The proportion of non-degenerate complementary subspaces in classical spaces

S.P. Glasby¹, Ferdinand Ihringer² and Sam Mattheus^{3,4}

¹Center for the Mathematics and Symmetry and Computation, University of Western Australia, Perth, 6009, Australia.

²Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium.

³Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium.

⁴Department of Mathematics, University of California San Diego, United States.

> Contributing authors: Stephen.Glasby@uwa.edu.au; Ferdinand.Ihringer@ugent.be; SMattheus@ucsd.edu;

Abstract

Given positive integers e_1, e_2 , let X_i denote the set of e_i -dimensional subspaces of a fixed finite vector space $V = (\mathbb{F}_q)^{e_1+e_2}$. Let Y_i be a non-empty subset of X_i and let $\alpha_i = |Y_i|/|X_i|$. We give a positive lower bound, depending only on $\alpha_1, \alpha_2, e_1, e_2, q$, for the proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ which intersect trivially. As an application, we bound the proportion of pairs of non-degenerate subspaces of complementary dimensions in a finite classical space that intersect trivially. This problem is motivated by an algorithm for recognizing classical groups. By using techniques from algebraic graph theory, we are able to handle orthogonal groups over the field of order 2, a case which had eluded Niemeyer, Praeger, and the first author.

 ${\bf Keywords:} \ {\rm expander} \ {\rm mixing} \ {\rm lemma}, \ {\rm finite} \ {\rm classical} \ {\rm group}, \ {\rm opposition} \ {\rm graph}$

1 Introduction

In this paper we use techniques from algebraic graph theory to solve a problem that arose from computational group theory. More precisely, we use the expander mixing lemma for bipartite graphs to establish bounds that are useful for algorithms which 'recognise' classical groups acting on their natural module, a central and difficult computational problem. The nature of this recognition problem is sketched in Section 1.2.

Let \mathbb{F} be a finite field, let e_1, e_2 be positive integers and let $V = \mathbb{F}^{e_1+e_2}$ be an $(e_1 + e_2)$ -dimensional \mathbb{F} -space endowed with a non-degenerate quadratic, symplectic or hermitian form. We bound the probability that a non-degenerate e_1 -subspace S_1 of V, and a non-degenerate e_2 -subspace S_2 of V, intersect trivially that is, satisfy $S_1 \cap S_2 = \{0\}$. Except for orthogonal spaces with q = 2, this problem was solved in [10, Theorem 1.1], using a combinatorial doublecounting argument [10, §3]. The following theorem gives sharper bounds, without exception, and is proved via relatively straightforward calculations involving the second largest eigenvalue of a graph, see Section 1.1.

Theorem 1.1 Let $V = \mathbb{F}^{e_1+e_2}$ be a non-degenerate orthogonal, symplectic or hermitian space where \mathbb{F}, e_1, e_2 are given in Table 1. Let Y_i be the set of all non-degenerate e_i -subspaces of V (of a fixed type $\sigma_i \in \{-,+\}$ in the orthogonal case). Then the proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is at least $1 - \frac{c}{|\mathbb{F}|}$ where c is given in Table 1. We may take $c = \frac{3}{2}$ if $(e_1, e_2, q) \neq (1, 1, 2)$.

form on $V = \mathbb{F}^{e_1 + e_2}$ \mathbb{F} e_i -subspaces Y_i e_1 e_2 c \mathbb{F}_q orthogonal of non-degenerate of $3/_{2}$ even even type $\varepsilon \in \{-,+\}$ type $\sigma_i \in \{-,+\}$ 10/7 < 3/2 \mathbb{F}_q non-degenerate even even symplectic \mathbb{F}_{q^2} ≥ 1 ≥ 1 hermitian non-degenerate 3/2 $(e_1, e_2, q) \neq (1, 1, 2)$

Table 1 Choices for \mathbb{F} , e_1 , e_2 , form on V, Y_i and c in Theorem 1.1

The non-degenerate subspaces of a symplectic space have even dimension so that e_1 and e_2 are both even in the second line of Table 1. In the case that V is a non-degenerate orthogonal space the e_i are also both even; however this is for a different reason. The authors of [11], in forthcoming work, describe an algorithm for recognising classical groups, and the papers [10, 11] provide the necessary background. In the algorithmic application the subspace S_i is the image of $g_i - 1$ for some element g_i of the orthogonal group on V, and S_i is non-degenerate of minus type by [11, Lemma 3.8(b)]. Hence the e_i are even in the first line of Table 1, as claimed. In the unitary case, we have c = 2 when $(e_1, e_2, q) = (1, 1, 2)$ and c = 1.26when $e_1, e_2 \ge 2$, see Theorem 5.2. The bounds listed in Table 1 all satisfy $1 - \frac{3}{2|\mathbb{F}|} \ge \frac{1}{4}$, and they facilitate a uniform analysis, for all fields, of a randomized algorithm for recognising classical groups. In contrast, the values of c in [10, Table 1] are 2.69 (for $q \ge 3$), 1.67 and 1.8 in the orthogonal, symplectic and unitary cases, respectively. Further, our methods are somewhat stronger and easier to apply than those in [10], and offer hope for extensions, see Section 6.

1.1 *q*-Kneser graphs

Let $V = (\mathbb{F}_q)^d$ be a *d*-dimensional vector space over the field with q elements. Let e_1, e_2 be positive integers. For i = 1, 2, denote by X_i the set of e_i -dimensional subspaces of V. We refer to an element $S_i \in X_i$ as an e_i -subspace or an e_i -space. Let Γ_{d,e_1,e_2} be the bipartite graph whose vertex set is the disjoint union $X_1 \cup X_2$ (where we take two disjoint copies of the set of e_1 -spaces if $e_1 = e_2$), and where two vertices $(S_1, S_2) \in X_1 \times X_2$ are adjacent whenever S_1 and S_2 intersect trivially. The condition $S_1 \cap S_2 = \{0\}$ is equivalently to dim $(S_1 + S_2) = e_1 + e_2$.

The q-Kneser graph qK(d, e) has been previously studied, for example, see [2, 5]. The vertices of qK(d, e) comprise e-subspaces of $V = (\mathbb{F}_q)^d$ and $\{S_1, S_2\}$ is an edge if $S_1 \cap S_2 = \{0\}$. If qK(d, e) has adjacency matrix A, then the bipartite double of qK(d, e) has adjacency matrix $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ and is isomorphic to $\Gamma_{d,e,e}$. The spectrum of qK(d, e) (i.e, the set of eigenvalues of A) is known, and hence too for its bipartite double $\Gamma_{d,e,e}$, and can be obtained from Delsarte [7, Theorem 10] or Eisfeld [8, Theorem 2.7]. The spectrum of Γ_{d,e_1,e_2} when $e_1 \neq e_2$ is more complicated. Brouwer [3] gives the spectrum of $\Gamma_{3,1,2}$; also Suda and Tanaka [16] study "cross-independent" sets in Γ_{d,e_1,e_2} with $d \ge 2e_1, 2e_2$. However, for our applications we want $d = e_1 + e_2$, as this is the key case for [10] which underpins [11].

We henceforth assume that $d = e_1 + e_2$, and write $\Gamma_{e_1,e_2} = \Gamma_{d,e_1,e_2}$. Since $\Gamma_{e_1,e_2} \cong \Gamma_{e_2,e_1}$, we shall assume additionally, without loss of generality, that $e_1 \ge e_2$.

For each e_1 -subspace S_1 of $(\mathbb{F}_q)^{e_1+e_2}$, there are $q^{e_1e_2}$ choices for an e_2 subspace S_2 with $S_1 \cap S_2 = \{0\}$. Similarly, for each e_2 -subspace S_2 there are $q^{e_2e_1}$ choices for an e_1 -subspace S_1 with $S_1 \cap S_2 = \{0\}$. Hence the graph Γ_{e_1,e_2} is $q^{e_1e_2}$ -regular. The following result is proved in Section 2, it determines the distinct eigenvalues of Γ_{e_1,e_2} , but not their multiplicities.

Proposition 1.2 Suppose that $e_1 \ge e_2 \ge 1$ and $d = e_1 + e_2$. The distinct eigenvalues of the bipartite graph Γ_{e_1,e_2} are $\lambda_0 > \cdots > \lambda_{e_2} > -\lambda_{e_2} > \cdots > -\lambda_0$ where $\lambda_j = q^{m_j}$ for $0 \le j \le e_2$ and $m_j = e_1e_2 - \frac{j(d+1-j)}{2}$.

The number $\begin{bmatrix} a \\ b \end{bmatrix}_a$ of b-subspaces of the a-dimensional vector space $(\mathbb{F}_q)^a$ is

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=0}^{b-1} \frac{q^{a-i}-1}{q^{b-i}-1} = \prod_{i=0}^{b-1} \frac{q^{a-i-1}+\dots+q+1}{q^{b-i-1}+\dots+q+1} = \prod_{i=1}^b \frac{q^{a-i+1}-1}{q^i-1}.$$

The second middle product shows that $\lim_{q\to 1} {a \brack b}_q = \prod_{i=0}^{b-1} \frac{a-i}{b-i} = {a \choose b}$, and ${a \brack b}_q \sim q^{b(a-b)}$ as $q \to \infty$. The next result is proved in Section 3 using Proposition 1.2, and the expander mixing lemma for regular bipartite graphs, see Lemma 3.1.

Proposition 1.3 Suppose that $e_1 \ge e_2 \ge 1$ and $d = e_1 + e_2$. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ be non-empty. Put $\alpha_i = |Y_i|/|X_i|$ for $i \in \{1, 2\}$. Then $\alpha_1 \alpha_2 > 0$ and

$$\frac{|\{(S_1, S_2) \in Y_1 \times Y_2 : S_1 \cap S_2 = 0\}|}{|Y_1| \cdot |Y_2|} \ge \frac{q^{e_1 e_2}}{\begin{bmatrix} d \\ e_1 \end{bmatrix}_q} \left(1 - \sqrt{(\frac{1}{\alpha_1} - 1)(\frac{1}{\alpha_2} - 1)q^{-\frac{d}{2}}}\right).$$

Suppose that $\min\{\alpha_1, \alpha_2\} \ge \alpha > 0$ and $\omega_q(e) = \prod_{i=1}^e (1 - q^{-i})$. Then

$$\frac{|\{(S_1, S_2) \in Y_1 \times Y_2 : S_1 \cap S_2 = 0\}|}{|Y_1| \cdot |Y_2|} > \omega_q(e_2) \left(1 - \left(\frac{1}{\alpha} - 1\right)q^{-\frac{d}{2}}\right)$$

1.2 Recognising classical groups and outline of the paper

A group G satisfying $\operatorname{SL}_d(q) \leq G \leq \operatorname{GL}_d(q)$ is generated by a set \mathcal{X} of elementary matrices, corresponding to elementary row operations. Furthermore, given an element $g \in G$ there is an efficient algorithm (e.g. Gaussian elimination) which writes g as a word in \mathcal{X} . However, in computational problems Gmay be generated by a set \mathcal{Y} of arbitrary-looking matrices, and 'recognising' G involves writing each element of \mathcal{X} as a word in \mathcal{Y} . A particularly helpful special case is when G is generated by a set $\mathcal{Y}' = \{g_1, g_2\}$ of two matrices with $S_1 = \operatorname{im}(g_1 - 1)$ and $S_2 = \operatorname{im}(g_2 - 1)$ non-degenerate complementary subspaces, and a key problem is to write each element of \mathcal{X} as a word in \mathcal{Y}' . This problem is practically difficult, as is the analogous problem for classical groups, and its solution uses random selections in G and the natural G-module $V = (\mathbb{F}_q)^d$. The authors of [10, 11] describe in forthcoming work an algorithm to solve this word problem, and the translation from a problem in group theory to a geometric problem is described, in part, in [10, 11]. Further context and details are given in [11, Section 3].

In Section 2, we determine the distinct eigenvalues of Γ_{e_1,e_2} by proving Proposition 1.2. The proof relies on an explicit algorithm in [6] based on the seminal work of Brouwer [3]. In Section 3, the role of the second largest eigenvalue λ_2 of Γ_{e_1,e_2} is elucidated in the expander mixing lemma for bipartite graphs: we give a short proof in Lemma 3.1. In addition, we prove Proposition 1.3 which shows that bounds (lower and upper) can be determined simply by computing two ratios α_1 and α_2 . Bounds for the orthogonal case are computed in Section 4, for the symplectic and unitary cases in Section 5, and computing α_1, α_2 is key. The orthogonal case is hardest because of the types of the (even dimensional) non-degenerate subspaces S_1 and S_2 . Finally, Section 6 discusses the general case $d > e_1 + e_2$.

2 Eigenvalues

The graph Γ_{e_1,e_2} can be described in the spherical building of type A_{d-1} , corresponding to the classical group $PSL_d(q)$. Adjacency in Γ_{e_1,e_2} corresponds to opposition in A_{d-1} (that is, an e_1 -space and an e_2 -space in A_{d-1} are opposite in a building-theoretical sense precisely when they are complementary, see [3] and [6, Lemma 3.7]). The Coxeter diagram for A_{d-1} is shown in Fig. 1. Brouwer observed in [3, Theorem 1.1] that for any opposition graph of a spherical building over \mathbb{F}_q , its eigenvalues are powers of q. Implicitly, [3] describes an algorithm to calculate the eigenvalues of graphs such as Γ_{e_1,e_2} . This algorithm is explicitly stated in [6, Algorithms 1, 2], which we sometimes refer to simply as Algorithms 1, 2.



Fig. 1 The Coxeter diagram of A_{d-1} where $d = e_1 + e_2$ and $e_1 \ge e_2$.

The key observation in [3] is that we can calculate the eigenvalues of the oppositeness relation from the irreducible characters of the Coxeter group associated with the building. In the case of A_{d-1} this means that we can calculate the eigenvalues of any opposition graph in A_{d-1} such as Γ_{e_1,e_2} from the irreducible characters of the symmetric group Sym(d).

The symmetric group is viewed in this setting as a Coxeter group with the set of adjacent transpositions $\{s_1, \ldots, s_{d-1}\}$ as its set of generators S, where $s_i = (i, i+1)$ for $i \in \{1, \ldots, d-1\}$.

The unique longest word in Sym(d) with respect to the Coxeter generators is denoted w_0 . Its length is $\binom{d}{2}$ and

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & d \\ d & d - 1 & \cdots & 1 \end{pmatrix} = \prod_{i=1}^{d-1} s_{d-1} \cdots s_{i+1} s_i.$$

For instance, $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ when d = 4.

We follow [6, Algorithms 1, 2] to calculate the eigenvalues of Γ_{e_1,e_2} up to sign. This suffices since the graph Γ_{e_1,e_2} is bipartite, and λ is an eigenvalue of Γ_{e_1,e_2} if and only if $-\lambda$ is an eigenvalue.

We apply Algorithm 2 to the building of type A_{d-1} with Coxeter group W = Sym(d). An e_1 -space is a partial flag of type $\{e_1\}$, its cotype is $J = \{1, \ldots, d-1\} \setminus \{e_1\}$. Set $W_J = \langle s_i : i \in J \rangle$. Then $W_J \cong \text{Sym}(e_1) \times \text{Sym}(e_2)$ as $e_2 = d - e_1$.

A partition μ of d, denoted $\mu \vdash d$, is a sequence $[\mu_1, \ldots, \mu_k]$ of positive integers with $\mu_1 \ge \cdots \ge \mu_k > 0$ and $\sum_{i=1}^k \mu_i = d$. The irreducible complex characters of Sym(d) have the form χ_{μ} for a unique $\mu \vdash d$. The parts of the conjugate partition μ^* of μ satisfy $\mu_i^* = |\{j \mid \mu_i \ge i\}|$. We define two invariants $a(\mu)$ and $a^*(\mu)$ as

$$a(\mu) = \sum_{i=1}^{k} (i-1)\mu_i$$
, and $a^*(\mu) = \sum_{i=1}^{k} \frac{\mu_i(\mu_i - 1)}{2} = \sum_{i=1}^{k} {\binom{\mu_i}{2}},$ (1)

and note that $a^*(\mu) = a(\mu^*)$, see [9, §§5.4.1, 5.4.2] and c.f. [6, Proposition 3.3].

Proposition 2.1 ([9, Proposition 5.4.11]) Let $d \ge 1$. Let χ_{μ} denote a character of Sym(d) corresponding to the partition μ of d. Then

$$\binom{d}{2}\frac{\chi_{\mu}(r)}{\chi_{\mu}(1)} = a^{*}(\mu) - a(\mu), \quad where \ r \in \operatorname{Sym}(d) \ is \ a \ transposition.$$

Following [6, Algorithm 1], we denote by R a set of representatives of the conjugacy classes containing the generators in S. Then observe that Rcomprises one transposition r since the conjugacy class s_i^W comprises all transpositions in W = Sym(d) for any $i \in \{1, \dots, d-1\}$, so that $|r^W| = \binom{d}{2}$. Furthermore, the structure constant q_s in Algorithm 1 equals q by the comment after [6, Proposition 2.4]. In summary, the output of Algorithm 1 is the eigenvalue λ_{μ} , where $\lambda_{\mu}^2 = q^{e_{\mu}}$ and the value of $e_{\mu} = {d \choose 2} \left(1 + \frac{\chi_{\mu}(s)}{\chi_{\mu}(1)}\right)$ is independent of the choice of $s \in S$.

Algorithm 2 applied to W = Sym(d) can be described as follows. It is convenient to compute the eigenvalue λ_{μ} of χ_{μ} up to sign, as remarked above.

- 1. Decompose the induced character $\operatorname{ind}_{W_I}^W(1_{W_J})$ as a sum $\sum \chi_{\mu}$ of irreducible characters of W, and determine the relevant partitions μ of d.
- 2. For each μ appearing in Step 1, calculate using Proposition 2.1 the exponent $e_{\mu} = {d \choose 2} \left(1 + \frac{\chi_{\mu}(r)}{\chi_{\mu}(1)}\right)$ where r is a transposition.
- 3. Calculate the length $\ell = \binom{e_1}{2} + \binom{e_2}{2}$ of the longest word in W_J , see below.

4. The eigenvalues of Γ_{e_1,e_2} are now $\pm q^{e_{\mu}/2-\ell}$ with μ determined in Step 1.

This description concurs with that of Algorithm 2 in [6], except that in Step 2, for the output of Algorithm 1 we use buildings of type A_{d-1} and Proposition 2.1.

Proof of Proposition 1.2 Step 1 of Algorithm 2 determines, via Frobenius reciprocity, the irreducible characters of W = Sym(d) that do not vanish when restricted to W_J . Precisely, we apply Pieri's rule [9, Corollary 6.1.7] to find the decomposition $\operatorname{ind}_{W_J}^W(1_{W_J}) = \sum_{j=0}^{e_1} \chi_{[d-j,j]}$. This completes Step 1 of Algorithm 2. For Step 2 of Algorithm 2, we apply Proposition 2.1 to each character $\chi_{[d-j,j]}$

of Sym(d). Write $\mu = [d - j, j]$ where $d - j \ge e_1 \ge e_2 \ge j$. (When j = 0, we identify

 $\mu_2 = [d, 0]$ with $\mu_2 = [d]$.) The functions $a(\mu)$ and $a^*(\mu)$ in (1) are:

$$a(\mu) = j$$
 and $a^*(\mu) = \begin{pmatrix} d-j\\ 2 \end{pmatrix} + \begin{pmatrix} j\\ 2 \end{pmatrix}$ for $0 \le j \le e_2$.

Hence, by Proposition 2.1,

$$e_{\mu} = \binom{d}{2} + \binom{d-j}{2} + \binom{j}{2} - j = d^2 - d + j^2 - jd - j.$$

This completes Step 2 of Algorithm 2.

In Step 3, the length of the longest word ℓ in $W_J = \text{Sym}(e_1) \times \text{Sym}(e_2)$ is $\binom{e_1}{2} + \binom{e_2}{2}$. Thus $\ell = \frac{d^2 - d - 2e_1 e_2}{2}$ and, by Step 4, the eigenvalue corresponding to χ_{μ} is $\pm q^{m_j}$ where

$$m_j = \frac{e_{\mu}}{2} - \ell = \frac{j^2 - jd - j + 2e_1e_2}{2} = e_1e_2 - \frac{j(d+1-j)}{2}.$$

3 The Density Bound

We state a version of the expander mixing lemma for regular, bipartite graphs. This is stated in [12, Theorem 5.1] using the language of block designs. Its proof is short, so we include it here. Given a graph Γ , we write $E(Y_1, Y_2)$ for the number of edges between subsets Y_1 and Y_2 of the set of vertices of Γ .

Lemma 3.1 Let Γ be a k-regular bipartite graph with vertex set $X_1 \cup X_2$. Let $Y_i \subseteq X_i$ with $\alpha_i := |Y_i|/|X_i|$ for $i \in \{1, 2\}$. Suppose that the eigenvalues of the adjacency matrix of Γ are $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{|X_1|+|X_2|}$. Then

$$\left|\frac{E(Y_1, Y_2)}{E(X_1, X_2)} - \alpha_1 \alpha_2\right| \leqslant \frac{\lambda_2}{k} \sqrt{\alpha_1 \alpha_2 (1 - \alpha_1)(1 - \alpha_2)}.$$

Since the graph Γ has $|X_1|k = k|X_2|$ edges by k-regularity, it follows that $|X_1| = |X_2|$. Our proof follows [12] and uses *interlacing*: Recall that the *quotient matrix* $B = (b_{ij})$ of a partition $P_1 \cup \cdots \cup P_m$ of the vertex set of Γ into m parts is an $(m \times m)$ -matrix where $b_{ij} = E(P_i, P_j)/|P_i|$. The eigenvalues of B *interlace* those of the adjacency matrix A of Γ by [12, Theorem 2.1], that is if the spectrum of A is $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and the spectrum of B is $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m$, then $\lambda_i \ge \mu_i \ge \lambda_{n-m+i}$ holds for all $i \in \{1, \ldots, m\}$.

Proof of Lemma 3.1. If $\alpha_1 = 0$, then Y_1 is empty. Hence $E(Y_1, Y_2) = 0$ and the bound holds with equality. If $\alpha_1 = 1$, then $Y_1 = X_1$ and by k-regularity

$$\frac{E(Y_1, Y_2)}{E(X_1, X_2)} = \frac{k|Y_2|}{k|X_2|} = \alpha_2,$$

and again the bound holds with equality. Similar arguments show that the bound holds for $\alpha_2 \in \{0, 1\}$. Henceforth assume that $0 < \alpha_1, \alpha_2 < 1$.

Write $D := E(X_1, X_2)$ and $E := E(Y_1, Y_2)$. Let $B = (b_{i,j})$ be the quotient matrix relative to the partition $Y_1 \stackrel{.}{\cup} (X_1 \setminus Y_1) \stackrel{.}{\cup} Y_2 \stackrel{.}{\cup} (X_2 \setminus Y_2)$ of the vertex set of Γ . Then

$$b_{1,3} + b_{1,4} = \frac{E(Y_1, X_2)}{|Y_1|} = k, \ b_{1,3} = \frac{E(Y_1, Y_2)}{|Y_1|} = \frac{E}{\alpha_1 |X_1|}, \text{ and similarly}$$

$$b_{2,3} + b_{2,4} = \frac{E(X_1 \setminus Y_1, X_2)}{|X_1 \setminus Y_1|} = k,$$

$$b_{2,3} = \frac{E(X_1 \setminus Y_1, Y_2)}{|X_1 \setminus Y_1|} = \frac{E(X_1, Y_2) - E}{(1 - \alpha_1) |X_1|} = \frac{\alpha_2 D - E}{(1 - \alpha_1) |X_1|}.$$

Thus

$$B = \begin{pmatrix} 0 & 0 & \frac{E}{\alpha_1 |X_1|} & k - \frac{E}{\alpha_1 |X_1|} \\ 0 & 0 & \frac{\alpha_2 D - E}{(1 - \alpha_1) |X_1|} & k - \frac{\alpha_2 D - E}{(1 - \alpha_1) |X_1|} \\ \frac{E}{\alpha_2 |X_2|} & k - \frac{E}{\alpha_2 |X_2|} & 0 & 0 \\ \frac{\alpha_1 D - E}{(1 - \alpha_2) |X_2|} & k - \frac{\alpha_1 D - E}{(1 - \alpha_2) |X_2|} & 0 & 0 \end{pmatrix}.$$
 (2)

Set $\delta = |X_1||X_2|\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)$. Using a computer, we find that $\det(tI-B) = (t^2 - k^2)(t^2 - \gamma^2/\delta)$ where $\gamma = E - D\alpha_1\alpha_2$.

As $|X_1| = |X_2|$, the eigenvalues $\mu_1 \ge \mu_2 \ge \mu_3 \ge \mu_4$ of B equal $\pm k, \pm \mu$ where

$$\mu := \frac{|E - D\alpha_1 \alpha_2|}{|X_1| \sqrt{\alpha_1 \alpha_2 (1 - \alpha_1)(1 - \alpha_2)}}$$

However, Γ is k-regular, so its largest eigenvalue λ_1 is k by [4, Proposition 1.3.8]. It follows from interlacing that $\mu_1 = k$ and $\mu_2 = \mu$. In addition, interlacing implies that $\mu_2 \leq \lambda_2$, that is

$$\frac{|E(Y_1, Y_2) - E(X_1, X_2)\alpha_1\alpha_2|}{|X_1|\sqrt{\alpha_1\alpha_2(1 - \alpha_1)(1 - \alpha_2)}} \leqslant \lambda_2$$

Using $|X_1| = E(X_1, X_2)/k$ proves the assertion.

Proof of Proposition 1.3 For $i \in \{1, 2\}$, let X_i denote the set of e_i -subspaces of V. It follows from Proposition 1.2 that $\lambda_1 = q^{e_1 e_2} = k$ and $\lambda_2 = q^{e_1 e_2 - d/2}$. Taking $\Gamma = \Gamma_{e_1, e_2}$ in Lemma 3.1 gives

$$\frac{E(Y_1, Y_2)}{E(X_1, X_2)} \ge \alpha_1 \alpha_2 - \sqrt{\alpha_1 \alpha_2 (1 - \alpha_1) (1 - \alpha_2)} q^{-\frac{d}{2}}.$$

Since $E(X_1, X_2) = k \cdot |X_1|$, we have

$$\frac{E(Y_1, Y_2)}{|Y_1| \cdot |Y_2|} = \frac{|X_1|^2}{|Y_1| \cdot |Y_2|} \cdot \frac{E(Y_1, Y_2)}{|X_1|^2} = (\alpha_1 \alpha_2)^{-1} \cdot \frac{k}{|X_1|} \cdot \frac{E(Y_1, Y_2)}{E(X_1, X_2)} \\
\geqslant \frac{k}{|X_1|} \cdot \left(1 - \sqrt{\frac{(1 - \alpha_1)(1 - \alpha_2)}{\alpha_1 \alpha_2}} q^{-\frac{d}{2}}\right) \\
= \frac{k}{|X_1|} \cdot \left(1 - \sqrt{(\frac{1}{\alpha_1} - 1)(\frac{1}{\alpha_2} - 1)} q^{-\frac{d}{2}}\right).$$

The first claimed inequality follows by rewriting $k/|X_1|$ because

$$\frac{k}{|X_1|} = \frac{q^{e_1e_2}}{\binom{d}{e_1}_q} = \frac{\omega_q(e_1)\omega_q(e_2)}{\omega_q(e_1 + e_2)} > \omega_q(e_2).$$

The second inequality now follows from $\sqrt{(\frac{1}{\alpha_1}-1)(\frac{1}{\alpha_2}-1)} \leq \frac{1}{\alpha}-1$.

Corollary 3.2 The second bound in Proposition 1.3 implies that

$$\frac{|\{(S_1, S_2) \in Y_1 \times Y_2 : S_1 \cap S_2 = 0\}|}{|Y_1| \cdot |Y_2|} > \left(1 - \frac{3}{2}q^{-1}\right) \left(1 - \left(\frac{1}{\alpha} - 1\right)q^{-d/2}\right).$$

Proof Since $\omega_q(\infty) > 1 - q^{-1} - q^{-2}$ by [15, Lemma 3.5], it follows that

$$\omega_q(e_2) > \omega_q(\infty) > 1 - q^{-1} - q^{-2} \ge 1 - \frac{3}{2q}$$

The result now follows from Proposition 1.3.

4 Proof in the orthogonal case

In this section, we prove the orthogonal bound in Theorem 1.1.

Theorem 4.1 Suppose that $e_1, e_2 \ge 2$ are even and $V = (\mathbb{F}_q)^{e_1+e_2}$ is an $(e_1 + e_2)$ dimensional vector space equipped with a non-degenerate quadratic form of type $\varepsilon \in \{-,+\}$. For $i \in \{1,2\}$ and for $\sigma_i \in \{-,+\}$, let Y_i denote the set of all non-degenerate e_i -spaces of type σ_i . The proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is at least $1 - \frac{3}{2q}$ for all $e_1, e_2 \ge 2$, $\varepsilon, \sigma_1, \sigma_2 \in \{-,+\}$ and all prime-powers $q \ge 2$.

Proof Let $e_1 = 2m_1$, $e_2 = 2m_2$ and let $V = (\mathbb{F}_q)^d$ be a non-degenerate orthogonal space of type $\varepsilon \in \{-, +\}$ where $d = e_1 + e_2$. We will assume, without loss of generality, that $e_2 \leq e_1$ and hence that $m_2 \leq m_1$. Denote the isometry group of V by $\mathrm{GO}_d^{\varepsilon}(q)$. We use the formula for $|\mathrm{GO}_{2m}^{\sigma}(q)|$ in [17, p. 141], where $\sigma \in \{+, -\}$ is identified with 1, -1, respectively. Since $q^{2m} - 1 = (q^m - \sigma)(q^m + \sigma)$, we have

$$|\mathrm{GO}_{2m}^{\sigma}(q)| = 2q^{m(m-1)}(q^m - \sigma) \prod_{i=1}^{m-1} (q^{2i} - 1) = \frac{2q^{m(2m-1)}}{1 + \sigma q^{-m}} \prod_{i=1}^m (1 - q^{-2i}).$$

Hence $|\operatorname{GO}_{2m}^{\sigma}(q)| \sim 2q^{m(2m-1)}$ as $q \to \infty$. Recall that $\omega_{q^2}(m) = \prod_{i=1}^m (1-q^{-2i})$.

Let Y_1 denote the set of non-degenerate e_1 -spaces in V of type σ_1 . The stabilizer of $S_1 \in Y_1$ in $\mathrm{GO}_d^{\varepsilon}(q)$ is $\mathrm{GO}_{e_1}^{\sigma_1}(q) \times \mathrm{GO}_{e_2}^{\varepsilon\sigma_1}(q)$ since S_1^{\perp} has type $\varepsilon\sigma_1$ by [14, Lemma 2.5.11(ii)]. It follows from the Orbit-Stabilizer Lemma that

$$|Y_1| = \frac{|\mathrm{GO}_d^{\varepsilon}(q)|}{|\mathrm{GO}_{e_1}^{\sigma_1}(q) \times \mathrm{GO}_{e_2}^{\varepsilon\sigma_1}(q)|} = \frac{q^{e_1e_2}(1+\sigma_1q^{-m_1})(1+\varepsilon\sigma_1q^{-m_2})}{2(1+\varepsilon q^{-m_1-m_2})} \frac{\omega_{q^2}(m_1+m_2)}{\omega_{q^2}(m_1)\omega_{q^2}(m_2)}$$

Hence $|Y_1| \sim \frac{1}{2}q^{e_1e_2}$ as $q \to \infty$. We shall write

$$\begin{bmatrix} d \\ e_1 \end{bmatrix}_q = \frac{q^{e_1 e_2}}{B_q(e_1, e_2)} \quad \text{where} \quad B_q(e_1, e_2) := \frac{\omega_q(e_1)\omega_q(e_2)}{\omega_q(e_1 + e_2)}.$$

Then

$$|Y_1| = \frac{q^{e_1 e_2} \lambda(\sigma_1, \varepsilon)}{B_{q^2}(m_1, m_2)} \quad \text{where} \quad \lambda(\sigma_1, \varepsilon) = \frac{(1 + \sigma_1 q^{-m_1})(1 + \varepsilon \sigma_1 q^{-m_2})}{2(1 + \varepsilon q^{-m_1 - m_2})}$$

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Hence

$$\alpha_1 = \frac{|Y_1|}{\begin{bmatrix} d\\ e_1 \end{bmatrix}_q} = \frac{\lambda(\sigma_1, \varepsilon)B_q(e_1, e_2)}{B_{q^2}(m_1, m_2)}$$

Proposition 1.3 gives the lower bound

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_q(e_1, e_2) \left(1 - \sqrt{\left(\frac{1}{\alpha_1} - 1\right)\left(\frac{1}{\alpha_2} - 1\right)}q^{-d/2}\right)$$

A computer program [13] checks that the above bound is greater than $1 - \frac{1.5}{q}$ for all $q \leq 5$ and $1 \leq m_2 \leq m_1 \leq 6$ except when $q \leq 5$ and $m_1 = m_2 = 1$, or (q, m_2, m_1) equals (2, 1, 2), (2, 1, 3), (2, 2, 2). For these seven exceptions, we used the GAP package FINING [1] to do an exact count. The GAP/FINING code [13] verifies that the lower bound $1 - \frac{1.5}{q}$ holds in these cases. Thus when $q \leq 5$, we will henceforth assume that $m_1 + m_2 \geq 7$, and hence that $d = 2m_1 + 2m_2 \geq 14$.

Observe now that $\lambda(\sigma_1, \varepsilon) \ge \lambda(-, +)$. Take a lower bound $\alpha_1 \ge \alpha$ and $\alpha_2 \ge \alpha$ where $\alpha := \lambda(-, +)B_q(e_1, e_2)B_{q^2}(m_1, m_2)^{-1}$. Proposition 1.3 gives the lower bound

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_q(e_1, e_2) \left(1 - \left(\frac{1}{\alpha} - 1\right)q^{-d/2}\right)
= B_q(e_1, e_2) \left(1 + q^{-d/2}\right) - \lambda(-, +)^{-1} B_{q^2}(m_1, m_2)q^{-d/2}.$$
(3)

We next consider the case $q \leq 5$ and $m_1 + m_2 \geq 7$. Since $B_{q^2}(m_1, m_2)$ is a decreasing function of m_2 , it follows that

$$B_{q^2}(m_1, m_2) \leq B_{q^2}(7 - m_2, m_2) \leq B_{q^2}(1, 6)$$

Furthermore, $B_q(e_1, e_2) \ge \omega_q(e_1) > \omega_q(\infty)$ and $1 + q^{-d/2} > 1$ so

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > \omega_q(\infty) - \lambda(-, +)^{-1} B_{q^2}(1, 6) q^{-7}.$$

Similar reasoning gives

$$\lambda(-,+)^{-1} = \frac{2(1+q^{-m_1-m_2})}{(1-q^{-m_1})(1-q^{-m_2})} \leqslant \frac{2(1+q^{-7})}{(1-q^{-m_1})(1-q^{-(7-m_1)})} \\ \leqslant \frac{2(1+q^{-7})}{(1-q^{-1})(1-q^{-6})}.$$

Using the more accurate lower bound $\omega_q(\infty) > 1 - q^{-1} - q^{-2} + q^{-5}$ from [15, Lemma 3.5], one can check by computer that

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > 1 - q^{-1} - q^{-2} + q^{-5} - \frac{2(1 + q^{-7})}{(1 - q^{-1})(1 - q^{-6})} B_{q^2}(1, 6)q^{-7} > 1 - \frac{3}{2q}$$

holds for $q \leq 5$.

Finally, suppose that $q \ge 7$ and $1 \le m_2 \le m_1$ holds. When $m_1 = m_2 = 1$, it follows from (3) and $1 + q^{-d/2} > 1$ that

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > \frac{1}{(1+q^{-1}+q^{-2})(1+q^{-2})} - \frac{2q^{-2}}{(1-q^{-1})^2}.$$

This is bound is dominated by the first term and is a decreasing function of q. Hence the bound is greater than $1 - \frac{1.5}{q}$ for all $q \ge 7$. It remains to consider the case $m_1 + m_2 \ge 3$ and hence $d \ge 6$. Arguing as above, and using $q \ge 7$, gives

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > 1 - q^{-1} - q^{-2} + q^{-5} - \frac{2(1+q^{-3})}{(1-q^{-1})(1-q^{-2})} B_{q^2}(1,2)q^{-3} > 1 - \frac{3}{2q}.$$

Thus in all cases the bound $1 - \frac{3}{2q}$ holds, as claimed.

5 Proof in the symplectic and unitary cases

In this section, we prove the symplectic and unitary bounds in Theorem 1.1.

Theorem 5.1 Suppose that $e_1, e_2 \ge 2$ are even and $V = (\mathbb{F}_q)^{e_1+e_2}$ is an $(e_1 + e_2)$ dimensional symplectic space. For $i \in \{1, 2\}$ let Y_i denote the set of all non-degenerate e_i -spaces of V. The proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is at least $1 - \frac{10}{7q}$ for all e_1, e_2 and all prime-powers $q \ge 2$.

Proof Let $e_1 = 2m_1$, $e_2 = 2m_2$ and let $V = (\mathbb{F}_q)^d$ be a non-degenerate symplectic space where $d = e_1 + e_2$. As before, we shall assume, without loss of generality, that $2 \leq e_2 \leq e_1$ and hence that $1 \leq m_2 \leq m_1$. Let $m = m_1 + m_2$. The isometry group $\operatorname{Sp}_{2m}(q)$ of V has order $q^{m^2} \prod_{i=1}^m (q^{2i} - 1) = q^{2m^2 + m} \omega_{q^2}(m)$ by [17, p. 70]. Let Y_i denote the set of non-degenerate e_i -spaces in V. Clearly $|Y_1| = |Y_2|$. The

stabilizer of $S_1 \in Y_1$ in $\operatorname{Sp}(V)$ is $\operatorname{Sp}(S_1) \times \operatorname{Sp}(S_1^{\perp})$. Therefore

$$\begin{aligned} |Y_2| &= |Y_1| = \frac{|\mathrm{Sp}_{e_1+e_2}(q)|}{|\mathrm{Sp}_{e_1}(q) \times \mathrm{Sp}_{e_2}(q)|} = \frac{q^{e_1e_2}\omega_{q^2}(m_1+m_2)}{\omega_{q^2}(m_1)\omega_{q^2}(m_2)} \\ &= \frac{q^{e_1e_2}}{B_{q^2}(m_1,m_2)} \quad \text{where } B_q(e_1,e_2) = \frac{\omega_q(e_1)\omega_q(e_2)}{\omega_q(e_1+e_2)}. \end{aligned}$$

Since $|X_1| = \begin{bmatrix} d \\ e_1 \end{bmatrix}_a = \begin{bmatrix} d \\ e_2 \end{bmatrix}_a = |X_2| = q^{e_1 e_2} / B_q(e_1, e_2)$, we have

$$\alpha_1 = \frac{|Y_1|}{|X_1|} = \frac{q^{e_1e_2}}{B_{q^2}(m_1, m_2)} \cdot \frac{B_q(e_1, e_2)}{q^{e_1e_2}} = \frac{B_q(e_1, e_2)}{B_{q^2}(m_1, m_2)}$$

Proposition 1.3 gives the lower bound

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_q(e_1, e_2) \left(1 - \left(\frac{1}{\alpha_1} - 1\right) q^{-d/2} \right)$$
$$\ge B_q(e_1, e_2) - \left(B_{q^2}(m_1, m_2) - B_q(e_1, e_2) \right) q^{-d/2}$$
$$> B_q(e_1, e_2) - B_{q^2}(m_1, m_2) q^{-d/2}.$$

Note that $B_q(e_1, e_2) \ge \omega_q(e_1) > \omega_q(\infty) > 1 - q^{-1} - q^{-2}$ by [15, Lemma 3.5]. Also, $1 > B_{q^2}(m_1, m_2)$ and $d = e_1 + e_2 \ge 4$. If $q \ge 5$, we have

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge 1 - q^{-1} - q^{-2} - q^{-2} > 1 - \frac{10}{7}q^{-1}.$$

For $q \in \{2, 3, 4\}$ we have $\omega_2(\infty) > 0.288$, $\omega_3(\infty) > 0.56$ and $\omega_4(\infty) > 0.688$. Thus when $q \in \{2, 3, 4\}$ and $d \ge 20$, we have

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge \omega_q(\infty) - q^{-10} > 1 - \frac{10}{7}q^{-1}.$$

Finally, if $q \in \{2, 3, 4\}$ and $d = e_1 + e_2 < 20$, then a computer program shows that the last inequality below is satisfied

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_q(e_1, e_2)(1 + q^{-d/2}) - B_{q^2}(m_1, m_2)q^{-d/2} > 1 - \frac{10}{7}q^{-1}.$$

Theorem 5.2 Suppose $V = (\mathbb{F}_{q^2})^{e_1+e_2}$ is an (e_1+e_2) -dimensional hermitian space where $e_1, e_2 \ge 1$. For $i \in \{1, 2\}$, let Y_i denote the set of all non-degenerate e_i -spaces of V. The proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is at least $1 - \frac{c}{q^2}$ where c = 2 when $(e_1, e_2, q) = (1, 1, 2)$, $c = \frac{3}{2}$ when $\min\{e_1, e_2\} = 1$ and $(e_1, e_2, q) \ne (1, 1, 2)$, and c = 1.26 otherwise.

Proof Let $V = (\mathbb{F}_{q^2})^d$ be a non-degenerate unitary space where $d = e_1 + e_2$. Let $\operatorname{GU}_d(q)$ denote the isometry group of V. We have $|\operatorname{GU}_d(q)| \sim q^{d^2}$ as $q \to \infty$, more precisely $|\operatorname{GU}_d(q)| = q^{d(d-1)/2} \prod_{i=1}^d (q^i - (-1)^i) = q^{d^2} \omega_{-q}(d)$ by [17, p. 118] where

$$\omega_{-q}(d) = \prod_{i=1}^{d} (1 - (-q)^{-i}).$$

The proportion of pairs $(S_1, S_2) \in Y_1 \times Y_2$ for which $S_1 \cap S_2 = \{0\}$ is unchanged if we swap the subscripts. Thus we can henceforth assume that $1 \leq e_2 \leq e_1$. Proposition 1.3 gives poor bounds when $e_2 = 1$. When $e_2 = 1$, it turns out to be simple to overestimate the complementary proportion using ideas in the proof of [10, Theorem 4.1]. The proportion of non-degenerate e_1 -subspaces (where $e_1 = d - 1$) that contain a given 1-subspace is at most c_1/q^2 where $c_1 := \frac{1+q^{-e_1}}{1-q^{-1-e_1}}$ by [10, p. 9]. This equals $\frac{2}{q^2}$ if $(e_1, e_2, q) = (1, 1, 2)$. If $e_1 \ge 2$, we have $c_1 \le \frac{1+2^{-2}}{1-2^{-3}} < \frac{3}{2}$, and if $q \ge 3$ we have $c_1 \le \frac{1+q^{-1}}{1-q^{-2}} = \frac{1}{1-q^{-1}} \le \frac{3}{2}$. Thus when $e_1 \ge 2$ and $e_2 = 1$ the complementary proportion is at most $\frac{3}{2q^2}$. This proves the claim when $e_2 = 1$. We henceforth assume that $2 \le e_2 \le e_1$.

The stabilizer of
$$S_1 \in Y_1$$
 in $\mathrm{GU}(V)$ is $\mathrm{GU}(S_1) \times \mathrm{GU}(S_1^{\perp})$. Hence
 $|Y_1| = |Y_2| = \frac{|\mathrm{GU}_{e_1+e_2}(q)|}{|\mathrm{GU}_{e_1}(q) \times \mathrm{GU}_{e_2}(q)|} = \frac{q^{2e_1e_2}\omega_{-q}(e_1+e_2)}{\omega_{-q}(e_1)\omega_{-q}(e_2)} = \frac{q^{2e_1e_2}}{B_{-q}(e_1,e_2)}.$

Replacing q with q^2 shows that $|X_1| = |X_2| = (q^2)^{e_1 e_2} / B_{q^2}(e_1, e_2)$, and hence

$$\alpha_1 = \frac{|Y_1|}{|X_1|} = \frac{B_{q^2}(e_1, e_2)}{B_{-q}(e_1, e_2)}$$

Proposition 1.3, with q replaced with q^2 , gives the lower bound

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_{q^2}(e_1, e_2) \left(1 - \left(\frac{1}{\alpha_1} - 1\right) q^{-d} \right)
= B_{q^2}(e_1, e_2)(1 + q^{-d}) - B_{-q}(e_1, e_2)q^{-d}
> B_{q^2}(e_1, e_2) - B_{-q}(e_1, e_2)q^{-d}.$$
(4)

Note that $B_{q^2}(e_1, e_2) \ge \omega_{q^2}(e_1) > \omega_{q^2}(\infty) > 1 - q^{-2} - q^{-4}$ by [15, Lemma 3.5]. To find an upper bound for $B_{-q}(e_1, e_2) = \prod_{i=2}^{e_2} \frac{(1 - (-q)^{-i})}{(1 - (-q)^{-i})}$, we will use

To find an upper bound for
$$B_{-q}(e_1, e_2) = \prod_{i=1}^{e_2} \frac{(1-(-q))^{-1}}{(1-(-q)^{-e_1-i})}$$
, we will use

$$(1 - q^{-2i})(1 + q^{-(2i+1)}) < 1 < (1 + q^{-(2j-1)})(1 - q^{-2j}).$$
(5)
m (5) that Π^{e_2} $(1 - (-q)^{-i}) < 1 + q^{-1}$ for all $q_2 > 1$ and

It follows from (5) that $\prod_{i=1}^{e_2} (1 - (-q)^{-i}) \leq 1 + q^{-1}$ for all $e_2 \geq 1$ and

 $\prod_{i=1}^{e_2} (1-(-q)^{-e_1-i}) \geqslant \begin{cases} 1 & \text{if } e_1 \text{ even, } e_2 \text{ even,} \\ 1+q^{-e_1-e_2} & \text{if } e_1 \text{ even, } e_2 \text{ odd,} \\ 1-q^{-e_1-1} & \text{if } e_1 \text{ odd, } e_2 \text{ even,} \\ (1-q^{-1-e_1})(1-q^{-e_1-e_2}) & \text{if } e_1 \text{ odd, } e_2 \text{ odd.} \end{cases}$

Hence the above product is greater than or equal to $(1 - q^{-4})(1 - q^{-6})$ for all $2 \leq e_2 \leq e_1$. Therefore $B_{-q}(e_1, e_2) \leq \frac{1+q^{-1}}{(1-q^{-4})(1-q^{-6})}$.

Since $d \ge 4$, it follows from (4) that

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > 1 - q^{-2} - q^{-4} - \frac{(1+q^{-1})q^{-4}}{(1-q^{-4})(1-q^{-6})}$$

This is greater than $1 - \frac{1.26}{q^2}$ for all $q \ge 4$.

Suppose now that $q \in \{2, 3\}$ and $d \ge 10$. Then

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} > \omega_{q^2}(\infty) - \frac{(1+q^{-1})q^{-10}}{(1-q^{-4})(1-q^{-6})}$$

and since $\omega_4(\infty) > 0.6885$, $\omega_9(\infty) > 0.876$, the above bound is greater than $1 - \frac{1.26}{q^2}$ for $q \in \{2, 3\}$. It remains to consider $q \leq 3$ and $2 \leq e_2 \leq e_1$ where $e_1 + e_2 < 10$. In this case, a simple computer program verifies that

$$\frac{E(Y_1, Y_2)}{|Y_1||Y_2|} \ge B_{q^2}(e_1, e_2)(1 + q^{-d}) - B_{-q}(e_1, e_2)q^{-d} > 1 - \frac{1.26}{q^2}.$$

6 Future Work

A more general problem when $e_1 + e_2 < d$ is considered in [11]. Here $V = \mathbb{F}^d$ is a finite non-degenerate classical space and a lower bound of the form $1 - \frac{c}{|\mathbb{F}|}$ is sought for the proportion of non-degenerate pairs (S_1, S_2) satisfying $\dim(S_i) = e_i$ and $S_1 \cap S_2 = \{0\}$. The bound given in [11, Theorem 1.1] has the form $1 - \frac{c}{|\mathbb{F}|}$ in the symplectic case, and the orthogonal case for q > 2, but only $1 - \frac{c}{|\mathbb{F}|^{1/2}}$ in the unitary case. If one could compute the second eigenvalue λ_2 of the bipartite graph Γ_{d,e_1,e_2} when $e_1 + e_2 < d$, *c.f.* Proposition 1.2, then it may be possible to obtain sharper lower bounds of the form $1 - \frac{c}{|\mathbb{F}|}$ via Lemma 3.1, in all cases.

Acknowledgment

We thank the referee for their very helpful comments. The first author is supported by the Australian Research Council Discovery Grant DP190100450. The second author is supported by a postdoctoral fellowship of the Research Foundation – Flanders (FWO).

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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