# The number of irreducible polynomials over finite fields with vanishing trace and reciprocal trace 

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#### Abstract

We present the formula for the number of monic irreducible polynomials of degree $n$ over the finite field $\mathbb{F}_{q}$ where the coefficients of $x^{n-1}$ and $x$ vanish for $n \geq 3$. In particular, we give a relation between rational points of algebraic curves over finite fields and the number of elements $a \in \mathbb{F}_{q^{n}}$ for which $\operatorname{Trace}(a)=0$ and $\operatorname{Trace}\left(a^{-1}\right)=0$. Besides, we apply the formula to give an upper bound on the number of distinct constructions of a family of sequences with good family complexity and cross-correlation measure.


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## 1 Introduction

Let $r$ be a positive integer, $p$ be a prime number and $q=p^{r}, \mathbb{F}_{q}$ be the finite field with $q$ elements and let $I_{q}(n)$ denote the number of monic irreducible polynomials of degree $n$ over $\mathbb{F}_{q}[x]$. It is a well-known formula given by Gauss [6] that

$$
I_{q}(n)=\frac{1}{n} \sum_{d \mid n} \mu(d) q^{n / d}
$$

Let $I_{q}\left(n, \gamma_{1}, \ldots, \gamma_{k}\right)$ denote the number of monic irreducible polynomials over $\mathbb{F}_{q}$ of degree $n$ whose first $k$ coefficients following the leading one is prescribed to $\gamma_{1}, \ldots, \gamma_{k} \in \mathbb{F}_{q}$, respectively. Carlitz in [4] showed that

$$
I_{q}(n, \gamma)=\frac{1}{q n} \sum_{d \mid n, p \nmid d} \mu(d) q^{n / d}
$$

Kuz'min [10, 11] considered the case of two prescribed coefficients and gave the formula for $I_{q}\left(n, \gamma_{1}, \gamma_{2}\right)$. Yucas and Mullen determined the formula for $I_{2}\left(n, \gamma_{1}, \gamma_{2}, \gamma_{3}\right)$ when $n$ is even [22], later Yucas and Fitzgerald determined the formula for $I_{2}\left(n, \gamma_{1}, \gamma_{2}, \gamma_{3}\right)$ when $n$ is odd [5]. Also, Yucas [21] gave an alternative proof of Carlit'z formula. Ahmadi et al. in [2] gave the formula for $I_{2^{r}}(n, 0,0)$ for all $r \geq 1$. Most recently, Granger present direct and indirect methods for solving the prescribed traces problem for $q=2$ and $n$ odd. And then in [7] he applied these methods for $I_{q}\left(n, \gamma_{1}, \gamma_{2}, \ldots, \gamma_{l}\right)$ and $l \geq 7$. Also he obtained explicit formulas for $l=3$ where $q=3$. Let $\bar{I}_{q}\left(n, \gamma_{1}, \gamma_{2}\right)$ denote the number of monic irreducible polynomials over $\mathbb{F}_{q}$ of degree $n$ with the coefficients of $x^{n-1}$ and $x$ being the prescribed values $\gamma_{1}, \gamma_{2}$, respectively. In this paper we give the formula for $\bar{I}_{q}(n, 0,0)$. Besides, we use this formula to present an upper bound on the number of distinct families with good pseudorandom measures such as family complexity and cross-correlation.
The paper is organized as follows. We present some definitions and previous results in Section 2. In Section 3 we present the concept of $L$-polynomial of algebraic curves over $F_{q}$ and its connection to the number of rational points on the algebraic curve. In Section 4 , we present our main result and prove the formula on the number of irreducible polynomials with vanishing trace and reciprocal trace. In Section 5 we give examples and tables for $q=4$ and $q=9$. In Section 6 we give a result on the number of distinct families of pseudorandom sequences with good family complexity and cross-correlation measure.

## 2 Preliminaries

For $a \in \mathbb{F}_{q^{n}}$, let the characteristic polynomial of $a$ over $\mathbb{F}_{q}$ be

$$
\prod_{i=0}^{n-1}\left(x-a^{q^{i}}\right)=x^{n}-a_{n-1} x^{n-1}+\cdots+(-1)^{n-1} a_{1} x+(-1)^{n} a_{0}
$$

Then we define trace and reciprocal-trace of $a \in \mathbb{F}_{q^{n}}$ to the base field $\mathbb{F}_{q}$ as $\operatorname{Tr}(a):=a_{n-1}$ and $\mathrm{r} \operatorname{Tr}(a):=a_{1} / a_{0}$, respectively. Hence, we have

$$
\operatorname{Tr}(a)=\sum_{i=0}^{n-1} a^{q^{i}} \text { and } \operatorname{r} \operatorname{Tr}(a)=\sum_{i=0}^{n-1} a^{-q^{i}}
$$

Let $f(x)=x^{n}-c_{n-1} x^{n-1}+\cdots+(-1)^{n-1} c_{1} x+(-1)^{n} c_{0} \in \mathbb{F}_{q}[x]$ be an irreducible polynomial over $\mathbb{F}_{q}$. Similarly, we define trace and reciprocal-trace of $f \in \mathbb{F}_{q^{n}}[x]$ as $\operatorname{Tr}(f):=c_{n-1}$ and $\operatorname{rTr}(f):=c_{1} / c_{0}$, respectively.

For $\gamma_{1}, \gamma_{2} \in \mathbb{F}_{q}$, let $F_{q}\left(n, \gamma_{1}, \gamma_{2}\right)$ be the number of elements $a \in \mathbb{F}_{q}^{n}$ for which $\operatorname{Tr}(a)=\gamma_{1}$ and $\mathrm{r} \operatorname{Tr}(a)=\gamma_{2}$. In this paper we will first consider the values of $F_{q}\left(n, \gamma_{1}, \gamma_{2}\right)$ and give its formula for $\gamma_{1}=0$ and $\gamma_{2}=0$. Before that, we give some definitions and preliminary results. We begin with the definition of Möbius function.

Definition 1. [15, Definition 2.1.22] The Möbius $\mu$ function is defined on the set of positive integers by

$$
\mu(m)= \begin{cases}1 & \text { if } m=1 \\ (-1)^{k} & \text { if } m=m_{1} m_{2} \ldots m_{k} \text { where the } m_{i} \text { are distinct primes } \\ 0 & \text { if } p^{2} \text { divides } m \text { for some prime } p\end{cases}
$$

Lemma 1. [13, Theorem 2.25] Let $F$ be a finite extension of $K=\mathbb{F}_{q}$. Then for $a \in F$ we have $\operatorname{Tr}(a)=0$ if and only if $a=y^{q}-y$ for some $y \in F$.

We note that for a positive integer $n$ with a positive divisor $d$ and $P$ be a polynomial of degree $n / d$, the following trivially holds

$$
\begin{equation*}
\operatorname{Tr}\left(P^{d}\right)=d \cdot \operatorname{Tr}(P) \text { and } \mathrm{r} \operatorname{Tr}\left(P^{d}\right)=d \cdot \mathrm{r} \operatorname{Tr}(P) \tag{1}
\end{equation*}
$$

We now present an analog result of [2, Theorem 1] in the following theorem. Since the proof is not direct, we give it here.
Theorem 1. Let $n \geq 2$ be an integer. Then

$$
\bar{I}_{q}(n, 0,0)=\frac{1}{n} \sum_{d \mid n, p \nmid d} \mu(d)\left(F_{q}(n / d, 0,0)-[p \text { divides } n] q^{n / p d}\right)
$$

Proof. We have

$$
\begin{aligned}
F_{q}(n, 0,0)= & \left|\bigcup_{\beta \in \mathbb{F}_{q^{n}}, \operatorname{Tr}(\beta)=0, \mathrm{rTr}(\beta)=0} \operatorname{Min}(\beta)\right| \\
= & \left|\bigcup_{d \mid n} \frac{n}{d}\left\{P \in \operatorname{Irr}\left(\frac{n}{d}\right): d \cdot \operatorname{Tr}(P)=0, d \cdot \mathrm{r} \operatorname{Tr}(P)=0\right\}\right| \\
= & {[p \text { divides } n]\left|\bigcup_{d|n, p| d} \frac{n}{d}\left\{P \in \operatorname{Irr}\left(\frac{n}{d}\right)\right\}\right| } \\
& +\left|\bigcup_{d \mid n, p \nmid d} \frac{n}{d}\left\{P \in \operatorname{Irr}\left(\frac{n}{d}\right): \operatorname{Tr}(P)=0, \operatorname{rTr}(P)=0\right\}\right| \\
= & {[p \text { divides } n] \sum_{d|n, p| d} \frac{n}{d} \bar{I}_{q}\left(\frac{n}{d}\right)+\sum_{d \mid n, p \nmid d} \frac{n}{d} \bar{I}_{q}\left(\frac{n}{d}, 0,0\right) } \\
= & {[p \text { divides } n] q^{n / p}+\sum_{d \mid n, p \nmid d} \frac{n}{d} \bar{I}_{q}\left(\frac{n}{d}, 0,0\right), }
\end{aligned}
$$

where the third equality follows from (1). Therefore,

$$
\bar{I}_{q}(n, 0,0)=\frac{1}{n} \sum_{d \mid n, p \nmid d}\left(F_{q}(n / d, 0,0)-[p \text { divides } n] q^{n / p d}\right) .
$$

## 3 L-Polynomial

In this chapter we define the $L$-Polynomial of curves over a finite field. Also we give a wellknown formula for the number of rational points on algebraic curves over the finite fields.

Definition 2. Let $q=p^{r}$ where $p$ is a prime number. Let $C=C\left(\mathbb{F}_{q}\right)$ be a (projective, smooth, absolutely irreducible) algebraic curve of genus g defined over $\mathbb{F}_{q}$. Consider the $L$-polynomial of the curve $C$ over $\mathbb{F}_{q}$ defined by

$$
L_{C}(t)=\exp \left(\sum_{n=1}^{\infty}\left(\# C\left(\mathbb{F}_{q^{n}}\right)-q^{n}-1\right) \frac{t^{n}}{n}\right)
$$

where $\# C\left(\mathbb{F}_{q^{n}}\right)$ denotes the number of $\mathbb{F}_{q^{n}}$-rational points of $C$.
Also $L_{C}(t)$ is defined as follows:

$$
L_{C}(t)=\sum_{i=0}^{2 g} c_{i} t^{i}
$$

where $c_{i} \in \mathbb{Z}$ and $g$ is the genus of $C$. For instance, for genus 1 , the $L$-polynomial given by $L_{C}(t)=q t^{2}+c_{1} t+1$, where $c_{1}=\# C\left(F_{q}\right)-(q+1)$. In general, the coefficients of the $L$-polynomial are determined by $\# C\left(\mathbb{F}_{q^{n}}\right)$ for $n=1,2, \ldots, g$. Let $\alpha_{1}, \ldots, \alpha_{2 g}$ be the roots of the reciprocal of the $L$-polynomial of $C$ over $\mathbb{F}_{q}$. Then

$$
L_{C}(t)=\prod_{i=1}^{2 g}\left(1-\alpha_{i}^{n} t\right)
$$

We also have that

$$
\begin{equation*}
\# C\left(\mathbb{F}_{q^{n}}\right)=\left(q^{n}+1\right)-\sum_{i=1}^{2 g}\left(\alpha_{i}\right)^{n} \tag{2}
\end{equation*}
$$

for all $n \geq 1$, where $\left|\alpha_{i}\right|=\sqrt{q}$.

## 4 Finding the values $F_{q}(n, 0,0)$

In this section we will find the numbers $F_{q}(n, 0,0)$ where $q$ is an even prime power and $n$ is a positive integer. We relate these numbers with $q-1$ elliptic curves which are related with trace.

Since calculating the number of $\mathbb{F}_{q}$-rational points of an elliptic curve is enough to find all the number of $\mathbb{F}_{q^{n}}$-rational points, the given formula for $F_{q}(n, 0,0)$ is fast to compute. Since these curves are related with trace, we can prefer to write an algorithm using the trace forms.
We note that exact method can be applied for $F_{q}\left(n, t_{1}, t_{2}\right)$ where $q$ is an any prime power and $t_{1}, t_{2} \in \mathbb{F}_{q}$.
Let $q$ be an even prime power and $n$ be a positive integer. For functions $q_{1}, q_{2}: \mathbb{F}_{q^{n}} \rightarrow \mathbb{F}_{q}$ define related $N\left(t_{1}, t_{2}\right)$ be the number of elements in $\mathbb{F}_{q^{n}}$ satisfying $q_{1}(x)=t_{1}$ and $q_{2}(x)=t_{2}$. For a function $f: \mathbb{F}_{q^{n}} \rightarrow \mathbb{F}_{q}$ define $Z(f)$ be the number of elements in $\mathbb{F}_{q^{n}}$ satisfying $f(x)=0$.

Lemma 2. [2, Lemma 6] Let $q_{1}, q_{2}: \mathbb{F}_{q^{n}} \rightarrow \mathbb{F}_{q}$ be any functions. Then

$$
N(0,0)=\frac{1}{q}\left(Z\left(q_{1}\right)+\sum_{\alpha \in \mathbb{F}_{q}} Z\left(\alpha q_{1}-q_{2}\right)-q^{n}\right) .
$$

Proof. It follows by the following equalities.

$$
\begin{aligned}
q^{n}=\sum_{\alpha, \beta \in \mathbb{F}_{q}} N(\alpha, \beta) & =\sum_{\beta \in \mathbb{F}_{q}} N(0, \beta)+\sum_{\beta \in \mathbb{F}_{q}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} N(\alpha, \beta) \\
& =Z\left(q_{1}\right)+\sum_{\beta \in \mathbb{F}_{q}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} N(\alpha, \alpha \beta) \\
& =Z\left(q_{1}\right)+\sum_{\alpha, \beta \in \mathbb{F}_{q}} N(\beta, \alpha \beta)-q N(0,0) \\
& =Z\left(q_{1}\right)+\sum_{\alpha \in \mathbb{F}_{q}} Z\left(\alpha q_{1}-q_{2}\right)-q N(0,0) .
\end{aligned}
$$

Lemma 3. Let $q_{1}(x)=\operatorname{Tr}(x)$ and $q_{2}(x)=r \operatorname{Tr}(x)$ be functions from $\mathbb{F}_{q^{n}}$ to $\mathbb{F}_{q}$. The number of $\mathbb{F}_{q^{n}}$-rational points of $x\left(y^{q}-y\right)=\alpha x^{2}-1$ equals to $q Z\left(\alpha q_{1}-q_{2}\right)-q+2$.
Proof. The projective curve $x y^{q}-x y z^{q-1}=\alpha x^{2} z^{q-1}-z^{q+1}$ has two infinity points (1:0:0) and $(0: 1: 0)$ and has no extra solution when $x=0$. If $x \neq 0$, then the points on $x\left(y^{q}-y\right)=$ $\alpha x^{2}-1$ are related with the the set of zeros of $\operatorname{Tr}\left(\alpha x-x^{-1}\right)$. If $x$ is a such zero, then there exists $y \in \mathbb{F}_{q^{n}}$ such that all the points $(x, y+c)$ are on the curve where $c \in \mathbb{F}_{q^{n}}$. Therefore, $\mathbb{F}_{q^{n}}$-rational points of $x\left(y^{q}-y\right)=\alpha x^{2}-1$ equals to

$$
2+q\left(Z\left(\alpha q_{1}-q_{2}\right)-1\right)=q Z\left(\alpha q_{1}-q_{2}\right)-q+2
$$

Lemma 4. Assume that $q$ is an even prime power. Let $\alpha \in \mathbb{F}_{q}^{\times}$. The number of $\mathbb{F}_{q^{n}}$-rational points of the curves $x\left(y^{q}+y\right)=\alpha x^{2}+1$ and $x\left(y^{q}+y\right)=x^{2}+1$ over $\mathbb{F}_{q}$ are same.

Proof. Since order of $\alpha$ is odd, there exist $n$ such that $2 n+1$ is the order of $\alpha$. The transformation $(x, y) \rightarrow\left(\alpha^{n} x, \alpha^{-n} y\right)$ on $x\left(y^{q}+y\right)=\alpha x^{2}+1$ gives $x\left(y^{q}+y\right)=x^{2}+1$.

The following lemma follows by Lemma 8 in [2].
Lemma 5. Assume that $q$ is an even prime power. The curve $C: x\left(y^{q}+y\right)=x^{2}+1$ over $\mathbb{F}_{q}$ is the fiber product of the curves $C_{\alpha}: x\left(y^{2}+y\right)=\alpha\left(x^{2}+1\right)$ over $\mathbb{F}_{q}$ where $\alpha \in \mathbb{F}_{q}^{\times}$. Therefore,

$$
\# C\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)=\sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(\# C_{\alpha}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)\right)
$$

The following lemma follows by an analogue of Lemma 8 in [2] to all primes.
Lemma 6. Assume that $q$ is prime $p$-power. Let $\alpha \in \mathbb{F}_{q}^{\times}$. The curve $C_{\alpha}: x\left(y^{q}-y\right)=\alpha x^{2}-1$ over $\mathbb{F}_{q}$ is the fiber product of the curves $C_{\alpha, \beta}: x\left(y^{p}-y\right)=\beta\left(\alpha x^{2}-1\right)$ over $\mathbb{F}_{q}$ where $\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}$as a representative set in $F_{q}^{\times}$. Therefore,

$$
\# C_{\alpha}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)=\sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}}\left(\# C_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)\right) .
$$

Theorem 2. Assume that $q$ is an even prime power. Let $C_{\alpha}: x\left(y^{2}+y\right)=\alpha\left(x^{2}+1\right)$ be curves over $\mathbb{F}_{q}$ for $\alpha \in \mathbb{F}_{q}^{\times}$. Define $S_{\alpha}\left(\mathbb{F}_{q^{n}}\right)=\# C_{\alpha}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)$. Then

$$
F_{q}(n, 0,0)=q^{n-2}+\frac{q-1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(S_{\alpha}\left(\mathbb{F}_{q^{n}}\right)+1\right)
$$

Proof. Let $q_{1}(x)=\operatorname{Tr}(x)$ and $q_{2}(x)=r \operatorname{Tr}(x)$ be functions from $\mathbb{F}_{q^{n}}$ to $\mathbb{F}_{q}$. By Lemma 2

$$
\begin{aligned}
q F_{q}(n, 0,0) & =Z\left(q_{1}\right)+Z\left(q_{2}\right)+\sum_{\alpha \in \mathbb{F}_{q}^{\times}} Z\left(\alpha q_{1}+q_{2}\right)-q^{n} \\
& =q^{n-1}+q^{n-1}+\sum_{\alpha \in \mathbb{F}_{q}^{\times}} Z\left(\alpha q_{1}+q_{2}\right)-q^{n} \\
& =q^{n-1}+\sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(Z\left(\alpha q_{1}+q_{2}\right)-q^{n-1}\right) .
\end{aligned}
$$

By Lemma 3 and Lemma 4

$$
\begin{aligned}
q F_{q}(n, 0,0) & =q^{n-1}+\sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(\frac{\# C\left(\mathbb{F}_{q^{n}}\right)+q-2}{q}-q^{n-1}\right) \\
& =q^{n-1}+\frac{q-1}{q}\left(\# C\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)+q-1\right)
\end{aligned}
$$

By Lemma 5

$$
\begin{aligned}
F_{q}(n, 0,0) & =q^{n-2}+\frac{q-1}{q^{2}}\left(\left(\sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(\# C_{\alpha}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)\right)\right)+q-1\right) \\
& =q^{n-2}+\frac{q-1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(S_{\alpha}\left(\mathbb{F}_{q^{n}}\right)+1\right) .
\end{aligned}
$$

Similarly, we can prove the following theorem. We will skip similar calculation details.
Theorem 3. Assume that $q$ is prime $p$-power. Let $C_{\alpha, \beta}: x\left(y^{p}-y\right)=\beta\left(\alpha x^{2}-1\right)$ be curves over $\mathbb{F}_{q}$ for $\alpha \in \mathbb{F}_{q}^{\times}$and $\beta \in F_{q}^{\times} / F_{p}^{\times}$as representative set in $F_{q}^{\times}$. Define $S_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)=\# C_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)-$ $\left(q^{n}+1\right)$. Then

$$
F_{q}(n, 0,0)=q^{n-2}+\frac{(q-1)^{2}}{q^{2}}+\frac{1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} \sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}} S_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)
$$

Proof. Let $q_{1}(x)=\operatorname{Tr}(x)$ and $q_{2}(x)=r \operatorname{Tr}(x)$ be functions from $\mathbb{F}_{q^{n}}$ to $\mathbb{F}_{q}$. By Lemma 2

$$
q F_{q}(n, 0,0)=q^{n-1}+\sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(Z\left(\alpha q_{1}-q_{2}\right)-q^{n-1}\right) .
$$

By Lemma 3 and Lemma 6

$$
\begin{aligned}
F_{q}(n, 0,0) & =q^{n-2}+\frac{(q-1)^{2}}{q^{2}}+\frac{1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(\# C_{\alpha}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)\right) \\
& =q^{n-2}+\frac{(q-1)^{2}}{q^{2}}+\frac{1}{q} \sum_{\alpha \in \mathbb{F}_{q}^{\times}}\left(\sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}}\left(\# C_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)-\left(q^{n}+1\right)\right)\right) \\
& =q^{n-2}+\frac{(q-1)^{2}}{q^{2}}+\frac{1}{q^{2}} \sum_{\alpha \in \mathbb{F}_{q}^{\times}} \sum_{\beta \in \mathbb{F}_{q}^{\times} / \mathbb{F}_{p}^{\times}} S_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right) .
\end{aligned}
$$

Remark. We note that the curve $C_{\alpha}: x\left(y^{2}+y\right)=\alpha\left(x^{2}+1\right)$ over $\mathbb{F}_{2^{r}}$ for $\alpha \in \mathbb{F}_{q}^{\times}$is non-singular. Therefore by using genus-degree formula it has genus 1 . On the other hand, for an odd prime power $q=p^{r}$, the curve $C_{\alpha, \beta}: x\left(y^{p}-y\right)=\beta\left(\alpha x^{2}-1\right)$ over $\mathbb{F}_{q}$ for $\alpha \in \mathbb{F}_{q}^{\times}$and $\beta \in F_{q}^{\times} / F_{p}^{\times}$ has genus $p-1$. This can be seen form [17, Theorem 3.7.8] as the Artin-Schreier extension of the rational function field $\mathbb{F}_{q}(x)$ defined by $y^{p}-y=\beta\left(\alpha x^{2}-1\right) / x$ has only ramified rational places $x$ and $1 / x$ of $\mathbb{F}_{q}(x)$.

## 5 Examples

In this section, we illustrate Theorems 1 and 3 for $q=4$ and $q=9$, respectively.
Example 1. Let $q=4$ and $n=5$. Let $C_{\alpha, \beta}: x\left(y^{2}-y\right)=\alpha\left(x^{2}+1\right)$ be curves over $\mathbb{F}_{4}$ for $\alpha \in \mathbb{F}_{4}^{\times}$. Let $S_{\alpha}\left(\mathbb{F}_{q^{n}}\right)$ be defined as in Theorem[2, Then by using Magma [3] we get

$$
\begin{equation*}
\sum_{\alpha \in \mathbb{F}_{4}^{\times}} S_{\alpha}\left(\mathbb{F}_{4^{5}}+1\right)=-176 . \tag{3}
\end{equation*}
$$

Then Theorem 2 gives

$$
\begin{equation*}
F_{4}(5,0,0)=64-\frac{528}{16}=31 \tag{4}
\end{equation*}
$$

On the other hand, we, in Table 1, tabulate the number of elements in $\mathbb{F}_{4^{n}}$ with both vanishing trace and reciprocal trace. We get the values in Table 1 by exhaustive counting. Note that (4) complies with the corresponding value in the Table 1 .

Table 1: The values of $F_{4}(n, 0,0)$.

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{4}(n, 0,0)$ | 7 | 16 | 31 | 268 | 1135 | 4096 | 16279 | 64684 |

Now we calculate the number of monic irreducible polynomials of degree 5 in $\mathbb{F}_{4}[x]$ with vanishing trace and reciprocal trace. By Theorem 1 we have

$$
\begin{equation*}
\bar{I}_{4}(5,0,0)=\frac{1}{5}\left(\mu ( 5 ) \left(F_{4}(1,0,0)+\mu(1)\left(F_{4}(1,0,0)\right) .\right.\right. \tag{5}
\end{equation*}
$$

Besides, Theorem gives $F_{4}(1,0,0)=1$ and $F_{4}(5,0,0)=31$ as above. Therefore we get

$$
\begin{equation*}
\bar{I}_{4}(5,0,0)=6 . \tag{6}
\end{equation*}
$$

Similarly, we counted exhaustively the number $\bar{I}_{4}(n, 0,0)$ of irreducible polynomials for $n=$ $3,4, \ldots, 10$ and tabulated them in Table 2, We see that (6) complies with the value given in Table 2

Table 2: The number $\bar{I}_{4}(n, 0,0)$ of monic irreducible polynomials

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $I_{4}(n, 0,0)$ | 0 | 0 | 6 | 34 | 162 | 480 | 1808 | 6366 |

Example 2. Let $q=9$ and $n=5$. Let $C_{\alpha, \beta}: x\left(y^{3}-y\right)=\beta\left(\alpha x^{2}-1\right)$ be curves over $\mathbb{F}_{9}$, where $\alpha \in \mathbb{F}_{9}^{\times}$and the elements $\beta$ are the representatives of the quotient group $F_{9}^{\times} / F_{3}^{\times}$. Let $S_{\alpha, \beta}\left(\mathbb{F}_{q^{n}}\right)$ be defined as in Theorem 3. Then by using Magma [3] we get

$$
\begin{equation*}
\sum_{\alpha \in \mathbb{F}_{9}^{\times}} \sum_{\beta \in \mathbb{F}_{9}^{\times} / \mathbb{F}_{3}^{\times}} S_{\alpha, \beta}\left(\mathbb{F}_{9^{5}}\right)=5768 . \tag{7}
\end{equation*}
$$

Then by Theorem 3 we have

$$
\begin{equation*}
F_{9}(5,0,0)=729+\frac{64+5768}{81}=801 . \tag{8}
\end{equation*}
$$

We see that (8) is equal to the value that we obtain by counting the number of elements in $\mathbb{F}_{9}$ with vanishing trace and reciprocal trace, see Table 3.

Table 3: The values of $F_{9}(n, 0,0)$.

| n | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{9}(n, 0,0)$ | 9 | 9 | 89 | 801 | 6561 | 57904 |

We know by Theorem 11 that the number of monic irreducible polynomials of degree 5 over $\mathbb{F}_{9}[x]$ with vanishing trace and reciprocal trace satisfies

$$
\begin{equation*}
\bar{I}_{9}(5,0,0)=\frac{1}{5}\left(\mu ( 5 ) \left(F_{9}(1,0,0)+\mu(1)\left(F_{9}(5,0,0)\right) .\right.\right. \tag{9}
\end{equation*}
$$

By Theorem 3 we get $F_{9}(1,0,0)=1$ and $F_{9}(5,0,0)=801$. Therefore we obtain

$$
\begin{equation*}
\bar{I}_{9}(5,0,0)=160 \tag{10}
\end{equation*}
$$

Then also we see that the values in Table 4 and (10) are equal.
Table 4: The number $\bar{I}_{9}(n, 0,0)$ of monic irreducible polynomials

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $I_{9}(n, 0,0)$ | 0 | 0 | 160 | 1080 | 8272 | 66500 | 530592 |

## 6 Pseudorandom sequences

Pseudorandom sequence is a sequence of numbers generated deterministically and looks random. The quality of a pseudorandom sequence are screened not only by statistical test packages (for example L'Ecuyer's TESTU01 [12], Marsaglia's Diehard [14] or the NIST battery [16]) but also by theoretical results on certain measures of pseudorandomness, see [8, 19] and references therein.
In some applications such as cryptography we need a large family of good pseudorandom sequences and we need to provide some bounds on several figures of merit [18]. In this section we consider the family complexity (short $f$-complexity) and the cross-correlation measure of order $\ell$ of families of sequences. We start with their definitions and then we define a family of sequences with good $f$-complexity and the cross-correlation measure. In this section we give an upper bound on the number of distinct families by using Theorems 1 and 3. Ahlswede et al. [1] introduced the $f$-complexity as follows.

Definition 3. The $f$-complexity $C(\mathcal{F})$ of a family $\mathcal{F}$ of binary sequences $E_{N} \in\{-1,+1\}^{N}$ of length $N$ is the greatest integer $j \geq 0$ such that for any $1 \leq i_{1}<i_{2}<\cdots<i_{j} \leq N$ and any $\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{j} \in\{-1,+1\}$ there is a sequence $E_{N}=\left\{e_{1}, e_{2}, \ldots, e_{N}\right\} \in \mathcal{F}$ with

$$
e_{i_{1}}=\epsilon_{1}, e_{i_{2}}=\epsilon_{2}, \ldots, e_{i_{j}}=\epsilon_{j} .
$$

It is easy to see that $2^{C(\mathcal{F})} \leq|\mathcal{F}|$, where $|\mathcal{F}|$ denotes the size of the family $\mathcal{F}$. Gyarmati et al. [9] introduced the cross-correlation measure of order $\ell$.

Definition 4. The cross-correlation measure of order $\ell$ of a family $\mathcal{F}$ of binary sequences $E_{i, N}=\left(e_{i, 1}, e_{i, 2}, \ldots, e_{i, N}\right) \in\{-1+1\}^{N}, i=1,2, \ldots, F$, is defined as

$$
\Phi_{\ell}(\mathcal{F})=\max _{M, D, I}\left|\sum_{n=1}^{M} e_{i_{1}, n+d_{1}} \cdots e_{i_{\ell}, n+d_{\ell}}\right|
$$

where $D$ denotes an $\ell$ tuple $\left(d_{1}, d_{2}, \ldots, d_{\ell}\right)$ of integers such that $0 \leq d_{1} \leq d_{2} \leq \cdots \leq d_{\ell}<$ $M+d_{\ell} \leq N$ and $d_{i} \neq d_{j}$ if $E_{i, N}=E_{j, N}$ for $i \neq j$ and $I$ denotes an $\ell$ tuple $\left(i_{1}, i_{2}, \ldots, i_{\ell}\right) \in$ $\{1,2, \ldots, F\}^{\ell}$.
In [20], a family of sequences of Legendre symbols generated from some irreducible polynomials with high family complexity and small cross-correlation measure up to a large order $\ell$ was given. Similarly, it was shown that its dual family has good measures. Let $p>2$ be a prime number, $n \geq 5$ and $\Omega_{p, n}$ be a set of irreducible polynomials over $\mathbb{F}_{p}$ of degree $n$ defined as

$$
\Omega_{p, n}=\left\{f(x)=x^{n}+a_{2} x^{n-2}+a_{3} x^{n-3}+\cdots+a_{n-2} x^{2}+a_{n} \in \mathbb{F}_{p}[x], a_{2}, a_{3} \neq 0\right\} .
$$

Let $f \in \Omega_{p, n}, f_{i}(X)=i^{n} f(X / i)$ for $i \in\{1,2, \ldots, p-1\}$ and $\mathcal{F}_{f}$ be a family of binary sequences defined as

$$
\begin{equation*}
\mathcal{F}_{f}=\left\{\left(\frac{f_{i}(j)}{p}\right)_{j=1}^{p-1}: i=1, \ldots, p-1\right\} \tag{11}
\end{equation*}
$$

and $\overline{\mathcal{F}_{f}}$ be the dual of $\mathcal{F}_{f}$. Then it is shown in [20] that they have good cross-correlation measure and family complexity as

$$
\Phi_{k}\left(\mathcal{F}_{f}\right) \ll n k p^{1 / 2} \log p \text { and } \Phi_{k}\left(\overline{\mathcal{F}_{f}}\right) \ll n k p^{1 / 2} \log p
$$

for each integer $k \in\{1,2, \ldots, p-1\}$ and

$$
C\left(\mathcal{F}_{f}\right) \geq\left(\frac{1}{2}-o(1)\right) \frac{\log \left(p / n^{2}\right)}{\log 2} \text { and } C\left(\overline{\mathcal{F}_{f}}\right) \geq\left(\frac{1}{2}-o(1)\right) \frac{\log \left(p / n^{2}\right)}{\log 2}
$$

We have the family size $\left|\mathcal{F}_{f}\right|=p$ for the family given in (11). On the other hand, the number $\#\left\{\mathcal{F}_{f} \mid f \in \Omega_{p, n}\right\}$ of distinct families that can be constructed as in (11) not known. Here, we give a partial solution for this problem, that is, an upper bound on the number of distinct families.
Corollary 1. Let $C_{\alpha}: x\left(y^{p}+y\right)=\alpha\left(x^{2}+1\right)$ be curves over $\mathbb{F}_{p}$ for $\alpha \in \mathbb{F}_{p}^{\times}$. Define $S_{\alpha}\left(\mathbb{F}_{p^{n}}\right)=$ $\# C_{\alpha}\left(\mathbb{F}_{p^{n}}\right)-\left(p^{n}+1\right)$. Then

$$
\#\left\{\mathcal{F}_{f} \mid f \in \Omega_{p, n}\right\}<\frac{1}{n} \sum_{d \mid n, p \nmid d} \mu(d)\left(F_{p}(n / d, 0,0)-[p \text { divides } n] p^{n / p d}\right)
$$

where

$$
F_{p}(n, 0,0)=p^{n-2}+\frac{(p-1)^{2}}{p^{2}}+\frac{1}{p^{2}} \sum_{\alpha \in \mathbb{F}_{p}^{\times}} S_{\alpha}\left(\mathbb{F}_{p^{n}}\right)
$$

Proof. The family $\mathcal{F}$ is constructed by using irreducible polynomials $f \in \bar{I}_{p}(n, 0,0)$. Hence we have the case $q=p^{r}$ for $r=1$. By Theorem 1, we get the number of irreducible polynomials in terms of $F_{p}(n, 0,0)$. On the other hand, as $q=p$, Theorem 3 gives the result.

## 7 Conclusion

In this paper, we proved the formula for number $\bar{I}_{q}(n, 0,0)$ of irreducible polynomial of degree $n$ over the finite field $\mathbb{F}_{q}, q=p^{r}$, such that the terms $x^{n-1}$ and $x$ vanish. Our formula reduces the problem of finding $\bar{I}_{q}(n, 0,0)$ into getting the roots of the L-polynomial of the corresponding algebraic curve defined over $\mathbb{F}_{q}$. The latter is an easier problem as the genus of the curve is $p-1$. In particular, they are elliptic curves when $q=2^{r}$ and the L-polynomial has only two roots.

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