# Hadamard matrices related to a certain series of ternary self-dual codes

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#### Abstract

In 2013, Nebe and Villar gave a series of ternary self-dual codes of length 2(p+1) for a prime p congruent to 5 modulo 8. As a consequence, the third ternary extremal self-dual code of length 60 was found. We show that the ternary self-dual code contains codewords which form a Hadamard matrix of order 2(p+1) when p is congruent to 5 modulo 24. In addition, it is shown that the ternary self-dual code is generated by the rows of the Hadamard matrix. We also demonstrate that the third ternary extremal self-dual code of length 60 contains at least two inequivalent Hadamard matrices.

#### 1 Introduction

Self-dual codes are one of the most interesting classes of codes. This interest is justified by many combinatorial objects and algebraic objects related to self-dual codes (see e.g., [15]). A Hadamard matrix is a kind of orthogonal matrix appearing in many research areas of Mathematics and practical applications (see e.g., [16] and [17]). One of the interesting and successful applications

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of Hadamard matrices is their use as codes. In particular, a special class of Hadamard matrices can give rise to self-dual codes as their row spaces. In this paper, we are interested in Hadamard matrices related to ternary self-dual codes found by Nebe and Villar [11].

A ternary self-dual code C of length n is an [n, n/2] code over the finite field of order 3 satisfying  $C = C^{\perp}$ , where  $C^{\perp}$  is the dual code of C. A ternary self-dual code of length n exists if and only if n is divisible by four. It was shown in [10] that the minimum weight d of a ternary self-dual code of length n is bounded by  $d \leq 3\lfloor n/12 \rfloor + 3$ . If  $d = 3\lfloor n/12 \rfloor + 3$ , then the code is called extremal. For  $n \in \{4, 8, 12, \ldots, 64\}$ , it is known that there is a ternary extremal self-dual code of length n (see [7, Table 6]). The ternary extended quadratic residue codes and the Pless symmetry codes are well known families of ternary (self-dual) codes. It is known that the ternary extended quadratic residue code  $QR_{60}$  of length 60 and the Pless symmetry code  $P_{60}$  of length 60 are ternary extremal self-dual codes (see [15, Table XII]). In 2013, Nebe and Villar [11] gave a series of ternary self-dual codes of length 2(p+1) for all primes  $p \equiv 5 \pmod{8}$ . As a consequence, the third ternary extremal self-dual code of length 60 was found.

A Hadamard matrix H of order n is an  $n \times n$  matrix whose entries are from  $\{1, -1\}$  such that  $HH^T = nI_n$ , where  $H^T$  is the transpose of H and  $I_n$  is the identity matrix of order n. It is known that the order n is necessarily 1, 2, or a multiple of 4. Recently, Tonchev [19] studied Hadamard matrices of order n formed by codewords of weight n in ternary extremal self-dual codes of length n, especially the extended quadratic residue codes and the Pless symmetry codes. From the construction, the extended quadratic residue code contains a type I Paley-Hadamard matrix. The Pless symmetry code contains a type II Paley-Hadamard matrix [13]. Tonchev [19] showed that the Pless symmetry code of length 36 contains exactly two inequivalent Hadamard matrices of order 36. This motivates us to study the existence of Hadamard matrices of order n formed by codewords of weight n in ternary self-dual codes found by Nebe and Villar [11].

The paper is organized as follows. In Section 2, definitions, notations and basic results are given. Especially, we review the construction of ternary self-dual codes  $NV^{(a)}(p)$  in [11] of length 2(p+1), where p is a prime with  $p \equiv 5 \pmod{8}$  and  $a \in \{1, -1\}$ . In Section 3, we show that  $NV^{(a)}(p)$  contains 2(p+1) codewords of weight 2(p+1) which form a Hadamard matrix  $H_{NV^{(a)}(p)}$  of order 2(p+1) for any prime  $p \equiv 5 \pmod{24}$  and  $a \in \{1, -1\}$  (see Theorem 2, which is our main theorem of this paper). We also give

characterizations of the Hadamard matrices  $H_{NV^{(a)}(p)}$  of order 2(p+1). In particular, it is shown that the ternary self-dual code  $NV^{(a)}(p)$  is generated by the rows of the Hadamard matrix  $H_{NV^{(a)}(p)}$ . This gives an alternative construction of the ternary self-dual code  $NV^{(a)}(p)$ . By Theorem 2, the third ternary extremal self-dual code  $NV^{(1)}(29)$  of length 60, which was found in [11], contains a Hadamard matrix of order 60. In Section 4, our computer search shows that  $NV^{(1)}(29)$  contains one more Hadamard matrix of order 60. Finally, in Section 5, we demonstrate that the currently known three ternary extremal self-dual codes of length 60 are constructed as four-negacirculant codes.

#### 2 Preliminaries

In this section, we give definitions and some known results of ternary selfdual codes and Hadamard matrices used in this paper. Especially, we give details for the construction of ternary self-dual codes  $NV^{(a)}(p)$  in [11] of length 2(p+1), where p is a prime with  $p \equiv 5 \pmod{8}$  and  $a \in \{1, -1\}$ .

#### 2.1 Ternary self-dual codes

Let  $\mathbb{F}_3 = \{0, 1, 2\}$  denote the finite field of order 3. A ternary [n, k] code C is a k-dimensional vector subspace of  $\mathbb{F}_3^n$ . All codes in this paper are ternary. The parameter n is called the *length* of C. A generator matrix of C is a  $k \times n$  matrix whose rows are a basis of C. The weight  $\operatorname{wt}(x)$  of a vector x of  $\mathbb{F}_3^n$  is the number of non-zero components of x. A vector of C is called a codeword. The minimum non-zero weight of all codewords in C is called the minimum weight of C. The weight enumerator of C is given by  $\sum_{c \in C} y^{\operatorname{wt}(c)} \in \mathbb{Z}[y]$ . The dual code  $C^{\perp}$  of a ternary code C of length n is defined as  $C^{\perp} = C$ 

The dual code  $C^{\perp}$  of a ternary code C of length n is defined as  $C^{\perp} = \{x \in \mathbb{F}_3^n \mid x \cdot y = 0 \text{ for all } y \in C\}$ , where  $x \cdot y$  is the standard inner product. A ternary code C is self-dual if  $C = C^{\perp}$ . A ternary self-dual code of length n exists if and only if n is divisible by four. Two ternary codes C and C' are equivalent if there is a monomial matrix P over  $\mathbb{F}_3$  with  $C' = C \cdot P$ , where  $C \cdot P = \{xP \mid x \in C\}$ . We denote two equivalent ternary codes C and D by  $C \cong D$ . All ternary self-dual codes were classified in [4], [6], [9] and [14] for lengths up to 24.

#### 2.2 Ternary extremal self-dual codes

It was shown in [10] that the minimum weight d of a ternary self-dual code of length n is bounded by  $d \leq 3\lfloor n/12 \rfloor + 3$ . If  $d = 3\lfloor n/12 \rfloor + 3$ , then the code is called *extremal*. For  $n \in \{4, 8, 12, \ldots, 64\}$ , it is known that there is a ternary extremal self-dual code of length n (see [7, Table 6]). By the Assmus–Mattson theorem [1], the supports of codewords of minimum weight in a ternary extremal self-dual code of length divisible by 12 form a 5-design. This is a reason for our interest in ternary extremal self-dual codes of length divisible by 12.

The weight enumerator of a ternary extremal self-dual code of length n is uniquely determined for each n [10]. The number  $A_n$  of codewords of weight n in a ternary extremal self-dual code of length n is listed in Table 1 for n = 12, 24, 36, 48, 60 (see [19]). Note that  $A_n = 2n$  for n = 12, 24, 48.

Table 1: Numbers  $A_n$  of codewords of weight n

$\overline{n}$	12	24	36	48	60
$A_n$	24	48	888	96	41184

The ternary extended quadratic residue codes and the Pless symmetry codes are well known families of ternary (self-dual) codes. More precisely, the extended quadratic residue code  $QR_{p+1}$  of length p+1 is a ternary self-dual code when p is a prime such that  $p \equiv -1 \pmod{12}$  (see [8, Chapter 6]). The Pless symmetry code  $P_{2q+2}$  of length 2q+2 is a ternary self-dual code when q is a prime power such that  $q \equiv -1 \pmod{6}$  [13] (see also [8, Chapter 10]). The extended quadratic residue codes  $QR_n$  and the Pless symmetry codes  $P_n$  yield ternary extremal self-dual codes when  $n \leq 60$  (see [15]). More precisely,  $P_{36}$  is the currently known ternary extremal self-dual code of length 36,  $QR_{48}$  and  $P_{48}$  are the currently known ternary extremal self-dual codes of length 48. In addition,  $QR_{60}$  and  $P_{60}$  are ternary extremal self-dual codes of length 60.

#### 2.3 Ternary self-dual codes given in [11]

In 2013, Nebe and Villar [11] gave a new series of ternary self-dual codes  $NV^{(a)}(p)$  of length 2(p+1) for all primes  $p \equiv 5 \pmod 8$  and  $a \in \{1, -1\}$  (see also [3, Section 4] for the details). Here, we review the construction of the ternary self-dual codes  $NV^{(a)}(p)$ .

Suppose that  $p \equiv 5 \pmod{8}$ . Let  $\mathbb{F}_p = \{0, 1, \dots, p-1\}$  denote the finite field of order p. Let  $\chi$  denote the quadratic character of  $\mathbb{F}_p$ . Define two  $p \times p$  matrices  $R_X = (r_{X_{a,b}})$  and  $R_Y = (r_{Y_{a,b}})$  as follows

$$r_{Xa,b} = \begin{cases} 0, & \text{if } a = b \text{ or } b - a \text{ is not a nonzero square in } \mathbb{F}_p, \\ \chi(c), & \text{if } b - a \text{ is a nonzero square } c^2 \text{ in } \mathbb{F}_p, \end{cases}$$

$$r_{Ya,b} = \begin{cases} 0, & \text{if } a = b \text{ or } 2(b - a) \text{ is not a nonzero square in } \mathbb{F}_p, \\ \chi(c), & \text{if } 2(b - a) \text{ is a nonzero square } c^2 \text{ in } \mathbb{F}_p, \end{cases}$$

where rows and columns of  $R_X$  and  $R_Y$  are indexed by the elements of  $\mathbb{F}_p$  with a fixed ordering. Then define two  $(p+1)\times(p+1)$  matrices X and Y as follows

$$X = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & R_X & \\ -1 & & & \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R_Y & \\ 0 & & & \end{pmatrix}.$$

In addition, define two  $2(p+1) \times 2(p+1)$  matrices as follows

$$B_w = \begin{pmatrix} X & Y \\ -Y^T & X^T \end{pmatrix}$$
 and  $B_{\epsilon w} = \begin{pmatrix} -Y^T & X^T \\ -X & -Y \end{pmatrix}$ .

Throughout this paper, let  $I_n$  denote the identity matrix of order n. For a = 1 and -1, let  $NV^{(a)}(p)$  denote the ternary code generated by the matrix M, where

$$M = \begin{cases} aI_{2(p+1)} + B_w, & \text{if } p \equiv 5 \pmod{24}, \\ aI_{2(p+1)} + B_w + B_{\epsilon w}, & \text{if } p \equiv 13 \pmod{24}. \end{cases}$$

Then  $NV^{(a)}(p)$  is self-dual [11] (see also [3, Theorem 8]).

**Proposition 1** (Nebe and Villar [11]).  $NV^{(1)}(29) \cong NV^{(-1)}(29)$  and  $QR_{60} \ncong NV^{(1)}(29) \ncong P_{60}$ .

The above proposition means that  $NV^{(1)}(29)$  is the third ternary extremal self-dual code of length 60. In this paper, we denote the code  $NV^{(1)}(29)$  by  $NV_{60}$ .

#### 2.4 Hadamard matrices and results in [19]

A Hadamard matrix H of order n is an  $n \times n$  matrix whose entries are from  $\{1, -1\}$  such that  $HH^T = nI_n$ , where  $H^T$  is the transpose of H. It is known that the order n is necessarily 1, 2, or a multiple of 4. A Hadamard matrix H of order n is called skew if  $H = A + I_n$ , where  $A = -A^T$ . Two Hadamard matrices H and K are said to be equivalent if there is (1, -1, 0)-monomial matrices P and Q with K = PHQ. An automorphism of a Hadamard matrix H is an equivalence of H to itself, i.e., a pair (P, Q) of monomial matrices P and Q such that H = PHQ. The set of all automorphisms of H forms a group, called the automorphism group of H, under the component-wise product:  $(P_1, Q_1)(P_2, Q_2) = (P_1P_2, Q_1Q_2)$ .

Recently, Tonchev [19] studied Hadamard matrices of order n formed by codewords of weight n in ternary extremal self-dual codes of length n, especially the extended quadratic residue codes and the Pless symmetry codes. In the context of Hadamard matrices, we consider the element 0, 1, 2 of  $\mathbb{F}_3$  as 0, 1, -1 of  $\mathbb{Z}$ , throughout this paper. It is trivial that  $n \equiv 0 \pmod{12}$  if a ternary (extremal) self-dual code of length n contains a Hadamard matrix formed by codewords of weight n. This is another reason for our interest in ternary extremal self-dual codes of length divisible by 12.

From the construction, the extended quadratic residue code contains a type I Paley-Hadamard matrix. The Pless symmetry code contains a type II Paley-Hadamard matrix [13]. Tonchev [19] showed that  $P_{36}$  contains exactly two inequivalent Hadamard matrices of order 36. In addition, Tonchev [19] gave a natural question, namely, is there any other ternary extremal self-dual code of length 36, 48, or 60 which contains a Hadamard matrix? This motivates us to study the existence of Hadamard matrices of order 2(p+1) formed by codewords of weight 2(p+1) in the ternary self-dual codes  $NV^{(a)}(p)$  found by Nebe and Villar [11].

## 3 Hadamard matrices related to $NV^{(a)}(p)$

Throughout this section, suppose that p is a prime with  $p \equiv 5 \pmod{24}$ . In this section, we show that  $NV^{(a)}(p)$  contains 2(p+1) codewords of weight 2(p+1) which form a Hadamard matrix  $H_{NV^{(a)}(p)}$  of order 2(p+1) for  $a \in \{1,-1\}$ . We also give characterizations of the Hadamard matrices  $H_{NV^{(a)}(p)}$  of order 2(p+1).

Let X and Y be the  $(p+1) \times (p+1)$  matrices as defined in Section 2.3. As described there, for a=1 and -1, the ternary code  $NV^{(a)}(p)$  generated by the following matrix

$$aI_{2(p+1)} + \begin{pmatrix} X & Y \\ -Y^T & X^T \end{pmatrix}$$

is a ternary self-dual code [11] (see also [3, Theorem 8]).

The following is our main theorem of this paper.

**Theorem 2.** Suppose that  $p \equiv 5 \pmod{24}$  and  $a \in \{1, -1\}$ . Then the ternary self-dual code  $NV^{(a)}(p)$  of length 2(p+1) contains 2(p+1) codewords of weight 2(p+1) which form a Hadamard matrix of order 2(p+1).

*Proof.* Since  $NV^{(a)}(p)$  is generated by the following matrix

$$\begin{pmatrix} X + aI_{p+1} & Y \\ -Y^T & X^T + aI_{p+1} \end{pmatrix},$$

the rows of the following two matrices

( 
$$X-Y^T+aI_{p+1}$$
  $Y+X^T+aI_{p+1}$  ) and   
 (  $-Y^T-X-aI_{p+1}$   $X^T-Y+aI_{p+1}$  )

are codewords of  $NV^{(a)}(p)$ . From the definition of X and Y, the 2(p+1) codewords has weight 2(p+1). In addition, we regard the following matrix as a  $\mathbb{Z}$ -matrix

$$H_{NV^{(a)}(p)} = \begin{pmatrix} X - Y^T + aI_{p+1} & Y + X^T + aI_{p+1} \\ -Y^T - X - aI_{p+1} & X^T - Y + aI_{p+1} \end{pmatrix}.$$
(1)

Since it is known [3] that

$$X^{T} = -X, Y^{T} = -Y, XY = YX \text{ and } X^{2} + Y^{2} = -pI_{n+1},$$

 $H_{NV^{(a)}(p)}$  is a Hadamard matrix of order 2(p+1).

Now we give characterizations of Hadamard matrices  $H_{NV^{(a)}(p)}$  of order 2(p+1).

**Proposition 3.** Let  $H_{NV^{(a)}(p)}$  denote the Hadamard matrix given in (1). Then  $aH_{NV^{(a)}(p)}$  is a skew Hadamard matrix for a=1 and -1.

*Proof.* The claim follows from that  $H_{NV^{(a)}(p)} + H_{NV^{(a)}(p)}^T = 2aI_{2(p+1)}$ .

Note that  $H_{NV^{(a)}(p)}$  has the form  $H_{NV^{(a)}(p)}=\left(egin{array}{cc}A&B\\-B^T&A^T\end{array}
ight),$  where

$$A = X - Y^{T} + aI_{p+1} = \begin{pmatrix} a & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & A' & & \\ -1 & & & \end{pmatrix}$$
 and 
$$B = Y + X^{T} + aI_{p+1} = \begin{pmatrix} a & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & B' & & \\ 1 & & & \end{pmatrix},$$

for some  $p \times p$  matrices A' and B'. Let  $\omega$  be a fixed primitive element of  $\mathbb{F}_p$ . Define

$$C_i = \omega^i \langle \omega^4 \rangle, i = 0, 1, 2, 3, \tag{2}$$

which are the cosets of the multiplicative subgroup of index 4 of  $\mathbb{F}_p$ . Note that  $C_1$  and  $C_3$  are interchanged if we choose  $\omega^{-1}$  as a primitive element instead of  $\omega$ . Since  $p \equiv 5 \pmod 8$ , we have  $-1 \in C_2$  and  $2 \in NQ = C_1 \cup C_3$ , where NQ denotes the set of nonsquares in  $\mathbb{F}_p$ . Hence,  $-2 \in C_1$  or  $-2 \in C_3$ , depending on the choice of  $\omega$ . We denote by  $C_{\epsilon}$  the coset containing -2, that is,

$$-2 \in C_{\epsilon}. \tag{3}$$

Then  $A' = (a'_{s,t})$  and  $B' = (b'_{s,t})$  can be written as

$$a'_{s,t} = \begin{cases} a, & \text{if } t = s, \\ 1, & \text{if } t - s \in C_0 \cup C_{\epsilon}, \\ -1, & \text{if } t - s \in C_2 \cup C_{\epsilon+2}, \end{cases}$$

$$b'_{s,t} = \begin{cases} a, & \text{if } t = s, \\ 1, & \text{if } t - s \in C_2 \cup C_{\epsilon}, \\ -1, & \text{if } t - s \in C_0 \cup C_{\epsilon+2}. \end{cases}$$

$$(4)$$

Throughout this section, we reduce the subscript of  $C_i$  modulo 4.

Let G be an additively written abelian group of order v. Two subsets  $D_1$  and  $D_2$  of G with  $k = |D_1| = |D_2|$  are called  $(v, k, \lambda)$  supplementary

difference sets if the list of differences x - y,  $x, y \in D_i$ , i = 1, 2, represents every nonzero element of G exactly  $\lambda$  times. Fixing an ordering for the elements of G, we define a matrix  $M = (m_{i,j})$  by

$$m_{i,j} = \begin{cases} 1, & \text{if } j - i \in X, \\ -1, & \text{if } j - i \notin X, \end{cases}$$

for  $X \subset G$ . The matrix M is called a type-1 matrix of X.

The following construction of Hadamard matrices easily follows from [16, Corollary 4.5 (i) and Lemma 4.8].

**Lemma 4.** Let  $D_i$ , i = 1, 2, be (2m + 1, m, m - 1) supplementary difference sets in an abelian group G of order v = 2m + 1. Furthermore, let  $M_1$  (resp.  $M_2$ ) be the type-1 matrix of  $D_1$  (resp.  $D_2$ ). Then

$$H(D_1, D_2) = \begin{pmatrix} 1 & 1 & \mathbf{1}_v & -\mathbf{1}_v \\ -1 & 1 & -\mathbf{1}_v & -\mathbf{1}_v \\ -\mathbf{1}_v^T & \mathbf{1}_v^T & -M_1 & -M_2 \\ \mathbf{1}_v^T & \mathbf{1}_v^T & M_2^T & -M_1^T \end{pmatrix}$$

is a Hadamard matrix of order 4(m+1), where  $\mathbf{1}_v$  denotes the all-one vector of length v.

It is known [18] that for any prime  $p \equiv 5 \pmod{24}$  and any i = 0, 1, 2, 3, the sets  $C_i \cup C_{i+1}$  and  $C_{i+1} \cup C_{i+2}$  are (p, (p-1)/2, (p-3)/2) supplementary difference sets in the additive group of  $\mathbb{F}_p$ , where  $C_i$ 's are defined as in (2) (more generally, the same claim holds for any prime power  $p \equiv 5 \pmod{8}$ ). By Lemma 4,  $H(C_i \cup C_{i+1}, C_{i+1} \cup C_{i+2})$  is a Hadamard matrix of order 2(p+1). Furthermore, from (4), the following theorem holds.

**Theorem 5.** Let  $H_{NV^{(a)}(p)}$  denote the Hadamard matrix defined as in (1). Then  $H_{NV^{(1)}(p)}$  and  $H_{NV^{(-1)}(p)}$  are equivalent to  $H(C_2 \cup C_{2+\epsilon}, C_0 \cup C_{\epsilon+2})$  and  $H(C_0 \cup C_{\epsilon}, C_2 \cup C_{\epsilon})$ , respectively, where  $C_{\epsilon}$  is defined as in (3).

Although the proof of the following proposition is somewhat trivial, we give it for the sake of completeness.

**Proposition 6.** The ternary self-dual code  $NV^{(a)}(p)$  is generated by the rows of the Hadamard matrix  $H_{NV^{(a)}(p)}$  defined as in (1).

*Proof.* It is sufficient to show that  $\operatorname{rank}_3(H_{NV^{(a)}(p)}) = p+1$ . Since  $H_{NV^{(a)}(p)} + H_{NV^{(a)}(p)}^T = 2aI_{2(p+1)}$ , we have

$$2(p+1) = \operatorname{rank}_{3}(2aI_{2(p+1)}) = \operatorname{rank}_{3}(H_{NV^{(a)}(p)} + H_{NV^{(a)}(p)}^{T})$$
  

$$\leq 2\operatorname{rank}_{3}(H_{NV^{(a)}(p)}),$$

i.e.,  $p+1 \leq \operatorname{rank}_3(H_{NV^{(a)}(p)})$ . On the other hand, since  $p+1 \equiv 0 \pmod 3$ ,  $H_{NV^{(a)}(p)}H_{NV^{(a)}(p)}^T \equiv O \pmod 3$ , where O denotes the  $2(p+1) \times 2(p+1)$  zero matrix. This implies that  $\operatorname{rank}_3(H_{NV^{(a)}(p)}) \leq p+1$ . This completes the proof.

The above proposition gives an alternative construction of the ternary self-dual code  $NV^{(a)}(p)$ .

#### 4 Hadamard matrices related to $NV_{60}$

By Theorem 2, the third ternary extremal self-dual code  $NV_{60}$  of length 60, which was found in [11], contains a Hadamard matrix of order 60. In this section, our computer search found one more Hadamard matrix of order 60 in  $NV_{60}$ . All computer calculations in this section were done by programs in the language C and programs in MAGMA [2].

Any ternary extremal self-dual code of length 60 contains 41184 codewords of weight 60 (see Table 1). Let  $W_{60}$  be the set of 41184 codewords of weight 60 in  $NV_{60}$ . It is trivial that there is a set  $W_{60}^+$  consisting of 20592 codewords of weight 60 such that

$$W_{60} = W_{60}^+ \cup \{2x \mid x \in W_{60}^+\}.$$

Let  $\rho$  be a map from  $\mathbb{F}_3$  to  $\mathbb{Z}$  sending 0, 1, 2 to 0, 1, -1, respectively. Define the following set

$$W_{60}^{\mathbb{Z}} = \{ (\rho(x_1), \rho(x_2), \dots, \rho(x_{60})) \mid (x_1, x_2, \dots, x_{60}) \in W_{60}^+ \} \subset \mathbb{Z}^{60}.$$

Then we define the simple undirected graph  $\Gamma$ , whose set of vertices is the set  $W_{60}^{\mathbb{Z}}$  and two vertices x and y are adjacent if x and y are orthogonal, noting that  $x, y \in \mathbb{Z}^{60}$ . Clearly, a 60-clique in  $\Gamma$  gives a Hadamard matrix. In addition, in order to find a Hadamard matrix, it is sufficient to consider only  $W_{60}^{\mathbb{Z}}$  as the set of vertices of  $\Gamma$ . Due to the computational complexity,

by the above approach, our computer search was able to find two 60-cliques, which imply two inequivalent Hadamard matrices  $H_{NV,1}$  and  $H_{NV,2}$ . The computation for finding cliques was performed using the clique finding algorithm CLIQUER [12]. The computation for verifying the inequivalence of  $H_{NV,1}$  and  $H_{NV,2}$  was done by the MAGMA function IsHadamardEquivalent. Therefore, we have the following proposition.

**Proposition 7.** The third ternary extremal self-dual code  $NV_{60}$  of length 60 contains at least two inequivalent Hadamard matrices of order 60 having as rows codewords of weight 60.

We verified by MAGMA that  $H_{NV,1}$  and  $H_{NV^{(1)}(29)}$  are equivalent, and  $H_{NV,1}$  and  $H_{NV,2}$  have automorphism groups of orders 24360 and 812, respectively. These were done by the MAGMA functions IsHadamardEquivalent and HadamardAutomorphismGroup, respectively.

Now we display the Hadamard matrix  $H_{NV,2}$ . Here, instead of this matrix, we display its binary Hadamard matrix  $B_{NV,2} = (H_{NV,2} + J)/2$ , where J is the  $60 \times 60$  all-one matrix. Let  $r_i$  denote the i-th row of  $B_{NV,2}$ . To save space, the vectors  $r_1, r_2, \ldots, r_{60}$  are written in octal using 0 = (0, 0, 0),  $1 = (0, 0, 1), \ldots, 7 = (1, 1, 1)$  in Figure 1. For example, the first row of  $H_{NV,2}$ 

It is worthwhile to determine whether C contains a Hadamard matrix which is not equivalent to a Paley-Hadamard matrix for  $C = QR_{60}$  and  $P_{60}$ .

### 5 Four-negacirculant codes of length 60

In this section, we demonstrate that the currently known ternary extremal self-dual codes of length 60 are constructed as four-negacirculant codes. All computer calculations in this section were done by programs in MAGMA [2].

An  $n \times n$  negacirculant matrix has the following form

$$\begin{pmatrix}
r_0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\
2r_{n-1} & r_0 & r_1 & \cdots & r_{n-3} & r_{n-2} \\
2r_{n-2} & 2r_{n-1} & r_0 & \cdots & r_{n-4} & r_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
2r_1 & 2r_2 & 2r_3 & \cdots & 2r_{n-1} & r_0
\end{pmatrix}.$$

762032465106141355640221363475131260237102747172112540476254744065152314302733500427471721225404762504427471722625404762724762032427206141357246517440135564143005716364221301174531710152324730605667200717211057047625453007504427473712625404104427471712625404761057163642053011745311057163641453011745651744065116414302736515237101273350306064762032463206141355646517440635564143021363475044260237126263710152321030605667632476203226720614136203246517214135564116364221361174531260171636422130117453121721105716362545301160324651741413556414574406515201430273352110571636254530117421363475041260237126221363475031260237125523710152335030605655232476200566720614523710152335030605665232476203166720614151744065152414302733515237101533350306052747172110140476254550152324760605667206476203246520614135564651744065156414302746515237100273350306316364221301174531263172110571076254530132110571632254530117440651523703027335034324651744013556414334717211050047625453347504427423712625403504427471312625404741523247622056672061363475044220237126253642213634345312602340651523713027335030

Figure 1: Binary Hadamard matrix  $B_{NV,2}$ 

Let A and B be  $n \times n$  negacirculant matrices. A ternary [4n, 2n] code having the following generator matrix

$$\begin{pmatrix}
I_{2n} & A & B \\
2B^T & A^T
\end{pmatrix}$$
(5)

is called a *four-negacirculant* code. Many ternary extremal self-dual four-negacirculant codes are known (see e.g., [5]).

Let  $C_1, C_2$  and  $C_3$  be the ternary four-negacirculant codes of length 60, having generator matrices of form (5), where the pairs  $(r_A, r_B)$  of the first

rows  $r_A$  and  $r_B$  of the negacirculant matrices A and B are as follows

```
((1,1,0,2,1,1,1,2,2,2,0,1,0,0,2),(2,0,0,2,1,0,0,1,2,2,0,1,0,2,2)),\\((1,1,2,2,1,2,2,1,1,1,2,1,2,1,2),(2,2,1,2,2,0,2,2,1,2,2,2,2,1,1)),\\((1,0,0,1,1,2,2,0,2,1,1,0,0,0,2),(1,2,0,0,2,2,1,1,0,0,0,0,2,2,0)),
```

respectively. We verified by MAGMA that  $C_1 \cong QR_{60}$ ,  $C_2 \cong P_{60}$  and  $C_3 \cong NV_{60}$ . This was done by the MAGMA function IsIsomorphic. Hence, we have the following proposition.

**Proposition 8.** For each C of the codes  $QR_{60}$ ,  $P_{60}$  and  $NV_{60}$ , there is a four-negacirculant code D such that  $C \cong D$ .

It is worthwhile to determine whether there is a new ternary extremal four-negacirculant self-dual code of length 60.

Remark 9. Two ternary extremal self-dual codes  $D_{60,1}$  and  $D_{60,2}$  of length 60 were constructed in [5]. We verified by MAGMA that  $D_{60,1} \cong QR_{60}$  and  $D_{60,2} \cong P_{60}$ . This was also done by the MAGMA function IsIsomorphic.

Remark 10. Recently, it has been shown in [3] that there are exactly three inequivalent ternary extremal self-dual codes of length 60 having an automorphism of order 29. On the other hand, since each of  $QR_{60}$ ,  $P_{60}$  and  $NV_{60}$  has an automorphism of order 29, the three codes found in [3] are  $QR_{60}$ ,  $P_{60}$  and  $NV_{60}$ .

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### References

- [1] E.F. Assmus Jr. and H.F. Mattson Jr., New 5-designs, J. Combinatorial Theory 6 (1969), 122–151.
- [2] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.
- [3] S. Bouyuklieva, J. de la Cruz and D. Villar, Extremal binary and ternary codes of length 60 with an automorphism of order 29 and a generalization, *Mathematics* **10** (2022), 748.

- [4] J.H. Conway, V. Pless and N.J.A. Sloane, Self-dual codes over GF(3) and GF(4) of length not exceeding 16, *IEEE Trans. Inform. Theory* **25** (1979), 312–322.
- [5] M. Harada, W. Holzmann, H. Kharaghani and M. Khorvash, Extremal ternary self-dual codes constructed from negacirculant matrices, *Graphs Combin.* 23 (2007), 401–417.
- [6] M. Harada and A. Munemasa, A complete classification of ternary selfdual codes of length 24, J. Combin. Theory Ser. A 116 (2009), 1063– 1072.
- [7] W.C. Huffman, On the classification and enumeration of self-dual codes, Finite Fields Appl. 11 (2005), 451–490.
- [8] W.C. Huffman and V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, Cambridge, 2003.
- [9] C.L. Mallows, V. Pless and N.J.A. Sloane, Self-dual codes over GF(3),  $SIAM\ J.\ Appl.\ Math.\ 31\ (1976),\ 649-666.$
- [10] C.L. Mallows and N.J.A. Sloane, An upper bound for self-dual codes, Inform. Control 22 (1973), 188–200.
- [11] G. Nebe and D. Villar, An analogue of the Pless symmetry codes, Seventh International Workshop on Optimal Codes and Related Topics, Bulgaria, pp. 158–163 (2013).
- [12] S. Niskanen and P.R.J. Östergård, Cliquer User's Guide, Version 1.0, Helsinki University of Technology Communications Laboratory Technical Report T48 (2003).
- [13] V. Pless, Symmetry codes over GF(3) and new five-designs, J. Combin. Theory Ser. A 12 (1972), 119–142.
- [14] V. Pless, N.J.A. Sloane and H.N. Ward, Ternary codes of minimum weight 6 and the classification of the self-dual codes of length 20, *IEEE Trans. Inform. Theory* **26** (1980), 305–316.
- [15] E. Rains and N.J.A. Sloane, "Self-dual codes," Handbook of Coding Theory, V.S. Pless and W.C. Huffman (Editors), Elsevier, Amsterdam, pp. 177–294, 1998.

- [16] J. Seberry, Orthogonal Designs: Hadamard Matrices, Quadratic Forms and Algebras, Springer, Cham, 2017.
- [17] J. Seberry and M. Yamada, *Hadamard Matrices: Constructions using Number Theory and Linear Algebra*, Wiley, NJ, 2020.
- [18] G. Szekeres, Tournaments and Hadamard matrices, Enseign. Math. (2) 15 (1969), 269–278.
- [19] V.D. Tonchev, On Pless symmetry codes, ternary QR codes, and related Hadamard matrices and designs, *Des. Codes Cryptogr.*, (to appear) https://doi.org/10.1007/s10623-021-00941-0.