The number of irreducible polynomials over finite fields with vanishing trace and reciprocal trace

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Abstract

We present the formula for the number of monic irreducible polynomials of degree n over the finite field \mathbb{F}_q where the coefficients of x^{n-1} and x vanish for $n \ge 3$. In particular, we give a relation between rational points of algebraic curves over finite fields and the number of elements $a \in \mathbb{F}_{q^n}$ for which $\operatorname{Trace}(a) = 0$ and $\operatorname{Trace}(a^{-1}) = 0$. Besides, we apply the formula to give an upper bound on the number of distinct constructions of a family of sequences with good family complexity and cross-correlation measure.

Keywords: Irreducible polynomials, Finite fields, Trace function, Algebraic curves, Pseudorandom sequences

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1 Introduction

Let r be a positive integer, p be a prime number and $q = p^r$, \mathbb{F}_q be the finite field with q elements and let $I_q(n)$ denote the number of monic irreducible polynomials of degree n over $\mathbb{F}_q[x]$. It is a well-known formula given by Gauss [6] that

$$I_q(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}$$

Let $I_q(n, \gamma_1, \ldots, \gamma_k)$ denote the number of monic irreducible polynomials over \mathbb{F}_q of degree n whose first k coefficients following the leading one is prescribed to $\gamma_1, \ldots, \gamma_k \in \mathbb{F}_q$, respectively. Carlitz in [4] showed that

$$I_q(n,\gamma) = \frac{1}{qn} \sum_{d|n,p \nmid d} \mu(d) q^{n/d}.$$

Kuz'min [10, 11] considered the case of two prescribed coefficients and gave the formula for $I_q(n, \gamma_1, \gamma_2)$. Yucas and Mullen determined the formula for $I_2(n, \gamma_1, \gamma_2, \gamma_3)$ when n is even [22], later Yucas and Fitzgerald determined the formula for $I_2(n, \gamma_1, \gamma_2, \gamma_3)$ when n is odd [5]. Also, Yucas [21] gave an alternative proof of Carlit'z formula. Ahmadi et al. in [2] gave the formula for $I_{2r}(n, 0, 0)$ for all $r \ge 1$. Most recently, Granger present direct and indirect methods for solving the prescribed traces problem for q = 2 and n odd. And then in [7] he applied these methods for $I_q(n, \gamma_1, \gamma_2, \ldots, \gamma_l)$ and $l \ge 7$. Also he obtained explicit formulas for l = 3 where q = 3. Let $\overline{I}_q(n, \gamma_1, \gamma_2, \ldots, \gamma_l)$ denote the number of monic irreducible polynomials over \mathbb{F}_q of degree n with the coefficients of x^{n-1} and x being the prescribed values γ_1, γ_2 , respectively. In this paper we give the formula for $\overline{I}_q(n, 0, 0)$. Besides, we use this formula to present an upper bound on the number of distinct families with good pseudorandom measures such as family complexity and cross-correlation.

The paper is organized as follows. We present some definitions and previous results in Section 2. In Section 3 we present the concept of L-polynomial of algebraic curves over F_q and its connection to the number of rational points on the algebraic curve. In Section 4, we present our main result and prove the formula on the number of irreducible polynomials with vanishing trace and reciprocal trace. In Section 5 we give examples and tables for q = 4 and q = 9. In Section 6 we give a result on the number of distinct families of pseudorandom sequences with good family complexity and cross-correlation measure.

2 **Preliminaries**

For $a \in \mathbb{F}_{q^n}$, let the characteristic polynomial of a over \mathbb{F}_q be

$$\prod_{i=0}^{n-1} (x - a^{q^i}) = x^n - a_{n-1}x^{n-1} + \dots + (-1)^{n-1}a_1x + (-1)^n a_0$$

Then we define trace and reciprocal-trace of $a \in \mathbb{F}_{q^n}$ to the base field \mathbb{F}_q as $\operatorname{Tr}(a) := a_{n-1}$ and $\operatorname{rTr}(a) := a_1/a_0$, respectively. Hence, we have

$$\operatorname{Tr}(a) = \sum_{i=0}^{n-1} a^{q^i} \text{ and } \operatorname{rTr}(a) = \sum_{i=0}^{n-1} a^{-q^i}$$

Let $f(x) = x^n - c_{n-1}x^{n-1} + \cdots + (-1)^{n-1}c_1x + (-1)^n c_0 \in \mathbb{F}_q[x]$ be an irreducible polynomial over \mathbb{F}_q . Similarly, we define trace and reciprocal-trace of $f \in \mathbb{F}_{q^n}[x]$ as $\operatorname{Tr}(f) := c_{n-1}$ and $\operatorname{rTr}(f) := c_1/c_0$, respectively. For $\gamma_1, \gamma_2 \in \mathbb{F}_q$, let $F_q(n, \gamma_1, \gamma_2)$ be the number of elements $a \in \mathbb{F}_q^n$ for which $\operatorname{Tr}(a) = \gamma_1$ and $\operatorname{rTr}(a) = \gamma_2$. In this paper we will first consider the values of $F_q(n, \gamma_1, \gamma_2)$ and give its formula for $\gamma_1 = 0$ and $\gamma_2 = 0$. Before that, we give some definitions and preliminary results. We begin with the definition of Möbius function.

Definition 1. [15, Definition 2.1.22] The Möbius μ function is defined on the set of positive integers by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1\\ (-1)^k & \text{if } m = m_1 m_2 \dots m_k \text{ where the } m_i \text{ are distinct primes}\\ 0 & \text{if } p^2 \text{ divides } m \text{ for some prime } p \end{cases}$$

Lemma 1. [13, Theorem 2.25] Let F be a finite extension of $K = \mathbb{F}_q$. Then for $a \in F$ we have Tr(a) = 0 if and only if $a = y^q - y$ for some $y \in F$.

We note that for a positive integer n with a positive divisor d and P be a polynomial of degree n/d, the following trivially holds

$$\operatorname{Tr}(P^d) = d \cdot \operatorname{Tr}(P) \text{ and } \operatorname{rTr}(P^d) = d \cdot \operatorname{rTr}(P).$$
 (1)

We now present an analog result of [2, Theorem 1] in the following theorem. Since the proof is not direct, we give it here.

Theorem 1. Let $n \ge 2$ be an integer. Then

$$\bar{I}_q(n,0,0) = \frac{1}{n} \sum_{d|n,p \nmid d} \mu(d) \left(F_q(n/d,0,0) - [p \text{ divides } n] q^{n/pd} \right).$$

Proof. We have

$$\begin{split} F_q(n,0,0) &= \left| \bigcup_{\beta \in \mathbb{F}_{q^n}, \operatorname{Tr}(\beta)=0, \operatorname{rTr}(\beta)=0} \operatorname{Min}(\beta) \right| \\ &= \left| \bigcup_{d|n} \frac{n}{d} \left\{ P \in \operatorname{Irr}\left(\frac{n}{d}\right) : d \cdot \operatorname{Tr}(P) = 0, \ d \cdot \operatorname{rTr}(P) = 0 \right\} \right| \\ &= \left[p \text{ divides } n \right] \left| \bigcup_{d|n, \ p|d} \frac{n}{d} \left\{ P \in \operatorname{Irr}\left(\frac{n}{d}\right) \right\} \right| \\ &+ \left| \bigcup_{d|n, \ p|d} \frac{n}{d} \left\{ P \in \operatorname{Irr}\left(\frac{n}{d}\right) : \operatorname{Tr}(P) = 0, \ \operatorname{rTr}(P) = 0 \right\} \right| \\ &= \left[p \text{ divides } n \right] \sum_{d|n, \ p|d} \frac{n}{d} \overline{I}_q\left(\frac{n}{d}\right) + \sum_{d|n, \ p|d} \frac{n}{d} \overline{I}_q\left(\frac{n}{d}, 0, 0\right) \\ &= \left[p \text{ divides } n \right] q^{n/p} + \sum_{d|n, \ p|d} \frac{n}{d} \overline{I}_q\left(\frac{n}{d}, 0, 0\right), \end{split}$$

where the third equality follows from (1). Therefore,

$$\bar{I}_q(n,0,0) = \frac{1}{n} \sum_{d|n, p \nmid d} \left(F_q(n/d,0,0) - [p \text{ divides } n] q^{n/pd} \right).$$

3 L-Polynomial

In this chapter we define the *L*-Polynomial of curves over a finite field. Also we give a wellknown formula for the number of rational points on algebraic curves over the finite fields.

Definition 2. Let $q = p^r$ where p is a prime number. Let $C = C(\mathbb{F}_q)$ be a (projective, smooth, absolutely irreducible) algebraic curve of genus g defined over \mathbb{F}_q . Consider the L-polynomial of the curve C over \mathbb{F}_q defined by

$$L_C(t) = \exp\left(\sum_{n=1}^{\infty} (\#C(\mathbb{F}_{q^n}) - q^n - 1)\frac{t^n}{n}\right)$$

where $\#C(\mathbb{F}_{q^n})$ denotes the number of \mathbb{F}_{q^n} -rational points of C.

Also $L_C(t)$ is defined as follows:

$$L_C(t) = \sum_{i=0}^{2g} c_i t^i$$

where $c_i \in \mathbb{Z}$ and g is the genus of C. For instance, for genus 1, the L-polynomial given by $L_C(t) = qt^2 + c_1t + 1$, where $c_1 = \#C(F_q) - (q+1)$. In general, the coefficients of the L-polynomial are determined by $\#C(\mathbb{F}_{q^n})$ for $n = 1, 2, \ldots, g$. Let $\alpha_1, \ldots, \alpha_{2g}$ be the roots of the reciprocal of the L-polynomial of C over \mathbb{F}_q . Then

$$L_C(t) = \prod_{i=1}^{2g} (1 - \alpha_i^n t)$$

We also have that

$$#C(\mathbb{F}_{q^n}) = (q^n + 1) - \sum_{i=1}^{2g} (\alpha_i)^n$$
(2)

for all $n \ge 1$, where $|\alpha_i| = \sqrt{q}$.

4 Finding the values $F_q(n, 0, 0)$

In this section we will find the numbers $F_q(n, 0, 0)$ where q is an even prime power and n is a positive integer. We relate these numbers with q - 1 elliptic curves which are related with trace.

Since calculating the number of \mathbb{F}_q -rational points of an elliptic curve is enough to find all the number of \mathbb{F}_{q^n} -rational points, the given formula for $F_q(n, 0, 0)$ is fast to compute. Since these curves are related with trace, we can prefer to write an algorithm using the trace forms.

We note that exact method can be applied for $F_q(n, t_1, t_2)$ where q is an any prime power and $t_1, t_2 \in \mathbb{F}_q$.

Let q be an even prime power and n be a positive integer. For functions $q_1, q_2 : \mathbb{F}_{q^n} \to \mathbb{F}_q$ define related $N(t_1, t_2)$ be the number of elements in \mathbb{F}_{q^n} satisfying $q_1(x) = t_1$ and $q_2(x) = t_2$. For a function $f : \mathbb{F}_{q^n} \to \mathbb{F}_q$ define Z(f) be the number of elements in \mathbb{F}_{q^n} satisfying f(x) = 0.

Lemma 2. [2, Lemma 6] Let $q_1, q_2 : \mathbb{F}_{q^n} \to \mathbb{F}_q$ be any functions. Then

$$N(0,0) = \frac{1}{q} \left(Z(q_1) + \sum_{\alpha \in \mathbb{F}_q} Z(\alpha q_1 - q_2) - q^n \right).$$

Proof. It follows by the following equalities.

$$q^{n} = \sum_{\alpha,\beta\in\mathbb{F}_{q}} N(\alpha,\beta) = \sum_{\beta\in\mathbb{F}_{q}} N(0,\beta) + \sum_{\beta\in\mathbb{F}_{q}} \sum_{\alpha\in\mathbb{F}_{q}^{\times}} N(\alpha,\beta)$$
$$= Z(q_{1}) + \sum_{\beta\in\mathbb{F}_{q}} \sum_{\alpha\in\mathbb{F}_{q}^{\times}} N(\alpha,\alpha\beta)$$
$$= Z(q_{1}) + \sum_{\alpha,\beta\in\mathbb{F}_{q}} N(\beta,\alpha\beta) - qN(0,0)$$
$$= Z(q_{1}) + \sum_{\alpha\in\mathbb{F}_{q}} Z(\alpha q_{1} - q_{2}) - qN(0,0).$$

Lemma 3. Let $q_1(x) = \text{Tr}(x)$ and $q_2(x) = \text{rTr}(x)$ be functions from \mathbb{F}_{q^n} to \mathbb{F}_q . The number of \mathbb{F}_{q^n} -rational points of $x(y^q - y) = \alpha x^2 - 1$ equals to $qZ(\alpha q_1 - q_2) - q + 2$.

Proof. The projective curve $xy^q - xyz^{q-1} = \alpha x^2 z^{q-1} - z^{q+1}$ has two infinity points (1:0:0) and (0:1:0) and has no extra solution when x = 0. If $x \neq 0$, then the points on $x(y^q - y) = \alpha x^2 - 1$ are related with the the set of zeros of $\text{Tr}(\alpha x - x^{-1})$. If x is a such zero, then there exists $y \in \mathbb{F}_{q^n}$ such that all the points (x, y + c) are on the curve where $c \in \mathbb{F}_{q^n}$. Therefore, \mathbb{F}_{q^n} -rational points of $x(y^q - y) = \alpha x^2 - 1$ equals to

$$2 + q(Z(\alpha q_1 - q_2) - 1) = qZ(\alpha q_1 - q_2) - q + 2.$$

Lemma 4. Assume that q is an even prime power. Let $\alpha \in \mathbb{F}_q^{\times}$. The number of \mathbb{F}_{q^n} -rational points of the curves $x(y^q + y) = \alpha x^2 + 1$ and $x(y^q + y) = x^2 + 1$ over \mathbb{F}_q are same.

Proof. Since order of α is odd, there exist n such that 2n + 1 is the order of α . The transformation $(x, y) \to (\alpha^n x, \alpha^{-n} y)$ on $x(y^q + y) = \alpha x^2 + 1$ gives $x(y^q + y) = x^2 + 1$.

The following lemma follows by Lemma 8 in [2].

Lemma 5. Assume that q is an even prime power. The curve $C : x(y^q + y) = x^2 + 1$ over \mathbb{F}_q is the fiber product of the curves $C_{\alpha} : x(y^2 + y) = \alpha(x^2 + 1)$ over \mathbb{F}_q where $\alpha \in \mathbb{F}_q^{\times}$. Therefore,

$$#C(\mathbb{F}_{q^n}) - (q^n + 1) = \sum_{\alpha \in \mathbb{F}_q^{\times}} (#C_{\alpha}(\mathbb{F}_{q^n}) - (q^n + 1)).$$

The following lemma follows by an analogue of Lemma 8 in [2] to all primes.

Lemma 6. Assume that q is prime p-power. Let $\alpha \in \mathbb{F}_q^{\times}$. The curve $C_{\alpha} : x(y^q - y) = \alpha x^2 - 1$ over \mathbb{F}_q is the fiber product of the curves $C_{\alpha,\beta} : x(y^p - y) = \beta(\alpha x^2 - 1)$ over \mathbb{F}_q where $\beta \in \mathbb{F}_q^{\times}/\mathbb{F}_p^{\times}$ as a representative set in F_q^{\times} . Therefore,

$$#C_{\alpha}(\mathbb{F}_{q^n}) - (q^n + 1) = \sum_{\beta \in \mathbb{F}_q^{\times}/\mathbb{F}_p^{\times}} \left(#C_{\alpha,\beta}(\mathbb{F}_{q^n}) - (q^n + 1) \right).$$

Theorem 2. Assume that q is an even prime power. Let $C_{\alpha} : x(y^2 + y) = \alpha(x^2 + 1)$ be curves over \mathbb{F}_q for $\alpha \in \mathbb{F}_q^{\times}$. Define $S_{\alpha}(\mathbb{F}_{q^n}) = \#C_{\alpha}(\mathbb{F}_{q^n}) - (q^n + 1)$. Then

$$F_q(n,0,0) = q^{n-2} + \frac{q-1}{q^2} \sum_{\alpha \in \mathbb{F}_q^{\times}} \left(S_\alpha(\mathbb{F}_{q^n}) + 1 \right).$$

Proof. Let $q_1(x) = \text{Tr}(x)$ and $q_2(x) = \text{rTr}(x)$ be functions from \mathbb{F}_{q^n} to \mathbb{F}_q . By Lemma 2

$$qF_q(n,0,0) = Z(q_1) + Z(q_2) + \sum_{\alpha \in \mathbb{F}_q^{\times}} Z(\alpha q_1 + q_2) - q_1$$
$$= q^{n-1} + q^{n-1} + \sum_{\alpha \in \mathbb{F}_q^{\times}} Z(\alpha q_1 + q_2) - q^n$$
$$= q^{n-1} + \sum_{\alpha \in \mathbb{F}_q^{\times}} \left(Z(\alpha q_1 + q_2) - q^{n-1} \right).$$

By Lemma 3 and Lemma 4

$$qF_q(n,0,0) = q^{n-1} + \sum_{\alpha \in \mathbb{F}_q^{\times}} \left(\frac{\#C(\mathbb{F}_{q^n}) + q - 2}{q} - q^{n-1} \right)$$
$$= q^{n-1} + \frac{q-1}{q} \left(\#C(\mathbb{F}_{q^n}) - (q^n + 1) + q - 1 \right)$$

By Lemma 5

$$F_q(n,0,0) = q^{n-2} + \frac{q-1}{q^2} \left(\left(\sum_{\alpha \in \mathbb{F}_q^{\times}} \left(\# C_\alpha(\mathbb{F}_{q^n}) - (q^n+1) \right) \right) + q - 1 \right) \\ = q^{n-2} + \frac{q-1}{q^2} \sum_{\alpha \in \mathbb{F}_q^{\times}} \left(S_\alpha(\mathbb{F}_{q^n}) + 1 \right).$$

Similarly, we can prove the following theorem. We will skip similar calculation details.

Theorem 3. Assume that q is prime p-power. Let $C_{\alpha,\beta} : x(y^p - y) = \beta(\alpha x^2 - 1)$ be curves over \mathbb{F}_q for $\alpha \in \mathbb{F}_q^{\times}$ and $\beta \in F_q^{\times}/F_p^{\times}$ as representative set in F_q^{\times} . Define $S_{\alpha,\beta}(\mathbb{F}_{q^n}) = \#C_{\alpha,\beta}(\mathbb{F}_{q^n}) - (q^n + 1)$. Then

$$F_q(n,0,0) = q^{n-2} + \frac{(q-1)^2}{q^2} + \frac{1}{q^2} \sum_{\alpha \in \mathbb{F}_q^{\times}} \sum_{\beta \in \mathbb{F}_q^{\times} / \mathbb{F}_p^{\times}} S_{\alpha,\beta}(\mathbb{F}_{q^n})$$

Proof. Let $q_1(x) = \text{Tr}(x)$ and $q_2(x) = \text{rTr}(x)$ be functions from \mathbb{F}_{q^n} to \mathbb{F}_q . By Lemma 2

$$qF_q(n,0,0) = q^{n-1} + \sum_{\alpha \in \mathbb{F}_q^{\times}} \left(Z(\alpha q_1 - q_2) - q^{n-1} \right).$$

By Lemma 3 and Lemma 6

$$F_{q}(n,0,0) = q^{n-2} + \frac{(q-1)^{2}}{q^{2}} + \frac{1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} \left(\#C_{\alpha}(\mathbb{F}_{q^{n}}) - (q^{n}+1) \right)$$
$$= q^{n-2} + \frac{(q-1)^{2}}{q^{2}} + \frac{1}{q} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} \left(\sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}} \left(\#C_{\alpha,\beta}(\mathbb{F}_{q^{n}}) - (q^{n}+1) \right) \right)$$
$$= q^{n-2} + \frac{(q-1)^{2}}{q^{2}} + \frac{1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} \sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}} S_{\alpha,\beta}(\mathbb{F}_{q^{n}}).$$

Remark. We note that the curve $C_{\alpha}: x(y^2+y) = \alpha(x^2+1)$ over \mathbb{F}_{2^r} for $\alpha \in \mathbb{F}_q^{\times}$ is non-singular. Therefore by using genus-degree formula it has genus 1. On the other hand, for an odd prime power $q = p^r$, the curve $C_{\alpha,\beta}: x(y^p - y) = \beta(\alpha x^2 - 1)$ over \mathbb{F}_q for $\alpha \in \mathbb{F}_q^{\times}$ and $\beta \in F_q^{\times}/F_p^{\times}$ has genus p - 1. This can be seen form [17, Theorem 3.7.8] as the Artin-Schreier extension of the rational function field $\mathbb{F}_q(x)$ defined by $y^p - y = \beta(\alpha x^2 - 1)/x$ has only ramified rational places x and 1/x of $\mathbb{F}_q(x)$.

5 Examples

In this section, we illustrate Theorems 1 and 3 for q = 4 and q = 9, respectively.

Example 1. Let q = 4 and n = 5. Let $C_{\alpha,\beta} : x(y^2 - y) = \alpha(x^2 + 1)$ be curves over \mathbb{F}_4 for $\alpha \in \mathbb{F}_4^{\times}$. Let $S_{\alpha}(\mathbb{F}_{q^n})$ be defined as in Theorem 2. Then by using Magma [3] we get

$$\sum_{\alpha \in \mathbb{F}_{4}^{\times}} S_{\alpha}(\mathbb{F}_{4^{5}} + 1) = -176.$$
(3)

Then Theorem 2 gives

$$F_4(5,0,0) = 64 - \frac{528}{16} = 31.$$
(4)

On the other hand, we, in Table 1, tabulate the number of elements in \mathbb{F}_{4^n} with both vanishing trace and reciprocal trace. We get the values in Table 1 by exhaustive counting. Note that (4) complies with the corresponding value in the Table 1.

Table 1: The values of $F_4(n, 0, 0)$.

n	3	4	5	6	7	8	9	10
$F_4(n, 0, 0)$	7	16	31	268	1135	4096	16279	64684

Now we calculate the number of monic irreducible polynomials of degree 5 in $\mathbb{F}_4[x]$ with vanishing trace and reciprocal trace. By Theorem 1 we have

$$\bar{I}_4(5,0,0) = \frac{1}{5} \big(\mu(5)(F_4(1,0,0) + \mu(1)(F_4(1,0,0)) \big).$$
(5)

Besides, Theorem 2 gives $F_4(1,0,0) = 1$ and $F_4(5,0,0) = 31$ as above. Therefore we get

$$\bar{I}_4(5,0,0) = 6. \tag{6}$$

Similarly, we counted exhaustively the number $\overline{I}_4(n, 0, 0)$ of irreducible polynomials for $n = 3, 4, \ldots, 10$ and tabulated them in Table 2. We see that (6) complies with the value given in Table 2.

Ta	ble	e 2:	The	numb	er I	$_{4}($	[n,	0,	0)	01	f mor	nic	irrec	luci	ib	le	pol	ly	no	mi	al	S
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n	3	4	5	6	7	8	9	10
$ar{I}_4(n,0,0)$	0	0	6	34	162	480	1808	6366

Example 2. Let q = 9 and n = 5. Let $C_{\alpha,\beta} : x(y^3 - y) = \beta(\alpha x^2 - 1)$ be curves over \mathbb{F}_9 , where $\alpha \in \mathbb{F}_9^{\times}$ and the elements β are the representatives of the quotient group $F_9^{\times}/F_3^{\times}$. Let $S_{\alpha,\beta}(\mathbb{F}_{q^n})$ be defined as in Theorem 3. Then by using Magma [3] we get

$$\sum_{\alpha \in \mathbb{F}_9^{\times}} \sum_{\beta \in \mathbb{F}_9^{\times}/\mathbb{F}_3^{\times}} S_{\alpha,\beta}(\mathbb{F}_{9^5}) = 5768.$$
(7)

Then by Theorem 3 we have

$$F_9(5,0,0) = 729 + \frac{64 + 5768}{81} = 801.$$
(8)

We see that (8) is equal to the value that we obtain by counting the number of elements in \mathbb{F}_9 with vanishing trace and reciprocal trace, see Table 3.

Table 3: The values of $F_9(n, 0, 0)$.

n	3	4	5	6	7	8
$F_9(n, 0, 0)$	9	9	89	801	6561	57904

We know by Theorem 1 that the number of monic irreducible polynomials of degree 5 over $\mathbb{F}_9[x]$ with vanishing trace and reciprocal trace satisfies

$$\bar{I}_9(5,0,0) = \frac{1}{5} \big(\mu(5)(F_9(1,0,0) + \mu(1)(F_9(5,0,0)) \big).$$
(9)

By Theorem 3 we get $F_9(1, 0, 0) = 1$ and $F_9(5, 0, 0) = 801$. Therefore we obtain

$$\bar{I}_9(5,0,0) = 160.$$
 (10)

Then also we see that the values in Table 4 and (10) are equal.

Table 4: The number	$I_9(n, 0, 0)$) of monic	irreducible	polynomials
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n	3	4	5	6	7	8	9
$\overline{I}_9(n,0,0)$	0	0	160	1080	8272	66500	530592

6 Pseudorandom sequences

Pseudorandom sequence is a sequence of numbers generated deterministically and looks random. The quality of a pseudorandom sequence are screened not only by statistical test packages (for example L'Ecuyer's TESTU01 [12], Marsaglia's Diehard [14] or the NIST battery [16]) but also by theoretical results on certain measures of pseudorandomness, see [8, 19] and references therein.

In some applications such as cryptography we need a large family of good pseudorandom sequences and we need to provide some bounds on several figures of merit [18]. In this section we consider the family complexity (short *f*-complexity) and the cross-correlation measure of order ℓ of families of sequences. We start with their definitions and then we define a family of sequences with good *f*-complexity and the cross-correlation measure. In this section we give an upper bound on the number of distinct families by using Theorems 1 and 3. Ahlswede et al. [1] introduced the *f*-complexity as follows.

Definition 3. The *f*-complexity $C(\mathcal{F})$ of a family \mathcal{F} of binary sequences $E_N \in \{-1, +1\}^N$ of length N is the greatest integer $j \ge 0$ such that for any $1 \le i_1 < i_2 < \cdots < i_j \le N$ and any $\epsilon_1, \epsilon_2, \ldots, \epsilon_j \in \{-1, +1\}$ there is a sequence $E_N = \{e_1, e_2, \ldots, e_N\} \in \mathcal{F}$ with

$$e_{i_1} = \epsilon_1, e_{i_2} = \epsilon_2, \dots, e_{i_j} = \epsilon_j.$$

It is easy to see that $2^{C(\mathcal{F})} \leq |\mathcal{F}|$, where $|\mathcal{F}|$ denotes the size of the family \mathcal{F} . Gyarmati et al. [9] introduced the cross-correlation measure of order ℓ . **Definition 4.** The cross-correlation measure of order ℓ of a family \mathcal{F} of binary sequences $E_{i,N} = (e_{i,1}, e_{i,2}, \ldots, e_{i,N}) \in \{-1+1\}^N$, $i = 1, 2, \ldots, F$, is defined as

$$\Phi_{\ell}(\mathcal{F}) = \max_{M,D,I} \left| \sum_{n=1}^{M} e_{i_1,n+d_1} \cdots e_{i_{\ell},n+d_{\ell}} \right|,$$

where D denotes an ℓ tuple $(d_1, d_2, \ldots, d_\ell)$ of integers such that $0 \le d_1 \le d_2 \le \cdots \le d_\ell < M + d_\ell \le N$ and $d_i \ne d_j$ if $E_{i,N} = E_{j,N}$ for $i \ne j$ and I denotes an ℓ tuple $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, \ldots, F\}^{\ell}$.

In [20], a family of sequences of Legendre symbols generated from some irreducible polynomials with high family complexity and small cross-correlation measure up to a large order ℓ was given. Similarly, it was shown that its dual family has good measures. Let p > 2 be a prime number, $n \ge 5$ and $\Omega_{p,n}$ be a set of irreducible polynomials over \mathbb{F}_p of degree n defined as

$$\Omega_{p,n} = \{ f(x) = x^n + a_2 x^{n-2} + a_3 x^{n-3} + \dots + a_{n-2} x^2 + a_n \in \mathbb{F}_p[x], a_2, a_3 \neq 0 \}.$$

Let $f \in \Omega_{p,n}$, $f_i(X) = i^n f(X/i)$ for $i \in \{1, 2, ..., p-1\}$ and \mathcal{F}_f be a family of binary sequences defined as

$$\mathcal{F}_{f} = \left\{ \left(\frac{f_{i}(j)}{p}\right)_{j=1}^{p-1} : i = 1, \dots, p-1 \right\},$$
(11)

and $\overline{\mathcal{F}_f}$ be the dual of \mathcal{F}_f . Then it is shown in [20] that they have good cross-correlation measure and family complexity as

$$\Phi_k(\mathcal{F}_f) \ll nkp^{1/2}\log p$$
 and $\Phi_k(\overline{\mathcal{F}_f}) \ll nkp^{1/2}\log p$

for each integer $k \in \{1, 2, \dots, p-1\}$ and

$$C(\mathcal{F}_f) \ge \left(\frac{1}{2} - o(1)\right) \frac{\log(p/n^2)}{\log 2} \text{ and } C(\overline{\mathcal{F}_f}) \ge \left(\frac{1}{2} - o(1)\right) \frac{\log(p/n^2)}{\log 2}.$$

We have the family size $|\mathcal{F}_f| = p$ for the family given in (11). On the other hand, the number $\#\{\mathcal{F}_f | f \in \Omega_{p,n}\}$ of distinct families that can be constructed as in (11) not known. Here, we give a partial solution for this problem, that is, an upper bound on the number of distinct families.

Corollary 1. Let $C_{\alpha} : x(y^p + y) = \alpha(x^2 + 1)$ be curves over \mathbb{F}_p for $\alpha \in \mathbb{F}_p^{\times}$. Define $S_{\alpha}(\mathbb{F}_{p^n}) = \#C_{\alpha}(\mathbb{F}_{p^n}) - (p^n + 1)$. Then

$$\#\{\mathcal{F}_{f}|f \in \Omega_{p,n}\} < \frac{1}{n} \sum_{d|n,p|d} \mu(d) \left(F_{p}(n/d,0,0) - [p \text{ divides } n]p^{n/pd}\right),$$

where

$$F_p(n,0,0) = p^{n-2} + \frac{(p-1)^2}{p^2} + \frac{1}{p^2} \sum_{\alpha \in \mathbb{F}_p^{\times}} S_{\alpha}(\mathbb{F}_{p^n})$$

Proof. The family \mathcal{F} is constructed by using irreducible polynomials $f \in \overline{I}_p(n, 0, 0)$. Hence we have the case $q = p^r$ for r = 1. By Theorem 1, we get the number of irreducible polynomials in terms of $F_p(n, 0, 0)$. On the other hand, as q = p, Theorem 3 gives the result.

7 Conclusion

In this paper, we proved the formula for number $\bar{I}_q(n, 0, 0)$ of irreducible polynomial of degree n over the finite field \mathbb{F}_q , $q = p^r$, such that the terms x^{n-1} and x vanish. Our formula reduces the problem of finding $\bar{I}_q(n, 0, 0)$ into getting the roots of the L-polynomial of the corresponding algebraic curve defined over \mathbb{F}_q . The latter is an easier problem as the genus of the curve is p-1. In particular, they are elliptic curves when $q = 2^r$ and the L-polynomial has only two roots.

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