

# STRONGLY REGULAR CONFIGURATIONS

MARIÉN ABREU<sup>1</sup>, MARTIN FUNK<sup>1</sup>, VEDRAN KRČADINAC<sup>2</sup>,  
AND DOMENICO LABBATE<sup>1</sup>

<sup>1</sup>*Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Potenza, Italy*

<sup>2</sup>*Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, HR-10000 Zagreb, Croatia*

**ABSTRACT.** We study combinatorial configurations with the associated point and line graphs being strongly regular. Examples not belonging to known classes such as partial geometries and their generalizations or elliptic semiplanes are constructed. Necessary existence conditions are proved and a table of feasible parameters of such configurations with at most 200 points is presented. Non-existence of some configurations with feasible parameters is proved.

## 1. INTRODUCTION

A (*combinatorial*) *configuration* is a finite partial linear space with constant point and line degrees. If there are  $v$  points of degree  $r$  and  $b$  lines of degree  $k$ , the parameters are written  $(v_r, b_k)$ . If  $v = b$ , or equivalently  $r = k$ , the configuration is called *symmetric* and the parameters are written  $(v_k)$ . Throughout the paper we assume  $k \geq 3$  and  $r \geq 3$ .

The *point graph* of a configuration has the  $v$  points as vertices, with two vertices being adjacent if the points are collinear. The *line graph* is defined dually: the  $b$  lines of the configuration are the vertices, and

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*E-mail address:* marien.abreu@unibas.it, martin.funk@unibas.it, vedran.krčadinac@math.hr, domenico.labbate@unibas.it.

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adjacency is concurrence.<sup>1</sup> The point and line graphs are regular of degrees  $r(k-1)$  and  $k(r-1)$ , respectively. A graph is called *strongly regular* with parameters  $SRG(n, d, \lambda, \mu)$  if it has  $n$  vertices, is regular of degree  $d$ , and every two vertices have  $\lambda$  common neighbors if they are adjacent and  $\mu$  common neighbors if they are not adjacent. We are interested in configurations with both associated graphs being strongly regular.

A prominent family of such configurations are the *partial geometries*  $pg(s, t, \alpha)$ , introduced by R. C. Bose [3]. These are configurations with  $r = t + 1$  and  $k = s + 1$  such that for every non-incident point-line pair  $(P, \ell)$ , there are exactly  $\alpha$  points on  $\ell$  collinear with  $P$ . The point graph is a

$$SRG\left(\frac{(s+1)(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right), \quad (1)$$

and the line graph is a

$$SRG\left(\frac{(t+1)(st+\alpha)}{\alpha}, t(s+1), t-1+s(\alpha-1), \alpha(s+1)\right). \quad (2)$$

Partial geometries include Steiner 2-designs  $pg(s, t, s+1)$  and their duals  $pg(s, t, t+1)$ , Bruck nets  $pg(s, t, t)$  [9, 10] and their duals  $pg(s, t, s)$  (transversal designs), and generalized quadrangles  $pg(s, t, 1)$  as special cases.

If  $v \neq b$ , partial geometries are the only configurations with both associated graphs strongly regular. This follows from [6, Theorem 1.2]:

**Theorem 1.1.** *Let the point graph of a  $(v_r, b_k)$  configuration be strongly regular. Then the configuration is a partial geometry or  $v \leq b$ . If  $v = b$ , then  $\det(A + kI)$  is a square, where  $A$  is the adjacency matrix of the point graph.*

If  $v = b$ , there are such configurations that are not partial geometries. The smallest examples are  $(10_3)$  configurations with associated graphs  $SRG(10, 6, 3, 4)$  (the complement of the Petersen graph). One such configuration is the Desargues configuration, which is a semipartial geometry for  $\alpha = 2$  and  $\mu = 4$  (see Section 3 for the definition). There is another such configuration not belonging to the known generalizations of partial geometries such as semipartial geometries [15, 20, 16] and strongly regular  $(\alpha, \beta)$ -geometries [28], represented in Figure 1.

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<sup>1</sup>We will always use the term *line graph* in this sense, and not in the sense of graph theory (the line graph  $L(G)$  of a graph  $G$ ).

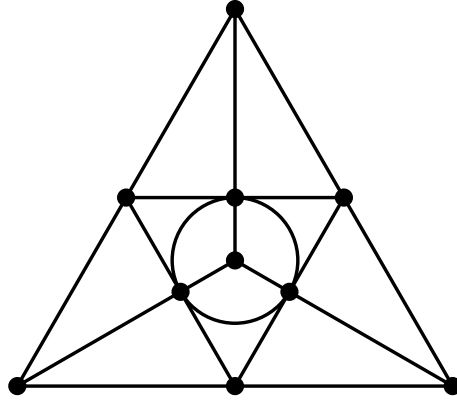


FIGURE 1. A strongly regular configuration that is not an  $(\alpha, \beta)$ -geometry.

In this paper we study combinatorial configurations similar to this one. In Section 2 we give the definition of a strongly regular configuration. The concept unifies known classes such as (semi)partial geometries and elliptic semiplanes [21] with several sporadic examples from the literature (see Remark on page 37 of [6] and Section 7.2 of [7]). We focus on strongly regular configurations that are proper and primitive, not belonging to the known classes. We prove two necessary conditions on the parameters of strongly regular configurations, stronger than conditions on the parameters of the associated strongly regular graphs.

In Section 3 we present families of strongly regular configurations. A family associated with Moore graphs and a family constructed from the points and planes of the finite projective space  $PG(4, q)$  have the same parameters as semipartial geometries. We prove that there are strongly regular configurations with these parameters that are not semipartial geometries. A third family constructed from finite projective planes has parameters not compatible with semipartial geometries.

Section 4 contains constructions of strongly regular configurations from difference sets in groups. Some of the configurations from the previous section can also be constructed in this way. We perform an exhaustive search in groups of order  $v \leq 200$  and find three other parameter sets  $(v_k; \lambda, \mu)$  for which strongly regular configurations exist. Configurations with a fourth parameter set are constructed in a different manner.

In the final Section 5 we present a table of feasible parameters of strongly regular configurations with  $v \leq 200$ . An on-line version of the

table with links to the actual configurations is available on the web page

<https://web.math.pmf.unizg.hr/~krcko/results/srconf.html>

We perform complete classifications of configurations with small parameters and prove non-existence for infinitely many feasible parameter sets corresponding to rook graphs.

## 2. DEFINITIONS AND CONDITIONS ON THE PARAMETERS

In view of the motivation presented in the Introduction, we make the following definition.

**Definition 2.1.** *A symmetric configuration will be called a strongly regular configuration with parameters  $(v_k; \lambda, \mu)$  if the associated point graph is a  $SRG(v, k(k-1), \lambda, \mu)$ .*

We can prove that the line graph is also strongly regular with the same parameters. We use the following lemma from [12].

**Lemma 2.2.** *Suppose that the point graph of a  $(v_r, b_k)$  configuration is strongly regular with parameters (1) corresponding to a partial geometry  $pg(s, t, \alpha)$ . Then the configuration is a  $pg(s, t, \alpha)$ .*

**Theorem 2.3.** *Given a strongly regular  $(v_k; \lambda, \mu)$  configuration, the associated line graph is a  $SRG(v, k(k-1), \lambda, \mu)$ .*

*Proof.* A graph is strongly regular with parameters  $SRG(v, d, \lambda, \mu)$  if and only if its adjacency matrix  $A$  satisfies

$$A^2 = dI + \lambda A + \mu(J - I - A). \quad (3)$$

Here  $I$  and  $J$  are the  $v \times v$  identity matrix and the all-one matrix. Let  $N$  be the incidence matrix of the configuration. Then,  $A = NN^t - kI$  and  $B = N^tN - kI$  are adjacency matrices of the point and line graphs, respectively. By (3), we have

$$NN^tNN^t + (\mu - \lambda - 2k)NN^t + (k(\lambda - \mu + 1) + \mu)I - \mu J = 0.$$

If the incidence matrix  $N$  is non-singular, we can multiply by  $N^{-1}$  from the left and by  $N$  from the right. Using  $N^{-1}J = \frac{1}{k}J$  and  $JN = kJ$ , we get

$$N^tNN^tN + (\mu - \lambda - 2k)N^tN + (k(\lambda - \mu + 1) + \mu)I - \mu J = 0.$$

This is equation (3) for the matrix  $B$ , and therefore the line graph is also strongly regular with the same parameters.

Now assume that the incidence matrix  $N$  is singular. Then the matrix  $N$  has eigenvalue 0 and the matrix  $A$  has eigenvalue  $-k$ . Thus,  $k^2 - (\mu - \lambda)k + \mu - k(k-1) = 0$  holds and  $k$  divides  $\mu$ . Denoting  $\alpha = \mu/k$ ,

we see that the parameters of the point graph correspond to a partial geometry (1). By Lemma 2.2, the configuration is a partial geometry and the line graph is also strongly regular with parameters (2). The same argument was used in the proof of [6, Theorem 1.2].  $\square$

We shall call strongly regular configurations with non-singular incidence matrices *proper*. The previous proof shows that improper configurations must be partial geometries. The parameters of a strongly regular configuration are not independent. A necessary condition for the existence of a  $SRG(v, k(k-1), \lambda, \mu)$  is

$$(v-1-k(k-1))\mu = k(k-1)(k(k-1)-1-\lambda).$$

From this,  $v$  can be expressed from  $k$ ,  $\lambda$ , and  $\mu$ , provided  $\mu \neq 0$ . There are many other necessary conditions on the parameters of a  $SRG(v, k(k-1), \lambda, \mu)$ . The adjacency matrix has eigenvalue  $k(k-1)$  with multiplicity 1 and two more eigenvalues

$$r, s = \frac{1}{2} \left( \lambda - \mu \pm \sqrt{(\lambda - \mu)^2 - 4(\mu - k(k-1))} \right) \quad (4)$$

with respective multiplicities

$$f, g = \frac{1}{2} \left( v - 1 \mp \frac{(r+s)(v-1) + 2k(k-1)}{r-s} \right). \quad (5)$$

The multiplicities are integers, giving divisibility conditions on the parameters. If  $f \neq g$ , the eigenvalues  $r, s$  are also integers. See [8] for further necessary conditions and [4] for tables of parameters of strongly regular graphs with up-to-date information on their existence. The parameters  $(v_k; \lambda, \mu)$  of a strongly regular configuration will be considered *feasible* if the associated strongly regular graphs exist or their existence cannot be ruled out. On top of that, we assume two more necessary conditions on the parameters. The first condition follows from [6, Theorem 1.2]:  $\det(A+kI) = \det(NN^t) = (\det N)^2$  is a square. The matrix  $A+kI$  has eigenvalues  $k^2, r+k, s+k$  with multiplicities 1,  $f, g$  and the determinant can be computed from the parameters.

**Proposition 2.4.** *If a strongly regular  $(v_k; \lambda, \mu)$  configuration exists, then  $(r+k)^f(s+k)^g$  is the square of an integer, where  $r, s, f, g$  are given by (4) and (5).*

For example, the condition rules out strongly regular  $(28_4; 6, 4)$  configurations, although  $SRG(28, 12, 6, 4)$  graphs exist. Equations (4) and (5) give  $r = 4, s = -2, f = 7, g = 20$ , and  $2^{41}$  is not a square. The second condition follows from a counting argument.

**Theorem 2.5.** *If a strongly regular  $(v_k; \lambda, \mu)$  configuration exists, then  $(v-k)(\lambda+1) \geq k(k-1)^3$ . Equality holds if and only if the configuration is a partial geometry.*

*Proof.* Fix a line  $\ell$  and for any point  $P$  not on  $\ell$ , denote by  $\alpha_P$  the number of lines through  $P$  concurrent with  $\ell$ . Count the number of flags  $(P, \ell_1)$  with  $\ell_1$  concurrent with  $\ell$  in two ways to obtain

$$\sum \alpha_P = k(k-1)^2.$$

Similarly, counting triples  $(P, \ell_1, \ell_2)$ , where  $\ell_1 \neq \ell_2$  are lines through  $P$  concurrent with  $\ell$ , gives

$$\sum \alpha_P(\alpha_P - 1) = k(k-1)(\lambda - (k-2)).$$

The sums are taken over all  $P \notin \ell$ . The average  $\alpha_P$  is  $\alpha = \frac{k(k-1)^2}{v-k}$ . Now we can compute

$$\begin{aligned} 0 &\leq \sum (\alpha_P - \alpha)^2 = \sum \alpha_P(\alpha_P - 1) + (1 - 2\alpha) \sum \alpha_P + (v-k)\alpha^2 = \\ &= k(k-1)(\lambda - k + 2) + \left(1 - 2\frac{k(k-1)^2}{v-k}\right) k(k-1)^2 + \frac{k^2(k-1)^4}{v-k} = \\ &= k(k-1) \left( \lambda + 1 - \frac{k(k-1)^3}{v-k} \right). \end{aligned}$$

From this we see that  $(v-k)(\lambda+1) \geq k(k-1)^3$  holds, with equality if and only if  $\alpha_P = \alpha$  for all  $P \notin \ell$ , i.e. the configuration is a partial geometry.  $\square$

For example, the parameters  $(81_5; 1, 6)$  do not satisfy Theorem 2.5 and strongly regular configurations with these parameters don't exist, although a  $SRG(81, 20, 1, 6)$  graph does. An equivalent form of the inequality  $k(\mu - \lambda - 1) \leq \mu$  follows from Hoffman's bound on the size of cliques in strongly regular graphs; see [5, Section 1.3]. Theorem 2.5 characterizes proper strongly regular configurations by their parameters.

**Corollary 2.6.** *A strongly regular  $(v_k; \lambda, \mu)$  configuration that is not a projective plane is proper if and only if  $(v-k)(\lambda+1) > k(k-1)^3$ .*

Projective planes of order  $n$  are partial geometries  $pg(n, n, n+1)$  and satisfy Theorem 2.5 with equality, but have non-singular incidence matrices. The associated point and line graphs are complete. More generally, we now consider the case when the associated graphs are imprimitive, i.e.  $\mu = 0$  or  $\mu = k(k-1)$  holds. In the first case the graphs are disjoint unions of complete graphs  $m \cdot K_{n^2+n+1}$  and the configuration is a disjoint union of  $m$  projective planes of order  $n$ .

This case can be characterized as strongly regular configurations with collinearity of points being an equivalence relation.

The second imprimitive case  $\mu = k(k - 1)$  is complementary: non-collinearity of points is an equivalence relation and the associated graphs are complete multipartite. Strongly regular configurations with these properties are known as *elliptic semiplanes*. Dembowski [21] defined a *finite semiplane* as a partial linear space with parallelism of lines and non-collinearity of points being equivalence relations. A semiplane is of *order*  $n$  if the largest degree of a point or line is  $n + 1$ . Dembowski proved that the set of all degrees is either  $\{n - 1, n, n + 1\}$ ,  $\{n, n + 1\}$ , or  $\{n + 1\}$ , and called semiplanes *hyperbolic*, *parabolic*, or *elliptic* accordingly. Elliptic semiplanes are precisely the strongly regular configurations with  $\mu = k(k - 1)$ . Most known elliptic semiplanes are of the form  $\mathcal{P} - B$ , where  $\mathcal{P}$  is a projective plane of order  $n$ , and  $B$  is a closed Baer subset. The only known exceptions are Baker's semiplane [1] with parameters  $(45_7; 39, 42)$  and Mathon's semiplane [34] with parameters  $(135_{12}; 129, 132)$ .

In the sequel we focus on strongly regular configurations that are proper and *primitive*, i.e. such that neither collinearity nor non-collinearity of points are equivalence relations. This is equivalent with  $0 < \mu < k(k - 1)$ . Table 3 contains all feasible parameters of such configurations with  $v \leq 200$ . We first present constructions of proper and primitive strongly regular configurations.

### 3. FAMILIES OF STRONGLY REGULAR CONFIGURATIONS

An  $(\alpha, \beta)$ -*geometry* [19] is a  $(v_r, b_k)$  configuration such that for every non-incident point-line pair  $(P, \ell)$ , there are either  $\alpha$  or  $\beta$  points on  $\ell$  collinear with  $P$ . Thus, a partial geometry is an  $(\alpha, \beta)$ -geometry with  $\alpha = \beta$ . If  $\alpha \neq \beta$ , the point graph is not necessarily strongly regular. Geometries with this additional property are called *strongly regular  $(\alpha, \beta)$ -geometries* and are studied in [28]. An important special case are the *semipartial geometries*, introduced in [15]. They are  $(0, \alpha)$ -geometries such that for every pair of non-collinear points, there are exactly  $\mu$  points collinear with both. The parameters are written  $(s, t, \alpha, \mu)$ , where  $r = t + 1$  and  $k = s + 1$  are the point and line degrees, and the point graph is a

$$SRG \left( 1 + \frac{s(t+1)(\mu + t(s+1-\alpha))}{\mu}, s(t+1), s-1+t(\alpha-1), \mu \right).$$

Strongly regular  $(\alpha, \beta)$ -geometries with  $v = b$  are strongly regular configurations by Definition 2.1. Our introductory example in Figure 1 is not an  $(\alpha, \beta)$ -geometry, although the parameters correspond to a

semipartial geometry. If  $\ell$  is the line represented as a circle, there are points  $P$  outside  $\ell$  with 1, 2, or 3 points on  $\ell$  collinear with  $P$ . This example is part of a family associated with Moore graphs of diameter two, i.e. strongly regular graphs with  $\lambda = 0$  and  $\mu = 1$ .

Moore graphs have parameters  $SRG(k^2 + 1, k, 0, 1)$  with  $k \in \{2, 3, 7, 57\}$  [29]. There is a unique graph for  $k = 2$  (the pentagon),  $k = 3$  (the Petersen graph), and  $k = 7$  (the Hoffman-Singleton graph), while for  $k = 57$  the existence of such a graph is unknown. The incidence structure with points being vertices of a  $SRG(k^2 + 1, k, 0, 1)$  and lines being neighborhoods of single vertices is a semipartial geometry with  $s = t = \alpha = k - 1$  and  $\mu = (k - 1)^2$  [15]. The point graph is the complementary  $SRG(k^2 + 1, k(k - 1), k(k - 2), (k - 1)^2)$ . Hence, this incidence structure is a strongly regular  $(v_k; \lambda, \mu)$  configuration with  $v = k^2 + 1$ ,  $\lambda = k(k - 2)$ , and  $\mu = (k - 1)^2$ .

For  $k = 3$ , the semipartial geometry is the Desargues configuration and there is one other  $(10_3; 3, 4)$  configuration given in Figure 1. For  $k = 7$ , the semipartial geometry has full automorphism group  $PSU(3, 5) : \mathbb{Z}_2$  of order 252000 acting flag-transitively. We found 210 other  $(50_7; 35, 36)$  configurations that are not  $(\alpha, \beta)$ -geometries. The semipartial geometry and 110 of the new examples are self-dual and the remaining ones form 50 dual pairs.

**Proposition 3.1.** *There are at least 211 non-isomorphic  $(50_7; 35, 36)$  configurations, one of which is a semipartial geometry. Orders of their full automorphism groups are given in Table 1.*

| Aut    | #Cf | Aut | #Cf | Aut | #Cf | Aut | #Cf | Aut | #Cf |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 252000 | 1   | 120 | 1   | 40  | 1   | 20  | 6   | 6   | 13  |
| 2520   | 1   | 96  | 1   | 36  | 1   | 16  | 3   | 4   | 15  |
| 1440   | 1   | 72  | 1   | 32  | 1   | 12  | 1   | 3   | 18  |
| 720    | 1   | 48  | 1   | 24  | 6   | 10  | 1   | 2   | 46  |
| 240    | 1   | 42  | 1   | 21  | 2   | 8   | 11  | 1   | 76  |

TABLE 1. Distribution of  $(50_7; 35, 36)$  configurations by order of full automorphism group.

The configurations of Proposition 3.1 are available through the on-line version of Table 3. They were constructed computationally, by prescribing automorphism groups and switching submatrices of the incidence matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$



We used GAP [23] and our own programs written in C. To check for isomorphism and compute full automorphism groups, we used nauty [36]. The construction method for configurations with prescribed automorphism groups is similar to constructions of quasi-symmetric designs in [33, 31] and relies on the clique-finding program Cliquer [39].

Another family of semipartial geometries is family (g) from [15], denoted by  $LP(n, q)$  in [20, 16]. The points of  $LP(n, q)$  are lines of the projective space  $PG(n, q)$ ,  $n \geq 3$ . The lines of  $LP(n, q)$  are 2-planes of  $PG(n, q)$  and incidence is inclusion. Then,  $LP(n, q)$  is a semipartial geometry with  $s = q(q+1)$ ,  $t = \frac{q^{n-1}-1}{q-1} - 1$ ,  $\alpha = q+1$ , and  $\mu = (q+1)^2$ . It is a partial geometry if and only if  $n = 3$ . Moreover,  $v = b$  holds if and only if  $n = 4$ . Thus,  $LP(4, q)$  is a  $(v_k; \lambda, \mu)$  configuration with

$$\begin{aligned} v &= (q^2 + 1)(q^4 + q^3 + q^2 + q + 1), \quad k = q^2 + q + 1, \\ \lambda &= q^3 + 2q^2 + q - 1, \quad \mu = (q + 1)^2. \end{aligned} \tag{6}$$

It is self-dual and has full automorphism group  $P\Gamma L(5, q)$ .

We now describe transformations of  $LP(4, q)$  into strongly regular configurations that are not semipartial geometries. We refer to them as *polarity transformations*; they are similar to the construction of polarity designs in [30]. Let  $H_0$  be a hyperplane of  $PG(4, q)$ . As a subgeometry,  $H_0$  is isomorphic to  $PG(3, q)$  and admits a polarity  $\pi$ , i.e. an inclusion-reversing involution. The polarity permutes the set of projective lines contained in  $H_0$  and exchanges the set of points in  $H_0$  with the set of planes in  $H_0$ . We modify incidence of the elements of  $LP(4, q)$  contained in  $H_0$ : a point  $L$  (projective line contained in  $H_0$ ) is incident with a line  $p$  (projective plane contained in  $H_0$ ) if  $\pi(L) \subseteq p$ . For the remaining pairs  $(L, p)$ , with  $L$  or  $p$  not contained in  $H_0$ , incidence remains unaltered. We claim that the new incidence structure  $LP(4, q)^\pi$  is a  $(v_k; \lambda, \mu)$  configuration with parameters (6).

The point and line degrees clearly remain the same and there is at most one line through every pair of points. The point graphs of  $LP(4, q)^\pi$  and  $LP(4, q)$  are identical. This follows from the next lemma.

**Lemma 3.2.** *Two projective lines of  $PG(n, q)$  are coplanar if and only if they intersect.*

If  $L_1$  and  $L_2$  are projective lines of  $H_0$ , then  $\pi(L_1)$ ,  $\pi(L_2)$  are contained in a plane  $p$  if and only if  $L_1$ ,  $L_2$  intersect in the point  $\pi(p)$  and hence, by Lemma 3.2, are contained in some plane  $p'$ . The line graph of  $LP(4, q)^\pi$  is changed, but remains strongly regular because of Theorem 2.3.

To see that the new configuration  $LP(4, q)^\pi$  is not a semipartial geometry, take a plane  $p$  in  $H_0$  and a projective line  $L$  intersecting  $H_0$

in the point  $\pi(p)$ . Then,  $(L, p)$  is a non-incident point-line pair of  $LP(4, q)^\pi$ . If  $\pi(M) \subseteq p$ , then  $M$  contains  $\pi(p)$  and is coplanar with  $L$ , i.e. collinear as a point of the configuration. Hence, all  $q^2 + q + 1$  points on  $p$  are collinear with  $L$ , whereas in a semipartial geometry the number is always 0 or  $\alpha = q + 1$ . The configurations  $LP(4, q)$  and  $LP(4, q)^\pi$  are therefore not isomorphic. Configurations obtained by transforming  $LP(4, q)$  with different polarities are all isomorphic, because the composition of two polarities is an isomorphism.

We define a dual transformation of  $LP(4, q)$  in the following manner. Take a point  $P_0$  of  $PG(4, q)$  and consider the quotient geometry of lines, planes and solids containing  $P_0$ . It is isomorphic to  $PG(3, q)$  and admits a polarity  $\pi'$  permuting the planes through  $P_0$  and exchanging the lines and solids through  $P_0$ . We modify incidence in  $LP(4, q)$  for projective lines  $L$  and planes  $p$  through  $P_0$ : they are incident if  $L \subseteq \pi'(p)$ . The new configuration  $LP(4, q)_{\pi'}$  is isomorphic to the dual of  $LP(4, q)^\pi$  and therefore strongly regular with parameters (6), but not a semipartial geometry. The line graphs of  $LP(4, q)_{\pi'}$  and  $LP(4, q)$  are identical, while the point graph of  $LP(4, q)_{\pi'}$  is changed.

A fourth  $(v_k; \lambda, \mu)$  configuration is obtained if we take a non-incident point-hyperplane pair  $P_0, H_0$  of  $PG(4, q)$  and apply both transformations. The lines and planes in  $H_0$  are different from the lines and planes through  $P_0$ , so incidence is changed in disjoint parts of the configuration. The resulting configuration  $LP(4, q)_{\pi'}^\pi$  has the same line graph as  $LP(4, q)^\pi$  and the same point graph as  $LP(4, q)_{\pi'}$  and is self-dual. This proves the following theorem.

**Theorem 3.3.** *For every prime power  $q$ , there are at least four strongly regular  $(v_k; \lambda, \mu)$  configuration with parameters (6). One of them is the semipartial geometry  $LP(4, q)$  and the others are not semipartial geometries.*

We now present an infinite family of strongly regular configurations with parameters different from semipartial geometries. The construction works by deleting a suitable subset from a projective plane, similarly as constructions of elliptic semiplanes.

**Theorem 3.4.** *Let  $\mathcal{P}$  be a projective plane of order  $n \geq 5$  and  $A, B, C$  be three non-collinear points. By deleting all points on the lines  $AB, AC, BC$  and all lines through the points  $A, B, C$ , there remains a strongly regular  $(v_k; \lambda, \mu)$  configuration with  $v = (n - 1)^2$ ,  $k = n - 2$ ,  $\lambda = (n - 4)^2 + 1$ , and  $\mu = (n - 3)(n - 4)$ . This configuration is not an  $(\alpha, \beta)$ -geometry.*

*Proof.* The number of points and lines in the remaining configuration is  $v = n^2 + n + 1 - 3 - 3(n - 1) = (n - 1)^2$  and they are of degree  $k = n - 2$ . Let  $P$  and  $Q$  be two remaining points that are collinear, i.e. are not on a line of  $\mathcal{P}$  through  $A$ ,  $B$  or  $C$ . Then the points non-collinear with  $P$  are the remaining points on the lines  $AP$ ,  $BP$ ,  $CP$ . There are  $3(n - 2)$  such points, and as many for  $Q$ . The points non-collinear with both  $P$  and  $Q$  are the intersections of one of the lines  $AP$ ,  $BP$ ,  $CP$  with one of the lines  $AQ$ ,  $BQ$ ,  $CQ$ ; there are 6 such points. By inclusion-exclusion, the number of points in the remaining configuration collinear with both  $P$  and  $Q$  is  $\lambda = (n - 1)^2 - 2 - 6(n - 2) + 6 = (n - 4)^2 + 1$ . If the points  $P$  and  $Q$  are non-collinear, a similar count shows that the number of points in the remaining configuration collinear with both  $P$  and  $Q$  is  $\mu = (n - 3)(n - 4)$ .

Let  $(P, \ell)$  be a non-incident point-line pair of the remaining configuration. We now count the points on  $\ell$  collinear with  $P$ . Let  $A'$ ,  $B'$ ,  $C'$  be the intersections of  $BC$ ,  $AC$ ,  $AB$  with  $\ell$ . These are the deleted points of  $\ell$ . If the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent,  $P$  lies on 0, 1 or 3 of these lines. Then there are  $n - 5$ ,  $n - 4$  or  $n - 2$  points on  $\ell$  collinear with  $P$ . In this case, the points  $A$ ,  $B$ ,  $C$ ,  $A'$ ,  $B'$ ,  $C'$  and the common point of  $AA'$ ,  $BB'$ ,  $CC'$  form a Fano subplane, so this can only occur if  $n$  is even or  $\mathcal{P}$  is non-Desarguesian. On the other hand, if the lines  $AA'$ ,  $BB'$ ,  $CC'$  are not concurrent,  $P$  lies on 0, 1 or 2 of these lines and there are  $n - 5$ ,  $n - 4$  or  $n - 3$  points on  $\ell$  collinear with  $P$ . In both cases there are three possibilities for the number of points on  $\ell$  collinear with  $P$ , so the configuration is not an  $(\alpha, \beta)$ -geometry.  $\square$

The associated graphs have parameters

$$SRG((n - 1)^2, (n - 2)(n - 3), (n - 4)^2 + 1, (n - 3)(n - 4)).$$

These are pseudo Latin square graphs  $LS_{n-3}(n - 1)$ , see [8, Section 8.4.2]. For  $n = 5$ , we get the Shrikhande graph [44] which is not a Latin square graph. For  $n = 7$ , the graphs have parameters  $LS_4(6)$  and are not Latin square graphs because there are no orthogonal Latin squares of order 6.

In the smallest case  $n = 5$ , the  $(16_3; 2, 2)$  configuration of Theorem 3.4 can be extended to a  $(16_4; 8, 12)$  configuration by adding a point to every line. This is a  $(4, 4)$ -net and can be embedded in the projective plane of order 4. This is an interesting transformation of the projective plane of order 5 into the projective plane of order 4, but it does not generalize to  $n > 5$ .

In the Desarguesian projective plane  $PG(2, q)$ , all triangles  $\{A, B, C\}$  are equivalent and Theorem 3.4 gives just one strongly regular configuration up to isomorphism, being self-dual. The smallest non-Desarguesian projective planes are of order 9: the Hall plane, its dual, and the self-dual Hughes plane. The Hall plane contains six inequivalent triangles and as many non-isomorphic  $(64_7; 26, 30)$  configurations arise from Theorem 3.4. These configurations are not self-dual. Of course, they are duals of the configurations derived from the dual Hall plane. The Hughes plane contains 16 inequivalent triangles. The corresponding configurations are not isomorphic; 10 are self-dual and there are 3 dual pairs. Information on the orders of full automorphism groups of these configurations is given in Table 2.

| Plane      | Aut | #Cf | Plane  | Aut | #Cf |
|------------|-----|-----|--------|-----|-----|
| $PG(2, 9)$ | 768 | 1   | Hughes | 144 | 1   |
| Hall       | 768 | 1   |        | 48  | 1   |
|            | 96  | 2   |        | 32  | 1   |
|            | 12  | 2   |        | 18  | 1   |
|            | 6   | 1   |        | 12  | 3   |
| Dual Hall  | 768 | 1   |        | 6   | 4   |
|            | 96  | 2   |        | 4   | 3   |
|            | 12  | 2   |        | 2   | 1   |
|            | 6   | 1   |        | 1   | 1   |

TABLE 2. Distribution of  $(64_7; 26, 30)$  configurations by order of full automorphism group.

Configurations obtained from different projective planes of order 9 are not isomorphic. Hence, the total number of  $(64_7; 26, 30)$  configurations arising from Theorem 3.4 is 29. We could not find any other examples with these parameters. This, together with the uniqueness results of Section 5 (Corollary 5.3 and Proposition 5.6), seems to suggest that every strongly regular configuration with parameters from Theorem 3.4 can be uniquely embedded in a projective plane of order  $n$ , but we do not have a proof.

#### 4. STRONG DEFICIENT DIFFERENCE SETS

Next we present constructions of strongly regular configurations using difference sets. Let  $G$  be a group of order  $v$ . A subset  $D \subseteq G$  of size  $k$  is a *deficient difference set* if for every  $x \in G \setminus \{1\}$ , there is at

most one pair  $(d_1, d_2) \in D \times D$  such that  $x = d_1^{-1}d_2$ . Shortly, the left differences  $d_1^{-1}d_2$  must all be distinct. This is equivalent with the right differences  $d_1d_2^{-1}$  being distinct. The elements of  $G$  as points and the *development*  $\text{dev } D = \{gD \mid g \in G\}$  as lines form a symmetric  $(v_k)$  configuration. The configuration has  $G$  an automorphism group acting regularly on the points and lines [22, 38]. In the cyclic case  $G = \mathbb{Z}_v$ , deficient difference sets are also called *modular Golomb rulers* [11].

Let  $\Delta(D) = \{d_1^{-1}d_2 \mid d_1, d_2 \in D, d_1 \neq d_2\}$  be the set of left differences of  $D$ . This is a subset of  $G \setminus \{1\}$  of size  $k(k-1)$ . For a group element  $x \neq 1$ , denote by  $n(x) = |\Delta(D) \cap x\Delta(D)|$ . Suppose that  $n(x) = \lambda$  for every  $x \in \Delta(D)$ , and  $n(x) = \mu$  for every  $x \notin \Delta(D)$ . We shall call a subset  $D$  with this property a *strong deficient difference set (SDDS)* for  $(v_k; \lambda, \mu)$ .

**Theorem 4.1.** *Let  $G$  be a group and  $D \subseteq G$  a strong deficient difference set for  $(v_k; \lambda, \mu)$ . Then,  $(G, \text{dev } D)$  is a strongly regular  $(v_k; \lambda, \mu)$  configuration with  $G$  as an automorphism group acting regularly on the points and lines. Conversely, any strongly regular  $(v_k; \lambda, \mu)$  configuration with an automorphism group  $G$  acting regularly on the points and lines can be obtained from a SDDS in  $G$ .*

*Proof.* Two points  $x, y \in G$  are collinear if and only if  $x^{-1}y \in \Delta(D)$ . Let us count the number of points  $z \in G \setminus \{x, y\}$  collinear with both  $x$  and  $y$ . This is equivalent with  $x^{-1}z \in \Delta(D)$  and  $y^{-1}z \in \Delta(D)$ , or  $z \in x\Delta(D) \cap y\Delta(D)$ , or  $x^{-1}z \in \Delta(D) \cap x^{-1}y\Delta(D)$ . The number of such points  $z$  is  $\lambda$  if  $x^{-1}y \in \Delta(D)$ , i.e. if  $x$  and  $y$  are collinear, and  $\mu$  otherwise. Hence, the point graph is strongly regular with parameters  $\text{SRG}(v, k(k-1), \lambda, \mu)$ .

Conversely, assume a  $(v_k; \lambda, \mu)$  configuration possesses an automorphism group  $G$  acting regularly. Then the points can be identified with the elements of  $G$  and every block is a deficient difference set generating this configuration. The argument above shows that it is in fact a  $(v_k; \lambda, \mu)$  SDDS.  $\square$

Configurations constructed from  $\text{PG}(2, q)$  by Theorem 3.4 can be obtained from strong deficient difference sets in the group  $G = \mathbb{F}_q^* \times \mathbb{F}_q^*$ . Here,  $\mathbb{F}_q^*$  denotes the multiplicative group of the finite field  $\mathbb{F}_q$ , isomorphic to the cyclic group  $\mathbb{Z}_{q-1}$ . If two of the points  $\{A, B, C\}$  are chosen on the “line at infinity” and the third point as the “origin”  $(0, 0)$ , points of the configuration can be identified with pairs  $(x, y)$  with  $x, y \in \mathbb{F}_q^*$ . Lines are sets of points satisfying equations of the form  $y = ax + b$ ,  $a, b \in \mathbb{F}_q^*$ . Hence, e.g.  $D = \{(x, x+1) \mid x \in \mathbb{F}_q^* \setminus \{-1\}\}$  is a SDDS for  $(v_k; \lambda, \mu)$  with  $v = (q-1)^2$ ,  $k = q-2$ ,  $\lambda = (q-4)^2 + 1$ , and

$\mu = (q-3)(q-4)$ . The full automorphism group of the configuration is  $((\mathbb{F}_q^* \times \mathbb{F}_q^*) : \text{Aut}(\mathbb{F}_q)) : S_3$ , where  $\text{Aut}(\mathbb{F}_q)$  are the field automorphisms, and  $S_3$  corresponds to collineations of  $PG(2, q)$  exchanging vertices of the triangle  $\{A, B, C\}$ .

The two  $(64_7; 26, 30)$  configurations with full automorphism groups of order 768 constructed from the Hall plane and its dual (see Table 2) can be obtained from SDDS's in the group  $G = Q_8 \times Q_8$ , where  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  is the quaternion group with usual multiplication (e.g.  $i^2 = j^2 = k^2 = -1, ij = k$ ). The difference set

$$D_1 = \{(1, 1), (i, -k), (j, k), (k, -j), (-i, j), (-j, i), (-k, -i)\}$$

gives the configuration constructed from the Hall plane and

$$D_2 = \{(1, 1), (i, -k), (j, j), (k, -j), (-i, -i), (-j, i), (-k, k)\}$$

gives the dual configuration. The Hall plane of order 9 and its dual are coordinatized by the quaternionic near-field. The first configuration arises from Theorem 3.4 when two of the points  $\{A, B, C\}$  are chosen on the translation line of the Hall plane, and the second configuration when one of the points is the translation point of the dual Hall plane.

We performed an exhaustive computer search for strong deficient difference sets with parameters corresponding to proper and primitive strongly regular configurations in groups of order  $v \leq 200$ , using the GAP library of small groups [23]. Apart from the examples just described, we found four other examples not corresponding to Theorem 3.4. The configurations constructed from these SDDS's have flag-transitive automorphism groups. Here are their descriptions.

**Example 4.2.** *SDDS's for  $(13_3; 2, 3)$  exist in the cyclic group  $\mathbb{Z}_{13}$ . There is one SDDS fixed by the multiplier 3:  $\{7, 8, 11\}$ . The development has full automorphism group  $\mathbb{Z}_{13} : \mathbb{Z}_3$  acting flag-transitively.*

This is the only cyclic strongly regular configuration we found. It can be embedded in the projective plane of order 3 by adding a point to every line.

**Example 4.3.** *SDDS's for  $(96_5; 4, 4)$  exist in the groups  $\mathbb{Z}_4 \times S_4$ ,  $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4) : \mathbb{Z}_2$ ,  $D_8 \times A_4$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$ . Here is one SDDS in  $\mathbb{Z}_4 \times S_4$ :*

$$\{(0, id), (1, (1, 4)(2, 3)), (1, (1, 3, 4, 2)), (1, (1, 4, 3)), (2, (1, 2, 4))\}.$$

*The developments are all isomorphic and give one self-dual configuration. The full automorphism group is  $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) : A_6) : \mathbb{Z}_2$  of order 11520 and acts flag-transitively.*

The associated graphs have parameters  $SRG(96, 20, 4, 4)$ . Many such graphs are known, see [7, 25]. The graph with the largest automorphism group of order 138240 is the point graph of the generalized quadrangle  $pg(5, 3, 1)$ . The graph of the  $(96_5; 4, 4)$  configuration has full automorphism group of order 11520. In [7], this graph is denoted by  $K''$  and the configuration is mentioned as a “partial linear space with five points per line and five lines on each point”.

**Example 4.4.** *SDDS's for  $(120_8; 28, 24)$  exist in the symmetric group  $S_5$ , e.g.*

$$\{id, (1, 2, 5, 3, 4), (1, 3, 4, 2, 5), (1, 5, 3, 2, 4), (1, 4)(2, 3, 5), \\ (1, 4, 5, 2), (1, 2, 4), (1, 2, 5)\}.$$

*Up to isomorphism one self-dual strongly regular configuration arises. The full automorphism group is isomorphic to the alternating group  $A_5$  of size 20160 and acts flag-transitively.*

This  $(120_8; 28, 24)$  configuration was constructed in [6] by embedding the  $pg(7, 8, 4)$  of [18, 12] into a Steiner 2- $(120, 8, 1)$  design. The 135 lines of the  $pg(7, 8, 4)$  and the 120 lines of the configuration cover every pair of the 120 points exactly once and form a design. The point graphs of the  $pg(7, 8, 4)$  and the  $(120_8; 28, 24)$  configuration are complementary with parameters  $SRG(120, 63, 30, 36)$  and  $SRG(120, 56, 28, 24)$ , respectively.

The  $pg(7, 8, 4)$  is part of an infinite family constructed from the hyperbolic quadric in  $PG(4n - 1, 2)$  [18]. The family is denoted by  $PQ^+(2n - 1, 2)$  and has parameters  $pg(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$ . These parameters fit a hypothetical  $(v_k; \lambda, \mu)$  configuration with

$$v = 2^{2n-1}(2^{2n} - 1), k = 2^{2n-1}, \lambda = 2^{2n-2}(2^{2n-1} - 1), \mu = 2^{2n-1}(2^{2n-2} - 1)$$

to make a 2- $(v, k, 1)$  design, but in [6, Theorem 2.1] it was proved that this is not possible for  $n > 2$ . Non-isomorphic partial geometries with the same parameters were constructed in [35, 17] that could possibly be embedded in Steiner 2-designs.

**Example 4.5.** *SDDS's for  $(155_7; 17, 9)$  exist in the group  $G = \mathbb{Z}_{31} : \mathbb{Z}_5$ . Let  $G$  be represented as permutations of  $\mathbb{Z}_{31}$  generated by  $f : x \mapsto x + 1 \pmod{31}$  and  $g : x \mapsto 2x \pmod{31}$ . Then,  $\{id, f^{12}g^4, f^{15}g, f^{18}, f^{20}g^2, f^{26}g^3, f^{30}\}$  is a SDDS. One self-dual strongly regular configuration arises, isomorphic to the semipartial geometry  $LP(4, 2)$ . The full automorphism group  $P\Gamma L(5, 2)$  is of order 9999360 and acts flag-transitively.*

The configurations obtained from  $LP(4, 2)$  by polarity transformations cannot be constructed from SDDS because their full automorphism groups are not transitive. The dual pair  $LP(4, 2)^\pi$  and  $LP(4, 2)_{\pi'}$  have full automorphism groups of order 322560 isomorphic to  $(\mathbb{Z}_2)^4 : P\Gamma L(4, 2)$ . The group acts in orbits of size 35, 120 on the points and 15, 140 on the lines of  $LP(4, 2)^\pi$ , and vice versa for  $LP(4, 2)_{\pi'}$ . The self-dual configuration  $LP(4, 2)_{\pi}^\pi$  has full automorphism group of order 20160 isomorphic to  $P\Gamma L(4, 2)$  acting in orbits of size 15, 35, 105.

Our final examples of strongly regular configurations can also not be obtained from SDDS's. They don't admit automorphism groups acting regularly, although some have flag-transitive automorphism groups.

**Example 4.6.** *There are at least four non-isomorphic  $(63_6; 13, 15)$  configurations. Two of them are self-dual with full automorphism groups  $PSU(3, 3) : \mathbb{Z}_2$  of order 12096 acting flag-transitively. Furthermore, there is a dual pair with full automorphism groups  $(SL(2, 3) : \mathbb{Z}_4) : \mathbb{Z}_2$  of order 192 acting in orbits of size 1, 6, 24, 32.*

The two self-dual  $(63_6; 13, 15)$  configurations are related to the smallest generalized hexagon  $GH(2, 2)$  (see [24, Section 5.7]). This is a  $(63_3)$  configuration with point and line graphs of girth 12 and diameter 6. The graphs are distance regular, but not strongly regular. A strongly regular  $(63_6; 13, 15)$  configuration can be constructed similarly as a semipartial geometry from a Moore graph: the new configuration has the same points as  $GH(2, 2)$ , and lines of the new configuration are sets of 6 points collinear with a given point of  $GH(2, 2)$ . The point graph of this  $(63_6)$  configuration is a  $SRG(63, 30, 13, 15)$ . The other self-dual  $(63_6; 13, 15)$  configuration is constructed in the same way from the dual of  $GH(2, 2)$ . We discovered the dual pair of non-transitive  $(63_6; 13, 15)$  configurations computationally, by prescribing automorphism groups.

## 5. A TABLE OF FEASIBLE PARAMETERS

In the final section we present a table of feasible parameters of strongly regular configurations with  $v \leq 200$ . A. E. Brouwer's table of strongly regular graphs [4] contains 437 parameter sets  $SRG(v, d, \lambda, \mu)$  with  $v \leq 200$ . It is known that strongly regular graphs do not exist in 62 cases. Among the remaining 375 cases, we look for those with  $d = k(k - 1)$  for some  $k \geq 3$ . This way we get 64 parameter sets  $(v_k; \lambda, \mu)$ .

Eleven of the 64 parameter sets do not satisfy Theorem 2.5. Six satisfy the theorem with equality and correspond to partial geometries  $pg(2, 2, 1)$ ,  $pg(3, 3, 1)$ ,  $pg(6, 6, 4)$ ,  $pg(5, 5, 2)$ ,  $pg(4, 4, 1)$ , and  $pg(5, 5, 1)$ .



The  $pg(q, q, 1)$  with  $q = 2, 3, 4, 5$  are the classical generalized quadrangles  $W(q)$  and their duals, see [41]. Two non-isomorphic  $pg(5, 5, 2)$ 's are known [46, 14, 32], whereas the existence of a  $pg(6, 6, 4)$  is open. Six of the remaining 47 parameter sets are eliminated by Proposition 2.4.

Thus, there are 41 feasible parameter sets  $(v_k; \lambda, \mu)$  of proper and primitive strongly regular configurations with  $v \leq 200$ . The parameters are listed in Table 3 along with information on the numbers of strongly regular configurations (#Cf) and self-dual strongly regular configurations (#SCf) up to isomorphism. A number in **boldface** indicates that this is the exact number, otherwise it is a lower bound.

In the smallest case  $(10_3; 3, 4)$ , there are altogether ten combinatorial  $(10_3)$  configurations denoted by  $(10_3)_i$ ,  $i = 1, \dots, 10$  in [27, Section 2.2]. Two of them are strongly regular: the Desargues configuration  $(10_3)_1$  and the configuration  $(10_3)_4$  depicted in Figure 1. Interestingly,  $(10_3)_4$  is the only one of the ten  $(10_3)$  configurations that cannot be drawn with straight lines, i.e. that is not a *geometric configuration* (see [27, 42]). In the next two cases  $(13_3; 2, 3)$  and  $(16_3; 2, 2)$ , the total numbers of  $(13_3)$  and  $(16_3)$  configurations are also known: 2036 [26] and 3 004 881 [2], respectively. Since the number of combinatorial  $(v_k)$  configurations grows rapidly with  $v$ , a better approach to classifying strongly regular configurations is through the associated graphs.

Suppose that a strongly regular graph  $\Gamma$  with parameters  $SRG(v, k(k-1), \lambda, \mu)$  is the point graph of a  $(v_k; \lambda, \mu)$  configuration. Every line of the configuration gives a clique of size  $k$  in  $\Gamma$ . Thus, there must be  $v$  such cliques with every pair of them intersecting in at most one point. Given the graph  $\Gamma$ , we define the *clique graph*  $\mathcal{C}(\Gamma)$  with vertices being  $k$ -cliques in  $\Gamma$ . Vertices of  $\mathcal{C}(\Gamma)$  are adjacent if the cliques intersect in at most one point. The task is to find the cliques of size  $v$  in  $\mathcal{C}(\Gamma)$ .

Up to isomorphism, there is a unique graph  $SRG(13, 6, 2, 3)$ , the Paley graph. The vertices of  $\Gamma$  are elements of the finite field  $\mathbb{F}_{13}$  with two vertices being adjacent if their difference is a quadratic residue. Using Cliquer [39], we found 26 cliques of size 3 in  $\Gamma$ . The clique graph  $\mathcal{C}(\Gamma)$  has 26 vertices and 286 edges. Using Cliquer once more, we found exactly two cliques of size 13 in  $\mathcal{C}(\Gamma)$ , corresponding to isomorphic  $(13_3; 2, 3)$  configurations. This proves that the cyclic configuration constructed in Example 4.2 is unique.

**Proposition 5.1.** *There is one  $(13_3; 2, 3)$  configuration up to isomorphism.*

There are two graphs with parameters  $SRG(16, 6, 2, 2)$ . One of them is the Shrikhande graph [44] with full automorphism group of order 192. Similarly as for the previous parameters, we found 32 cliques of size 3

| No. | $(v_k; \lambda, \mu)$ | #Cf      | #SCf     | Comments        |
|-----|-----------------------|----------|----------|-----------------|
| 1   | $(10_3; 3, 4)$        | <b>2</b> | <b>2</b> |                 |
| 2   | $(13_3; 2, 3)$        | <b>1</b> | <b>1</b> | Proposition 5.1 |
| 3   | $(16_3; 2, 2)$        | <b>1</b> | <b>1</b> | Proposition 5.3 |
| 4   | $(25_4; 5, 6)$        | <b>0</b> | <b>0</b> | Proposition 5.5 |
| 5   | $(36_5; 10, 12)$      | <b>1</b> | <b>1</b> | Proposition 5.6 |
| 6   | $(41_5; 9, 10)$       | ?        | ?        |                 |
| 7   | $(45_4; 3, 3)$        | <b>0</b> | <b>0</b> | Proposition 5.7 |
| 8   | $(49_4; 5, 2)$        | <b>0</b> | <b>0</b> | Corollary 5.4   |
| 9   | $(49_6; 17, 20)$      | 1        | 1        | Theorem 3.4     |
| 10  | $(50_7; 35, 36)$      | 211      | 111      | Proposition 3.1 |
| 11  | $(61_6; 14, 15)$      | ?        | ?        |                 |
| 12  | $(63_6; 13, 15)$      | 4        | 2        | Example 4.6     |
| 13  | $(64_7; 26, 30)$      | 29       | 11       | Theorem 3.4     |
| 14  | $(81_8; 37, 42)$      | ?        | ?        |                 |
| 15  | $(85_6; 11, 10)$      | ?        | ?        |                 |
| 16  | $(85_7; 20, 21)$      | ?        | ?        |                 |
| 17  | $(96_5; 4, 4)$        | 1        | 1        | Example 4.3     |
| 18  | $(99_7; 21, 15)$      | ?        | ?        |                 |
| 19  | $(100_9; 50, 56)$     | 1        | 1        | Theorem 3.4     |
| 20  | $(105_9; 51, 45)$     | ?        | ?        |                 |
| 21  | $(113_8; 27, 28)$     | ?        | ?        |                 |
| 22  | $(120_8; 28, 24)$     | 1        | 1        | Example 4.4     |
| 23  | $(121_5; 9, 2)$       | <b>0</b> | <b>0</b> | Corollary 5.4   |
| 24  | $(121_6; 11, 6)$      | ?        | ?        |                 |
| 25  | $(121_9; 43, 42)$     | ?        | ?        |                 |
| 26  | $(121_{10}; 65, 72)$  | ?        | ?        |                 |
| 27  | $(125_9; 45, 36)$     | ?        | ?        |                 |
| 28  | $(136_6; 15, 4)$      | ?        | ?        |                 |
| 29  | $(136_9; 36, 40)$     | ?        | ?        |                 |
| 30  | $(144_{11}; 82, 90)$  | 1        | 1        | Theorem 3.4     |

TABLE 3. Feasible parameters of proper primitive strongly regular configurations.

| No. | $(v_k; \lambda, \mu)$  | #Cf | #SCf | Comments    |
|-----|------------------------|-----|------|-------------|
| 31  | $(145_9; 35, 36)$      | ?   | ?    |             |
| 32  | $(153_8; 19, 21)$      | ?   | ?    |             |
| 33  | $(155_7; 17, 9)$       | 4   | 2    | Theorem 3.3 |
| 34  | $(169_9; 31, 30)$      | ?   | ?    |             |
| 35  | $(169_{12}; 101, 110)$ | ?   | ?    |             |
| 36  | $(171_{11}; 73, 66)$   | ?   | ?    |             |
| 37  | $(175_6; 5, 5)$        | ?   | ?    |             |
| 38  | $(181_{10}; 44, 45)$   | ?   | ?    |             |
| 39  | $(196_{10}; 40, 42)$   | ?   | ?    |             |
| 40  | $(196_{13}; 122, 132)$ | ?   | ?    |             |
| 41  | $(196_{13}; 125, 120)$ | ?   | ?    |             |

TABLE 3. Feasible parameters of proper primitive strongly regular configurations (continued).

in  $\Gamma$  and two cliques of size 16 in  $\mathcal{C}(\Gamma)$ , corresponding to isomorphic  $(16_3; 2, 2)$  configurations.

The other  $SRG(16, 6, 2, 2)$  has full automorphism group of order 1152. This is the  $4 \times 4$  *rook graph*, sometimes also called the *lattice graph* or *grid graph*. Vertices of the  $n \times n$  rook graph  $R_n$  are pairs  $(x, y)$  with  $x, y \in \{1, \dots, n\}$ . Two vertices  $(x_1, y_1), (x_2, y_2)$  are adjacent if  $x_1 = x_2$  or  $y_1 = y_2$  holds. The graph  $R_n$  is strongly regular with parameters  $SRG(n^2, 2(n-1), n-2, 2)$  and has  $2n$  maximal cliques of size  $n$ , being sets of vertices with a fixed coordinate. Any clique of size at least 2 is contained in exactly one of these maximal cliques. If  $R_n$  is the point graph of a  $(v_k; \lambda, \mu)$  configuration, then  $2(n-1) = k(k-1)$  holds. This is equivalent with  $n = \binom{k}{2} + 1$  and the configuration would have parameters

$$v = \left( \binom{k}{2} + 1 \right)^2, \quad \lambda = \binom{k}{2} - 1, \quad \mu = 2. \quad (7)$$

We now prove that this cannot occur.

**Theorem 5.2.** *The  $n \times n$  rook graph is not the point graph of a strongly regular configuration.*

*Proof.* Lines of the configuration would give a set  $\mathcal{C}$  of  $v = n^2$  cliques of size  $k$  in  $R_n$ , pairwise intersecting in at most one vertex. A maximal clique of size  $n = \binom{k}{2} + 1$  contains no more than  $\frac{n(n-1)}{k(k-1)}$  of the cliques

in  $\mathcal{C}$ , because each of the  $n(n-1)$  pairs of distinct vertices is contained in at most one  $k$ -clique, and a  $k$ -clique covers  $k(k-1)$  pairs. Therefore,  $\mathcal{C}$  is not larger than  $2n \cdot \frac{n(n-1)}{k(k-1)}$ . This is equal to  $v = n^2$ , and therefore the cliques of  $\mathcal{C}$  contained in a given  $n$ -clique cover every pair of its  $n$  vertices exactly once. In this way we get a Steiner 2- $(n, k, 1)$  design. If  $r = \frac{n-1}{k-1}$  is the replication number of the design, Fisher's inequality  $r \geq k$  gives  $n-1 \geq k(k-1)$ , a contradiction with  $n = \binom{k}{2} + 1$ .  $\square$

Together with the discussion about the Shrikhande graph, this proves that the strongly regular configuration constructed from  $PG(2, 5)$  by Theorem 3.4 is unique.

**Corollary 5.3.** *There is one  $(16_3; 2, 2)$  configuration up to isomorphism.*

Furthermore, Theorem 5.2 eliminates infinitely many feasible parameter sets of strongly regular configurations.

**Corollary 5.4.** *Strongly regular configurations with parameters (7) do not exist for  $k > 3$ .*

*Proof.* In [44], Shrikhande proved that for  $n > 4$  the only strongly regular graph with parameters  $SRG(n^2, 2(n-1), n-2, 2)$  is the  $n \times n$  rook graph.  $\square$

We can eliminate two more parameter sets  $(v_k; \lambda, \mu)$  and prove uniqueness for another computationally, when all  $SRG(v, k(k-1), \lambda, \mu)$  graphs are known.

**Proposition 5.5.** *Strongly regular  $(25_4; 5, 6)$  configurations do not exist.*

*Proof.* Up to isomorphism, there are exactly 15 strongly regular graphs with parameters  $SRG(25, 12, 5, 6)$  [40, 43]. The adjacency matrices are available on E. Spence's web page [45]. Cliquer found from 73 to 90 cliques of size 4 in these graphs, but none of the corresponding clique graphs  $\mathcal{C}(\Gamma)$  contain a clique of size 25.  $\square$

**Proposition 5.6.** *There is one  $(36_5; 10, 12)$  configuration up to isomorphism.*

*Proof.* There are exactly 32 548 graphs  $SRG(36, 20, 10, 12)$  [37]. Adjacency matrices of the complementary graphs are available on the web page [45]. Using Cliquer, we found that the  $SRG(36, 20, 10, 12)$  graphs  $\Gamma$  contain from 132 to 336 cliques of size 5. Only one of the corresponding clique graphs  $\mathcal{C}(\Gamma)$  contains a clique of size 36. This happens when the complementary graph  $\bar{\Gamma}$  with parameters  $SRG(36, 15, 6, 6)$  is

the graph constructed from the cyclic Latin square of order 6. Two strongly regular configurations arise, both isomorphic to the configuration constructed from  $PG(2, 7)$  by Theorem 3.4.  $\square$

**Proposition 5.7.** *Strongly regular  $(45_4; 3, 3)$  configurations do not exist.*

*Proof.* There are 78 graphs  $SRG(45, 12, 3, 3)$  [13]. Adjacency matrices are available on [45]. The graphs contain from 12 to 135 cliques of size 4 and the corresponding clique graphs do not contain cliques of size 45.  $\square$

It is also known that graphs with parameters  $SRG(50, 42, 35, 36)$  are unique, i.e. isomorphic to the complement of the Hoffman-Singleton graph. This graph has 2708 150 cliques of size 7 and we could not classify all cliques of size 50 in  $\mathcal{C}(\Gamma)$ . There may be other  $(50_7; 35, 36)$  configurations apart from the 211 examples of Proposition 3.1.

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