# Constructions of MDS symbol-pair codes with minimum distance seven or eight 

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#### Abstract

Symbol-pair codes are proposed to guard against pair-errors in symbol-pair read channels. The minimum symbol-pair distance plays a vital role in determining the error-correcting capability and the constructions of symbol-pair codes with largest possible minimum symbol-pair distance is of great importance. Maximum distance separable (MDS ) symbol-pair codes are optimal in the sense that such codes can acheive the Singleton bound. In this paper, for length $5 p$, two new classes of MDS symbol-pair codes with minimum symbol-pair distance seven or eight are constructed by utilizing repeated-root cyclic codes over $\mathbb{F}_{p}$, where $p$ is a prime. In addition, we derive a class of MDS symbol-pair codes with minimum symbol-pair distance seven and length $4 p$.


Keywords: MDS symbol-pair codes, minimum symbol-pair distance, constacyclic codes, repeated-root cyclic codes

## 1 Introduction

With the development of modern high density data storage systems, symbol-pair code was proposed by Cassuto and Blaum to combat against pair-errors over symbol-pair read channels in [1, 2]. They also showed that a code $\mathcal{C}$ with minimum symbol-pair distance $d_{p}$ can correct up to $\left\lfloor\left(d_{p}-1\right) / 2\right\rfloor$ symbol-pair errors [1,2]. Later, Cassuto and Litsyn [3] showed that codes

[^0]for correcting pair-errors exist with strictly higher rates compared to codes for the Hamming metric with the same relative distance. In [6], Chee, Kiah and Wang established a Singletontype bound on symbol-pair codes. Similar to classical codes, symbol-pair codes meeting the Singleton-type bound are called MDS symbol-pair codes and the error-correcting capability of MDS symbol-pair codes is optimal. Later, Ding, Zhang and Ge extended the Singleton-type bound to the $b$-symbol case in 9 .

Many attempts have been made in the constructions of MDS symbol-pair codes. In [17], Kai, Zhu and Li provided MDS symbol-pair codes with length $q^{2}+q+1$ through constacyclic codes over $\mathbb{F}_{q}$. Later, Li and Ge [19] generalized the results in [17] and they also constructed a number of MDS symbol-pair codes with minimum symbol-pair distance seven by analyzing certain linear fractional transformations. Shortly afterwards, Chen, Lin and Liu [7] constructed several MDS symbol-pair codes with length $3 p$ from repeated-root cyclic codes over $\mathbb{F}_{p}$. In 2018, Ding et al. [8] obtained some MDS symbol-pair codes over $\mathbb{F}_{q}$ with larger minimum symbol-pair distance based on elliptic curves and the lengths of these codes are bounded by $q+2 \sqrt{q}$. In the same year, Kai et al. [18] constructed three classes of MDS symbol-pair codes using repeated-root constacyclic codes over $\mathbb{F}_{p}$, see Table $\mathbb{1}$. Recently, some new results on constructing symbol-pair codes were presented in [12,14,21]. Moreover, some decoding algorithms of symbol-pair codes were proposed by various researchers in [15,20, 25, 27, 28] and the symbol-pair weight distributions of some linear codes over finite fields were studied in [10, 11, 13, 22, 26] and the references therein.

In Table [1, we summarize some known MDS symbol-pair codes from constacyclic codes. Observe that there exists only one class of codes with length $5 p$ and minimum symbol-pair distance five in Table (1. The constructions of symbol-pair codes with comparatively large minimum symbol-pair distance is an interesting topic. This paper focuses on the further constructions of MDS symbol-pair codes with length $5 p$. Precisely, two new classes of MDS symbol-pair codes with minimum symbol-pair distance seven or eight are constructed by utilizing repeated-root cyclic codes over $\mathbb{F}_{p}$. In addition, for $n=4 p$, we derive a class of MDS symbol-pair codes with $d_{p}=7$, which generalizes the result in [18].

The remainder of this paper is organized as follows. In Section 2, we introduce some basic notation and results on symbol-pair codes and constacyclic codes. By exploiting repeated-root cyclic codes, for length $5 p$, two new classes of MDS symbol-pair codes with minimum symbolpair distance seven or eight are constructed in Section 3.1 and a class of MDS symbol-pair codes with length $4 p$ is presented in Section 3.2. Section 4 concludes the paper.

Table 1: Some known MDS symbol-pair codes from constacyclic codes

| Values of $\left(n, d_{p}\right)_{q}$ | Conditions | References |
| :---: | :---: | :---: |
| $(n, 5)_{q}$ | $n \mid\left(q^{2}+q+1\right)$ | $[17],[19]$ |
| $(n, 6)_{q}$ | $n \mid\left(q^{2}+1\right)$ | $[17],[19]$ |
| $(n, 6)_{q}$ | $n \mid\left(q^{2}-1\right), n$ odd or $n$ even and $v_{2}(n)<v_{2}\left(q^{2}-1\right)$ | $[19]$ |
| $(n, 6)_{q}$ | $q \geq 3, n \geq q+4, n \mid\left(q^{2}-1\right)$ | $[7]$ |
| $(l p, 5)_{p}$ | $p \geq 5, l>2, \operatorname{gcd}(l, p)=1, l \mid(p-1)$ | $[7]$ |
| $\left(p^{2}+p, 6\right)_{p}$ | $p \geq 3$ | $[18]$ |
| $\left(2 p^{2}-2 p, 6\right)_{p}$ | $p \geq 3$ | $[18]$ |
| $(3 p, 6)_{p}$ | $p \geq 5$ | $[7]$ |
| $(3 p, 7)_{p}$ | $p \geq 5$ | $[7]$ |
| $(3 p, 8)_{p}$ | $3 \mid(p-1)$ | $[7]$ |
| $(3 p, 10)_{p}$ | $3 \mid(p-1)$ | $[21]$ |
| $(3 p, 12)_{p}$ | $3 \mid(p-1)$ | $[21]$ |
| $(4 p, 7)_{p}$ | $p \equiv 3(\bmod 4)$ | $[18]$ |
| $(4 p, 7)_{p}$ | $p \equiv 1(\bmod 4)$ | Theorem $[1$ |
| $(5 p, 7)_{p}$ | $5 \mid(p-1), p \neq 41$ | Theorem 1$]$ |
| $(5 p, 8)_{p}$ | $5 \mid(p-1)$ | Theorem 2 |

where $q$ is a power of a prime $p$.

## 2 Preliminaries

In this section, we introduce some notation and auxiliary tools on symbol-pair codes and constacyclic codes, which will be used to prove our main results in the sequel.

### 2.1 Symbol-pair codes

Let $\mathbb{F}_{q}$ be the finite field with $q$ elements, where $q$ is a prime power. Let $n$ be a positive integer and $\mathbf{x}=\left(x_{0}, x_{1}, \cdots, x_{n-1}\right)$ be a vector in $\mathbb{F}_{q}^{n}$. Then the symbol-pair read vector of $\mathbf{x}$ is

$$
\pi(\mathbf{x})=\left(\left(x_{0}, x_{1}\right),\left(x_{1}, x_{2}\right), \cdots,\left(x_{n-2}, x_{n-1}\right),\left(x_{n-1}, x_{0}\right)\right)
$$

Obviously, each vector $\mathbf{x} \in \mathbb{F}_{q}^{n}$ has a unique pair representation $\pi(\mathbf{x})$. Recall that the Hamming weight of $\mathbf{x}$ is

$$
w_{H}(\mathbf{x})=\left|\left\{i \in \mathbb{Z}_{n} \mid x_{i} \neq 0\right\}\right|
$$

where $\mathbb{Z}_{n}$ denotes the residue class ring $\mathbb{Z} / n \mathbb{Z}$. Correspondingly, the symbol-pair weight of $\mathbf{x}$ is

$$
w_{p}(\mathbf{x})=\left|\left\{i \in \mathbb{Z}_{n} \mid\left(x_{i}, x_{i+1}\right) \neq(0,0)\right\}\right| .
$$

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_{q}^{n}$, the symbol-pair distance from $\mathbf{x}$ to $\mathbf{y}$ is defined as

$$
d_{p}(\mathbf{x}, \mathbf{y})=\left|\left\{i \in \mathbb{Z}_{n} \mid\left(x_{i}, x_{i+1}\right) \neq\left(y_{i}, y_{i+1}\right)\right\}\right| .
$$

A code $\mathcal{C}$ over $\mathbb{F}_{q}$ of length $n$ is a nonempty subsets of $\mathbb{F}_{q}^{n}$. Elements of $\mathcal{C}$ are called codewords in $\mathcal{C}$. The minimum symbol-pair distance of $\mathcal{C}$ is

$$
d_{p}(\mathcal{C})=\min \left\{d_{p}(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}\right\}
$$

and we refer such a code as an $\left(n, d_{p}(\mathcal{C})\right)_{q}$ symbol-pair code. A well-known relationship between $d_{H}(\mathcal{C})$ and $d_{p}(\mathcal{C})$ in [1,2] states that for any $0<d_{H}(\mathcal{C})<n$,

$$
d_{H}(\mathcal{C})+1 \leq d_{p}(\mathcal{C}) \leq 2 \cdot d_{H}(\mathcal{C})
$$

The following lemma reveals a connection between the symbol-pair distance and the Hamming distance of a code $\mathcal{C}$.

Lemma 1. ( [1, 2]) For any $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ with $\mathbf{x}=\left(x_{0}, \cdots, x_{n-1}\right)$ and $\mathbf{y}=\left(y_{0}, \cdots, y_{n-1}\right)$. Define $S=\left\{i \in \mathbb{Z}_{n} \mid x_{i} \neq y_{i}\right\}$. Let $S=\bigcup_{i=1}^{L} S_{i}$ be a partition of $S$, which satisfies:
(1) the elements of each subset $S_{i}$ are consecutive in the sense of modulo $n$;
(2) for any different $i, j \in[1, L]$ and $a \in S_{i}, b \in S_{j}$, a and $b$ are not consecutive.

Then

$$
d_{p}(\mathbf{x}, \mathbf{y})=d_{H}(\mathbf{x}, \mathbf{y})+L .
$$

In contrast to classical error-correcting codes, the size of symbol-pair codes satisfies the following Singleton bound.
Lemma 2. ([5) Let $q \geq 2$ and $2 \leq d_{p} \leq n$. If $\mathcal{C}$ is a symbol-pair code with length $n$ and minimum symbol-pair distance $d_{p}$, then $|\mathcal{C}| \leq q^{n-d_{p}+2}$.

The symbol-pair code achieving the Singleton bound is called a maximum distance separable (MDS ) symbol-pair code.

### 2.2 Constacyclic codes

In this subsection, we introduce some notation of constacyclic codes. For a fixed nonzero element $\eta$ of $\mathbb{F}_{q}$, the $\eta$-constacyclic shift $\tau_{\eta}$ on $\mathbb{F}_{q}^{n}$ is

$$
\tau_{\eta}\left(x_{0}, x_{1}, \cdots, x_{n-1}\right)=\left(\eta x_{n-1}, x_{0}, \cdots, x_{n-2}\right)
$$

A linear code $\mathcal{C}$ is called an $\eta$-constacyclic code if $\tau_{\eta}(\mathbf{c}) \in \mathcal{C}$ for any codeword $\mathbf{c} \in \mathcal{C}$. An $\eta$ constacyclic code is a cyclic code if $\eta=1$ and a negacyclic code if $\eta=-1$. It should be noted that each codeword $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in \mathcal{C}$ is identical to its polynomial representation

$$
c(x)=c_{0}+c_{1} x+\cdots+c_{n-1} x^{n-1} .
$$

For convenience, we always regard the codeword $\mathbf{c}$ in $\mathcal{C}$ as the corresponding polynomial $c(x)$ in this paper. Notice that a linear code $\mathcal{C}$ is an $\eta$-constacyclic code if and only if it is an ideal of the principle ideal ring $\mathbb{F}_{q}[x] /\left\langle x^{n}-\eta\right\rangle$. As a consequence, there exists a unique monic divisor $g(x) \in \mathbb{F}_{q}[x]$ of $x^{n}-\eta$ such that

$$
\mathcal{C}=\langle g(x)\rangle=\left\{f(x) g(x)\left(\bmod \left(x^{n}-\eta\right)\right) \mid f(x) \in \mathbb{F}_{q}[x]\right\}
$$

The polynomial $g(x)$ is called the generator polynomial of $\mathcal{C}$ and the dimension of $\mathcal{C}$ is $n-k$, where $k$ is the degree of $g(x)$.

Recall that a $q$-ary $\eta$-constacyclic code of length $n$ is a simple-root constacyclic code if $\operatorname{gcd}(n, q)=1$ and a repeated-root constacyclic code if $p \mid n$, where $p$ is the characteristic of $\mathbb{F}_{q}$. Simple-root constacyclic codes can be characterized by their defining sets [16, 23]. Compared to simple-root cyclic codes, repeated-root cyclic codes are no longer characterized by its set of zeros. Let $\mathcal{C}=\langle g(x)\rangle$ be a repeated-root cyclic code of length $l p^{e}$ over $\mathbb{F}_{q}$, where $l$ and $e$ are positive integers with $\operatorname{gcd}(l, p)=1$. It is shown in [4] that the minimum Hamming distance of $\mathcal{C}$ can be derived from $d_{H}\left(\overline{\mathcal{C}}_{t}\right)$. Here $\overline{\mathcal{C}}_{t}$ is a simple-root cyclic code fully determined by $\mathcal{C}$ as follows.

More precisely, assume that

$$
g(x)=\prod_{i=1}^{r} m_{i}(x)^{e_{i}}
$$

where each $m_{i}(x)$ is a monic irreducible polynomial over $\mathbb{F}_{q}$ and $e_{i}$ are positive integers. For a fixed $t$ with $0 \leq t \leq p^{e}-1, \overline{\mathcal{C}}_{t}$ is defined to be a simple-root cyclic code of length $l$ over $\mathbb{F}_{q}$ with the generator polynomial

$$
\bar{g}_{t}(x)=\prod_{1 \leq i \leq r, e_{i}>t} m_{i}(x)
$$

If $\bar{g}_{t}(x)=x^{l}-1$, then $\overline{\mathcal{C}}_{t}$ contains only the all-zero codeword and we set $d_{H}\left(\overline{\mathcal{C}}_{t}\right)=\infty$. If each $e_{i} \leq t$, then $\bar{g}_{t}(x)=1$ and $d_{H}\left(\overline{\mathcal{C}}_{t}\right)=1$.

The following lemma reveals that the minimum Hamming distance of repeated-root cyclic codes can be determined by the polynomial algebra, which will be applied to derive the minimum Hamming distance of codes in this paper.

Lemma 3. ( [4]) Let $\mathcal{C}$ be a repeated-root cyclic code of length lpe over $\mathbb{F}_{q}$, where $l$ and e are positive integers with $\operatorname{gcd}(l, p)=1$. Then

$$
\begin{equation*}
d_{H}(\mathcal{C})=\min \left\{P_{t} \cdot d_{H}\left(\overline{\mathcal{C}}_{t}\right) \mid 0 \leq t \leq p^{e}-1\right\} \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{t}=w_{H}\left((x-1)^{t}\right)=\prod_{i}\left(t_{i}+1\right) \tag{2}
\end{equation*}
$$

with $t_{i}$ 's being the coefficients of the radix-p expansion of $t$.

In this paper, we will employ repeated-root cyclic codes to construct new MDS symbol-pair codes. The following lemmas are very useful.

Lemma 4. ( [7]) Let $\mathcal{C}$ be an $\left[n, k, d_{H}(\mathcal{C})\right]$ constacyclic code over $\mathbb{F}_{q}$ with $2 \leq d_{H}(\mathcal{C})<n$. Then we have $d_{p}(\mathcal{C}) \geq d_{H}(\mathcal{C})+2$ if and only if $\mathcal{C}$ is not an MDS code, i.e., $k<n-d_{H}(\mathcal{C})+1$.

Lemma 5. Let $\mathcal{C}=\langle g(x)\rangle$ be a repeated-root cyclic code of length lpe over $\mathbb{F}_{q}$ and $c(x)=$ $\left(x^{l}-1\right)^{t} v(x)$ a codeword in $\mathcal{C}$ with Hamming weight $d_{H}(\mathcal{C})$, where $l$ and $e$ are positive integers with $\operatorname{gcd}(l, p)=1,0 \leq t \leq p^{e}-1$ and $\left(x^{l}-1\right) \nmid v(x)$. Then

$$
w_{H}(c(x))=P_{t} \cdot N_{v}
$$

where $P_{t}$ is defined as (2) in Lemma 3 and $N_{v}=w_{H}\left(v(x) \bmod \left(x^{l}-1\right)\right)$.

Proof Denote $\bar{v}(x)=\left(v(x) \bmod \left(x^{l}-1\right)\right)$ and

$$
\bar{c}_{t}(x)=\left(\left(x^{l}-1\right)^{t} \cdot \bar{v}(x)^{p^{e}} \bmod \left(x^{l p^{e}}-1\right)\right)
$$

Assume that

$$
g(x)=\prod_{i=1}^{r} m_{i}(x)^{e_{i}}
$$

and

$$
\bar{g}_{t}(x)=\prod_{1 \leq i \leq r, e_{i}>t} m_{i}(x) .
$$

It follows from $x^{l p^{e}}-1=\left(x^{l}-1\right)^{p^{e}},\left(x^{l}-1\right) \nmid v(x)$ and $g(x) \mid c(x)$ that $\bar{g}_{t}(x) \mid \bar{v}(x)$. Combining with $t<p^{e}$, one can obtain that for any $1 \leq i \leq r$,
i) if $e_{i}>t$, then $m_{i}(x) \mid \bar{v}(x)$ and $m_{i}(x)$ is a factor of $\bar{c}_{t}(x)$ with multiplicity at least $p^{e}$;
ii) if $e_{i} \leq t$, then $m_{i}(x)$ is a factor of $\bar{c}_{t}(x)$ with multiplicity at least $t$.

Hence $g(x) \mid \bar{c}_{t}(x)$.

Meanwhile, due to $\operatorname{deg}(\bar{v}(x))<l$, there must exist a root of $x^{l}-1$ whose multiplicity in $\bar{c}_{t}(x)$ is exactly $t$. This leads to $\left(x^{l p^{e}}-1\right) \nmid \bar{c}_{t}(x)$ and then $\bar{c}_{t}(x)$ is a nonzero codeword in $\mathcal{C}$. It can be verified that

$$
\begin{aligned}
w_{H}\left(\bar{c}_{t}(x)\right) & =w_{H}\left(\left(x^{l}-1\right)^{t} \cdot \bar{v}(x)^{p^{e}} \bmod \left(x^{l p^{e}}-1\right)\right) \\
& \leq w_{H}\left(\left(x^{l}-1\right)^{t} \cdot \bar{v}(x)^{p^{e}}\right) \leq w_{H}\left(\left(x^{l}-1\right)^{t}\right) \cdot w_{H}\left(\bar{v}(x)^{p^{e}}\right)=P_{t} \cdot N_{v}
\end{aligned}
$$

On the other hand, according to Theorem 6.3 in [24], we have

$$
w_{H}(c(x)) \geq w_{H}\left(\left(x^{l}-1\right)^{t}\right) \cdot w_{H}\left(v(x) \bmod \left(x^{l}-1\right)\right)=P_{t} \cdot N_{v} \geq w_{H}\left(\bar{c}_{t}(x)\right)
$$

Since $w_{H}(c(x))=d_{H}(\mathcal{C})$, one can immediately conclude that

$$
w_{H}(c(x))=w_{H}\left(\bar{c}_{t}(x)\right)=P_{t} \cdot N_{v}
$$

This completes the proof.
The following lemma will be frequently used to prove our results.
Lemma 6. Let $p$ be a prime power with $5 \mid(p-1)$, $\beta$ be a primitive 5 -th root of unity in $\mathbb{F}_{p}$ and $a_{i} \in \mathbb{F}_{p}^{*}$ for $1 \leq i \leq 3$. Then

$$
\begin{equation*}
\beta^{2}+3 \beta+1 \neq 0 \tag{3}
\end{equation*}
$$

and for $(i, j)=(2,3),(2,4)$ or $(3,4)$, the solution of the $\mathbb{F}_{p}$-linear system of equations

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0  \tag{4}\\
1+a_{1} \beta+a_{2} \beta^{i}+a_{3} \beta^{j}=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{2 i}+a_{3} \beta^{2 j}=0
\end{array}\right.
$$

is given as

| Value of $(i, j)$ | Corresponding solution $\left(a_{1}, a_{2}, a_{3}\right)$ |
| :--- | :--- |
| $(2,3)$ | $\left(-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \frac{\beta^{2}+\beta+1}{\beta^{3}},-\frac{1}{\beta^{3}}\right)$ |
| $(2,4)$ | $\left(-\frac{1}{\beta},-\frac{\beta}{\beta+1}, \frac{1}{\beta(\beta+1)}\right)$ |
| $(3,4)$ | $\left(\frac{\beta^{2}}{\beta+1},-\frac{1}{\beta+1},-\beta\right)$. |

Proof Assume that $\beta^{2}+3 \beta+1=0$. The fact $\beta$ is a primitive 5 -th root of unity indicates

$$
0=\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=-5(3 \beta+1)
$$

which yields $\beta^{2}=-3 \beta-1=0$, a contradiction.
If $(i, j)=(2,3)$, then (4) is transformed into

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0, \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3} \beta^{3}=0, \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3} \beta=0 .
\end{array}\right.
$$

This leads to

$$
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} .
$$

Similarly, we can derive the solutions of (4) for $(i, j)=(2,4)$ and $(3,4)$. This completes the proof.

## 3 Constructions of MDS Symbol-Pair Codes

In this section, we propose three new classes of MDS symbol-pair codes from repeated-root cyclic codes by analyzing the solutions of certain equations over $\mathbb{F}_{p}$. Firstly, for length $5 p$, two classes of MDS symbol-pair codes with minimum symbol-pair distance 7 or 8 are constructed respectively. In addition, for $n=4 p$, we derive a class of MDS symbol-pair codes with $d_{p}=7$.

From now on, we denote $c^{(k)}(x)$ by the $k$-th formal derivative of $c(x)$, where $k$ is a positive integer and $c(x) \in \mathbb{F}_{p}[x]$. Let $\star$ denote an element in $\mathbb{F}_{p}^{*}$ and $\mathbf{0}$ is the zero vector. Due to the linearity and the cyclic shift property of cyclic codes, we assume that the constant term of $c(x)$ occurred in this paper is always 1 .

### 3.1 MDS symbol-pair codes for $n=5 p$

In this subsection, two classes of MDS symbol-pair codes with length $5 p$ are constructed.
Now we present a class of MDS symbol-pair codes with minimum symbol-pair distance 7 for any prime $p$ with $5 \mid(p-1)$ and $p \neq 41$.

Theorem 1. Let $p$ be a prime with $5 \mid(p-1)$ and $p \neq 41$. Then there exists an $\operatorname{MDS}(5 p, 7)_{p}$ symbol-pair code.

Proof Let $\mathcal{C}$ be a repeated-root cyclic code of length $5 p$ over $\mathbb{F}_{p}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-\beta)\left(x-\beta^{2}\right)
$$

where $\beta$ is a primitive 5 -th root of unity in $\mathbb{F}_{p}$.

Note that $\mathcal{C}$ is a $[5 p, 5 p-5,4]$ cyclic code due to Lemma 3. Indeed, recall that $\bar{g}_{t}(x)$ is the generator polynomial of $\overline{\mathcal{C}}_{t}$. If $t=0$, then

$$
\bar{g}_{0}(x)=(x-1)(x-\beta)\left(x-\beta^{2}\right)
$$

and

$$
P_{0} \cdot d_{H}\left(\overline{\mathcal{C}}_{0}\right)=1 \cdot 4=4 .
$$

If $t=1$, then $\bar{g}_{1}(x)=x-1$ and

$$
P_{1} \cdot d_{H}\left(\overline{\mathcal{C}}_{1}\right)=2 \cdot 2=4 .
$$

If $t=2$, then $\bar{g}_{2}(x)=x-1$ and

$$
P_{2} \cdot d_{H}\left(\overline{\mathcal{C}}_{2}\right)=3 \cdot 2=6 .
$$

If $3 \leq t \leq p-1$, then $\bar{g}_{t}(x)=1$ and

$$
P_{t} \cdot d_{H}\left(\overline{\mathcal{C}}_{t}\right)=(t+1) \cdot 1=t+1 \geq 4
$$

With the aid of the equality (1) in Lemma 3, one can immediately get $d_{H}(\mathcal{C})=4$.
Since $\mathcal{C}$ is not MDS, by Lemma 4 , one can obtain that $d_{p}(\mathcal{C}) \geq 6$. Now we claim that there does not exist a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,6)$. On the contrary, without loss of generality, we assume

$$
c(x)=c_{0}+c_{1} x+c_{2} x^{2}+c_{3} x^{3}+c_{4} x^{4}
$$

where $c_{i} \in \mathbb{F}_{p}^{*}$ for any $0 \leq i \leq 4$. This is contradictory with

$$
\operatorname{deg}(g(x))=5, \quad \operatorname{deg}(c(x)) \geq \operatorname{deg}(g(x))
$$

Thus, there does not exist a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,6)$. To show that $\mathcal{C}$ is an $\operatorname{MDS}(5 p, 7)_{p}$ symbol-pair code, it is sufficient to verify that there does not exist a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$.

Let $c(x)$ be a codeword in $\mathcal{C}$ with Hamming weight 4. Suppose that $c(x)$ has the factorization $c(x)=\left(x^{5}-1\right)^{t} v(x)$, where $0 \leq t \leq p-1,\left(x^{5}-1\right) \nmid v(x)$ and

$$
v(x)=v_{0}\left(x^{5}\right)+x v_{1}\left(x^{5}\right)+x^{2} v_{2}\left(x^{5}\right)+x^{3} v_{3}\left(x^{5}\right)+x^{4} v_{4}\left(x^{5}\right) .
$$

It follows from Lemma 5 that

$$
4=w_{H}\left(\left(x^{5}-1\right)^{t}\right) \cdot w_{H}\left(v(x) \bmod \left(x^{5}-1\right)\right)=(1+t) N_{v}
$$

where $N_{v}=w_{H}\left(v(x) \bmod \left(x^{5}-1\right)\right)$. Then one can deduce that $\left(N_{v}, t\right)=(1,3),(2,1)$ or $(4,0)$.
If $\left(N_{v}, t\right)=(1,3)$, then it is obvious that the symbol-pair weight of $c(x)$ is greater than 6 .
If $\left(N_{v}, t\right)=(2,1)$ and $c(x)$ has symbol-pair weight 6 , then Lemma 1 indicates that its certain cyclic shift must have the form

$$
(\star, \star, \mathbf{0}, \star, \star, \mathbf{0}) .
$$

Let

$$
c(x)=1+a_{1} x+a_{2} x^{5 i}+a_{3} x^{5 i+1}
$$

for some positive integer $i$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$. It follows from $5 \mid(p-1)$ and $\operatorname{gcd}(i, p)=1$ that $p \nmid 5 i$. The fact $c(1)=c(\beta)=0$ induces that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0, \\
1+a_{1} \beta+a_{2}+a_{3} \beta=0
\end{array}\right.
$$

which implies $a_{1}=-a_{3}$ and $a_{2}=-1$. Then $c^{(1)}(1)=c^{(2)}(1)=0$ yields

$$
\left\{\begin{array}{l}
a_{1}-5 i-(5 i+1) a_{1}=0, \\
-5 i(5 i-1)-5 i(5 i+1) a_{1}=0 .
\end{array}\right.
$$

This indicates $a_{1}=-1$ and then $2=0$, a contradiction.
If $\left(N_{v}, t\right)=(4,0)$ and $c(x)$ has symbol-pair weight 6 , then its corresponding cyclic shift must have the form

$$
(\star, \star, \mathbf{0}, \star, \star, \mathbf{0})
$$

or

$$
(\star, \star, \star, \mathbf{0}, \star, \mathbf{0}) .
$$

In what follows, we discuss the above two cases one by one.
Case I For the case ( $\star, \star, \mathbf{0}, \star, \star, \mathbf{0})$, there are two subcases to be considered:

- For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+2}+a_{3} x^{5 i+3}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, it follows from $c(1)=c^{(1)}(1)=c^{(2)}(1)=0$ that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0  \tag{5}\\
a_{1}+(5 i+2) a_{2}+(5 i+3) a_{3}=0 \\
(5 i+2)(5 i+1) a_{2}+(5 i+3)(5 i+2) a_{3}=0
\end{array}\right.
$$

If $p \mid(5 i+2)$, then (5) implies that $a_{1}=-a_{3}$ and $a_{2}=-1$. Then $c(\beta)=c\left(\beta^{2}\right)=0$ yields

$$
\left\{\begin{array}{l}
1+a_{1} \beta-\beta^{2}-a_{1} \beta^{3}=0 \\
1+a_{1} \beta^{2}-\beta^{4}-a_{1} \beta^{6}=0
\end{array}\right.
$$

One can immediately obtain that

$$
a_{1}=\frac{\beta^{2}-1}{\beta-\beta^{3}}=\frac{\beta^{4}-1}{\beta^{2}-\beta^{6}} .
$$

This leads to $\beta=1$, a contradiction.
If $p \nmid(5 i+2)$, then (5) yields that $a_{1}=-a_{2}$ and $a_{3}=-1$. It follows from $c(\beta)=c\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
1+a_{1} \beta-a_{1} \beta^{2}-\beta^{3}=0 \\
1+a_{1} \beta^{2}-a_{1} \beta^{4}-\beta^{6}=0
\end{array}\right.
$$

Then one gets that

$$
a_{1}=\frac{\beta^{3}-1}{\beta-\beta^{2}}=\frac{\beta^{6}-1}{\beta^{2}-\beta^{4}}
$$

which induces

$$
\beta^{3}+1=\beta(\beta+1) .
$$

This implies $(\beta-1)\left(\beta^{2}-1\right)=0$, a contradiction.

- Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+3}+a_{3} x^{5 i+4}$ with $0 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3} \in$ $\mathbb{F}_{p}^{*}$. By arguments similar to the previous subcase of $c(x)=1+a_{1} x+a_{2} x^{5 i+2}+a_{3} x^{5 i+3}$, one can also derive a contradiction and we omit the proof here.

Case II For the remaining case $(\star, \star, \star, \mathbf{0}, \star, \mathbf{0})$, there are also two subcases to be discussed:

- Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+3}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$.

Notice that $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 indicates

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} . \tag{6}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(2)}(1)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+3) a_{3}=0  \tag{7}\\
2 a_{2}+(5 i+3)(5 i+2) a_{3}=0
\end{array}\right.
$$

Observe that (7) yields

$$
\left\{\begin{array}{l}
a_{1}=(5 i+3)(5 i+1) a_{3}  \tag{8}\\
(5 i+2) a_{1}+2(5 i+1) a_{2}=0
\end{array}\right.
$$

and the second equality in (7) indicates $p \nmid(5 i+2)$. Let $t=5 i+2$. By (6) and (8), one can immediately have

$$
\left\{\begin{array}{l}
t^{2}=\beta^{3}+\beta^{2}+\beta+1  \tag{9}\\
t(\beta-2)=2
\end{array}\right.
$$

The second equality in (9) indicates $\beta \neq 2$ and $t=-\frac{2}{\beta-2}$. By substituting the value of $t$ into the first equality in (9), one can obtain

$$
\frac{4 \beta}{(\beta-2)^{2}}=\left(\beta^{3}+\beta^{2}+\beta+1\right) \beta
$$

It follows from $\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=0$ that $\beta^{2}=-4$ and

$$
\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=13-3 \beta=0
$$

This leads to $\beta=\frac{13}{3}$ and then

$$
\beta^{2}=\frac{169}{9}=-4
$$

implies $5 \cdot 41=0$, which is contradictory with $5 \mid(p-1)$ and $p \neq 41$.

- For the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+4}$ with $0 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, it follows from $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 that

$$
\begin{equation*}
a_{1}=-\frac{1}{\beta}, \quad a_{2}=-\frac{\beta}{\beta+1}, \quad a_{3}=\frac{1}{\beta(\beta+1)} \tag{10}
\end{equation*}
$$

On the other hand, $c^{(1)}(1)=c^{(2)}(1)=0$ yields that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+4) a_{3}=0, \\
2 a_{2}+(5 i+4)(5 i+3) a_{3}=0
\end{array}\right.
$$

which is equivalent to

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+4) a_{3}=0 \\
a_{1}=(5 i+4)(5 i+2) a_{3}
\end{array}\right.
$$

Let $t=5 i+3$. Together with (10), one can immediately obtain that

$$
\left\{\begin{array}{l}
t=2 \beta^{2}+\beta \\
t^{2}+\beta=0
\end{array}\right.
$$

Then by substituting the value of $t$, one has

$$
\begin{equation*}
\beta^{2}=3 \beta+3 \tag{11}
\end{equation*}
$$

and

$$
0=\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=61 \beta+49
$$

If $p=61$, then $49=0$, which is impossible. If $p \neq 61$, then $\beta=-\frac{49}{61}$. It follows from (11) that

$$
\left(\frac{49}{61}\right)^{2}=3\left(-\frac{49}{61}+1\right)
$$

This implies $5 \cdot 41=0$, a contradiction similar as the previous subcase.

Therefore, $\mathcal{C}$ is an MDS $(5 p, 7)_{p}$ symbol-pair code. This completes the proof.
Another class of MDS symbol-pair codes with $n=5 p$ and $d_{p}=8$ is proposed as follows.
Theorem 2. Let $p$ be a prime with $5 \mid(p-1)$. Then there exists an $M D S(5 p, 8)_{p}$ symbol-pair code.

Proof Let $\mathcal{C}$ be a repeated-root cyclic code of length $5 p$ over $\mathbb{F}_{p}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-\beta)\left(x-\beta^{2}\right)^{2}
$$

where $\beta$ is a primitive 5 -th root of unity in $\mathbb{F}_{p}$. It can be verified that $\mathcal{C}$ is an $\operatorname{MDS}(5 p, 8)_{p}$ symbol-pair code by similar techniques used in the proof of Theorem 1 . Since the proof is lengthy and some of them seems a bit cumbersome, we present it in the Appendix.

Now we provide two examples to illustrate the constructions in Theorems 11 and 2,
Example 1. (1) Let $\mathcal{C}$ be a repeated-root cyclic code of length 55 over $\mathbb{F}_{11}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-3)\left(x-3^{2}\right)
$$

MAGMA experiments show that $\mathcal{C}$ is a $[55,50,4]$ code and the minimum symbol-pair distance of $\mathcal{C}$ is 7 , which satisfies our result in Theorem 1 .
(2) Let $\mathcal{C}$ be a repeated-root cyclic code of length 55 over $\mathbb{F}_{11}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-3)\left(x-3^{2}\right)^{2}
$$

By a MAGMA program, it can be checked that $\mathcal{C}$ is a $[55,49,4]$ code and the minimum symbolpair distance of $\mathcal{C}$ is 8 , which is consistent with our result in Theorem (2.

### 3.2 MDS symbol-pair codes for $n=4 p$

In this subsection, we shall construct a class of MDS symbol-pair codes with $d_{p}=7$, which generalizes Theorem 3.8 in [18].

Theorem 1. Let $p$ be an odd prime. Then there exists an MDS $(4 p, 7)_{p}$ symbol-pair code.
Proof The case $p \equiv 3(\bmod 4)$ has been settled, see Theorem 3.8 in [18]. For the case $p \equiv 1(\bmod 4)$, let $\mathcal{C}$ be a repeated-root cyclic code of length $4 p$ over $\mathbb{F}_{p}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-\omega)(x+\omega)
$$

where $\omega$ is a primitive 4 -th root of unity in $\mathbb{F}_{p}$. In the following, we will claim that for $p \equiv$ $1(\bmod 4)$, the code $\mathcal{C}$ is also an $\operatorname{MDS}(4 p, 7)_{p}$ symbol-pair code.

By Lemma 3, one can derive that the parameter of $\mathcal{C}$ is [ $4 p, 4 p-5,4]$. Since $\mathcal{C}$ is not MDS, by Lemma 4 , we get $d_{p}(\mathcal{C}) \geq 6$. With a similar argument as Theorem 1 , one can obtain that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,6)$. In order to show that $\mathcal{C}$ is an $\operatorname{MDS}(4 p, 7)_{p}$ symbol-pair code, we need to prove that there does not exist a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$.

Let $c(x)$ be a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$. Then Lemma $\mathbb{1}$ indicates that its certain cyclic shift must have the form

$$
(\star, \star, \star, \mathbf{0}, \star, \mathbf{0})
$$

or

$$
(\star, \star, \mathbf{0}, \star, \star, \mathbf{0}) .
$$

For the case $(\star, \star, \star, \mathbf{0}, \star, \mathbf{0})$, we assume that

$$
c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{l}
$$

for some positive integer $l$ with $4 \leq l \leq 4 p-2$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$. It follows from $c(1)=$ $c^{(1)}(1)=c^{(2)}(1)=0$ that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0 \\
a_{1}+2 a_{2}+l a_{3}=0 \\
2 a_{2}+l(l-1) a_{3}=0
\end{array}\right.
$$

This yields

$$
\begin{equation*}
a_{1}=-\frac{2 l}{l-1}, \quad a_{2}=\frac{l}{l-2}, \quad a_{3}=-\frac{2}{(l-1)(l-2)} . \tag{12}
\end{equation*}
$$

- If $l$ is even, then we have

$$
\left\{\begin{array}{l}
1+a_{1} \omega-a_{2}+a_{3} \omega^{l}=0 \\
1-a_{1} \omega-a_{2}+a_{3} \omega^{l}=0
\end{array}\right.
$$

since $c(\omega)=c(-\omega)=0$ and $\omega^{2}=-1$. It follows that $a_{1}=0$, which is impossible.

- If $l$ is odd, then $c(\omega)=c(-\omega)=0$ indicates that

$$
\left\{\begin{array}{l}
1+a_{1} \omega-a_{2}+a_{3} \omega^{l}=0 \\
1-a_{1} \omega-a_{2}-a_{3} \omega^{l}=0
\end{array}\right.
$$

This implies that $a_{2}=1$, which contradicts with the result in (12).

For the remaining case $(\star, \star, \mathbf{0}, \star, \star, \mathbf{0})$, we suppose that

$$
c(x)=1+a_{1} x+a_{2} x^{l}+a_{3} x^{l+1}
$$

for some positive integer $l$ with $3 \leq l \leq 4 p-3$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$. Then $c(1)=c^{(1)}(1)=$ $c^{(2)}(1)=0$ indicates that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0,  \tag{13}\\
a_{1}+l a_{2}+(l+1) a_{3}=0, \\
l(l-1) a_{2}+l(l+1) a_{3}=0 .
\end{array}\right.
$$

It follows from $c(\omega)=c(-\omega)=0$ that

$$
\left\{\begin{array}{l}
1+a_{1} \omega+a_{2} \omega^{l}+a_{3} \omega^{l+1}=0  \tag{14}\\
1-a_{1} \omega+a_{2}(-\omega)^{l}+a_{3}(-\omega)^{l+1}=0
\end{array}\right.
$$

Now we divide the proof into the following two subcases:

- For the subcase $p \mid l$, (13) yields that

$$
\begin{equation*}
a_{1}+a_{3}=0, \quad a_{2}=-1 \tag{15}
\end{equation*}
$$

If $l$ is even, then we have $l=2 p$ due to $3 \leq l \leq 4 p-3$. It follows from (14) and (15) that

$$
1=\omega^{l}=\omega^{2 p}=(-1)^{p}
$$

which is impossible. Similarly, if $l$ is odd, one can obtain that $\omega^{2 l}=1$, a contradiction.

- For the subcase $p \nmid l$, it follows from (13) that

$$
\begin{equation*}
a_{1}=-\frac{l+1}{l-1}, \quad a_{2}=\frac{l+1}{l-1}, \quad a_{3}=-1 \tag{16}
\end{equation*}
$$

If $l$ is even, then by (14) and (16), one can deduce that

$$
a_{1}=\omega^{l}, \quad 1+a_{2} \omega^{l}=0
$$

Then one can obtain that

$$
1=a_{1}^{2}=\left(-\frac{l+1}{l-1}\right)^{2}
$$

This implies $4 l=0$, a contradiction. By a similar manner, for odd $l$, one can derive that $\omega^{l+1}=\omega^{l-1}=1$, which is impossible.

Consequently, $\mathcal{C}$ is an $\operatorname{MDS}(4 p, 7)_{p}$ symbol-pair code. This proves the desired conclusion.

Now we give an example to illustrate the construction in Theorem 1 ,
Example 2. Let $\mathcal{C}$ be a repeated-root cyclic code of length 20 over $\mathbb{F}_{5}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-2)(x+2)
$$

It can be checked by MAGMA that $\mathcal{C}$ is a $[20,15,4]$ code and the minimum symbol-pair distance of $\mathcal{C}$ is 7, which coincides with our result in Theorem 1 .

## 4 Conclusions and future work

In this paper, three new classes of MDS symbol-pair codes over $\mathbb{F}_{p}$ with $p$ an odd prime were constructed from repeated-root cyclic codes. Firstly, for $n=5 p$, two classes of MDS symbol-pair codes with minimum symbol-pair distance seven or eight were presented. In addition, for length $n=4 p$, we derived a class of MDS symbol-pair codes with $d_{p}=7$ and our construction extends the result in [18]. Note that by utilizing repeated-root cyclic codes, one can construct MDS symbol-pair codes by transforming the problem into analyzing the solutions of certain equations over finite fields.

However, it seems impracticable to construct $(5 q, 7)_{p},(5 q, 8)_{p}$ and $(4 q, 7)_{p}$ MDS symbol-pair codes with $q$ being a power of $p$ using the techniques in Theorems 1-3. For instance, for the case $q=p^{2}, 5 \mid(q-1)$, let $\mathcal{C}$ be a repeated-root cyclic code of length $5 q$ over $\mathbb{F}_{q}$ with the generator polynomial of the form

$$
g(x)=(x-1)^{e_{1}}(x-\omega)^{e_{2}}\left(x-\omega^{2}\right)^{e_{3}}\left(x-\omega^{3}\right)^{e_{4}}\left(x-\omega^{4}\right)^{e_{5}}
$$

where $\omega$ is a primitive 5 -th root of unity in $\mathbb{F}_{q}$. It can be checked that $\mathcal{C}$ is not an MDS symbolpair code. It needs further study to construct MDS symbol-pair codes with larger minimum symbol-pair distance and length $l q$, where $q=p^{m}$ with $m>1$.

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## Appendix Proof of Theorem 2:

Recall that $\mathcal{C}$ is a repeated-root cyclic code of length $5 p$ over $\mathbb{F}_{p}$ with the generator polynomial

$$
g(x)=(x-1)^{3}(x-\beta)\left(x-\beta^{2}\right)^{2}
$$

where $\beta$ is a primitive 5 -th root of unity in $\mathbb{F}_{p}$. By Lemma 3, one can derive that the parameter of $\mathcal{C}$ is $[5 p, 5 p-6,4]$. Since $\mathcal{C}$ is not MDS, Lemma 4 yields that $d_{p}(\mathcal{C}) \geq 6$. Similar as Theorem 1 , one can derive that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,6)$ or $(6,7)$. To prove that $\mathcal{C}$ is an $\operatorname{MDS}(5 p, 8)_{p}$ symbol-pair code, it suffices to determine that there does not exist a codeword in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6),(4,7)$ or $(5,7)$.

Case I: $\quad\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$. Since $\mathcal{C}$ is the subcode of the code in Theorem 1 and the proof of Theorem 1 indicates that there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$ unless $p=41$. Now it is sufficient to show that for $p=41$, there does not exist a codeword $c(x)$ in $\mathcal{C}$ with $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(4,6)$. More precisely, we just need to consider Case II in Theorem 1. There are two subcases to be discussed:

- Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+3}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$. Notice that $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 induces

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} . \tag{17}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+3) a_{3}=0, \\
a_{1}+2 a_{2} \beta^{2}+(5 i+3) a_{3} \beta^{4}=0
\end{array}\right.
$$

which yields

$$
a_{1}\left(\beta^{4}-1\right)+2 a_{2}\left(\beta^{4}-\beta^{2}\right)=0
$$

Combining with (17), one can get $(\beta-1)^{2}=0$, a contradiction.

- For the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+4}$ with $0 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, by $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6, one can obtain that

$$
\begin{equation*}
a_{1}=-\frac{1}{\beta}, \quad a_{2}=-\frac{\beta}{\beta+1}, \quad a_{3}=\frac{1}{\beta(\beta+1)} . \tag{18}
\end{equation*}
$$

On the other hand, $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+4) a_{3}=0 \\
a_{1}+2 a_{2} \beta^{2}+(5 i+4) a_{3} \beta=0
\end{array}\right.
$$

which induces

$$
a_{1}(\beta-1)=2 a_{2}\left(\beta^{2}-\beta\right) .
$$

Together with (18), one can immediately obtain that

$$
2 \beta^{3}=\beta+1 .
$$

This leads to

$$
(\beta-1)\left(2 \beta^{2}+2 \beta+1\right)=0 .
$$

The fact $\beta$ is a primitive 5 -th root of unity implies that $2 \beta^{2}+2 \beta+1=0$ and then one has

$$
\beta^{2}+\beta=-\left(\beta^{2}+\beta+1\right)=\beta^{4}+\beta^{3}
$$

which is impossible.
 of $c(x)$ must have the form

$$
(\star, \star, \mathbf{0}, \star, \mathbf{0}, \star, \mathbf{0}) .
$$

Assume that $c(x)=\left(x^{5}-1\right)^{t} v(x)$, where $0 \leq t \leq p-1,\left(x^{5}-1\right) \nmid v(x)$ and

$$
v(x)=v_{0}\left(x^{5}\right)+x v_{1}\left(x^{5}\right)+x^{2} v_{2}\left(x^{5}\right)+x^{3} v_{3}\left(x^{5}\right)+x^{4} v_{4}\left(x^{5}\right) .
$$

Recall that $N_{v}=w_{H}\left(v(x) \bmod \left(x^{5}-1\right)\right)$. Then by Lemma 圆, one can deduce that

$$
4=w_{H}\left(\left(x^{5}-1\right)^{t}\right) \cdot w_{H}\left(v(x) \bmod \left(x^{5}-1\right)\right)=(1+t) N_{v} .
$$

If $\left(N_{v}, t\right)=(1,3)$, then it is easily seen that the symbol-pair weight of $c(x)$ is greater than 7.

If $\left(N_{v}, t\right)=(2,1)$, then there are three subcases to be discussed:
(1) For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i}+a_{3} x^{5 j}$ with $1 \leq i<j \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, it can be verified that

$$
\left\{\begin{array}{l}
1+a_{1} \beta+a_{2}+a_{3}=0 \\
1+a_{1} \beta^{2}+a_{2}+a_{3}=0
\end{array}\right.
$$

since $c(\beta)=c\left(\beta^{2}\right)=0$. Then one can obtain that $a_{1}=0$, a contradiction.
(2) For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+1}+a_{3} x^{5 j+1}$ with $1 \leq i<j \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, by $c(1)=c(\beta)=0$, one can get

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}=0, \\
1+a_{1} \beta+a_{2} \beta+a_{3} \beta=0 .
\end{array}\right.
$$

This implies that $\beta=1$, which is impossible.
(3) For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i}+a_{3} x^{5 j+1}$ with $1 \leq i<j \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, it follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+5 i a_{2}+(5 j+1) a_{3}=0 \\
a_{1}+5 i a_{2} \beta^{3}+(5 j+1) a_{3}=0
\end{array}\right.
$$

This leads to $\beta^{3}=1$, a contradiction.
If $\left(N_{v}, t\right)=(4,0)$, then there are also three subcases to be considered:
(1) For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+2}+a_{3} x^{5 j+3}$ with $1 \leq i<j \leq p-1$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, by Lemma 6 and $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$, one can derive that

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} . \tag{19}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+(5 i+2) a_{2}+(5 j+3) a_{3}=0, \\
a_{1}+(5 i+2) a_{2} \beta^{2}+(5 j+3) a_{3} \beta^{4}=0
\end{array}\right.
$$

which indicates

$$
\left\{\begin{array}{l}
\left(\beta^{4}-1\right) a_{1}+\left(\beta^{4}-\beta^{2}\right)(5 i+2) a_{2}=0 \\
\left(\beta^{2}-1\right)(5 i+2) a_{2}+(5 j+3)\left(\beta^{4}-1\right) a_{3}=0
\end{array}\right.
$$

Together with (19), one can immediately obtain that

$$
\left\{\begin{array}{l}
\beta^{2}+1=(5 i+2) \beta  \tag{20}\\
(5 i+2)\left(\beta^{2}+\beta+1\right)=(5 j+3)\left(\beta^{2}+1\right)
\end{array}\right.
$$

By substituting the value of $\beta^{2}+1$ in the first equality into the second equality of (20), we can get

$$
(5 i+2)(5 i+3) \beta=(5 i+2)(5 j+3) \beta
$$

which yields $i=j$ due to $p \nmid(5 i+2)$. This contradicts with $i<j$.
(2) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+2}+a_{3} x^{5 j+4}$ with $1 \leq i \leq j \leq p-2$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$. The fact $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 leads to

$$
\begin{equation*}
a_{1}=-\frac{1}{\beta}, \quad a_{2}=-\frac{\beta}{\beta+1}, \quad a_{3}=\frac{1}{\beta(\beta+1)} \tag{21}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}=\beta(5 i+2) a_{2} \\
(5 i+2)(\beta+1) a_{2}+(5 j+4) a_{3}=0
\end{array}\right.
$$

By substituting (21), one can immediately derive that

$$
\left\{\begin{array}{l}
\beta+1=(5 i+2) \beta^{3}  \tag{22}\\
(5 i+2) \beta^{2}(\beta+1)=5 j+4
\end{array}\right.
$$

This leads to $(5 i+2)^{2}=5 j+4$. Since it can be verified that $p \nmid(5 i+2)$, it follows from $c^{(2)}(1)=0$ that

$$
\begin{equation*}
\beta^{2}=(5 i+2)(5 i+3) . \tag{23}
\end{equation*}
$$

Then (21) and $c^{(1)}(1)=0$ indicates that

$$
\begin{equation*}
(5 i+2) \beta^{2}+\beta-(5 j+3)=0 \tag{24}
\end{equation*}
$$

Let $t=5 i+2$. Then one has $\beta+1=t \beta^{3}$ and $\beta^{2}=t(t+1)$ due to the first equality of (22) and (23). It follows from (24) that

$$
t^{2}(t+1)+\beta-\left(t^{2}-1\right)=0
$$

which implies $\beta+1=-t^{3}$. Combining with $\beta+1=t \beta^{3}$, we have $\beta^{3}=-t^{2}$. Since $\beta$ is a primitive 5 -th root of unity, one can derive

$$
\begin{aligned}
0 & =\beta^{4}+\beta^{3}+\beta^{2}+\beta+1 \\
& =(\beta+1)\left(\beta^{3}+1\right)+\beta^{2} \\
& =-t^{3}\left(-t^{2}+1\right)+t(t+1) \\
& =t(t+1)\left(t^{3}-t^{2}+1\right)
\end{aligned}
$$

It follows from $t(t+1)=\beta^{2} \neq 0$ that $t^{3}-t^{2}+1=0$. Then we obtain

$$
\beta=-t^{3}-1=-t^{2}=\beta^{3}
$$

which yields $\beta^{2}-1=0$, a contradiction.
(3) For the subcase $c(x)=1+a_{1} x+a_{2} x^{5 i+3}+a_{3} x^{5 j+4}$ with $0 \leq i<j \leq p-2$ and $a_{1}, a_{2}, a_{3} \in \mathbb{F}_{p}^{*}$, it follows from $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 that

$$
\begin{equation*}
a_{1}=\frac{\beta^{2}}{\beta+1}, \quad a_{2}=-\frac{1}{\beta+1}, \quad a_{3}=-\beta \tag{25}
\end{equation*}
$$

Since $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$, one can immediately get

$$
\left\{\begin{array}{l}
(5 i+3)\left(\beta^{4}-1\right) a_{2}+(5 j+4)(\beta-1) a_{3}=0 \\
a_{1}(\beta-1)=(5 i+3)\left(\beta^{4}-\beta\right) a_{2}
\end{array}\right.
$$

Together with (25), one can conclude that

$$
\left\{\begin{array}{l}
(5 i+3)\left(\beta^{2}+1\right)+(5 j+4) \beta=0 \\
(5 i+3)\left(\beta^{2}+\beta+1\right)+\beta=0
\end{array}\right.
$$

which indicates

$$
\left\{\begin{array}{l}
(5 i+3) \beta^{2}+(5 j+4) \beta+5 i+3=0 \\
(5 i+3) \beta^{2}+(5 i+4) \beta+5 i+3=0
\end{array}\right.
$$

It follows that $5(i-j)=0$, a contradiction.
Case III: $\left(w_{H}(c(x)), w_{p}(c(x))\right)=(5,7)$. In this case, we can assume that $c(x)$ is of the form

$$
(\mathbf{a}, \mathbf{0}, \mathbf{b}, \mathbf{0})
$$

where $\mathbf{a}, \mathbf{b}$ are row vectors with all entries of $\mathbf{a}, \mathbf{b}$ being nonzero. Then its certain cyclic shift must have the form

$$
(\star, \star, \star, \star, \mathbf{0}, \star, \mathbf{0})
$$

or

$$
(\star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0}) .
$$

- For $(\star, \star, \star, \star, \mathbf{0}, \star, \mathbf{0})$, there are five subcases to be considered:
(1) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{5 i}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It can be verified that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}=0 \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3} \beta^{3}+a_{4}=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3} \beta+a_{4}=0
\end{array}\right.
$$

since $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$. Then one can derive that $p \nmid\left(a_{4}+1\right)$. By Lemma 6, one can obtain

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}\left(a_{4}+1\right), a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}\left(a_{4}+1\right), a_{3}=-\frac{1}{\beta^{3}}\left(a_{4}+1\right) \tag{26}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+3 a_{3}+5 i a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+3 a_{3} \beta^{4}+5 i a_{4} \beta^{3}=0
\end{array}\right.
$$

which indicates

$$
\left(\beta^{3}-1\right) a_{1}+2\left(\beta^{3}-\beta^{2}\right) a_{2}+3\left(\beta^{3}-\beta^{4}\right) a_{3}=0
$$

Combining with (26), one can derive that

$$
-\left(\beta^{3}-1\right) \beta\left(\beta^{2}+\beta+1\right)+2 \beta^{2}(\beta-1)\left(\beta^{2}+\beta+1\right)+3 \beta^{3}(\beta-1)=0
$$

Since $\beta$ is a primitive 5 -th root of unity, by expanding the above equality, one can get $\beta^{2}+3 \beta+1=0$. This is contradictory with the inequality (3) in Lemma 6,
(2) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{5 i+1}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It follows from $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 that

$$
\begin{equation*}
a_{1}+a_{4}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} . \tag{27}
\end{equation*}
$$

Then $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ induces that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+3 a_{3}+(5 i+1) a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+3 a_{3} \beta^{4}+(5 i+1) a_{4}=0
\end{array}\right.
$$

This leads to

$$
2\left(\beta^{2}-1\right) a_{2}+3\left(\beta^{4}-1\right) a_{3}=0
$$

Together with (27), one can immediately get $(\beta-1)^{2}=0$, which is impossible.
(3) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{5 i+2}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. The fact $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 induces

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}+a_{4}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}=-\frac{1}{\beta^{3}} . \tag{28}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+3 a_{3}+(5 i+2) a_{4}=0, \\
a_{1}+2 a_{2} \beta^{2}+3 a_{3} \beta^{4}+(5 i+2) a_{4} \beta^{2}=0
\end{array}\right.
$$

which implies

$$
\left(\beta^{2}-1\right) a_{1}+3\left(\beta^{2}-\beta^{4}\right) a_{3}=0
$$

By substituting (28) into the above equality, we have $(\beta-1)^{2}=0$, a contradiction.
(4) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{5 i+3}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. By $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6, one has

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{3}+a_{4}=-\frac{1}{\beta^{3}} . \tag{29}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+3 a_{3}+(5 i+3) a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+3 a_{3} \beta^{4}+(5 i+3) a_{4} \beta^{4}=0
\end{array}\right.
$$

This yields

$$
\left(\beta^{4}-1\right) a_{1}+2\left(\beta^{4}-\beta^{2}\right) a_{2}=0
$$

Combining with (29), one can derive that $(\beta-1)^{2}=0$, which is impossible.
(5) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+a_{4} x^{5 i+4}$ with $1 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It can be verified that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}=0 \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3} \beta^{3}+a_{4} \beta^{4}=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3} \beta+a_{4} \beta^{3}=0
\end{array}\right.
$$

since $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$. Then one can obtain that

$$
\left\{\begin{array}{l}
a_{1}=-\beta^{3} a_{4}+\beta^{2}+\beta  \tag{30}\\
a_{2}=-\left(\beta^{4}+1\right) a_{4}-\beta-1 \\
a_{3}=-\left(\beta^{2}+\beta+1\right) a_{4}-\beta^{2}
\end{array}\right.
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+3 a_{3}+(5 i+4) a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+3 a_{3} \beta^{4}+(5 i+4) a_{4} \beta=0
\end{array}\right.
$$

which implies

$$
(\beta-1) a_{1}+2\left(\beta-\beta^{2}\right) a_{2}+3\left(\beta-\beta^{4}\right) a_{3}=0
$$

This is equivalent to

$$
a_{1}-2 \beta a_{2}-3 \beta\left(\beta^{2}+\beta+1\right) a_{3}=0
$$

Together with (30), one can immediately have

$$
\left(-\beta^{3}+2 \beta\left(\beta^{4}+1\right)+3 \beta\left(\beta^{2}+\beta+1\right)^{2}\right) a_{4}+\beta^{2}+\beta+2 \beta(\beta+1)+3 \beta^{3}\left(\beta^{2}+\beta+1\right)=0
$$

Then we get that

$$
-\beta^{3}+2 \beta\left(\beta^{4}+1\right)+3 \beta\left(\beta^{2}+\beta+1\right)^{2}=0
$$

due to $\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=0$ and $a_{4} \in \mathbb{F}_{p}^{*}$. By a straightforward computation, one has $\beta^{2}+3 \beta+1=0$. This contradicts with the inequality (3) in Lemma 6,

- For $(\star, \star, \star, \mathbf{0}, \star, \star, \mathbf{0})$, there are also five subcases to be considered:
(1) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i}+a_{4} x^{5 i+1}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It follows from $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}=0 \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3}+a_{4} \beta=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3}+a_{4} \beta^{2}=0
\end{array}\right.
$$

which implies

$$
\left\{\begin{array}{l}
\left(a_{1}+a_{4}\right)(\beta-1)+a_{2}\left(\beta^{2}-1\right)=0 \\
\left(a_{1}+a_{4}\right)\left(\beta^{2}-\beta\right)+a_{2}\left(\beta^{4}-\beta^{2}\right)=0
\end{array}\right.
$$

This indicates that $\beta\left(\beta^{2}-1\right) a_{2}=\left(\beta^{4}-\beta^{2}\right) a_{2}$. Hence $\beta=1$, a contradiction.
(2) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+1}+a_{4} x^{5 i+2}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It can be verified that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}=0 \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3} \beta+a_{4} \beta^{2}=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3} \beta^{2}+a_{4} \beta^{4}=0
\end{array}\right.
$$

since $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$. Then one can derive that

$$
\left\{\begin{array}{l}
\left(a_{2}+a_{4}\right)\left(\beta^{2}-\beta\right)=\beta-1 \\
\left(a_{2}+a_{4}\right)\left(\beta^{4}-\beta^{3}\right)=\beta-1
\end{array}\right.
$$

It follows that $\beta^{3}=\beta$, which is impossible.
(3) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+2}+a_{4} x^{5 i+3}$ with $1 \leq i \leq p-1$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. The fact $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ and Lemma 6 induces that

$$
\begin{equation*}
a_{1}=-\frac{\beta^{2}+\beta+1}{\beta^{2}}, \quad a_{2}+a_{3}=\frac{\beta^{2}+\beta+1}{\beta^{3}}, \quad a_{4}=-\frac{1}{\beta^{3}} . \tag{31}
\end{equation*}
$$

It follows from $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ that

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+2) a_{3}+(5 i+3) a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+(5 i+2) a_{3} \beta^{2}+(5 i+3) a_{4} \beta^{4}=0
\end{array}\right.
$$

This yields

$$
\left(\beta^{2}-1\right) a_{1}+(5 i+3)\left(\beta^{2}-\beta^{4}\right) a_{4}=0
$$

By substituting (31), one can deduce that

$$
\beta^{2}-(5 i+2) \beta+1=0 .
$$

Let $t=5 i+2$. Then $\beta^{2}=t \beta-1$ and

$$
\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=(t \beta-1)\left(t^{2}+t-1\right)=0
$$

It follows that $t^{2}+t=1$. By $c^{(2)}(1)=0$ and (31), we get

$$
5 i(t+1) a_{3}=(t+2) \beta+1
$$

The fact $c^{(1)}(1)=0$ indicates 5i $a_{3}=(2-t)(\beta+1)$. Hence

$$
(t+2) \beta+1=(t+1)(2-t)(\beta+1) .
$$

This leads to $t^{2} \beta-2 t=0$ due to $t^{2}+t=1$. It follows from $t \neq 0$ that $t \beta=2$ and $\beta^{2}=t \beta-1=1$, a contradiction.
(4) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+3}+a_{4} x^{5 i+4}$ with $1 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It can be checked that

$$
\left\{\begin{array}{l}
1+a_{1}+a_{2}+a_{3}+a_{4}=0 \\
1+a_{1} \beta+a_{2} \beta^{2}+a_{3} \beta^{3}+a_{4} \beta^{4}=0 \\
1+a_{1} \beta^{2}+a_{2} \beta^{4}+a_{3} \beta+a_{4} \beta^{3}=0
\end{array}\right.
$$

since $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$. Then one can derive that

$$
\left\{\begin{array}{l}
a_{1}=-\beta^{3} a_{4}+\beta^{2}+\beta  \tag{32}\\
a_{2}=-\left(\beta^{4}+1\right) a_{4}-\beta-1 \\
a_{3}=-\left(\beta^{2}+\beta+1\right) a_{4}-\beta^{2}
\end{array}\right.
$$

Let $t=5 i+2$. $\operatorname{By} c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ and (32), we have

$$
\left\{\begin{array}{l}
t \beta^{2}+\beta+2=\left((t-1) \beta^{4}+t \beta^{3}+t\right) a_{4}  \tag{33}\\
2 \beta^{2}+\beta+t=\left(t \beta^{2}+(t-1) \beta+t\right) a_{4}
\end{array}\right.
$$

Then

$$
\left(t \beta^{2}+\beta+2\right)\left(t \beta^{2}+(t-1) \beta+t\right)=\left(2 \beta^{2}+\beta+t\right)\left((t-1) \beta^{4}+t \beta^{3}+t\right)
$$

which implies

$$
\left(t^{2}+t-1\right)\left(\beta^{2}-1\right)=0
$$

Thus $t^{2}+t=1$. It follows from $c^{(2)}(1)=0$ that $2 a_{2}+a_{3}+(2 t+3) a_{4}=0$. Together with (32), one can immediately get

$$
\left(-\beta^{4}+\beta^{3}+2 t+1\right) a_{4}=\beta^{2}+2 \beta+2
$$

Combining with the second equality in (33), we can obtain

$$
\left(-\beta^{4}+\beta^{3}+2 t+1\right)\left(2 \beta^{2}+\beta+t\right)=\left(\beta^{2}+2 \beta+2\right)\left(t \beta^{2}+(t-1) \beta+t\right)
$$

By expanding the above equality, one can deduce

$$
\left(\beta^{2}-1\right) t+3 \beta^{2}+2=0
$$

which yields $t=\frac{3 \beta^{2}+2}{1-\beta^{2}}$. The fact $t^{2}+t-1=0$ induces

$$
\left(\frac{3 \beta^{2}+2}{1-\beta^{2}}\right)^{2}+\frac{3 \beta^{2}+2}{1-\beta^{2}}-1=0
$$

which is equivalent to

$$
\left(3 \beta^{2}+2\right)^{2}+\left(3 \beta^{2}+2\right)\left(1-\beta^{2}\right)-\left(1-\beta^{2}\right)^{2}=0
$$

It follows that

$$
\beta^{4}+3 \beta^{2}+1=0
$$

which indicates

$$
2 \beta^{2}-\beta^{3}-\beta=0
$$

due to $\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=0$. Hence $\beta(\beta-1)^{2}=0$, which is impossible.
(5) Consider the subcase $c(x)=1+a_{1} x+a_{2} x^{2}+a_{3} x^{5 i+4}+a_{4} x^{5 i+5}$ with $0 \leq i \leq p-2$ and $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{F}_{p}^{*}$. It follows from $c(1)=c(\beta)=c\left(\beta^{2}\right)=0$ that $p \nmid\left(a_{4}+1\right)$ and

$$
\begin{equation*}
a_{1}=-\frac{1}{\beta}\left(a_{4}+1\right), \quad a_{2}=-\frac{\beta}{\beta+1}\left(a_{4}+1\right), \quad a_{3}=\frac{1}{\beta(\beta+1)}\left(a_{4}+1\right) \tag{34}
\end{equation*}
$$

due to Lemma 6. The fact $c^{(1)}(1)=c^{(1)}\left(\beta^{2}\right)=0$ leads to

$$
\left\{\begin{array}{l}
a_{1}+2 a_{2}+(5 i+4) a_{3}+(5 i+5) a_{4}=0 \\
a_{1}+2 a_{2} \beta^{2}+(5 i+4) a_{3} \beta+(5 i+5) a_{4} \beta^{3}=0
\end{array}\right.
$$

which implies

$$
\left\{\begin{array}{l}
\left(\beta^{3}-1\right) a_{1}+2\left(\beta^{3}-\beta^{2}\right) a_{2}+(5 i+4)\left(\beta^{3}-\beta\right) a_{3}=0, \\
2(\beta+1) a_{2}+(5 i+4) a_{3}+(5 i+5)\left(\beta^{2}+\beta+1\right) a_{4}=0 .
\end{array}\right.
$$

By substituting (34), one can obtain that

$$
\begin{equation*}
\beta^{4}+(5 i+3) \beta^{3}-(5 i+3) \beta-1=0 \tag{35}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(-2 \beta^{2}(\beta+1)+5 i+4\right)\left(a_{4}+1\right)+(5 i+5) \beta(\beta+1)\left(\beta^{2}+\beta+1\right) a_{4}=0 . \tag{36}
\end{equation*}
$$

Let $t=5 i+3$. It follows from (35) that

$$
\beta^{4}-1+t\left(\beta^{3}-\beta\right)=\left(\beta^{2}-1\right)\left(\beta^{2}+1+t \beta\right)=0
$$

which yields $\beta^{2}=-t \beta-1$. Then we have

$$
0=\beta^{4}+\beta^{3}+\beta^{2}+\beta+1=-\left(t^{2}-t-1\right) \beta^{2}
$$

which indicates $t^{2}=t+1$ due to $\beta^{2} \neq 0$. It can be verified that

$$
\begin{aligned}
& -2 \beta^{2}(\beta+1)+5 i+4+(5 i+5) \beta(\beta+1)\left(\beta^{2}+\beta+1\right) \\
= & -2 \beta^{3}-2 \beta^{2}+t+1-(t+2)\left(\beta^{4}+\beta+2\right) \\
= & -2 t(\beta+1)+2(t \beta+1)+(t+2)(\beta+t)-(t+2) \beta-t-3=0 .
\end{aligned}
$$

Hence (36) and $a_{4} \in \mathbb{F}_{p}^{*}$ induces

$$
0=-2 \beta^{2}(\beta+1)+5 i+4=3-t
$$

which means that $t=3$ and $\beta^{2}=-3 \beta-1$, a contradiction with the inequality (3) in Lemma 6

As a consequence, $\mathcal{C}$ is an $\operatorname{MDS}(5 p, 8)_{p}$ symbol-pair code. The desired result follows.


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