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CORRECTION

Correction to: Cameron–Liebler sets of k-spaces in PG(n, q)

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Abstract
In this document we describe and correct a mistake made in the article [2]. We prove a new classification theorem.

1 Introduction

In 2019 the authors published the article [2] in which they introduced and investigated *Cameron–Liebler sets* of k-spaces in the projective geometry $\mathrm{PG}(n, q)$. In [2, Sect. 2] a characterisation theorem was proved and in [2, Sect. 3] several properties and examples of Cameron–Liebler sets of k-spaces were presented. Due to the characterisation theorem we can give several definitions of Cameron–Liebler sets, but for this paper the following is the most useful.

Definition 1.1 [2, Theorem 2.9 and Definition 2.10] A non-empty set $\mathcal{L}$ of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 2k + 1$, with characteristic vector χ, is a Cameron–Liebler set of k-spaces in $\mathrm{PG}(n, q)$ with parameter $x = |\mathcal{L}|\begin{bmatrix} n \\ k \end{bmatrix}^{-1}$ if and only if for every k-space π the number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi))\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$.

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In the fourth section of [2] a classification result for the Cameron–Liebler classes of k-spaces in $\mathrm{PG}(n, q)$ with $n \geq 3k + 2$ and with parameter at most $q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$ was given. It was however pointed out to us by Ferdinand Ihringer that there is a mistake in the proof of this result, more precisely in the proof of Lemma 4.6: there the coefficient $\frac{(1-\lfloor x \rfloor)x\lfloor x \rfloor}{2}$ is negative which makes one of the implications false.

In this correction we will present a new proof for a (slightly weaker) classification result. We classify the Cameron–Liebler sets of k-spaces in the projective space $\mathrm{PG}(n, q)$ with parameter at most $\frac{1}{\sqrt[8]{2}}q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.

We also mention a related result by Ferdinand Ihringer which was published in the meantime and which gives a result similar to our incorrect Lemma 2.6.

Lemma 1.2 [5, Lemma 19] *Let Y be a Cameron–Liebler set of k-spaces in* $\mathrm{PG}(n, q)$, $n \geq 2k+2$ *and* $q \geq 7$, *with parameter x. If* $x \leq q^{(n-2k-r)/3}$, *with* $0 \leq r < k+1$ *and* $n+r+1 \equiv 0 \pmod{k+1}$, *then Y contains at most x pairwise disjoint k-spaces.*

In the same article, Ferdinand Ihringer used this lemma to obtain the following classification result.

Theorem 1.3 [5, Theorem 7] *Let Y be a Cameron–Liebler set of k-spaces in* $\mathrm{PG}(n, q)$, $n \geq 2k+1$, *with parameter x. If* $16x \leq \min\{q^{(n-2k-r)/3}, q^{(n-k-l+2)/3}\}$, *with* $0 \leq r < k+1$, $n+r+1 \equiv 0 \pmod{k+1}$ *and l is the integer satisfying* $\frac{q^{l-1}-1}{q-1} < x \leq \frac{q^l-1}{q-1}$, *then* $x \leq 2$ *and Y is trivial.*

We note that on the one hand this result is valid for all values of $n \geq 2k+1$, while our result only applies for $n \geq 3k+2$. On the other hand the bound in our result (Theorem 2.17) improves the bound from Theorem 1.3.

Another important result for Cameron–Liebler sets of k-spaces in $\mathrm{PG}(n, q)$ is the following.

Remark 1.4 For $q \in \{2, 3, 4, 5\}$, a complete classification is known for Cameron–Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, see [3]. There, the authors show that the only Cameron–Liebler sets in this context are the trivial Cameron–Liebler sets, independent of the values of k and n.

Throughout the article we use *Gaussian binomial coefficients* $\begin{bmatrix} a \\ b \end{bmatrix}_q$ for $a, b \in \mathbb{N} \setminus \{0\}$, $a \geq b$, and prime power $q \geq 2$:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q^a-1)\cdots(q^{a-b+1}-1)}{(q^b-1)\cdots(q-1)}.$$

The Gaussian binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is equal to the number $(b-1)$-spaces in the projective space $\mathrm{PG}(a-1, q)$.

To simplify notation, we denote $q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$ by $f(q, n, k)$.

2 Corrected classification result

In [2, Lemma 4.1 and Theorem 4.3] we classified the Cameron–Liebler sets of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 3k+2$, with parameter $x \in]0, 2[$. We will now show that there are no

Cameron–Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 3k+2$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.

We recall Lemmas 2.1, 3.1(4), 3.4, 3.5, 3.6 and Theorem 4.4 from [2].

Lemma 2.1 [6, Sect. 170] *The number of j-spaces disjoint from a fixed m-space in* $\mathrm{PG}(n,q)$ *equals* $q^{(m+1)(j+1)}\begin{bmatrix} n-m \\ j+1 \end{bmatrix}$.

Lemma 2.2 *If $\mathcal{L}$ and $\mathcal{L}'$ are two Cameron–Liebler sets of k-spaces in* $\mathrm{PG}(n,q)$ *with parameters x and x' respectively, and $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{L} \setminus \mathcal{L}'$ is a Cameron–Liebler set of k-spaces with parameter $x - x'$.*

Notation 2.3 *Let n, k, i be integers, let q be a prime power, let x be a rational number and let $\mathcal{S}_0$ be a k-spread of a $(2k+1)$-space. We introduce the following notation.*

$$W_i(q,n,k) = \begin{cases} q^{2k^2+k+\frac{3i^2}{2}-\frac{i}{2}-3ik}\begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}\prod_{j=0}^{i}(q^{k-j+1}-1) & \text{if } i \geq 0 \\ q^{2(k+1)^2}\begin{bmatrix} n-2k-1 \\ k+1 \end{bmatrix} & \text{if } i = -1 \end{cases}$$

$$W(q,n,k) = \sum_{i=-1}^{k} W_i(q,n,k)$$

$$W_\Sigma(q,n,k) = \frac{1}{(q^{k+1}-1)^2}\sum_{i=0}^{k} W_i(q,n,k)(q^{i+1}-1)$$

$$W_{\bar{\Sigma}}(q,n,k) = \frac{1}{q^{n+1}-q^{2k+2}}\sum_{i=-1}^{k-1} W_i(q,n,k)(q^{k+1}-q^{i+1})$$

$$d_2(q,n,k,x,\mathcal{S}_0) = (W_\Sigma - W_{\bar{\Sigma}})|\mathcal{S}_0 \cap \mathcal{L}| - 2W_\Sigma + xW_{\bar{\Sigma}}$$

$$s_1(q,n,k,x) = x\begin{bmatrix} n \\ k \end{bmatrix} - (x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$$

$$s_2(q,n,k,x,\mathcal{S}_0) = x\begin{bmatrix} n \\ k \end{bmatrix} - 2(x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} + d_2(q,n,k,x,\mathcal{S}_0)$$

$$d_2'(q,n,k,x) = (x-2)W_\Sigma$$

$$s_2'(q,n,k,x) = x\begin{bmatrix} n \\ k \end{bmatrix} - 2(x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} + d_2'(q,n,k,x)$$

Lemma 2.4 *Let π and π' be two disjoint k-spaces in* $\mathrm{PG}(n,q)$ *with $\Sigma = \langle \pi, \pi' \rangle$, and let P be a point in $\Sigma \setminus (\pi \cup \pi')$ and let P' be a point not in Σ. Then the number of k-spaces disjoint from π and π' equals $W(q,n,k)$, the number of k-spaces disjoint from π and π' through P equals $W_\Sigma(q,n,k)$ and the number of k-spaces disjoint from π and π' through P' equals $W_{\bar{\Sigma}}(q,n,k)$.*

Lemma 2.5 *Let $\mathcal{L}$ be a Cameron–Liebler set of k-spaces in* $\mathrm{PG}(n,q)$ *with parameter x.*

1. *For every $\pi \in \mathcal{L}$, there are s_1 elements of $\mathcal{L}$ meeting π.*
2. *For skew $\pi, \pi' \in \mathcal{L}$ and a k-spread $\mathcal{S}_0$ in $\Sigma = \langle \pi, \pi' \rangle$, there exist exactly d_2 subspaces in $\mathcal{L}$ that are skew to both π and π' and there exist s_2 subspaces in $\mathcal{L}$ that meet both π and π'.*
3. *Define $d_2'(q,n,k,x) = (x-2)W_\Sigma$ and $s_2'(q,n,k,x) = x\begin{bmatrix} n \\ k \end{bmatrix} - 2(x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} + d_2'(q,n,k,x)$. If $n > 3k+1$, then $|\mathcal{S}_0 \cap \mathcal{L}| \leq x$ for every k-spread $\mathcal{S}_0$ in Σ. Moreover we have that $d_2(q,n,k,x,\mathcal{S}_0) \leq d_2'(q,n,k,x)$ and $s_2(q,n,k,x,\mathcal{S}_0) \leq s_2'(q,n,k,x)$.*

Lemma 2.6 *Let c, n, k be nonnegative integers with $n > 3k + 1$ and*

$$(c+1)s_1 - \binom{c+1}{2}s_2' > x\begin{bmatrix} n \\ k \end{bmatrix},$$

then no Cameron–Liebler set of k-spaces in $\mathrm{PG}(n, q)$ with parameter x contains $c + 1$ mutually skew k-spaces.

Theorem 2.7 [1, Theorem 1.4] *Let $k \geq 1$ be an integer. If $q \geq 3$ and $n \geq 2k + 2$, or if $q = 2$ and $n \geq 2k + 3$, then any family $\mathcal{F}$ of pairwise non-trivially intersecting k-spaces of $\mathrm{PG}(n, q)$, with $\cap_{F\in\mathcal{F}} F = \emptyset$ has size at most $\begin{bmatrix} n \\ k \end{bmatrix} - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{k+1}$.*

We prove the new theorem in a series of lemmas. Recall that the set of all k-spaces in a hyperplane in $\mathrm{PG}(n, q)$ is a Cameron–Liebler set of k-spaces with parameter $x = \frac{q^{n-k}-1}{q^{k+1}-1}$ (see [2, Example 3.3]) and note that $f(q, n, k) \in O(\sqrt{q^{n-2k}})$ while $\frac{q^{n-k}-1}{q^{k+1}-1} \in O(q^{n-2k-1})$.

Lemma 2.8 *For $n \geq 2k + 2$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix} > \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} > W_\Sigma .$$

If also $k \geq 2$, then

$$\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} > q^{nk-k^2} + q^{nk-k^2-1} + q^{nk-k^2-2} .$$

Proof The first inequality follows since $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of k-spaces through a fixed point in $\mathrm{PG}(n, q)$, $\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$ is the number of k-spaces through a fixed point disjoint from a given k-space not through that point (see Lemma 2.1), and W_Σ is the number of k-spaces through a fixed point and disjoint from two given k-spaces not through that point (see Lemma 2.4).

The second inequality, for $k \geq 2$, $n \geq 2k + 2$, follows from

$$\begin{aligned}
\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} &= \left(\prod_{i=0}^{k-3}\left(\frac{q^{n-k-1-i}-1}{q^{k-i}-1}\right)\right)\left(\frac{q^{n-2k+1}-1}{q-1}\frac{q^{n-2k}-1}{q^2-1}\right)q^{k^2+k} \\
&> q^{(n-2k-1)(k-2)}(q^{n-2k} + q^{n-2k-1} + q^{n-2k-2})q^{n-2k-2}q^{k^2+k} \\
&= q^{nk-k^2} + q^{nk-k^2-1} + q^{nk-k^2-2} .
\end{aligned}$$

Lemma 2.9 [4, Lemmas 2.1 and 2.2] *Let $n > k > 0$ be integers. If $q \geq 3$, then*

$$\left(1 + \frac{1}{q}\right)q^{k(n-k)} \leq \begin{bmatrix} n \\ k \end{bmatrix} \leq 2q^{k(n-k)} .$$

Notation 2.10 *We denote $\Delta(q, n, k) = \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$ and $C(q, n, k) = \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}$ from now on. Then, according to Lemma 2.5 we can write*

$$\begin{aligned}
s_1(q, n, k, x) &= xC(q, n, k) + \Delta(q, n, k) \quad \textit{and} \\
s_2'(q, n, k, x) &= xC(q, n, k) + (2 - x)\Delta(q, n, k) + (x - 2)W_\Sigma .
\end{aligned}$$

We denote $\Delta(q, n, k)$ and $C(q, n, k)$ by Δ and C if q, n and k are clear from the context.

Before proving a lemma on Δ and C, we give a result on the Gaussian binomial coefficients. First, we recall the (double) q-analogue of Pascal's rule:

$$q^b \begin{bmatrix} a-1 \\ b \end{bmatrix}_q + \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_q = \begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a-1 \\ b \end{bmatrix}_q + q^{a-b} \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_q . \tag{1}$$

Lemma 2.11 *For integers a, b, c with $0 \le b, c \le a$ we have that*

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \sum_{i=0}^{c} \begin{bmatrix} a-c \\ b-i \end{bmatrix}_q \begin{bmatrix} c \\ i \end{bmatrix}_q q^{(b-i)(c-i)} .$$

Here, we consider $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = 0$ if $y < 0$ or $y > x$.

Proof Induction on c. In the induction step we use the left equality in (1). □

Lemma 2.12 *If $n \ge 2k+1$ and $q \ge 3$, then*

$$W_\Sigma \le \Delta - \frac{C}{2} .$$

Proof First, using the definition of W_Σ as given in Lemma 2.4, we find

$$\begin{aligned} W_\Sigma &= \frac{1}{(q^{k+1}-1)^2} \sum_{i=0}^{k} (q^{i+1}-1) q^{2k^2+k+\frac{3}{2}i^2-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} \prod_{j=0}^{i} (q^{k-j+1}-1) \\ &= q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3}{2}i^2-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) . \end{aligned}$$

Here, the final product is considered 1 if $i = 0$ (the 'empty' product). Now, using the definitions of Δ and C as in Notation 2.10, the inequality in the statement of the lemma can be written as:

$$q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3}{2}i^2-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) \le \frac{3}{2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \frac{1}{2} \begin{bmatrix} n \\ k \end{bmatrix} . \tag{2}$$

For $k = 1$ it reduces to

$$q^3 \begin{bmatrix} n-3 \\ 1 \end{bmatrix} + q(q-1) \le \frac{3}{2} \begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2 - \frac{1}{2} \begin{bmatrix} n \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad \frac{q-1}{2} \ge 0 ,$$

which is true for all $q \ge 2$. So, we will from now on assume that $k \ge 2$.

Repeatedly applying the left equality in (1) we find that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = q^{k^2+k} \left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] + \sum_{i=0}^{k} q^{ik} \left[\begin{smallmatrix} n-i-1 \\ k-1 \end{smallmatrix}\right]$, so inequality (2) can be rewritten as

$$\begin{aligned} & q^{2k^2+k} \sum_{i=0}^{k} q^{\frac{3}{2}i^2-\frac{i}{2}-3ik} \begin{bmatrix} n-2k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \prod_{j=1}^{i} (q^{k-j+1}-1) + \frac{1}{2} \sum_{i=0}^{k} q^{ik} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} \\ & \le \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} . \end{aligned}$$

We now apply Lemma 2.11 on the right hand side of this inequality and we see that it is equivalent with

$$
\begin{aligned}
&q^{2k^2+k}\sum_{i=0}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}\prod_{j=1}^{i}(q^{k-j+1}-1)+\frac{1}{2}\sum_{i=0}^{k}q^{ik}\begin{bmatrix}n-i-1\\k-1\end{bmatrix}\\
&\qquad\le q^{k^2+k}\sum_{i=0}^{k}q^{(k-i)^2}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}\\
\Leftrightarrow\ &q^{2k^2+k}\sum_{i=1}^{k} q^{\frac{3i^2}{2}-\frac{i}{2}-3ik}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}\prod_{j=1}^{i}(q^{k-j+1}-1)+\frac{1}{2}\sum_{i=0}^{k}q^{ik}\begin{bmatrix}n-i-1\\k-1\end{bmatrix}\\
&\qquad\le q^{k^2+k}\sum_{i=1}^{k}q^{(k-i)^2}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}.
\end{aligned}
\tag{3}
$$

Now, we note that $\prod_{j=1}^{i}(q^{k-j+1}-1)\le q^{(i-1)(k+1)-\frac{i(i-1)}{2}}(q^{k-i+1}-1)$ for $i\ge 1$. So, in order to prove (3), it is sufficient to show that the following inequality is valid:

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{k}q^{ik}\begin{bmatrix}n-i-1\\k-1\end{bmatrix}\\
&\le q^{k^2+k}\sum_{i=1}^{k}\left(q^{(k-i)^2}-q^{(k-i)(k-i-1)-1}(q^{k-i+1}-1)\right)\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}\\
&=q^{k^2+k}\sum_{i=1}^{k}q^{(k-i)(k-i-1)-1}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}\\
&=q^{2k^2-2k+1}\begin{bmatrix}n-2k-1\\k-1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}+q^{k^2+k}\sum_{i=2}^{k}q^{(k-i)(k-i-1)-1}\begin{bmatrix}n-2k-1\\k-i\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}.
\end{aligned}
\tag{4}
$$

Applying Lemma 2.9 for $q\ge 3$ on the left hand side in (4) we find that

$$
\frac{1}{2}\sum_{i=0}^{k}q^{ik}\begin{bmatrix}n-i-1\\k-1\end{bmatrix}\le q^{(k-1)(n-k)}\sum_{i=0}^{k}q^{i}=q^{(k-1)(n-k)}\frac{q^{k+1}-1}{q-1}. \tag{5}
$$

Now applying Lemma 2.9 for $q\ge 3$ on the first term of the right hand side in (4) we find that

$$
q^{2k^2-2k+1}\begin{bmatrix}n-2k-1\\k-1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}\ge\left(1+\frac{1}{q}\right)q^{(k-1)(n-k)+1}\frac{q^k-1}{q-1}=(q+1)q^{(k-1)(n-k)}\frac{q^k-1}{q-1}. \tag{6}
$$

From (5) and (6) it follows that in order to prove (4), it is sufficient to show that the following inequality is valid:

$$
q^{k+1}-1\le(q+1)\left(q^k-1\right)\quad\Leftrightarrow\quad q^k\ge q.
$$

This statement is clearly true. □

Lemma 2.13 *If* $x \leq \frac{1}{\sqrt[8]{2}} f(q, n, k)$ *and* $n \geq 2k + 2$, *then* $\frac{\Delta}{C} > \sqrt[4]{2}x^2$.

Proof We want to prove that

$$\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} > \sqrt[4]{2}x^2 \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \right),$$

provided that $x \leq \frac{1}{\sqrt[8]{2}} f(q, n, k)$. We first look at the case $k \geq 2$. Given a k-space π in PG$(n-1, q)$, the number of $(k-1)$-spaces meeting π equals $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] - \left[\begin{smallmatrix} n-k-1 \\ k \end{smallmatrix}\right] q^{k^2+k}$ by Lemma 2.1. We know that this number is smaller than the product of the number of points $Q \in \pi$ and the number of $(k-1)$-spaces through Q. This implies that

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} &\leq \begin{bmatrix} k+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= \frac{q^{k+1}-1}{q-1} \cdot \frac{(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^{k-1}-1)\cdots(q-1)} \\ &< \frac{q^{nk-\frac{k^2}{2}-n+\frac{3k}{2}+1}}{(q-1)^{\frac{k^2}{2}-\frac{k}{2}+1}} . \end{aligned}$$

From this computation and the assumption on x it follows that

$$\begin{aligned} \sqrt[4]{2}x^2 \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \right) < (f(q,n,k))^2 \frac{q^{nk-\frac{k^2}{2}-n+\frac{3k}{2}+1}}{(q-1)^{\frac{k^2}{2}-\frac{k}{2}+1}} &= q^{nk-k^2-2}(q^2+q+1) \\ &\leq \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} , \end{aligned}$$

where the final inequality is given by Lemma 2.8 (which we can apply since $k \geq 2$).

Now we look at the case $k = 1$. We have to prove that

$$\begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2 > \sqrt[4]{2}x^2 \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-2 \\ 1 \end{bmatrix} q^2 \right) \quad \Leftrightarrow \quad \frac{q^{n-2}-1}{q^2-1} q^2 > \sqrt[4]{2}x^2 .$$

By the assumption on x it is sufficient to prove that

$$\frac{q^{n-2}-1}{q^2-1} q^2 > f(q, n, 1)^2 = q^{n-5}(q^3-1) \quad \Leftrightarrow \quad q^{n-2} + q^{n-3} - q^{n-5} - q^2 > 0 ,$$

which is clearly true since $n \geq 4$. □

Lemma 2.14 *Let* $\mathcal{L}$ *be a Cameron–Liebler set of* k*-spaces in* PG(n, q), $n \geq 3k + 2$, *with parameter* $2 \leq x \leq \frac{1}{\sqrt[8]{2}} f(q, n, k)$, *then* $\mathcal{L}$ *cannot contain* $\left\lfloor \frac{3}{2}x \right\rfloor$ *mutually disjoint* k*-spaces.*

Proof We apply Lemma 2.6 with $c + 1 = \left\lfloor \frac{3}{2}x \right\rfloor$ and have to show that

$$\left\lfloor \frac{3}{2}x \right\rfloor s_1 - \binom{\left\lfloor \frac{3}{2}x \right\rfloor}{2} s_2' > x \begin{bmatrix} n \\ k \end{bmatrix} .$$

Using Notation 2.10 and Lemma 2.12 we see that it is sufficient to prove that

$$\begin{aligned}
&\left\lfloor \frac{3}{2}x \right\rfloor (xC+\Delta) - x(\Delta+C) \\
&\qquad -\frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1\right)\left(xC-(x-2)\Delta+(x-2)\left(\Delta-\frac{C}{2}\right)\right) > 0 \\
\Leftrightarrow \quad & \Delta\left(\left\lfloor \frac{3}{2}x \right\rfloor - x\right) > C\left(x - \left\lfloor \frac{3}{2}x \right\rfloor x + \frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1\right)\left(\frac{x}{2}+1\right)\right) .
\end{aligned}$$

From Lemma 2.13, we know that $\frac{\Delta}{C} > \sqrt[4]{2}x^2$. Hence it is sufficient to prove that

$$\left(\left\lfloor \frac{3}{2}x \right\rfloor - x\right)\sqrt[4]{2}x^2 > x - \left\lfloor \frac{3}{2}x \right\rfloor x + \frac{1}{2}\left\lfloor \frac{3}{2}x \right\rfloor \left(\left\lfloor \frac{3}{2}x \right\rfloor - 1\right)\left(\frac{x}{2}+1\right) \tag{7}$$

for all admissible x. We denote $\frac{3}{2}x - \left\lfloor \frac{3}{2}x \right\rfloor$ by ε. Then, $0 \le \varepsilon < 1$. We rewrite (7) as

$$\begin{aligned}
&\left(\frac{3}{2}x - \varepsilon - x\right)\sqrt[4]{2}x^2 > x - \left(\frac{3}{2}x-\varepsilon\right)x + \frac{1}{2}\left(\frac{3}{2}x-\varepsilon\right)\left(\frac{3}{2}x-\varepsilon-1\right)\left(\frac{x}{2}+1\right) \\
\Leftrightarrow \quad & -\left(\frac{x+2}{4}\right)\varepsilon^2 + \left(\frac{(3-4\sqrt[4]{2})x^2+x-2}{4}\right)\varepsilon + \frac{(8\sqrt[4]{2}-9)x^3+12x^2-4x}{16} > 0 .
\end{aligned} \tag{8}$$

The nontrivial zero of the quadratic function $g(\varepsilon) = -\left(\frac{x+2}{4}\right)\varepsilon^2 + \left(\frac{(3-4\sqrt[4]{2})x^2+x-2}{4}\right)\varepsilon$ is smaller than 1 for any x, so $g(\varepsilon) > g(1)$ for any $\varepsilon \in [0, 1[$ regardless of x. So, to prove (8), it is sufficient to prove

$$\begin{aligned}
&\left(\frac{1}{2}\sqrt[4]{2} - \frac{9}{16}\right)x^3 + \left(\frac{3}{2} - \sqrt[4]{2}\right)x^2 - \frac{1}{4}x - 1 \ge 0 \\
\Leftrightarrow \quad & (x-2)\left((8\sqrt[4]{2}-9)x^2 + 6x + 8\right) \ge 0 ,
\end{aligned}$$

which is clearly true for $x \ge 2$. □

Lemma 2.15 *If* $2 \le x \le \frac{1}{\sqrt[8]{2}} f(q,n,k)$ *and* $n \ge 2k+2$ *and* $q \ge 3$, *then*

$$\begin{aligned}
&\frac{x-1}{\frac{3}{2}x-2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x-3\right)s_2' > x\begin{bmatrix} n \\ k \end{bmatrix} - x\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \quad \textit{and} \\
&\frac{x-1}{\frac{3}{2}x-2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x-3\right)s_2' > \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} + q^{k+1} .
\end{aligned}$$

Proof To prove the first inequality we rewrite it using Notation 2.10:

$$\frac{x-1}{\frac{3}{2}x-2}\Delta - \left(\frac{3}{2}x-3\right)(xC + (2-x)\Delta + (x-2)W_\Sigma) > xC .$$

Using Lemma 2.12 we see that it is sufficient to prove

$$\frac{x-1}{\frac{3}{2}x-2}\Delta > C\left(\frac{3}{4}x^2 + x - 3\right) .$$

From Lemma 2.13, we know that $\frac{\Delta}{C} > \sqrt[4]{2}x^2$. Hence it is sufficient to prove that

$$\frac{x-1}{\frac{3}{2}x-2}\sqrt[4]{2}x^2 > \left(\frac{3}{4}x^2+x-3\right) \quad \Leftrightarrow \quad \left(\sqrt[4]{2}-\frac{9}{8}\right)x^3 - \sqrt[4]{2}x^2 + \frac{13}{2}x - 6 > 0\,.$$

Using a computer algebra packet, we find that the last inequality is valid for all $x \geq 2$.

To prove the second inequality for $k \geq 2$ it is sufficient to prove that

$$x\begin{bmatrix} n \\ k \end{bmatrix} - x\begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} > \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k} + q^{k+1}$$
$$\Leftrightarrow \quad q^{k+1} < (x-1)\left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}q^{k^2+k}\right) = (x-1)\sum_{i=0}^{k} q^{ik}\begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix},$$

whereby we applied repeatedly the left equality in (1). We immediately see that

$$(x-1)\sum_{i=0}^{k} q^{ik}\begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix} > q^{k^2}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} > q^{(n-k)(k-1)+k} > q^{2k+2} > q^{k+1}\,.$$

For $k = 1$ we prove the second inequality directly. Note that $s'_2 = x + 2q$. The inequality reduces to

$$\frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 - \left(\frac{3}{2}x-3\right)(x+2q) > q^2+q+1$$
$$\Leftrightarrow \quad \frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 > \frac{3}{2}x^2 + 3(q-1)x + q^2 - 5q + 1\,. \tag{9}$$

Recall that $2 \leq x \leq \frac{1}{\sqrt[8]{2}}f(q, n, 1) = \frac{1}{\sqrt[8]{2}}q^{\frac{n-5}{2}}\sqrt{q^3-1} < q^{\frac{n-2}{2}}$. We look at the left hand side of (9) and find

$$\begin{aligned}\frac{x-1}{\frac{3}{2}x-2}\cdot\frac{q^{n-2}-1}{q-1}q^2 &= \left(\frac{2}{3}+\frac{2}{3(3x-4)}\right)\frac{q^{n-2}-1}{q-1}q^2 > \left(\frac{2}{3}+\frac{2}{9(x-1)}\right)\frac{q^{n-2}-1}{q-1}q^2 \\ &> \frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2 + \frac{2}{9\left(q^{\frac{n-2}{2}}-1\right)}\frac{q^{n-2}-1}{q-1}(q^2-1) \\ &= \frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2 + \frac{2}{9}\left(q^{\frac{n-2}{2}}+1\right)(q+1)\,.\end{aligned}$$

For the right hand side of (9) we find that

$$\begin{aligned}\frac{3}{2}x^2 + 3(q-1)x + q^2 - 5q + 1 &< \frac{3}{2\sqrt[4]{2}}q^{n-5}\left(q^3-1\right) + 3(q-1)q^{\frac{n-2}{2}} + q^2 - 5q + 1 \\ &< \frac{3}{2}q^{n-5}\left(q^3-1\right) + 3(q-1)q^{\frac{n-2}{2}} + q^2 - 5q + 1\,.\end{aligned}$$

So, to prove (9) it is sufficient to prove that

$$\frac{2}{3}\frac{q^{n-2}-1}{q-1}q^2 + \frac{2}{9}\left(q^{\frac{n-2}{2}}+1\right)(q+1) \geq \frac{3}{2}q^{n-5}\left(q^3-1\right) + 3(q-1)q^{\frac{n-2}{2}} + q^2 - 5q + 1$$
$$\Leftrightarrow \quad \frac{2}{3}q^{n-1} - \frac{5}{6}q^{n-2} + \frac{2}{3}\frac{q^{n-4}-1}{q-1}q^2 + \frac{3}{2}q^{n-5} - q^{\frac{n-2}{2}}\left(\frac{25}{9}q - \frac{29}{9}\right) - q^2 + \frac{47}{9}q - \frac{7}{9} \geq 0\,. \tag{10}$$

For $n = 4, 5$ we can check this to be true for all $q \geq 3$ using computer algebra software. For $n \geq 6$ we rewrite (10) as follows:

$$\frac{5}{18}(q-3)q^{n-2} + \frac{q^{\frac{n}{2}}}{18}\left(7q^{\frac{n-2}{2}} - 50\right) + \frac{2}{3}\frac{q^{n-4}-1}{q-1}q^2 + \left(\frac{29}{9}q^{\frac{n-2}{2}} - q^2\right) + \frac{47}{9}q + \left(\frac{3}{2}q^{n-5} - \frac{7}{9}\right) \geq 0\,.$$

Here each of the terms in the left hand side is positive for $q \geq 3$ since $n \geq 6$, which proves the second inequality in the statement for $k = 1$. □

Lemma 2.16 *If $\mathcal{L}$ is a Cameron–Liebler set of k-spaces in* $\mathrm{PG}(n, q)$*, $n \geq 3k+2$ and $q \geq 3$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} f(q, n, k)$, then $\mathcal{L}$ contains a point-pencil.*

Proof Let π be a k-space in $\mathcal{L}$ and let c be the maximum number of elements of $\mathcal{L}$ that are pairwise disjoint. By Definition 1.1, there are $(x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$ k-spaces in $\mathcal{L}$ disjoint from π. Within this collection of k-spaces, we find at most $c-1$ spaces $\sigma_1, \sigma_2, \ldots, \sigma_{c-1}$ that are pairwise disjoint. By Lemma 2.14, $c - 1 \leq \left\lfloor \frac{3}{2}x \right\rfloor - 2$. By the pigeonhole principle, we find an index i so that σ_i meets at least $\frac{x-1}{c-1}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} \geq \frac{x-1}{\left\lfloor \frac{3}{2}x \right\rfloor - 2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$ elements of $\mathcal{L}$ that are skew to π. We denote this collection of k-spaces disjoint from π and meeting σ_i in at least a point by $\mathcal{F}_i$.

Now we want to show that $\mathcal{F}_i$ contains a family of pairwise intersecting subspaces. For any σ_j with $j \neq i$, we find at most s'_2 elements that meet σ_i and σ_j. In this way, we find that there are at least $\frac{x-1}{\left\lfloor \frac{3}{2}x \right\rfloor - 2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - (c-2)s'_2 \geq \frac{x-1}{\frac{3}{2}x-2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x - 3\right) s'_2$ elements of $\mathcal{L}$ that meet σ_i, are disjoint from π and that are disjoint from σ_j for all $j \neq i$. We denote this subset of $\mathcal{F}_i \subseteq \mathcal{L}$ by $\mathcal{F}'_i$. This collection $\mathcal{F}'_i$ of k-spaces is a set of pairwise intersecting k-spaces: if two elements α, β in $\mathcal{F}'_i$ would be disjoint, then $(\{\sigma_1, \ldots, \sigma_{c-1}\} \setminus \{\sigma_i\}) \cup \{\alpha, \beta, \pi\}$ would be a collection of $c+1$ pairwise disjoint elements of $\mathcal{L}$, which is impossible since we supposed that c is size of the maximal set of pairwise disjoint k-space in $\mathcal{L}$. By Lemma 2.15 we have $\frac{x-1}{\frac{3}{2}x-2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - \left(\frac{3}{2}x - 3\right) s'_2 > \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} + q^{k+1}$ since $2 \leq x \leq \frac{1}{\sqrt[8]{2}} f(q, n, k)$. This implies that $\cap_{F \in \mathcal{F}'_i} F$ is not empty by Theorem 2.7; let P be a point contained in $\cap_{F \in \mathcal{F}'_i} F$. We conclude that $\mathcal{F}'_i$ is a part of the point-pencil through P.

We conclude by showing that $\mathcal{L}$ contains the whole point-pencil through P. If $\gamma \notin \mathcal{L}$ is a k-space through P, then γ meets at least $\frac{x-1}{\frac{3}{2}x-2}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} - (\frac{3}{2}x - 3)s'_2 > x\begin{bmatrix} n \\ k \end{bmatrix} - x\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$ elements of $\mathcal{F}'_i \subseteq \mathcal{L}$, where the inequality follows from Lemma 2.15. This contradicts Definition 1.1. □

Theorem 2.17 *There are no Cameron–Liebler sets of k-spaces in* $\mathrm{PG}(n, q)$*, $n \geq 3k+2$ and $q \geq 3$, with parameter $2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2} - \frac{k^2}{4} - \frac{3k}{4} - \frac{3}{2}} (q-1)^{\frac{k^2}{4} - \frac{k}{4} + \frac{1}{2}} \sqrt{q^2+q+1}$.*

Proof We prove this result using induction on x. By Lemma 2.16 we know that $\mathcal{L}$ contains the point-pencil $[P]_k$ through a point P. By Lemma 2.2, $\mathcal{L} \setminus [P]_k$ is a Cameron–Liebler set of k-spaces with parameter $x - 1$, which by the induction hypothesis (in case $x - 1 \geq 2$) or by [2, Lemma 4.1] (in case $1 < x - 1 < 2$) does not exist, or which is a point-pencil (in case $x - 1 = 1$) by [2, Theorem 4.3]. In the former case there is an immediate contradiction; in the latter case $\mathcal{L}$ contains two disjoint point-pencils of k-spaces, a contradiction. □

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References

1. Blokhuis A., Brouwer A.E., Chowdhury A., Frankl P., Mussche T., Patkós B., Szőnyi T.: A Hilton-Milner theorem for vector spaces. Electron. J. Comb. **17**, 71 (2010).
2. Blokhuis A., De Boeck M., D'haeseleer J.: Cameron-Liebler classes of k-spaces in PG(n, q). Des. Codes Cryptogr. **87**(8), 1839–1856 (2019).
3. Filmus Y., Ihringer F.: Boolean degree 1 functions on some classical association schemes. J. Comb. Theory Ser. A **162**, (2019).
4. Ihringer F.: Finite Geometry Intersecting Algebraic Combinatorics. PhD thesis, Justus-Liebig-Universität Gießen (2015).
5. Ihringer F.: Remarks on the Erdős Matching Conjecture for Vector Spaces. (2020).
6. Segre B.: Lectures on Modern Geometry (with an appendix by L. Lombardo-Radice). Consiglio Nazionale delle Ricerche, Monografie Mathematiche. Edizioni Cremonese, Roma, p. 479 (1961).