Uniqueness of an association scheme related to the Witt design on 11 points

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Abstract

It follows from Delsarte theory that the Witt 4-$(11,5,1)$ design gives rise to a Q-polynomial association scheme $\mathcal{W}$ defined on the set of its blocks. In this note we show that $\mathcal{W}$ is unique, i.e., defined up to isomorphism by its parameters.

1 Introduction

A t-(v,k,λ) design is a set of k-subsets (called **blocks**) of v points such that every t-subset is contained in exactly λ blocks. A t-design with $\lambda = 1$ is called a Steiner system, and the most celebrated ones have parameters 4-$(11,5,1)$, 5-$(12,6,1)$, 3-$(22,6,1)$, 4-$(23,7,1)$, and 5-$(24,8,1)$, which are often referred to as the Mathieu designs or the Witt systems. In particular, the 4-$(11,5,1)$ design W_{11} arises from a 4-transitive action of the Mathieu group M_{11} on 11 points, and its existence and uniqueness was first shown by Witt [3]. The design W_{11} has 66 blocks, and every two distinct blocks B and B' have 1, 2, or 3 points in common. Let us define a binary symmetric relation R_i on W_{11} by

$$(B,B') \in R_i \Leftrightarrow |B \cap B'| = 4 - i,$$

for $i = 1,2,3$ and let $R_0 = \{(B,B) \mid B \in W_{11}\}$. Then the pair $\mathcal{W} = (W_{11}, \{R_0, R_1, R_2, R_3\})$ is an association scheme (of 3 classes) by [7, Theorem 5.25]. Moreover, this scheme is Q-polynomial (see [2] for the definitions and more results about P,Q-polynomial association schemes). Williford [8, 9] compiled the tables of feasible parameters of primitive 3-class Q-polynomial association schemes (on up to 2800 vertices), where the uniqueness of the scheme $\mathcal{W}$ was left blank (see also [6, Appendix B]). In this note, we show that $\mathcal{W}$ is unique, i.e., it is determined up to isomorphism by its parameters. The proof is computer-assisted by Mathematica and relies on a spherical representation of the scheme [1].

2 The parameters of $\mathcal{W}$

Let us recall some standard facts from the theory of association schemes (see [2]). Let A_i denote the logical matrix of the relation R_i, for $i = 0,1,2,3$. Then A_0 is the identity matrix of size 66 and:

(1) $\sum_{i=0}^{3} A_i = J$, the square all-one matrix of size 66,

(2) $A_i^\top = A_i$ $(0 \le i \le 3)$,

(3) $A_i A_j = \sum_{k=0}^{3} p_{i,j}^k A_k$, where $p_{i,j}^k$ are nonnegative integers $(0 \le i,j,k \le 3)$, called the **intersection numbers** of the scheme, which we refer to as the parameters of the scheme.

The intersection numbers of $\mathcal{W}$, written in the form of matrices $(L_i)_{kj} = (p_{i,j}^k)$, are found in the tables by Williford [9]:

$$L_1 = \begin{bmatrix} 0 & 30 & 0 & 0 \\ 1 & 15 & 10 & 4 \\ 0 & 15 & 6 & 9 \\ 0 & 8 & 12 & 10 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 20 & 0 \\ 0 & 10 & 4 & 6 \\ 1 & 6 & 10 & 3 \\ 0 & 12 & 4 & 4 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 & 15 \\ 0 & 4 & 6 & 5 \\ 0 & 9 & 3 & 3 \\ 1 & 10 & 4 & 0 \end{bmatrix},$$

and they determine ([2, Theorem 4.1]) the first and second eigenmatrices P and $Q = \frac{1}{66}P^{-1}$ of $\mathcal{W}$:

$$P = \begin{bmatrix} 1 & 30 & 20 & 15 \\ 1 & 8 & -2 & -7 \\ 1 & -1 & -2 & 2 \\ 1 & -6 & 8 & -3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 10 & 44 & 11 \\ 1 & \frac{8}{3} & -\frac{22}{15} & -\frac{11}{5} \\ 1 & -1 & -\frac{22}{5} & \frac{22}{5} \\ 1 & -\frac{14}{3} & \frac{88}{15} & -\frac{11}{5} \end{bmatrix},$$

where the $(P)_{ij}$-entry $(0 \le i, j \le 3)$ is the eigenvalue of A_j on the i-th maximal common eigenspace of the matrices $A_0, \ldots, A_3$ (which commute and hence can be simultaneously diagonalized).

Theorem 2.1. *An association scheme with the above parameters is isomorphic to $\mathcal{W}$.*

In what follows, we assume that $\mathcal{X} = (X, \{R_0, R_1, R_2, R_3\})$ is an association scheme with the same parameters as $\mathcal{W}$. Let E_1 denote the orthogonal projection matrix onto the 1st maximal common eigenspace (of multiplicity $m_1 = Q_{01} = 10$) of the matrices A_i's of $\mathcal{X}$. Since E_1 is positive semidefinite, we may regard $\frac{|X|}{m_1}E_1$ as the Gram matrix of vectors in the unit sphere S^{m_1-1} in $\mathbb{R}^{m_1}$, and write $\frac{|X|}{m_1}E_1 = F_1F_1^\top$ where F_1 is a $|X| \times m_1$ matrix. We now identify a vertex $x \in X$ with the x-th row $\mathbf{x} \in \mathbb{R}^{m_1}$ of F_1, and such a map is said to be a **spherical representation** of the scheme $\mathcal{X}$ into its first eigenspace. Define the **angle set** $A(X) := \{\langle \mathbf{x}, \mathbf{y} \rangle \mid x, y \in X, x \neq y\}$ and observe that $A(X) = \left\{\frac{Q_{j1}}{Q_{01}} \mid 1 \le j \le 3\right\}$. Note that the map is injective, since $Q_{j1} \neq Q_{01}$ provided that $j \neq 0$; thus, it defines all relations of the scheme. Therefore, to prove Theorem 2.1 it suffices to show that the Gram matrix of the vertex set X embedded into the unit sphere S^9 in $\mathbb{R}^{10}$ is unique, up to orthogonal transformation.

3 Proof of Theorem 2.1

Observe that the graph (X, R_2) is strongly regular with parameters $(66, 20, 10, 4)$. Such a strongly regular graph is isomorphic to the triangular graph $T(12)$ by [5]; thus, we may identify the point set X with the vertex set of $T(12)$, which is $\binom{[12]}{2}$, where $[12] := \{1, \ldots, 12\}$, and R_2 with the edge set of $T(12)$, which is $\left\{\{v, v'\} \mid |v \cap v'| = 1, v, v' \in \binom{[12]}{2}\right\}$. The next lemma can easily be seen from this description of $T(12)$.

Lemma 3.1. *Each vertex of the triangular graph $T(12)$ is contained in two maximum cliques of order 11. Each vertex outside such a clique has exactly two neighbors in it.*

Note that a clique of order 11 in $T(12)$ is a **Delsarte clique**, as it attains the Delsarte bound [4, Proposition 1.3.2]. Fix a vertex $x = \{1, 2\}$ of $T(12)$. The two Delsarte cliques containing x are

$$C_1 = \{\{1, j\} \mid 2 \le j \le 12\} \quad \text{and} \quad C_2 = \{\{2, j\} \mid j = 1, 3 \le j \le 12\}.$$

Consider the image of C_1 in the spherical representation in $\mathbb{R}^{10}$: since C_1 is a clique, the angle between any two vectors is the same; thus, these 11 vectors form a regular simplex. Let V_1 be an 11×10 matrix whose row vectors correspond to the vertices of C_1. Up to orthogonal transformation, we may assume that the matrix V_1 has the following form:

$$V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{3\sqrt{11}}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & \frac{2\sqrt{55}}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & \frac{\sqrt{385}}{20} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & \frac{\sqrt{1155}}{35} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & \frac{\sqrt{33}}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & -\frac{\sqrt{33}}{30} & \frac{\sqrt{22}}{5} & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & -\frac{\sqrt{33}}{30} & -\frac{\sqrt{22}}{20} & \frac{\sqrt{330}}{20} & 0 & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & -\frac{\sqrt{33}}{30} & -\frac{\sqrt{22}}{20} & -\frac{\sqrt{330}}{60} & \frac{\sqrt{165}}{15} & 0 \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & -\frac{\sqrt{33}}{30} & -\frac{\sqrt{22}}{20} & -\frac{\sqrt{330}}{60} & -\frac{\sqrt{165}}{30} & \frac{\sqrt{55}}{10} \\ -\frac{1}{10} & -\frac{\sqrt{11}}{30} & -\frac{\sqrt{55}}{60} & -\frac{\sqrt{385}}{140} & -\frac{\sqrt{1155}}{210} & -\frac{\sqrt{33}}{30} & -\frac{\sqrt{22}}{20} & -\frac{\sqrt{330}}{60} & -\frac{\sqrt{165}}{30} & -\frac{\sqrt{55}}{10} \end{bmatrix},$$

where the first row corresponds to x.

Now the problem is to add the remaining 55 vectors such that together with the vectors from C_1 their Gram matrix is permutation equivalent to E_1. Since $A(X) = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = \frac{4}{15}, \alpha_2 = -\frac{1}{10}, \alpha_3 = -\frac{7}{15}$, every such a vector $\mathbf{u} \in S^9$ in question satisfies $V_1\mathbf{u}^\top \in \{\alpha_1, \alpha_2, \alpha_3\}^{10}$, and there are at most 3^{10} candidates for $\mathbf{u}$.

We first determine the coordinates of the vectors in the image of C_2. Let $z \in C_1 \setminus \{x\}$ and $y \in C_2$. Then $(x, z) \in R_2$, $(x, y) \in R_2$ and $(y, z) \in R_1 \cup R_2 \cup R_3$. Since $p^2_{2,1} = 6, p^2_{2,2} = 10, p^2_{2,3} = 3$ and, by Lemma 3.1, for each vertex $z \in C_1 \setminus \{x\}$, there is exactly one vertex $y \in C_2 \setminus \{x\}$ such that $\langle \mathbf{y}, \mathbf{z} \rangle = \alpha_2$, it follows that the vertices in C_2 are taken from the following set:

$$Y_1 = \left\{\mathbf{y} \in S^9 \mid V_1\mathbf{y}^\top = (\alpha_2, \mathbf{v})^\top, \mathbf{v} = \left(\{[\alpha_1]^6, [\alpha_2]^1, [\alpha_3]^3\}\right)\right\},$$

where $\left(\{[\alpha_1]^6, [\alpha_2]^1, [\alpha_3]^3\}\right)$ denotes a vector of length 10 having 6 entries equal to α_1, 1 entry α_2, and 3 entries α_3. Note that $|Y_1| = 840$. Consider a graph with vertex set Y_1 and edge set E_1 defined by $\{\mathbf{y}, \mathbf{y}'\} \in E_1$ if and only if $\langle \mathbf{y}, \mathbf{y}' \rangle = \alpha_2$. Then every clique of order 10 in the graph (Y_1, E_1) is a candidate for C_2. Let us define a 10×10 matrix $V_2 = C \cdot V_1$, where

$$C = \frac{1}{3}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix},$$

and $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$. We can prove the following lemma with the aid of a computer.

Lemma 3.2. *There exist exactly* 30240 *cliques of order* 10 *in the graph* (Y_1, E_1)*. For every such clique* C_2*, the Gram matrix of* $C_1 \cup C_2$ *is permutationally equivalent to that of the rows of* V.

By Lemma 3.2, we may fix the set Y_2 of the row vectors of V_2 representing C_2 and extend the above argument to other Delsarte cliques, which yields that $X \setminus (C_1 \cup C_2)$ can be represented by a subset of the following set:

$$Y = \left\{\mathbf{u} \in S^9 \mid V_1\mathbf{u}^\top = \mathbf{v}^\top, \mathbf{v} = \left(\{[\alpha_1]^6, [\alpha_2]^2, [\alpha_3]^3\}\right)\right\} \setminus Y_2.$$

Note that $|Y| = 4610$, which is somewhat smaller than 3^{10}. We determine the set Y and select only those of its vectors whose inner products with the vectors from Y_2 belong to $A(X)$. Let Z denote the set of such vectors. With the aid of a computer, we obtain $|Z| = 90$. We proceed by finding a maximal subset of Z such that the inner product of every two of its distinct vectors is in $A(X)$.

Lemma 3.3. *The set Z splits into $Z_1 \cup Z_2$ in such a way that $|Z_1| = |Z_2| = 45$, $Z_1 \cap Z_2 = \emptyset$, $A(Z_1) = A(Z_2) = \{\alpha_1, \alpha_2, \alpha_3\}$, and $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \notin \{\alpha_1, \alpha_2, \alpha_3\}$ for all $\mathbf{z}_1 \in Z_1$ and all $\mathbf{z}_2 \in Z_2$. Moreover, the Gram matrices $C_1 \cup C_2 \cup Z_1$ and $C_1 \cup C_2 \cup Z_2$ are permutationally equivalent.*

Thus, we may assume that the 66 vertices of X are represented by the row vectors from V_1, V_2, and Z_1, and we can directly verify that, together with the binary relations defined by their inner products, they form an association scheme of 3 classes with the same parameters as $\mathcal{W}$. This completes the proof of Theorem 2.1.

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