JACOBI POLYNOMIALS AND HARMONIC WEIGHT ENUMERATORS OF THE FIRST-ORDER REED-MULLER CODES AND THE EXTENDED HAMMING CODES

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ABSTRACT. In the present paper, we give harmonic weight enumerators and Jacobi polynomials for the first-order Reed–Muller codes and the extended Hamming codes. As a corollary, we show the nonexistence of combinatorial 4-designs in these codes.

1. Introduction

Let m be a positive integer, and set $V = \mathbb{F}_2^m$. The first-order Reed–Muller code RM(1,m) is defined as the subspace of \mathbb{F}_2^V consisting of affine linear functions:

$$RM(1,m) = \{(\lambda(x) + b)_{x \in V} \mid \lambda \in V^*, b \in \mathbb{F}_2\},\$$

where $V^* = \text{Hom}(V, \mathbb{F}_2)$. We remark that the weight enumerator of RM(1, m) is

$$x^{2^m} + (2^{m+1} - 2)x^{2^{m-1}}y^{2^{m-1}} + y^{2^m}.$$

It is well known that the dual code of RM(1, m) is isomorphic to an extended Hamming code H_{2^m} [7].

Let C = RM(1, m) or H_{2^m} , and $C_{\ell} := \{c \in C \mid \operatorname{wt}(c) = \ell\}$. In this paper, we call C_{ℓ} a shell of the code C whenever it is non-empty. Shells of RM(1, m) and H_{2^m} are known to support combinatorial 3-designs by the Assmus–Mattson theorem (see Theorem 2.1) or the transitivity argument (see [7, Ch. 13. §9]). More precisely, the set $\mathcal{B}(C_{\ell}) := \{\operatorname{supp}(x) \mid x \in C_{\ell}\}$ forms the set of blocks of a combinatorial 3-design.

In [9], H. Nakasora and the first named author gave the first nontrivial examples of a code that supports combinatorial t-designs for all weights obtained from the Assmus–Mattson theorem and that supports t'-designs for some weights with some t' > t (see also [3, 6, 8, 10]). Hence, it is natural to ask whether certain shells of RM(1,m) and H_{2^m} support combinatorial 4-designs. The aim of the present paper is to settle this problem by computing the Jacobi polynomials and harmonic weight enumerators for the first-order Reed–Muller codes and the extended Hamming codes. The definitions of

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the Jacobi polynomials and harmonic weight enumerators will be given in Section 2.

Theorem 1.1. Let C = RM(1, m) and $T = \{0, u_1, u_2, u_3\} \in \binom{V}{4}$.

(1) (a) If
$$u_1 + u_2 \neq u_3$$
, then

$$J_{C,T}(w,z,x,y) = w^{4}x^{2^{m}-4} + (2^{m-3}-1)w^{4}x^{2^{m-1}-4}y^{2^{m-1}}$$

$$+ 2^{m-1}w^{3}zx^{2^{m-1}-3}y^{2^{m-1}-1} + 3 \cdot 2^{m-2}w^{2}z^{2}x^{2^{m-1}-2}y^{2^{m-1}-2}$$

$$+ 2^{m-1}wz^{3}x^{2^{m-1}-1}y^{2^{m-1}-3} + (2^{m-3}-1)z^{4}x^{2^{m-1}}y^{2^{m-1}-4}$$

$$+ z^{4}y^{2^{m}-4}.$$

(b) If
$$u_1 + u_2 = u_3$$
, then

$$J_{C,T}(w, z, x, y) = w^{4}x^{2^{m}-4} + (2^{m-2} - 1)w^{4}x^{2^{m-1}-4}y^{2^{m-1}} + 3 \cdot 2^{m-1}w^{2}z^{2}x^{2^{m-1}-2}y^{2^{m-1}-2} + (2^{m-2} - 1)z^{4}x^{2^{m-1}}y^{2^{m-1}-4} + z^{4}y^{2^{m}-4}$$

(2) (a) If
$$u_1 + u_2 \neq u_3$$
, then

$$J_{C^{\perp},T}(w,z,x,y) = \frac{1}{2^{m+1}} \Big((w+z)^4 (x+y)^{2^m-4}$$

$$+ (2^{m-3}-1)(w+z)^4 (x+y)^{2^{m-1}-4} (x-y)^{2^{m-1}}$$

$$+ 2^{m-1} (w+z)^3 (w-z)(x+y)^{2^{m-1}-3} (x-y)^{2^{m-1}-1}$$

$$+ 3 \cdot 2^{m-2} (w+z)^2 (w-z)^2 (x+y)^{2^{m-1}-2} (x-y)^{2^{m-1}-2}$$

$$+ 2^{m-1} (w+z)(w-z)^3 (x+y)^{2^{m-1}-1} y^{2^{m-1}-3}$$

$$+ (2^{m-3}-1)(w-z)^4 (x+y)^{2^{m-1}} (x-y)^{2^{m-1}-4}$$

$$+ (w-z)^4 (x-y)^{2^m-4} \Big).$$

(b) If
$$u_1 + u_2 = u_3$$
, then

$$J_{C^{\perp},T}(w,z,x,y) = \frac{1}{2^{m+1}} \Big((w+z)^4 (x+y)^{2^m-4}$$

$$+ (2^{m-2}-1)(w+z)^4 (x+y)^{2^{m-1}-4} (x-y)^{2^{m-1}}$$

$$+ 3 \cdot 2^{m-1} (w+z)^2 (w-z)^2 (x+y)^{2^{m-1}-2} (x-y)^{2^{m-1}-2}$$

$$+ (2^{m-2}-1)(w-z)^4 (x+y)^{2^{m-1}} (x-y)^{2^{m-1}-4}$$

$$+ (w-z)^4 (x-y)^{2^m-4} \Big).$$

Theorem 1.2. Let C = RM(1,m) and $U \subset V$ be a three-dimensional subspace of V. We assume that f is a harmonic function of degree $k \in \mathbb{N}$

$$\begin{split} such \ that \ f &= 0 \ on \ \binom{V}{k} \setminus \binom{U}{k}. \ Then \ we \ have \\ w_{C,f}(x,y) &= 2^{m-3} x^{2^{m-1}} y^{2^{m-1}} \sum_{a \in H_8, \text{wt}(a) = 4} \widetilde{f}(a), \\ w_{C^{\perp},f}(x,y) &= (-1)^k (xy)^k 2^{2^{m-1} - 2} \\ & \left(\frac{x+y}{\sqrt{2}}\right)^{2^{m-1} - k} \left(\frac{x-y}{\sqrt{2}}\right)^{2^{m-1} - k} \sum_{a \in H_8, \text{wt}(a) = 4} \widetilde{f}(a). \end{split}$$

We show, as a corollary, the nonexistence of combinatorial 4-designs in these codes.

Corollary 1.3. Let C = RM(1, m) or H_{2^m} . Then for any $\ell \in \mathbb{N}$, C_ℓ is not a combinatorial 4-design.

This paper is organized as follows. In Section 2, we define and give some basic properties of codes, combinatorial t-designs, Jacobi polynomials, and harmonic weight enumerators used in this paper. In Sections 3, 4, and 5, we prove Theorems 1.1 and 1.2 and Corollary 1.3, respectively.

All computer calculations reported in this paper were done using MAGMA [4] and MATHEMATICA [12].

2. Preliminaries

2.1. Codes and combinatorial t-designs. A binary linear code C of length n is a linear subspace of \mathbb{F}_2^n . An inner product (x,y) on \mathbb{F}_2^n is given by

$$(x,y) = \sum_{i=1}^{n} x_i y_i,$$

where $x, y \in \mathbb{F}_2^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. The dual of a linear code C is defined as

$$C^{\perp} = \{ y \in \mathbb{F}_2^n \mid (x, y) = 0 \text{ for all } x \in C \}.$$

For $x \in \mathbb{F}_2^n$, the weight $\operatorname{wt}(x)$ is the number of its nonzero components.

A combinatorial t-design is a pair $\mathcal{D} = (\Omega, \mathcal{B})$, where Ω is a set of points of cardinality v, and \mathcal{B} is a collection of k-element subsets of Ω called blocks, with the property that any t points are contained in precisely λ blocks.

The support of a vector $x := (x_1, \ldots, x_n), x_i \in \mathbb{F}_2$ is the set of indices of its nonzero coordinates: $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. Let $\Omega := \{1, \ldots, n\}$ and $\mathcal{B}(C_\ell) := \{\operatorname{supp}(x) \mid x \in C_\ell\}$. Then for a code C of length n, we say that the shell C_ℓ is a combinatorial t-design if $(\Omega, \mathcal{B}(C_\ell))$ is a combinatorial t-design.

The following theorem is from Assmus and Mattson [1]. It is one of the most important theorems in coding theory and design theory:

Theorem 2.1 ([1]). Let C be a linear code of length n over \mathbb{F}_q with minimum weight d. Let C^{\perp} denote the dual code of C, with minimum weight d^{\perp} . Suppose that an integer t $(1 \leq t \leq n)$ is such that there are at most d-t

weights of C^{\perp} in $\{1, 2, ..., n-t\}$, or such that there are at most $d^{\perp} - t$ weights of C in $\{1, 2, ..., n-t\}$. Then the supports of the words of any fixed weight in C form a t-design (with possibly repeated blocks).

2.2. **Jacobi polynomials.** Let C be a binary code of length n and $T \subset [n] := \{1, \ldots, n\}$. Then the Jacobi polynomial of C with T is defined as follows [11]:

$$J_{C,T}(w,z,x,y) := \sum_{c \in C} w^{m_0(c)} z^{m_1(c)} x^{n_0(c)} y^{n_1(c)},$$

where for $c = (c_1, \ldots, c_n)$,

$$m_i(c) = |\{j \in T \mid c_j = i\}|,$$

 $n_i(c) = |\{j \in [n] \setminus T \mid c_j = i\}|.$

The following is a generalization of the classical MacWilliams identity:

Theorem 2.2 ([11]). Let C be a binary code of length n and $T \subset [n]$. Then we have

$$J_{C^{\perp},T}(w,z,x,y) = \frac{1}{|C|} J_{C,T}(w+z,w-z,x+y,x-y).$$

It is easy to see that C_{ℓ} is a combinatorial t-design if and only if the coefficient of $z^t x^{n-\ell} y^{\ell-t}$ in $J_{C,T}$ is independent of the choice of T with |T| = t.

2.3. Harmonic weight enumerators. In this subsection, we review the concept of harmonic weight enumerators.

Let $\Omega = \{1, 2, ..., n\}$ be a finite set (which will be the set of coordinates of the code) and let X be the set of its subsets, while, for each k = 0, 1, ..., n, let X_k be the set of its k-subsets. We denote by $\mathbb{R}X$ and $\mathbb{R}X_k$ the real vector spaces spanned by the elements of X and X_k , respectively. An element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{z \in X_L} f(z)z$$

and is identified with the real-valued function on X_k given by $z \mapsto f(z)$.

An element $f \in \mathbb{R}X_k$ can be extended to an element $f \in \mathbb{R}X$ by setting, for all $u \in X$,

$$\widetilde{f}(u) = \sum_{z \in X_k, z \subset u} f(z).$$

The differentiation γ is the operator on $\mathbb{R}X$ defined by linearity from

$$\gamma(z) = \sum_{y \in X_{k-1}, y \subset z} y$$

for all $z \in X_k$ and for all k = 0, 1, ..., n. An element $f \in \mathbb{R}X_k$ satisfying $\gamma(f) = 0$ is called a harmonic function of degree k, and we denote by Harm_k the set of all harmonic functions of degree k:

$$\operatorname{Harm}_k = \ker(\gamma|_{\mathbb{R}^{X_t}}).$$

Theorem 2.3 ([5, Theorem 7]). A set $\mathcal{B} \subset X_m$ (where $m \leq n$) of blocks is a t-design if and only if $\sum_{b \in \mathcal{B}} \widetilde{f}(b) = 0$ for all $f \in \operatorname{Harm}_k$, $1 \leq k \leq t$.

In [2], the harmonic weight enumerator associated with a binary linear code C was defined as follows.

Definition 2.4. Let C be a binary code of length n and let $f \in \operatorname{Harm}_k$. The harmonic weight enumerator associated with C and f is

$$w_{C,f}(x,y) = \sum_{c \in C} \widetilde{f}(c) x^{n-\text{wt}(c)} y^{\text{wt}(c)},$$

where we write $\widetilde{f}(\operatorname{supp}(c))$ as $\widetilde{f}(c)$ for short.

Bachoc proved the following MacWilliams-type equality.

Theorem 2.5 ([2]). Let C be a binary code of length n and $f \in \operatorname{Harm}_k$. Let $w_{C,f}(x,y)$ be the harmonic weight enumerator associated with C and f. Then

$$w_{C,f}(x,y) = (xy)^k Z_{C,f}(x,y),$$

where $Z_{C,f}$ is a homogeneous polynomial of degree n-2k and satisfies

$$Z_{C^{\perp},f}(x,y) = (-1)^k \frac{2^{n/2}}{|C|} Z_{C,f}\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

It follows from Theorem 2.3 that C_{ℓ} is a combinatorial t-design if and only if the coefficient of $x^{n-\ell}y^{\ell}$ in $w_{C,f}(x,y)$ vanishes.

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. Let $V = \mathbb{F}_2^m$ and C = RM(1, m). For $c = (\lambda(x) + b)_{x \in V}$ and $T \subset V$, we define

$$c|_T := (\lambda(x) + b)_{x \in T}.$$

Lemma 3.1. Let $T = \{0, u_1, u_2, u_3\} \in {V \choose 4}$.

(1) If $u_1 + u_2 \neq u_3$, then

$$|\{c \in C \setminus \{0, \mathbf{1}\} \mid \operatorname{wt}(c|_T) = i\}| = \begin{cases} 2^{m-3} - 1 & \text{if } i = 0, 4, \\ 2^{m-1} & \text{if } i = 1, 3, \\ 3 \cdot 2^{m-2} & \text{if } i = 2. \end{cases}$$

(2) If $u_1 + u_2 = u_3$, then

$$|\{c \in C \setminus \{0, \mathbf{1}\} \mid \operatorname{wt}(c|_T) = i\}| = \begin{cases} 2^{m-2} - 1 & \text{if } i = 0, 4, \\ 0 & \text{if } i = 1, 3, \\ 3 \cdot 2^{m-1} & \text{if } i = 2. \end{cases}$$

Proof. By identifying $(\lambda(x))_{x\in V}$ with $\lambda\in V^*$, we have $C=V^*\cup (V^*+1)$. For $i=0,1,\ldots,4$, define

$$a_i = |\{c \in C \setminus \{0, \mathbf{1}\} \mid \text{wt}(c|_T) = i\}|,$$

 $b_i = |\{c \in V^* \mid \text{wt}(c|_T) = i\}|.$

Then

$$\begin{split} b_0 &= |\{c \in V^* \mid c|_T = 0\}| \\ &= |u_1^{\perp} \cap u_2^{\perp} \cap u_3^{\perp}| \\ &= \begin{cases} 2^{m-3} & \text{if } u_1 + u_2 \neq u_3, \\ 2^{m-2} & \text{otherwise,} \end{cases} \\ b_1 &= |\{c \in V^* \mid \text{wt}(c|_T) = 1\}| \\ &= 3|\{c \in V^* \mid c(u_1) = 1, \ c(u_2) = c(u_3) = 0\}| \\ &= 3|u_2^{\perp} \cap u_3^{\perp} \setminus u_1^{\perp}| \\ &= \begin{cases} 3(|u_2^{\perp} \cap u_3^{\perp}| - |u_2^{\perp} \cap u_3^{\perp} \cap u_1^{\perp}|) & \text{if } u_1 + u_2 \neq u_3, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 3 \cdot 2^{m-3} & \text{if } u_1 + u_2 \neq u_3, \\ 0 & \text{otherwise,} \end{cases} \\ b_2 &= |\{c \in V^* \mid \text{wt}(c|_T) = 2\}| \\ &= 3|\{c \in V^* \mid c(u_1) = c(u_2) = 1, \ c(u_3) = 0\}| \\ &= 3|u_3^{\perp} \setminus (u_1^{\perp} \cup u_2^{\perp})| \\ &= 3(|u_3^{\perp}| - |u_3^{\perp} \cap u_1^{\perp}| - |u_3^{\perp} \cap u_2^{\perp}| + |u_3^{\perp} \cap u_1^{\perp} \cap u_2^{\perp}|) \\ &= \begin{cases} 3 \cdot 2^{m-3} & \text{if } u_1 + u_2 \neq u_3, \\ 3 \cdot 2^{m-2} & \text{otherwise,} \end{cases} \\ b_4 &= 0, \\ b_3 &= |V^*| - b_0 - b_1 - b_2 - b_4 \\ &= 2^m - b_0 - b_1 - b_2 \\ &= \begin{cases} 2^m - 7 \cdot 2^{m-3} & \text{if } u_1 + u_2 \neq u_3, \\ 2^m - 4 \cdot 2^{m-2} & \text{otherwise,} \end{cases} \\ &= \begin{cases} 2^{m-3} & \text{if } u_1 + u_2 \neq u_3, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \\ 0 & \text{otherwise.} \end{split}$$

Since

$$a_{i} = |\{c \in (V^{*} \setminus \{0\}) \cup ((V^{*} + \mathbf{1}) \setminus \{\mathbf{1}\}) \mid \operatorname{wt}(c|_{T}) = i\}|$$

$$= |\{c \in V^{*} \setminus \{0\} \mid \operatorname{wt}(c|_{T}) = i\}| + |\{c \in (V^{*} + \mathbf{1}) \setminus \{\mathbf{1}\} \mid \operatorname{wt}(c|_{T}) = i\}|$$

$$= b_{i} - \delta_{i,0} + |\{c \in V^{*} \setminus \{0\} \mid \operatorname{wt}(c|_{T}) = 4 - i\}|$$

$$= b_i - \delta_{i,0} + b_{4-i} - \delta_{i,4},$$

we obtain the desired results.

Proof of Theorem 1.1. Part (1) follows from Lemma 3.1, by noticing that $\operatorname{wt}(c) = 2^{m-1}$ for all $c \in C \setminus \{0, 1\}$. Part (2) follows from part (1) and Theorem 2.2.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let $V = \mathbb{F}_2^m$.

Lemma 4.1. Let C = RM(1, m) and $U \subset V$ be a three-dimensional subspace of V. Then $\{c|_{U} \mid c \in C\} = H_8$, and for each $a \in H_8$, we have

$$|\{c \in C_{2^{m-1}} \mid c|_U = a\}| = 2^{m-3}.$$

Proof. The first statement is immediate since $RM(1,3)=H_8$. As for the second statement, we may assume a has entry 0 at the coordinate $0 \in V$. Then the number of $c \in C_{2^{m-1}}$ with $c|_U = a$ is the same as the number of $\lambda \in V^*$ satisfying $\operatorname{Ker} \lambda \supset U$, which is 2^{m-3} .

Proof of Theorem 1.2. Since f is a harmonic function of degree $k \geq 1$, we have $\widetilde{f}(\emptyset) = 0$. Since $\gamma(f) = 0$ and

$$\sum_{y \in \binom{U}{k-1}} (\gamma(f))(y) = k\widetilde{f}(U),$$

we have $\widetilde{f}(U)=0$. Note that $\{c|_U\mid c\in C\}=H_8$ has codewords of weight 0,4,8 only. This implies that, for $c\in C,\ \widetilde{f}(c)=\widetilde{f}(c|_U)$ is nonzero only if $\operatorname{wt}(c|_U)=4$, or equivalently, $\operatorname{wt}(c)=2^{m-1}$. Then

$$w_{C,f}(x,y) = \sum_{c \in C, \text{ wt}(c)=2^{m-1}} \widetilde{f}(c|_U) x^{2^m - \text{wt}(c)} y^{\text{wt}(c)}$$

$$= x^{2^{m-1}} y^{2^{m-1}} \sum_{a \in H_8, \text{wt}(a)=4} |\{c \in C_{2^{m-1}} \mid c|_U = a\}| \widetilde{f}(a)$$

$$= 2^{m-3} x^{2^{m-1}} y^{2^{m-1}} \sum_{a \in H_8, \text{wt}(a)=4} \widetilde{f}(a)$$

by Lemma 4.1. Then we obtain (1). Using Theorem 2.5, we obtain (2). \Box

5. Proof of Corollary 1.3

In this section, we give a proof of Corollary 1.3.

Proof of Corollary 1.3. Let C = RM(1, m). We give two proofs.

The first proof relies on properties of Jacobi polynomials. Let $T_1 = \{0, u_1, u_2, u_3\} \in \binom{V}{4}$ and $T_2 = \{0, v_1, v_2, v_3\} \in \binom{V}{4}$. We assume that $u_1 + u_2 \neq u_3$ and $v_1 + v_2 = v_3$. By Theorem 1.1,

$$J_{C,T_1} - J_{C,T_2} = -2^{m-3}x^{2^{m-1}-4}y^{2^{m-1}-4}(wy - xz)^4.$$

The coefficient of $z^4x^{2^m-\ell}y^{\ell-4}$ in $J_{C,T_1}-J_{C,T_2}$ is non-zero whenever C_ℓ is non-empty. Hence, C_ℓ is not a 4-design.

By using Theorem 2.2, we have

$$J_{C^\perp,T_1}-J_{C^\perp,T_2}=-(x^2-y^2)^{2^{m-1}-4}(wy-xz)^4.$$

The coefficient of $z^4x^{2^m-\ell}y^{\ell-4}$ in $J_{C^{\perp},T_1}-J_{C^{\perp},T_2}$ is non-zero whenever $(C^{\perp})_{\ell}$ is non-empty. Hence, $(C^{\perp})_{\ell}$ is not a 4-design.

The second proof relies on properties of harmonic weight enumerators. Let $U \subset V$ be a three-dimensional subspace of V. Then $\{c|_{U} \mid c \in C\} = H_{8}$ by Lemma 4.1. Let $\mathcal{B} = \{\text{supp}(z) \mid z \in H_{8}, \text{ wt}(z) = 4\}$, and let τ be a transposition of two coordinates of H_{8} . Then \mathcal{B} is the set of 14 blocks of a 3-(8,4,1) design, and $|\mathcal{B} \cap \mathcal{B}^{\tau}| = 6$. Define

$$f = \sum_{z \in \mathcal{B}} z - \sum_{z \in \mathcal{B}^{\tau}} z.$$

Then $f \in \text{Harm}_4$, and

$$\sum_{\substack{a \in H_8 \\ \text{wt}(a)=4}} f(a) = |\mathcal{B}| - |\mathcal{B} \cap \mathcal{B}^{\tau}| = 8.$$

By Theorem 1.2 we have

$$w_{C,f}(x,y) = 8 \cdot 2^{m-3} x^{2^{m-1}} y^{2^{m-1}},$$

$$w_{C^{\perp},f}(x,y) = 8(xy)^4 (x^2 - y^2)^{2^{m-1} - 4},$$

and the coefficients of $x^{2^m-\ell}y^{\ell}$ in $w_{C,f}$ (resp. $w_{C^{\perp},f}$) do not vanish whenever C_{ℓ} (resp. $(C^{\perp})_{\ell}$) is non-empty. By Theorem 2.3, the proof is complete. \square

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STATEMENTS AND DECLARATIONS

Competing Interests. The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

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