# Eggs in finite projective spaces and unitals in translation planes

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**Abstract**

Inspired by the connection between ovoids and unitals arising from the Buekenhout construction in the André/Bruck-Bose representation of translation planes of dimension at most two over their kernel, and since eggs of $\mathrm{PG}(4m-1,q)$, $m \geq 1$, are a generalization of ovoids, we explore the relation between eggs and unitals in translation planes of higher dimension over their kernel. By investigating such a relationship, we construct a unital in the Dickson semifield plane of order $3^{10}$, which is represented in $\mathrm{PG}(20,3)$ by a

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cone whose base is a set of points constructed from the dual of the Penttila-Williams egg in $\mathrm{PG}(19, 3)$. This unital is not polar; so, up to the knowledge of the authors, it seems to be a new unital in such a plane.

**Keywords:** Unital, Blocking set, Egg, Projective plane

# 1 Introduction

Field reduction has become a theme of finite geometry which turned out very fruitful in the last few decades. Given a construction of an interesting object from a configuration in a vector space of dimension $r$ over a field of order $q^n$, the question is raised as to which objects give rise similar configurations in a vector space of dimension $rn$ over a field of order $q$.

The Buekenhout-Metz construction of unitals in finite translation planes [15, 33] (which gives all known unitals in Desarguesian planes) can be recontextualized in this fashion, with cones projecting an ovoid as a base in the André/Bruck-Bose representation of such planes.

It has long been known [14] that unitals are extremal in size among minimal blocking sets (at the other end than that most studied - large rather than small). The observation of Lunardon [29] at the turn of the millennium that changing the field gave access to many more subspaces, some of which were blocking sets, transformed the theory of blocking sets in the process giving rise to the idea of linear sets. Thus, the idea of Buekenhout and Metz was taken by Szőnyi et al. [40] and, later, by Mazzocca and Polverino [31] to provide further minimal blocking sets, using cones rather than subspaces.

For the construction by Tits of generalized quadrangles from (ovals and) ovoids, the configurations that arise by applying a field reduction are eggs, and the similar objects are translation generalized quadrangles. Thus, changing the field for ovoids and studying eggs gave the possibility of new translation generalized quadrangles, first realized in work of Kantor [25] from three decades past; the result of field reduction applied to the concept of an ovoid is an egg.

Motivated by the relationship between ovoids and unitals via the Buekenhout-Metz construction, and since eggs are generalization of ovoids, we explore possible relationships between eggs and unitals. Putting all the above ideas together, in this paper we construct a unital in the Dickson semifield plane of order $3^{10}$, which is represented in $\mathrm{PG}(20, 3)$ by a cone whose base is a set of points constructed from the dual of the Penttila-Williams egg in $\mathrm{PG}(19, 3)$. This unital does not arise from a polarity; so it is a new unital, up to the knowledge of the authors.

While field reduction is usually thought of in a projective setting, algebraic dimensions are more amenable to an introductory discussion of it, so we will take a vector space approach along all the paper.

## 2 Definitions and preliminary results

A *unital* in a finite projective plane $\pi$ of order $n^2$ is a set $\mathcal{U}$ of $n^3+1$ points such that every line of $\pi$ meets $\mathcal{U}$ in 1 or $n+1$ points. Therefore, $\mathcal{U}$ is equipped with a family of subsets, each of size $n+1$, such that every pair of distinct points of $\mathcal{U}$ is contained in exactly one subset of the family; such subsets are usually called *blocks*, and $\mathcal{U}$ turns out to be a 2-$(n^3+1, n+1, 1)$ design.

In a computer search, Brouwer [10, 11] found a large number of mutually non-isomorphic 2-$(28, 4, 1)$ designs. Only a few of these are embeddable in a projective plane of order 9 as unitals. One of the examples has been generalized by Grüning [23], who constructed a unital of order $q$ for any odd prime power in both the Hall plane and dual Hall plane of order $q$. An infinite family of non-Buekenhout unitals in the Hall planes of order $q^2$ have been constructed in [19]. Other infinite families of unitals in various square order planes are known to exist; see e.g. [1], [5], [6], [18], [36], [37]. The only known 2-$(n^3+1, n+1, 1)$ design with $n$ not a prime power is the one found in [4] and [30] where $n = 6$. For more on 2-$(n^3+1, n+1, 1)$ designs embeddable as unitals in projective plane, see [8].

In the Desarguesian projective plane $\mathrm{PG}(2, q^2)$, a unital can arise from a unitary polarity: the points of the unital are the absolute points, and the blocks are the intersections of the non-absolute lines of the polarity with $\mathcal{U}$. These unitals are called *classical* or *Hermitian unitals*. By a result of Seib [39], the absolute points of a unitary polarity in any square order projective plane form the point-set of a unital. Such unitals are called *polar unitals*. So, classical unitals of $\mathrm{PG}(2, q^2)$ are examples of polar unitals, and Ganley [21] showed that polar unitals exist in any Dickson commutative semifield plane of odd order.

A *finite semifield* is a finite set $\mathbb{S}$ with two binary operations $+$ and $*$, such that $(\mathbb{S}, +)$ is an abelian group and $(\mathbb{S} \setminus \{0\}, *)$ is a loop such that both distributive laws hold.

Let $\pi(\mathbb{S})$ be the point-line geometry whose points are the elements in $\mathbb{S} \times \mathbb{S}$ and in $\{(m) : m \in \mathbb{S} \cup \{\infty\}\}$, and the lines are the sets

$$[m, k] = \{(y, x) \in \mathbb{S} \times \mathbb{S} : m * x + y = k\} \cup \{(m)\},$$

$$[z] = \{(y, z) : y \in \mathbb{S}\} \cup \{(\infty)\}$$

and

$$[\infty] = \{(m) : m \in \mathbb{S}\} \cup \{(\infty)\}.$$

with $m, k, z \in \mathbb{S}$, and $\infty$ a symbol not in $\mathbb{S}$.

It turns out that $\pi(\mathbb{S})$ is a translation plane which is called the *semifield plane coordinatized by* $\mathbb{S}$. We refer to [9] and [17] for basic information on semifields and translation planes.

For any semifield $\mathbb{S}$, the subset $\mathcal{N}_l = \{a \in \mathbb{S} : a * (x * y) = (a * x) * y, \forall x, y \in \mathbb{S}\}$ is called the *left nucleus* of $\mathbb{S}$. Similarly, the *middle nucleus* $\mathcal{N}_m$ and the *right nucleus* $\mathcal{N}_r$ are defined. The set $\mathcal{K} = \{a \in \mathcal{N}_l \cap \mathcal{N}_m \cap \mathcal{N}_r : a * b = b * a, \forall b \in \mathbb{S}\}$ is called the *center* of $\mathbb{S}$. Each of these four structures is a field, and a finite semifield is a left vector space over its left nucleus and a two-sided vector space over its center [17]. Here, $\mathcal{K}$ is isomorphic to the kernel of the translation plane $\pi(\mathbb{S})$.

For any element $b$ of the semifield $\mathbb{S}$ with center $\mathcal{K}$, the map $\phi_b : x \in \mathbb{S} \mapsto xb \in \mathbb{S}$ is a linear map when $\mathbb{S}$ is considered over its left nucleus $\mathcal{N}_l$. It turns out that the set $\mathcal{C}_{\mathbb{S}} = \{\phi_b : b \in \mathbb{S}\}$ is a $\mathcal{K}$-vector subspace of the vector space of the $\mathcal{N}_l$-linear maps of $\mathbb{S}$. Since $\mathbb{S}$ is finite, we may assume $\mathcal{K} = \mathbb{F}_q$, $\mathcal{N}_l = \mathbb{F}_{q^n}$ and $\mathbb{S}$ is an $t$-dimensional left vector space over $\mathbb{F}_{q^n}$, for some positive integers $n$ and $t$.

Under the previous indentification, the set $\mathcal{C}_{\mathbb{S}}$ satisfies the following properties: (i) $\mathcal{C}_{\mathbb{S}}$ has $q^{nt}$ elements; (ii) $\mathcal{C}_{\mathbb{S}}$ contains the zero and the identity maps; (iii) $A - B$ is non-singular for all distinct $A, B \in \mathcal{C}_{\mathbb{S}}$. A set of linear maps of $V(t, q^n)$ satisfying the above properties is called a *spread set* of $V(t, q^n)$.

A $(t-1)$-spread of the $(r-1)$-dimensional projective space $\mathrm{PG}(r-1, q)$ over $\mathbb{F}_q$ is a set $\mathcal{S}$ of $(t-1)$-dimensional projective subspaces such that every point is contained in exactly one subspace of $\mathcal{S}$. It is known that a $(t-1)$-spread of $\mathrm{PG}(r-1, q)$ exists if and only if $t$ divides $r$ [17].

Let $\mathcal{C}$ be a spread set of $V(t, q^n) = \mathbb{F}_{q^n}^t$. In $\mathrm{PG}(2t-1, q^n)$ consider the subspaces

$$S_\tau = \{((x_1, \ldots, x_t)^\tau, x_1, \ldots, x_t) : x_i \in \mathbb{F}_{q^n}\},$$

for all $\tau \in \mathcal{C}$. Then, the set $\mathcal{S} = \{S_\tau : \tau \in \mathcal{C}\} \cup \{S_\infty\}$, with $S_\infty = \{(x_1, \ldots, x_t, 0, \ldots, 0) : x_i \in \mathbb{F}_{q^n}\}$ forms a $(t-1)$-spread of $\mathrm{PG}(2t-1, q^n)$.

Conversely, let $\mathcal{S}$ be a $(t-1)$-spread of $\mathrm{PG}(2t-1, q^n)$. Then, it is possible to choose homogeneus coordinates in $\mathrm{PG}(2t-1, q^n)$ such that there is a spread set $\mathcal{C}$ of $V(t, q^n)$ from which $\mathcal{S}$ is constructed as above. Thanks to the André/Bruck-Bose construction, the spread $\mathcal{S}$ defines a translation plane $\Pi(\mathcal{S})$ [3, 12, 13]. If the set $\mathcal{C}$ is closed under the sum, then there is a (finite) semifield $\mathbb{S}$ that coordinatizes $\Pi(\mathcal{S})$ such that $\mathcal{C} = \mathcal{C}_{\mathbb{S}}$; the left nucleus of $\mathbb{S}$ is $\mathbb{F}_{q^n}$ and $\mathbb{S}$ can be viewed as a $t$-dimensional left vector space over $\mathbb{F}_{q^n}$ [17]. In addition, if $\mathbb{F}_q$ is the largest subfield $K$ of $\mathbb{F}_{q^n}$ such that $\mathcal{C}$ is a $K$-vector subspace of the vector space of the $\mathbb{F}_{q^n}$-linear maps of $V(t, q^n)$, the center of $\mathbb{S}$ is $\mathbb{F}_q$. Therefore, there exists a canonical correspondence between translation planes coordinatized over a semifield $\mathbb{S}$ with dimension $t$ over its left nucleus $\mathbb{F}_{q^n}$ and center $\mathbb{F}_q$, and the $(t-1)$-spreads of $\mathrm{PG}(2t-1, q^n)$ arising from a spread set of $V(t, q^n)$, that is closed under the sum. Moreover, it is well-known that the resulting plane is Desarguesian if and only if $\mathcal{S}$ is a Desarguesian spread [13].

Buekenhout [15], and Metz [33] (by refining Buekenhout's idea), constructed unitals in any translation planes with dimension at most two over their kernel by using the André/Bruck-Bose representation of such planes. These unitals are cones of

PG$(4,q)$ projecting an ovoid in a 3-dimensional subspace of PG$(4,q)$ from a point at infinity. These unitals are called *Buekenhout-Metz unitals*. Since classical unitals can be obtained in this way, they fall in the class of Buekenhout-Metz unitals which, so far, are the only known unitals of PG$(2,q^2)$.

Many other authors have used the above representation of PG$(2,q^n)$ in PG$(2n,q)$ to study objects in the Desarguesian plane in order to determine whether this higher dimensional representation provides additional information about those objects in the plane. In particular, the projective plane PG$(2,q^4)$, modelled in PG$(8,q)$, has been considered in [7] to study the representation of classical unitals, and the representation of PG$(2,q^{2m})$ in PG$(4m,q)$, for $m>1$, have been considered to study other geometric objects of the plane; see [31, 32, 38, 40] just to cite some.

A *blocking set* in a projective plane $\pi$ is a set of points such that every line of $\pi$ has a non-empty intersection with the set. A blocking set is said to be *minimal* if through any of its points there is a line of $\pi$ intersecting it precisely in that point.

In the paper [14], Bruen and Thas proved that, when the order of the projective plane is a square, say $n^2$, then the size of a minimal blocking set is bounded by $n^3+1$. This size is reached if and only if the minimal blocking set is a unital.

In [31] the following geometric setting was introduced to construct large minimal blocking sets of PG$(2,q^{2m})$ from cones in its André/Bruck-Bose representation in PG$(4m,q)$. Let $\mathbf{z}$ be a fixed element of a $(2m-1)$-spread $\mathcal{S}$ of $\Sigma_\infty$ and $\mathcal{V}$ an $(m-1)$-dimensional subspace of $\mathbf{z}$. Let $\Gamma$ be a $(3m-1)$-dimensional subspace of $\Sigma_\infty$ disjoint from $\mathcal{V}$. For every $\mathbf{x}\in\mathcal{S}$, $\mathbf{x}\neq\mathbf{z}$, let $I(\mathbf{x})$ be the $(2m-1)$-dimensional subspace $\langle\mathbf{x},\mathcal{V}\rangle\cap\Gamma$. We denote by $\mathcal{I}(\mathcal{V})$ the set of all the subspaces $I(\mathbf{x})$, $\mathbf{x}\in\mathcal{S}$. Let $\Gamma'$ be an affine $3m$-dimensional subspace of PG$(4m,q)$ through $\Gamma$, and denote by $\mathcal{F}(\mathcal{V})$ the set of all affine $2m$-dimensional subspaces of $\Gamma'$ containing an element of $\mathcal{I}(\mathcal{V})$.

Let $\mathcal{F}$ be a family of $2m$-dimensional subspaces of $\Gamma'$. An $\mathcal{F}$-*blocking set* of $\Gamma'$ is a set $\mathcal{B}$ of affine points such that every element of $\mathcal{F}$ has a non-empty intersection with $\mathcal{B}$. The blocking set $\mathcal{B}$ is said to be *minimal* if through any point of $\mathcal{B}$ there is an element in $\mathcal{F}$ intersecting $\mathcal{B}$ precisely in that point.

By keeping the above geometric setting in mind, the following result, which is a sharpening of Corollary 3.3 in [31], is crucial for our succeeding considerations.

**Proposition 2.1.** *Let $\mathcal{B}$ be a set of affine points of $\Gamma'$ and*

$$\mathcal{B}^*=\bigcup_{P\in\mathcal{B}}\langle\mathcal{V},P\rangle\cup\{\mathbf{z}\}. \tag{1}$$

*If $\mathcal{B}$ is a minimal $\mathcal{F}(\mathcal{V})$-blocking set, then $\mathcal{B}^*$ is a minimal blocking set of size $|\mathcal{B}^*|=q^m|\mathcal{B}|+1$ in the translation plane $\Pi(\mathcal{S})$.*

*Proof.* Construction 2 in [31] works perfectly well under the milder hypothesis that $\mathcal{S}$ is any $(2m-1)$-spread of $\Sigma_\infty$. The details are left to the reader. ☐

By combining the above result of Bruen and Thas with Proposition 2.1, we get the following theorem.

**Theorem 2.2.** *Let $\mathcal{B}$ be a minimal $\mathcal{F}(\mathcal{V})$-blocking set of size $q^{2m}$. Then, the cone $\mathcal{B}^*$ defined in Proposition* 2.1 *is a unital in* $\Pi(\mathcal{S})$.

If $\mathcal{S}$ is a Desarguesian $(2m-1)$-spread of $\Sigma_\infty$, then there is a unique Desarguesian $(m-1)$-spread, say $\mathcal{T}$, that fills every element of $\mathcal{S}$, i.e., $\mathcal{T}$ induces a $(m-1)$-spread in each spread element of $\mathcal{S}$ [20]. The following result gives a characterization of Buekenhout-Metz unitals as cones in $\mathrm{PG}(4m, q)$.

**Proposition 2.3.** *[31] Let $\mathcal{S}$ be a Desarguesian $(2m-1)$-spread of $\Sigma_\infty$ and $\mathcal{B}$ a minimal $\mathcal{F}(\mathcal{V})$-blocking set of size $q^{2m}$. Then, the cone $\mathcal{B}^*$ is a Buekenhout-Metz unital in* $\mathrm{PG}(2, q^{2m})$ *if and only if $\mathcal{V}$ is an element of the spread $\mathcal{T}$.*

# 3 Unitals from eggs

An *egg* in $\mathrm{PG}(4m-1, q)$ is a set $\mathcal{E}$ of $q^{2m}+1$ pairwise disjoint $(m-1)-$dimensional subspaces such that any three egg elements span a $(3m-1)-$dimensional subspace. When $m = 1$, this definition recovers indeed the notion of ovoid in $\mathrm{PG}(3, q)$. Therefore, since the notion of an egg, introduced by J.A. Thas in [41], generalizes that of an ovoid, it make sense to investigate whether it is possible to mimic Buekenhout's construction to get unitals in translation planes with dimension over their kernel greater than two, by using eggs. Apart from the so-called *elementary eggs*, which are obtained by applying the field reduction to an ovoid in $\mathrm{PG}(3, q^m)$, there are few other known examples of eggs, namely, the Kantor-Knuth eggs, the Cohen-Ganley eggs and the (sporadic) Penttila-Williams egg; see [27] for an explicit description of these objects.

Let $\mathcal{E}$ be an egg in $\mathrm{PG}(4m-1, q)$. For every egg element $E$ there exists a unique $(3m-1)$-dimensional subspace, denoted by $E^*$, containing $E$ and disjoint from any other egg element; it is called the *tangent space of* $\mathcal{E}$ at $E$. Therefore, the egg $\mathcal{E}$ defines an egg in the dual space of $\mathrm{PG}(4m-1, q)$, called the *dual egg* of $\mathcal{E}$ and denoted by $\mathcal{E}^D$.

The following result is a corollary of Theorem 2.2.

**Theorem 3.1.** *Let $\mathcal{E}$ be an egg in* $\mathrm{PG}(4m-1, q)$, *and $E_\infty$ a fixed egg element. Let $\Gamma'$ be a $3m$-dimensional subsubspace of* $\mathrm{PG}(4m-1, q)$ *containing the tangent space $E^*_\infty$ at $E_\infty$. In $\Gamma'$ we consider the sets:*

$$\mathcal{B}_{\mathcal{E}} = \{E \cap \Gamma' : E \in \mathcal{E}, E \neq E_\infty\}$$

*and*

$$\mathcal{I}_{\mathcal{E}} = \{E^* \cap E^*_\infty : E \in \mathcal{E}, E \neq E_\infty\}.$$

*Let $\mathcal{F}_\mathcal{E}$ be the family of all affine $2m$-dimensional subspaces of $\Gamma'$ containing an element of $\mathcal{I}_\mathcal{E}$, and assume that $\mathcal{B}_\mathcal{E}$ is a minimal $\mathcal{F}_\mathcal{E}$-blocking set.*

*Embed $\Gamma'$ in $\mathrm{PG}(4m,q)$ in such a way that $E^*_\infty$ is a subspace of the hyperplane at infinity $\Sigma_\infty$ of $\mathrm{PG}(4m,q)$, and $\Gamma'$ is an affine subspace.*

*If there exist a $(2m-1)$-spread $\mathcal{S}$ of $\Sigma_\infty$ and a $(m-1)$-dimensional subspace $\mathcal{V}$ disjoint from $E^*_\infty$ and contained in a spread element $\mathbf{z}$ such that $\mathcal{I}_\mathcal{E} = \mathcal{I}(\mathcal{V})$, then the cone*

$$\mathcal{B}^* = \bigcup_{P \in \mathcal{B}_\mathcal{E}} \langle P, \mathcal{V} \rangle \cup \{\mathbf{z}\}$$

*is a unital in $\Pi(\mathcal{S})$.*

*Proof.* Here, $\mathcal{B}_\mathcal{E}$ is a set of $q^{2m}$ points of $\Gamma' \setminus E^*_\infty$, and hence it consists of affine points of $\mathrm{PG}(4m,q)$. Furthermore, every element in $\mathcal{I}_\mathcal{E}$ is a $(2m-1)$-dimensional subspace of $E^*_\infty$. By Theorem 2.2, if $\mathcal{F}_\mathcal{E}$ coincides with the family $\mathcal{F}(\mathcal{V})$ previously defined, then $\mathcal{B}^*$ is a unital in the semifield plane $\Pi(\mathcal{S})$. Since $\mathcal{F}_\mathcal{E}$ consists of all affine $2m$-dimensional subspaces of $\Gamma'$ through an element of $\mathcal{I}_\mathcal{E}$, we get that $\mathcal{F}_\mathcal{E} = \mathcal{F}(\mathcal{V})$ if and only if $\mathcal{I}_\mathcal{E} = \mathcal{I}(\mathcal{V})$. $\square$

An egg is said to be *good at an element* $E$ if every $(3m-1)$-dimensional subspace containing $E$ and at least two other egg elements, contains exactly $q^m+1$ egg elements [42].

Let $\mathcal{K}$ be the quadratic cone in $\mathrm{PG}(3,q^m)$ with equation $X_0X_1 = X_2^2$. A *flock* of $\mathcal{K}$ is a set of $q^m$ planes partitioning the cone minus its vertex $V = \langle(0,0,0,1)\rangle$ into disjoint conics. In accordance with this choice of coordinates, the planes of a flock of $\mathcal{K}$ can be written as $tX_0 + f(t)X_1 + g(t)X_2 + X_3 = 0$, for all $t \in \mathbb{F}_{q^m}$, for some $f,g : \mathbb{F}_{q^m} \to \mathbb{F}_{q^m}$. We denote this flock by $\mathcal{F}(f,g)$. If $f$ and $g$ are linear over a subfield of $\mathbb{F}_{q^m}$, then the flock is called a *semifield* flock. The maximal subfield with this property is called the *kernel* of the flock.

From now on, we assume that the kernel of a semifield flock $\mathcal{F}(f,g)$ is $\mathbb{F}_q$. This implies that the $f$ and $g$ are $\mathbb{F}_q$-linearized polynomials, i.e.

$$f(t) = \sum_{i=0}^{m-1} c_i t^{q^i}, \quad g(t) = \sum_{i=0}^{m-1} b_i t^{q^i},$$

for some $b_i, c_i \in \mathbb{F}_{q^m}$, $i = 0, \ldots, m-1$.

If a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ is fixed, then every $r$-ple $(x_1,\ldots,x_r) \in \mathbb{F}^r_{q^m}$ can be viewed as a $rm$-ple over $\mathbb{F}_q$, which will be denoted by $(x_1,\ldots,x_r)_q$. In the paper [27] it was shown that for every semifield flock $\mathcal{F}(f,g)$ there corresponds an egg in $\mathrm{PG}(4m-1,q)$ whose dual, say $\mathcal{E}$, is good at an element, which can be assumed to be $E_\infty$. Then, the elements and the tangent spaces of $\mathcal{E}$ have the following form, respectively:

$$\begin{aligned}
&E(a,b) = \{(t, -g_{(a,b)}(t), -at, -bt)_q : t \in \mathbb{F}_{q^m}\}, \text{ for all } a, b \in \mathbb{F}_{q^m},\\
&E_\infty = \{(0,t,0,0)_q : t \in \mathbb{F}_{q^m}\},\\
&E^*(a,b) = \{(t, h_{(a,b)}(r,s) + g_{(a,b)}(t), r, s)_q : t,r,s \in \mathbb{F}_{q^m}\}, \text{ for all } a,b \in \mathbb{F}_{q^m},\\
&E^*_\infty = \{(0,t,r,s)_q : t,r,s \in \mathbb{F}_{q^m}\},
\end{aligned} \tag{2}$$

with

$$g_{(a,b)}(t) = a^2 t + \sum_{i=0}^{m-1} (b_i ab + c_i b^2)^{1/q^i} t^{1/q^i},$$

and

$$h_{(a,b)}(r,s) = 2ar + \sum_{i=0}^{m-1} (b_i(as+br) + 2c_i bs)^{1/q^i}.$$

Because of the expression of the polynomials $g(a,b)$ and $h(a,b)$, such an egg will be denoted by $\mathcal{E}(\mathbf{b},\mathbf{c})$.

**Theorem 3.2.** *Let $\mathcal{E} = \mathcal{E}(\mathbf{b},\mathbf{c})$ be a good egg of* $\mathrm{PG}(4m-1,q)$*, which is good at $E_\infty$. Then, the set $\mathcal{B}_\mathcal{E}$ is a minimal $\mathcal{F}_\mathcal{E}$-blocking set in $\Gamma' = \mathrm{PG}(3m,q)$ if and only if $X^2 + \sum_{i=0}^{m-1}(b_i XY + c_i Y^2)^{1/q^i} + c = 0$ has a solution for all $c \in \mathbb{F}_{q^m}$.*

*Proof.* Let $\Gamma' = \{(u,t,r,s)_q : u \in \mathbb{F}_q \text{ and } r,s,t \in \mathbb{F}_{q^m}\}$. It is evident that $\Gamma'$ is a projective space of dimension $3m$ over $\mathbb{F}_q$ and it contains $E^*_\infty$. By taking into account the general form of the elements of $\mathcal{E} = \mathcal{E}(\mathbf{b},\mathbf{c})$, we get

$$\mathcal{B}_\mathcal{E} = \{\langle(1, -g_{(a,b)}(1), -a, -b)_q\rangle : a,b \in \mathbb{F}_{q^m}\}$$

and

$$\mathcal{I}_\mathcal{E} = \{I(a,b) : a,b \in \mathbb{F}_{q^m}\},$$

where

$$I(a,b) = E^*(a,b) \cap E^*_\infty = \{(0, h_{(a,b)}(r,s), r, s)_q : r,s \in \mathbb{F}_{q^m}\}. \tag{3}$$

All the affine $2m$-dimensional subspaces of $\Gamma'$ through an $I(a,b)$ are determined by joining it with an affine point of the affine $m$-dimensional subspace spanned by $E_\infty$ and $O = \langle(1,0,0,0,0)\rangle$. Therefore, the elements of $\mathcal{F}_\mathcal{E}$ have the form

$$F(a,b,c) = \{(u, uc + h_{(a,b)}(r,s), r, s)_q : u \in \mathbb{F}_q \text{ and } r,s \in \mathbb{F}_{q^m}\},$$

for all $a,b,c \in \mathbb{F}_{q^m}$.

A point $P(x,y) = \langle(1, -g_{(x,y)}(1), -x, -y)_q\rangle \in \mathcal{B}_\mathcal{E}$ lies in $F(a,b,c)$ if and only if

$$-g_{(x,y)}(1) = -h_{(a,b)}(x,y) + c$$

or, equivalently, if and only if $(x, y)$ is a solution of

$$X^2 + \sum_{i=0}^{m-1} (b_i XY + c_i Y^2)^{1/q^i} - 2aX - \sum_{i=0}^{m-1} (b_i(aY + bX) + 2c_i bY)^{1/q^i} + c = 0. \quad (4)$$

We refer to the polynomial on the left-hand side of the equation as $H_{(a,b,c)}(x, y)$. Since $\mathcal{E}$ is an egg, for any given $a, b \in \mathbb{F}_{q^m}$, the intersection of the tangent space $E^*(a, b)$ with $\Gamma'$ is the $2m$-dimensional subspace $F(a, b, c') \in \mathcal{F}_\mathcal{E}$, with $c' = g_{(a,b)}(1)$. Therefore, through the point $P(a, b) \in \mathcal{B}_\mathcal{E}$ there is the element $F(a, b, c') \in \mathcal{F}_\mathcal{E}$ intersecting $\mathcal{B}_\mathcal{E}$ precisely at $P(a, b)$. This implies that $\mathcal{B}_\mathcal{E}$ is a minimal $\mathcal{F}_\mathcal{E}$-blocking set if and only if Eq. (4) has a solution $(x, y) \in \mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$ for any given elements $a, b, c \in \mathbb{F}_{q^m}$.

From [26, Lemma 1.4], for any $a, b \in \mathbb{F}_{q^m}$, the linear collineation

$$\begin{array}{llll} \psi_{a,b}: & \mathrm{PG}(4m-1, q) & \longrightarrow & \mathrm{PG}(4m-1, q) \\ & \langle (u, t, r, s)_q \rangle & \mapsto & \langle (u, t + h_{(a,b)}(r, s) - g_{(a,b)}(u), r - ua, s - ub)_q \rangle \end{array}$$

fixes $E_\infty$ pointwise and maps $E(a', b')$ to $E(a'+a, b'+b)$. In addition, $\psi_{a,b}$ fixes $\Gamma'$, and hence $\mathcal{B}_\mathcal{E}$. A straightforward, though tedious, calculation shows that $\psi_{a,b}$ acts also on the set of tangent spaces by fixing $E^*_\infty$ setwise and mapping $E^*(a', b')$ to $E^*(a+a', b+b')$. This implies that $\psi_{a,b}$ fixes both $\mathcal{I}_\mathcal{E}$ and $\mathcal{F}_\mathcal{E}$ setwise; in particular, $F(a', b', c)$ is mapped to $F(a + a', b + b', c')$, with $c' = c - g_{a,b}(1) + h_{(a'+a,b'+b)}(a, b)$. This means that, because of the linearity of the second sum in Eq. (4), $H_{(a'+a,b'+b,c)}(x, y) = 0$ has a solution for all $c \in \mathbb{F}_{q^m}$ if and only if $H_{(a,b,c)}(x, y) = 0$ has a solution for all $c \in \mathbb{F}_{q^m}$. Therefore, $\mathcal{B}_\mathcal{E}$ is a minimal $\mathcal{F}_\mathcal{E}$-blocking set if and only if, for a fixed pair $(a, b) \in \mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$, Eq. (4) has at least one solution $(x, y) \in \mathbb{F}_{q^m} \times \mathbb{F}_{q^m}$, for all $c \in \mathbb{F}_{q^m}$. In particular, we can chose $(a, b) = (0, 0)$ so that Eq. (4) reduces to

$$X^2 + \sum_{i=0}^{m-1} (b_i XY + c_i Y^2)^{1/q^i} + c = 0. \quad (5)$$

□

# 4 A new unital in a Dickson commutative semifield plane

In [35], Penttila and Williams constructed an ovoid of the parabolic quadric $Q(4, 3^5)$ in $\mathrm{PG}(4, 3^5)$, i.e., a set $\mathcal{O}$ of $3^{10} + 1$ points having exactly one point on each generator of the quadric. Moreover, $\mathcal{O}$ is a translation ovoid, meaning that the points of $\mathcal{O}$ can be coordinatized by using functions that are additive over $\mathbb{F}_3$. According to a construction given in [28], such a translation ovoid corresponds to a semifield flock of the quadratic cone in $\mathrm{PG}(3,3^5)$, which, in turn, corresponds to a generalized quadrangle with parameters $(3^{10},3^5)$, whose point-line dual is a translation generalized quadrangle. By a result of Payne and Thas [34, 8.7.1], the latter generalized quadrangle is isomorphic to $T(\mathcal{E})$ for some egg $\mathcal{E}$ in $\mathrm{PG}(19,3)$. By Theorem 3.4 in [27], the dual egg of $\mathcal{E}$ forms a good egg $\mathcal{E}^D$ in $\mathrm{PG}(19,3)$. Whence, via the above correspondences, the Penttila-Williams ovoid of $Q(4,3^5)$ gives rise to a good (dual) egg in $\mathrm{PG}(19,3)$. In order to simplify the notation, we will refer to it as $\mathcal{E}=\mathcal{E}(\mathbf{b},\mathbf{c})$ with $\mathbf{b}=(0,1,0,0,0)$, $\mathbf{c}=(0,0,0,-1,0)$; see [27].

According to the expressions of the polynomials $g_{(a,b)}(t)$ and $h_{(a,b)}(r,s)$ in this case, the egg elements of $\mathcal{E}$ are defined by the polynomials

$$g_{(a,b)}(t)=a^2t-(b^2)^{3^2}t^{3^2}+(ab)^{3^4}t^{3^4} \tag{6}$$

and

$$h_{(a,b)}(r,s)=-ar+b^{3^2}s^{3^2}+(br+as)^{3^4}, \tag{7}$$

for all $a,b\in\mathbb{F}_{3^5}$.

Let $p$ be an odd prime and $\xi$ a non-square in $\mathbb{F}_{p^m}$. By [17, p.241], the multiplication defined by

$$(x,y)*(a,b)=(ax+\xi b^\alpha y^\alpha, bx+ay)$$

with $\alpha\in\mathrm{Aut}(\mathbb{F}_{p^m})$ not the identity, turns $\mathbb{F}_{p^m}^2$ into a Dickson commutative semifield of order $p^{2m}$ which we denote by $\mathbb{D}=\mathbb{D}(p^m,\xi,\alpha)$. In particular, its middle nucleus is $\mathcal{N}_m=\{(a,0):a\in\mathbb{F}_{p^m}\}$, and its left nucleus is $\mathcal{N}_l=\{(a,0):a\in\mathrm{Fix}(\alpha)\}$, coinciding with its center $\mathcal{K}$.

Now, let $p=3$ and $m=5$. For any pair $(a,b)\in\mathbb{F}_{3^5}^2$, we consider the following map

$$\tau_{(a,b)}:\quad (x,y)\quad\mapsto\quad (bx+ay,-ax+b^{3^2}y^{3^2}),$$

which defines the subspaces $S(a,b)=\{((x,y)^{\tau_{(a,b)}},x,y)_3:x,y\in\mathbb{F}_{3^5}\}$ of $\mathrm{PG}(19,3)$. Set $\mathcal{S}=\{S(a,b):a,b\in\mathbb{F}_{3^5}\}\cup\{S_\infty\}$, where $S_\infty=\{(x,y,0,0)_3:x,y\in\mathbb{F}_{3^5}\}$.

Let $\varphi$ be the linear map $\varphi:(x,y)\mapsto(-y,x)$. Then, the set $\{\varphi\tau_{(a,b)}:a,b\in\mathbb{F}_{3^5}\}$ is precisely the spread set of $\mathbb{F}_3^{10}$ associated with the Dickson commutative semifield $\mathbb{D}=\mathbb{D}(3^5,-1,3^2)$.

It turns out that $\mathcal{S}$ is a 9-spread of $\Sigma_\infty=\mathrm{PG}(19,3)$ and, by [2], the translation plane $\Pi(\mathcal{S})$ is isomorphic to the Dickson commutative semifield plane $\pi(\mathbb{D})$.

Let $\mathcal{V}=\{(t,-t^{3^4},0,0)_3:t\in\mathbb{F}_{3^5}\}$. Then, $\mathcal{V}$ is contained in the spread element $\mathbf{z}=S_\infty$ and it intersects trivially the subspace $\Gamma=E_\infty^*=\{(0,t,r,s)_3:t,r,s\in\mathbb{F}_{3^5}\}$. We also have

$$\langle S(a,b),\mathcal{V}\rangle=\{(br+as+t,-ar+b^{3^2}s^{3^2}-t^{3^4},r,s)_3:t,r,s\in\mathbb{F}_{3^5}\},$$

giving

$$\langle S(a,b),\mathcal{V}\rangle\cap\Gamma=\{(0,-ar+b^{3^2}s^{3^2}+(br+as)^{3^4},r,s)_3:r,s\in\mathbb{F}_{3^5}\}$$

which is precisely the subspace $I(a,b)$ defined by expression (3), with $h_{(a,b)}(r,s)$ as in (7).

**Proposition 4.1.** *The set $\mathcal{B}_{\mathcal{E}}$ defined by the Penttila-Williams egg $\mathcal{E} = \mathcal{E}(\mathbf{b}, \mathbf{c})$ is a minimal $\mathcal{F}(\mathcal{E})$-blocking set.*

*Proof.* By taking into account Theorem 3.2, $\mathcal{B}_{\mathcal{E}}$ is a minimal $\mathcal{F}_{\mathcal{E}}$-blocking set if and only if

$$X^2 + (XY)^{3^4} - (Y^2)^{3^2} = -c \tag{8}$$

has a solution for all $c \in \mathbb{F}_{q^m}$.

We distinguish two cases: $-c$ is a square in $\mathbb{F}_{q^m}$ or not. If $-c$ is a square, then $(\pm\sqrt{-c}, 0)$ are solutions of Eq. (8); if $-c$ is not a square, then $(0, \pm\sqrt{c^{3^3}})$ are solutions of Eq. (8). □

By Theorem 3.1, the cone

$$\mathcal{B}^*_{\mathcal{E}} = \{\langle(1, c, -g_{(a,b)}(1) - c^{3^4}, -a, -b)_3\rangle : a, b, c \in \mathbb{F}_{3^5}\} \cup \{S_\infty\},$$

with $g_{(a,b)}(t)$ as in (6), is a unital in the translation plane $\Pi(\mathcal{S})$.

Consider the collineation of $\mathrm{PG}(20,3)$ defined as $\overline{\varphi} : \langle(u, v, t, r, s)_3\rangle \mapsto \langle(u, -t, v, r, s)_3\rangle$. Then, $\Pi(\mathcal{S})^{\overline{\varphi}}$ represents the Dickson commutative semifield plane $\pi(\mathbb{D})$. It turns out that the set

$$\mathcal{U} = \{(g_{(a,b)}(1) + c^{3^4}, c, -a, -b) : a, b, c \in \mathbb{F}_{3^5}\} \cup \{(\infty)\}$$

is a unital in $\pi(\mathbb{D})$. Note that $\mathcal{U}$ cannot be a Buekenhout-Metz unital since $\pi(\mathbb{D})$ is a 10-dimensional translation plane over its kernel $\mathbb{F}_3$. On the other hand, as $\pi(\mathbb{D})$ admits unitary polarities [21], $\mathcal{U}$ might be a polar unital. The following result shows that this is not the case.

**Theorem 4.2.** *The unital $\mathcal{U}$ is not a polar unital in $\pi(\mathbb{D})$.*

*Proof.* Since the tangent space at the egg element $E(0,0)$ is $E^*(0,0)$, the tangent line of $\Pi(\mathcal{S})$ at the point $O = \langle(1,0,0,0,0)_3\rangle \in \mathcal{B}^*_{\mathcal{E}}$ is the subspace spanned by $S(0,0)$ and $O$. Then, the tangent line of $\pi(\mathbb{D})$ at $(0,0) \in \mathcal{U}$ is $[0,0]$.

From [24, Theorem 2.1], any unitary polarity of $\pi(\mathbb{D})$ mapping $(0,0)$ to $[0,0]$ is given by

$$\begin{array}{rccc} \rho_a : & (x_1, x_2, y_1, y_2) & \leftrightarrow & [ax_1, -ax_2, -y_1, y_2], \\ & (m_1, m_2) & \leftrightarrow & (a^{-1}m_1, -a^{-1}m_2) \\ & (\infty) & \leftrightarrow & [\infty]. \end{array}$$

for some non-zero $a \in \mathbb{F}_{3^5}$.

The unital $\mathcal{U}$ is a polar unital with respect to $\rho_a$, for some $a \in \mathbb{F}_{3^5}$, if and only if each of its points is an absolute point. Straightforward calculations show that the point $(1,1,0,0) \in \mathcal{U}$ is not incident with $\rho_a(1,1,0,0) = [a, -a, 0, 0]$ for all non-zero $a \in \mathbb{F}_{3^5}$, showing that $\mathcal{U}$ is not a polar unital. □

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