 1 .

- For the subcase of $(\star, \star, \star, \star, \star, \mathbf{0}, \star, \mathbf{0})$. Let $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l}$ with $6 \leq l \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{V}(l) \in\{0,1\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$, one can immediately obtain

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}+a_{5}=0 \\
1+a_{1} \omega+a_{2} \omega^{2}+a_{3}+a_{4} \omega+a_{5} \omega^{l}=0 \\
1+a_{1} \omega^{2}+a_{2} \omega+a_{3}+a_{4} \omega^{2}+a_{5} \omega^{2 l}=0
\end{array}\right.
$$

This leads to $a_{2}=0$, a contradiction.

- If $\mathcal{V}(l)=2$, then $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ indicates $a_{3}=0$, a contradiction.
- For the subcase of $(\star, \star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0})$. Let $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l}+a_{5} x^{l+1}$ with $5 \leq l \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{V}(l) \in\{0,2\}$, then it follows from $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ that $a_{1}=0$ or $a_{2}=0$, a contradiction.
- If $\mathcal{V}(l)=1$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one immediately has $a_{3}=0$, a contradiction.
- For the subcase of $(\star, \star, \star, \mathbf{0}, \star, \star, \star, \mathbf{0})$. Let $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l}+a_{4} x^{l+1}+a_{5} x^{l+2}$ with $4 \leq l \leq 3 p-4$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$. Then for any $4 \leq l \leq 3 p-4$, by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can derive $a_{i}=0$ for some $3 \leq i \leq 5$, a contradiction.

Secondly, assume that there exists a codeword $c(x)$ in $\mathcal{C}$ with Hamming weight 6 and symbolpair weight 9 . There are three subcases to be considered:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l_{1}}+a_{5} x^{l_{2}}$ with $5 \leq l_{1}<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right)=(1,2)$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can obtain that $a_{3}=0$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \neq(1,2)$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ indicates that $a_{1}=0$ or $a_{2}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{1}+1}+a_{5} x^{l_{2}}$ with $4 \leq l_{1}<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \in\{(0,2),(1,0),(2,1)\}$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can immediately derive $a_{i}=0$ for some $3 \leq i \leq 5$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \notin\{(0,2),(1,0),(2,1)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ implies that $1=0$ or $a_{i}=0$ for some $i \in\{1,2\}$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{l_{1}}+a_{3} x^{l_{1}+1}+a_{4} x^{l_{2}}+a_{5} x^{l_{2}+1}$ with $3 \leq l_{1}<l_{2} \leq$ $3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right)=(0,0)$, then one can deduce that $p \nmid l_{1} l_{2}\left(l_{2}-l_{1}\right)$. By $c(1)=c(\omega)=$ $c\left(\omega^{2}\right)=0$, one immediately gets

$$
1+a_{2}+a_{4}=a_{1}+a_{3}+a_{5}=0 .
$$

It follows from $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ that

$$
l_{1} a_{2}+l_{2} a_{4}=l_{1} a_{3}+l_{2} a_{5}=0
$$

Then one can conclude that $a_{2}+a_{3}=a_{4}+a_{5}=0$ due to $c^{(2)}(1)=0$. Hence the fact $c^{(3)}(1)=0$ indicates that $l_{2}\left(l_{2}-l_{1}\right) a_{5}=0$, a contradiction.

- If $\mathcal{C S}\left(l_{1}, l_{2}\right)=(1,2)$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can immediately deduce $a_{3}=0$ or $a_{5}=0$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \notin\{(0,0),(1,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ yields that $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.

Thirdly, suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 7 and symbol-pair weight 9 . Then we ought to discuss the following three subcases:

- For the subscase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{l}$ with $7 \leq l \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{V}(l)=0$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$, one can immediately obtain $a_{1}+a_{4}=$ 0 . It follows from $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ that $a_{1}+4 a_{4}=0$. Hence $3 a_{4}=0$, a contradiction.
- If $\mathcal{V}(l) \in\{1,2\}$, then $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ implies that $a_{3}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l}+a_{6} x^{l+1}$ with $6 \leq l \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{V}(l) \in\{0,1\}$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one has $a_{2}=0$ or $a_{3}=0$, a contradiction.
- If $\mathcal{V}(l)=2$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ indicates that $a_{4}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l}+a_{5} x^{l+1}+a_{6} x^{l+2}$ with $5 \leq l \leq 3 p-4$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$. For any $5 \leq l \leq 3 p-4, c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=$ $c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ yields that $a_{i}=0$ for some $4 \leq i \leq 6$, a contradiction.

Consequently, $\mathcal{C}$ is an MDS $(3 p, 10)_{p}$ symbol-pair code. This completes the proof.
In what follows, we construct a class of MDS symbol-pair codes with length $3 p$ and minimum symbol-pair distance 12, which is the maximum minimum symbol-pair distance for all known MDS symbol-pair codes from constacyclic codes.

Theorem 2. Let $p$ be an odd prime with $3 \mid(p-1)$. Then there exists an MDS $(3 p, 12)_{p}$ symbolpair code.

Proof. Let $\mathcal{C}$ be a repeated-root cyclic code of length $3 p$ over $\mathbb{F}_{p}$ with generator polynomial

$$
g(x)=(x-1)^{5}(x-\omega)^{3}\left(x-\omega^{2}\right)^{2}
$$

where $\omega$ is a primitive third root of unity in $\mathbb{F}_{p}$. It follows from Lemman㨁 that $\mathcal{C}$ is a $[3 p, 3 p-10,6]$ code. Since $\mathcal{C}$ is not MDS, by Lemma 4 , one can get $d_{p}(\mathcal{C}) \geq 8$. With a similar manner to the proof of Theorem [1 it can be verified that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(9,10)$ or $(10,11)$. Besides, the proof of Theorem 11 yields that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(6,8),(6,9),(7,8),(7,9)$ or $(8,9)$.

To determine that $\mathcal{C}$ is an $\operatorname{MDS}(3 p, 12)_{p}$ symbol-pair code, it is sufficient to derive that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(6,10),(6,11),(7,10)$, $(7,11),(8,10),(8,11)$ or $(9,11)$.

Firstly, we suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 6 and symbol-pair weight 10. Without loss of generality, we just consider the following two subcases:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{2}}+a_{5} x^{l_{3}}$ with $4 \leq l_{1}<l_{2}<l_{3} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}\right)=(0,1,2)$, then it can be checked that by $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$, one can immediately get $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}\right) \neq(0,1,2)$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=$ $c^{(1)}\left(\omega^{2}\right)=0$ yields that $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{l_{1}}+a_{3} x^{l_{1}+1}+a_{4} x^{l_{2}}+a_{5} x^{l_{3}}$ with $3 \leq l_{1}<l_{2}<$ $l_{3} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \in\{(0,0,0),(0,1,1),(0,2,2),(1,0,2),(1,2,0),(2,1,2),(2,2,1)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, it follows that $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \in\{(0,1,0),(0,0,1)\}$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=c^{(2)}(1)=$ $c^{(2)}(\omega)=0$, one can immediately obtain $a_{4}=0$ or $a_{5}=0$ since $p \nmid l_{3}\left(l_{3}-l_{1}\right)$ or $p \nmid l_{2}\left(l_{2}-l_{1}\right)$. This leads to a contradiction.
- For other conditions, it can be verified that $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ indicates $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 5$, which is impossible.

Secondly, we assume that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 6 and symbol-pair weight 11. Let $c(x)=1+a_{1} x+a_{2} x^{l_{1}}+a_{3} x^{l_{2}}+a_{4} x^{l_{3}}+a_{5} x^{l_{4}}$ with $3 \leq l_{1}<l_{2}<l_{3}<l_{4} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 5$.

- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}, l_{4}\right) \in\{(0,0,0,1),(0,1,1,1),(0,1,2,2)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=$ $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ indicates that $a_{i}=0$ for some $2 \leq i \leq 5$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}, l_{4}\right)=(0,0,1,1)$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=c^{(2)}(1)=$ $c^{(2)}(\omega)=0$, one has $a_{i}=0$ for some $2 \leq i \leq 5$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}, l_{4}\right) \notin\{(0,1,1,1),(0,1,2,2),(0,0,1,1),(0,0,0,1)\}$, then $c(1)=c(\omega)=$ $c\left(\omega^{2}\right)=0$ yields that $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.

Thirdly, we suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 7 and symbol-pair weight 10 . There are five subcases to be discussed:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l_{1}}+a_{6} x^{l_{2}}$ with $6 \leq l_{1}<$ $l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \in\{(0,0),(0,1),(1,1)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$, one can immediately deduce $a_{2}=0$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \in\{(1,2),(2,2)\}$, then $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ yields that $a_{3}=0$, a contradiction.
$-\operatorname{If} \mathcal{C S}\left(l_{1}, l_{2}\right)=(0,2)$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=$ 0 , one has $a_{4}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l_{1}}+a_{5} x^{l_{1}+1}+a_{6} x^{l_{2}}$ with $6 \leq$ $\left(l_{1}+1\right)<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \in\{(0,0),(0,1),(2,0),(2,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ implies that $a_{1}=0$ or $a_{2}=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \in\{(0,2),(1,0),(1,1),(1,2),(2,1)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=$ $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can immediately get $a_{i}=0$ for some $4 \leq i \leq 6$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{1}+1}+a_{5} x^{l_{1}+2}+a_{6} x^{l_{2}}$ with $6 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$. For any $6 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-2$, it follows from $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ that $a_{4}=0$ or $a_{5}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{1}+1}+a_{5} x^{l_{2}}+a_{6} x^{l_{2}+1}$ with $5 \leq\left(l_{1}+1\right)<l_{2} \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \in\{(0,0),(1,1),(2,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ indicates that $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 2$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \in\{(0,1),(0,2),(1,2)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=$ $c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can conclude that $a_{5}=0$ or $a_{6}=0$, a contradiction.

Fourthly, we assume that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 7 and symbol-pair weight 11. There are three subcases to be considered:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l_{1}}+a_{5} x^{l_{2}}+a_{6} x^{l_{3}}$ with $5 \leq l_{1}<$ $l_{2}<l_{3} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}\right) \in\{(0,1,2),(1,1,2),(1,2,2)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=$ $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can immediately have $a_{i}=0$ for some $1 \leq i \leq$ 6 , a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}, l_{3}\right) \notin\{(0,1,2),(1,1,2),(1,2,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ implies that $a_{1}=0$ or $a_{2}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{1}+1}+a_{5} x^{l_{2}}+a_{6} x^{l_{3}}$ with $5 \leq$ $\left(l_{1}+1\right)<l_{2}<l_{3} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \in\{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\}$, then by $c(1)=c(\omega)=$ $c\left(\omega^{2}\right)=0$, one can derive that $1=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \notin\{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=$ $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ induces that $a_{i}=0$ for some $1 \leq i \leq 6$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{l_{1}}+a_{3} x^{l_{1}+1}+a_{4} x^{l_{2}}+a_{5} x^{l_{2}+1}+a_{6} x^{l_{3}}$ with $4 \leq\left(l_{1}+1\right)<l_{2}<l_{3} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 6$.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \in\{(1,1,1),(1,1,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=0$ yields $1=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \in\{(0,0,0),(0,0,1)\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=$ $c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=c^{(2)}(1)=c^{(2)}(\omega)=0$, one can deduce that $a_{4}=0$ or $a_{5}=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}, l_{3}\right) \notin\{(0,0,0),(0,0,1),(1,1,1),(1,1,2)\}$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=$ $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ leads to $1=0$ or $a_{i}=0$ for some $1 \leq i \leq 6$, a contradiction.

Fifthly, we suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 8 and symbol-pair weight 10 . There are four subcases to be considered:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{6}+a_{7} x^{l}$ with $8 \leq l \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $8 \leq l \leq 3 p-2$, it can be verified that $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ induces that $a_{4}=0$ or $a_{5}=0, \mathrm{a}$ contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{l}+a_{7} x^{l+1}$ with $7 \leq l \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $7 \leq l \leq 3 p-3, c(1)=c(\omega)=$ $c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ implies that $a_{i}=0$ for some $3 \leq i \leq 5$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l}+a_{6} x^{l+1}+a_{7} x^{l+2}$ with $6 \leq l \leq 3 p-4$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $6 \leq l \leq 3 p-4$, it follows from $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ that $a_{i}=0$ for some $5 \leq i \leq 7$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l}+a_{5} x^{l+1}+a_{6} x^{l+2}+a_{7} x^{l+3}$ with $5 \leq l \leq 3 p-5$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $5 \leq l \leq 3 p-5$, by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can derive that $a_{5}=0$ or $a_{6}=0$, a contradiction.

Sixthly, we suppose that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 8 and symbol-pair weight 11. Without loss of generality, it suffices to consider the following five subcases:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{l_{1}}+a_{7} x^{l_{2}}$ with $7 \leq l_{1}<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right)=(1,2)$, then it can be verified that by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can obtain $a_{3}=0$, a contradiction.
- If $\mathcal{C S}\left(l_{1}, l_{2}\right) \neq(1,2)$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ indicates $a_{4}=0$ or $a_{5}=0$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l_{1}}+a_{6} x^{l_{1}+1}+a_{7} x^{l_{2}}$ with $7 \leq\left(l_{1}+1\right)<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$.
- If $\mathcal{V}\left(l_{1}, l_{2}\right)=(1,2)$, then by $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can get $a_{3}=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \neq(1,2)$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ implies $a_{i}=0$ for some $1 \leq i \leq 7$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l_{1}}+a_{5} x^{l_{1}+1}+a_{6} x^{l_{1}+2}+a_{7} x^{l_{2}}$ with $7 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $7 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-2$, it can be verified that by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can deduce $a_{i}=0$ for some $4 \leq i \leq 6$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{l_{1}}+a_{5} x^{l_{1}+1}+a_{6} x^{l_{2}}+a_{7} x^{l_{2}+1}$ with $6 \leq\left(l_{1}+1\right)<l_{2} \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$.
- If $\mathcal{V}\left(l_{1}, l_{2}\right)=(1,1)$, then $c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ yields $a_{3}=0$, a contradiction.
- If $\mathcal{V}\left(l_{1}, l_{2}\right) \neq(1,1)$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=$ 0 , one can derive $a_{i}=0$ for some $1 \leq i \leq 7$, a contradiction.
- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l_{1}}+a_{4} x^{l_{1}+1}+a_{5} x^{l_{1}+2}+a_{6} x^{l_{2}}+a_{7} x^{l_{2}+1}$ with $6 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 7$. For any $6 \leq\left(l_{1}+2\right)<l_{2} \leq 3 p-3$, it follows from $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ that $a_{i}=0$ for some $1 \leq i \leq 5$, a contradiction.

Finally, we assume that $c(x)$ is a codeword in $\mathcal{C}$ with Hamming weight 9 and symbol-pair weight 11. There are four subcases to be discussed:

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{6}+a_{7} x^{7}+a_{8} x^{l}$ with $9 \leq l \leq 3 p-2$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 8$.
- If $\mathcal{V}(l) \in\{0,1\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can obtain $a_{5}=0$, a contradiction.
- If $\mathcal{V}(l)=2$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ induces

$$
\left\{\begin{array}{l}
1+a_{3}+a_{6}=a_{1}+a_{4}+a_{7}=a_{2}+a_{5}+a_{8}=0, \\
3 a_{3}+6 a_{6}=a_{1}+4 a_{4}+7 a_{7}=2 a_{2}+5 a_{5}+l a_{8}=0
\end{array}\right.
$$

which indicates that

$$
\begin{equation*}
a_{2}=\frac{l-5}{3} a_{8}, \quad a_{3}=-2, \quad a_{4}=-2 a_{7}, \quad a_{5}=-\frac{l-2}{3} a_{8}, \quad a_{6}=1 . \tag{4}
\end{equation*}
$$

The fact $c^{(2)}(1)=c^{(2)}(\omega)=0$ indicates

$$
\left\{\begin{array}{l}
2 a_{2}+20 a_{5}+l(l-1) a_{8}+6 a_{3}+30 a_{6}+12 a_{4}+42 a_{7}=0, \\
2 a_{2}+20 a_{5}+l(l-1) a_{8}+\left(6 a_{3}+30 a_{6}\right) \omega+\left(12 a_{4}+42 a_{7}\right) \omega^{2}=0
\end{array}\right.
$$

which yields

$$
a_{7}=\omega \quad \text { and } \quad a_{8}=\frac{18 \omega^{2}}{(l-2)(l-5)}
$$

due to (4). It follows from $c^{(3)}(1)=0$ that

$$
6 a_{3}+24 a_{4}+60 a_{5}+120 a_{6}+210 a_{7}+l(l-1)(l-2) a_{8}=0
$$

which yields $l=2-3 \omega^{2} . \operatorname{By} c^{(4)}(1)=0$, one can get

$$
24 a_{4}+120 a_{5}+360 a_{6}+840 a_{7}+l(l-1)(l-2)(l-3) a_{8}=0 .
$$

This implies $\left(l^{2}+l-12\right) \omega+24=0$, which leads to

$$
\left(\left(2-3 \omega^{2}\right)^{2}+2-3 \omega^{2}-12\right) \omega+24=0 .
$$

Thus $-5 \omega=0$, a contradiction.

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{6}+a_{7} x^{l}+a_{8} x^{l+1}$ with $8 \leq l \leq 3 p-3$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 8$.
- If $\mathcal{V}(l) \in\{0,2\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can conclude $a_{4}=0$ or $a_{5}=0$, a contradiction.
- If $\mathcal{V}(l)=1$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ implies

$$
\begin{equation*}
a_{2}=\frac{l-4}{3} a_{8}, \quad a_{3}=-2, \quad a_{4}=-\frac{l-1}{3} a_{7}, \quad a_{5}=-\frac{l-1}{3} a_{8}, \quad a_{6}=1 . \tag{5}
\end{equation*}
$$

It can be verified by (5) and $c^{(2)}(1)=c^{(2)}(\omega)=0$ that

$$
a_{7}=\frac{18 \omega}{(l-1)(l-4)} \quad \text { and } \quad a_{8}=\frac{18 \omega^{2}}{(l-1)(l-4)}
$$

According to $c^{(3)}(1)=0$, one can obtain $l=-\omega^{2}-4 \omega$. Hence $c^{(4)}(1)=0$ yields $3 \omega^{2}=0$, a contradiction.

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{5}+a_{6} x^{l}+a_{7} x^{l+1}+a_{8} x^{l+2}$ with $7 \leq l \leq 3 p-4$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 8$.
- If $\mathcal{V}(l)=0$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one can derive

$$
\begin{equation*}
a_{2}=\frac{l-3}{3} a_{8}, \quad a_{3}=-\frac{l}{l-3}, \quad a_{4}=-\frac{l}{3} a_{7}, \quad a_{5}=-\frac{l}{3} a_{8}, \quad a_{6}=\frac{3}{l-3} \tag{6}
\end{equation*}
$$

By (6) and $c^{(2)}(1)=c^{(2)}(\omega)=0$, one has

$$
a_{7}=\frac{3 \omega}{l-3} \quad \text { and } \quad a_{8}=\frac{3 \omega^{2}}{l-3}
$$

Then $c^{(3)}(1)=0$ implies

$$
6 a_{3}+24 a_{4}+60 a_{5}+l(l-1)(l-2) a_{6}+l(l-1)(l+1) a_{7}+l(l+1)(l+2) a_{8}=0
$$

which yields $\omega^{2}=\omega$, a contradiction.

- If $\mathcal{V}(l)=1$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ indicates $a_{2}=\frac{l-4}{3} a_{7}, \quad a_{3}=-\frac{l+2}{l-1}, \quad a_{4}=-\frac{l-1}{3} a_{6}, \quad a_{5}=-\frac{l-1}{3} a_{7}, \quad a_{8}=\frac{3}{l-1}$.

It follows from (7) and $c^{(2)}(1)=c^{(2)}(\omega)=0$ that

$$
a_{6}=\frac{3(l+2) \omega}{(l-1)(l-4)} \quad \text { and } \quad a_{7}=\frac{3(l+2) \omega^{2}}{(l-1)(l-4)}
$$

By $c^{(3)}(1)=0$, one can deduce that $3 \omega^{2}=0$, a contradiction.

- If $\mathcal{V}(l)=2$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one has

$$
\begin{equation*}
a_{2}=\frac{l-5}{3} a_{6}, a_{3}=-\frac{l+1}{3} a_{7}, a_{4}=-\frac{l+1}{3} a_{8}, a_{5}=-\frac{l-2}{3} a_{6}, a_{7}=\frac{3}{l-2} . \tag{8}
\end{equation*}
$$

Due to (8) and $c^{(2)}(1)=c^{(2)}(\omega)=0$, one can immediately obtain that

$$
a_{6}=\frac{3(l+1) \omega^{2}}{(l-2)(l-5)} \quad \text { and } \quad a_{8}=\frac{3 \omega}{l-2}
$$

Thus $c^{(3)}(1)=0$ leads to $3=0$, a contradiction.

- For the subcase of $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{4}+a_{5} x^{l}+a_{6} x^{l+1}+a_{7} x^{l+2}+a_{8} x^{l+3}$ with $6 \leq l \leq 3 p-5$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for any $1 \leq i \leq 8$.
- If $\mathcal{V}(l) \in\{0,1\}$, then by $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$, one gets $a_{6}=0$ or $a_{7}=0$, a contradiction.
- If $\mathcal{V}(l)=2$, then $c(1)=c(\omega)=c\left(\omega^{2}\right)=c^{(1)}(1)=c^{(1)}(\omega)=c^{(1)}\left(\omega^{2}\right)=0$ induces

$$
\begin{equation*}
a_{2}=\frac{3}{l-2} a_{8}, a_{3}=-\frac{l+1}{3} a_{6}, a_{4}=-\frac{l+1}{3} a_{7}, a_{5}=-\frac{l+1}{l-2} a_{8}, a_{6}=\frac{3}{l-2} . \tag{9}
\end{equation*}
$$

By (9) and $c^{(2)}(1)=c^{(2)}(\omega)=0$, one can derive

$$
a_{7}=\frac{3 \omega}{l-2} \quad \text { and } \quad a_{8}=\omega^{2}
$$

It follows from $c^{(3)}(1)=0$ that $l=-3 \omega^{2}-1$. Then $c^{(4)}(1)=0$ yields $l \omega+2=$ $-\omega-1=0$, a contradiction.

As a result, $\mathcal{C}$ is an $\operatorname{MDS}(3 p, 12)_{p}$ symbol-pair code. The desired result follows.
Remark 1. Note that if $\mathcal{C}$ is a repeated-root cyclic code of length $3 p$ over $\mathbb{F}_{p}$ with generator polynomial

$$
g(x)=(x-1)^{4}(x-\omega)^{3}\left(x-\omega^{2}\right)^{2}
$$

where $\omega$ is a primitive third root of unity in $\mathbb{F}_{p}$. Due to Theorem 1 , we can conclude that $\mathcal{C}$ is an $A M D S(3 p, 10)_{p}$ symbol-pair code. Indeed, by Lemma 园, one can immediately get $d_{H}(\mathcal{C})=$ 5. Since $\left(x^{3}-1\right) \mid g(x)$ and $2<5=d_{H}(\mathcal{C})<3 p-(3 p-9)=9$, Lemma 5 indicates that $d_{p}(\mathcal{C}) \geq 8$. It follows from the proof of Theorem 1 that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,8),(5,9),(6,8),(6,9),(7,8),(7,9)$ or $(8,9)$. Therefore, the inequality (1) shows that $\mathcal{C}$ is an $A M D S(3 p, 10)_{p}$ symbol-pair code.

In what follows, we present two examples to illustrate the result in Theorems 1 and 2,
Example 1. (1) Let $\mathcal{C}$ be a repeated-root cyclic code of length 21 over $\mathbb{F}_{7}$ with generator polynomial

$$
g(x)=(x-1)^{4}(x-2)^{2}\left(x-2^{2}\right)^{2}
$$

By the computation software MAGMA, it can be verified that $\mathcal{C}$ is a $[21,13,5]$ code and the minimum symbol-pair distance of $\mathcal{C}$ is 10 , which coincides with our result in Theorem 1 .
(2) Let $\mathcal{C}$ be a repeated-root cyclic code of length 21 over $\mathbb{F}_{7}$ with generator polynomial

$$
g(x)=(x-1)^{5}(x-2)^{3}\left(x-2^{2}\right)^{2}
$$

MAGMA experiments yield that $\mathcal{C}$ is a $[21,11,6]$ code and the minimum symbol-pair distance of $\mathcal{C}$ is 12 , which is consistent with our result in Theorem (2.

## 4 Conclusions

In this paper, for $n=3 p$, we construct two new classes of MDS symbol-pair codes over $\mathbb{F}_{p}$ with $p$ an odd prime by employing repeated-root cyclic codes:

- $[3 p, 3 p-8,5]$ code with $d_{p}=10$;
- $[3 p, 3 p-10,6]$ code with $d_{p}=12$.

As mentioned in Table 1, these codes poss minimum symbol-pair distance bigger than all the known MDS symbol-pair codes from constacyclic codes. Note that alongside with larger minimum symbol-pair distance, much more cases need to be considered, which has not been explored.

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