



Multiple stage optimization driven group decision making method with interval linguistic fuzzy preference relations based on ordinal consistency and DEA cross-efficiency

Jinpei Liu^{1,2} · Zijie Qiang¹ · Peng Wu¹ · Pengcheng Du¹

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Abstract

Interval linguistic term (ILT) is highly useful to express decision-makers' (DMs') uncertain preferences in the decision-making process. This paper proposes a new group decision-making (GDM) method with interval linguistic fuzzy preference relations (ILFPRs) by integrating ordinal consistency improvement algorithm, cooperative game, Data Envelopment Analysis (DEA) cross-efficiency model, and stochastic simulation. Firstly, the ordinal consistency of ILFPR is developed. For improving the ordinal consistency of an ILFPR, a two-stage integer optimization model is presented to derive an ILFPR with ordinal consistency. Then, a weight-determination method for obtaining DMs' weights is presented based on cooperative games. Moreover, a DEA cross-efficiency model is presented to obtain the priorities of linguistic preference relation derived from ILFPR. Meanwhile, the expected ranking vector of ILFPR is obtained based on the DEA cross-efficiency model by integrating stochastic preference analysis and Monte Carlo stochastic simulation. Finally, a numerical example of emergency logistics selection illustrates the applicability and credibility of the proposed method.

Keywords Group decision making · Interval linguistic fuzzy preference relation · Ordinal consistency · Cooperative game · DEA cross-efficiency

✉ Peng Wu
littlepengwu@126.com

¹ School of Business, Anhui University, Hefei 230601, Anhui, China

² Department of Industrial and Systems Engineering, North Carolina State University, Raleigh, NC 27695, USA

1 Introduction

In group decision-making (GDM), the decision-maker (DM) gives his/her preference information by pairwise comparison of alternatives based on a certain attribute (or a certain criterion). Based on these preference information, preference relation (PR) is constructed (Meng et al., 2019a). In recent years, the study of GDM based on PR has become hotter and published extensive research. Consistency is performed to ensure that PRs are neither random nor illogical in pairwise comparisons. The research on PR mainly focuses on consistency, such as additive consistency (Meng et al., 2019a; Wu et al., 2020), multiplicative consistency (Wu & Tu, 2021), and ordinal consistency (Tanino, 1984; Wu & Tu, 2021; Xu et al., 2014).

Based on a minimum modification with 0–1 programming, Wu and Tu (2021) proposed a transitive linear programming model of multiplicative preference relation (MPR). Tanino (1984) proposed ordinal consistency, such as weak transitivity of fuzzy preference relation (FPR). Transitivity can be used to check the ordinal consistency of DMs' judgments. If PR is not transitive, the ranking results of alternatives may be misleading or unreasonable. Xu et al. (2014) used the score function and accuracy function to compare the interval elements, defined the weak transitivity of FPR, and proposed an algorithm to address the three-way cycle. Although the existing research has innovated the improved algorithm of additive consistency or multiplicative consistency of PRs, the modified PRs still contradict the order consistency and cannot achieve greater consistency. Therefore, it is necessary to propose a new adjustment method to make the modified PRs satisfy both additive consistency and ordinal consistency. In the following examples, the order inconsistency of PRs will be pointed out, and the order consistency adjustment algorithm will be studied.

For complex and uncertain decision-making problems, DM is more inclined to give qualitative decision-making information than quantitative decision information. As an effective preference structure, linguistic preference relation (LPR) can effectively express the preferences of DMs. In order to improve the consistency of LPR, different literature has proposed different consistency adjustment methods. Wu et al. (2020) defined an additive consistency index of LPR and an integer optimization model was proposed to derive an acceptably additive consistent LPR from one with unacceptably additive consistency.

In actual decision-making problems, due to the limitation of DM's knowledge and the complexity of decision-making, the accuracy of the preferences provided by DMs may not be guaranteed. However, the interval linguistic fuzzy preference relation (ILFPR), as an expression of preference information, can well express DMs' uncertain preference. In order to ensure the minimum number of adjustments and minimize the number of adjustments of iterative adjusting strategy for ILFPR, Meng et al. (2020) introduced a threshold of group consistency index and established some acceptable consistent iterative adjusting strategy models. Xu et al. (2019) presented a linear goal programming model to estimate the missing values of the incomplete uncertain linguistic preference relations (ULPR),

introduced a simple consistency level, and developed a consensus reaching process. Meng et al. (2019b) defined a novel additive consistency and group consensus index for ILFPRs, and then proposed a method for solving GDM problem with ILFPRs. Meng et al. (2016) built a consistency-based programming model to estimate the missing values to cope with the inconsistency ILFPRs. Garcia et al. (2012) defined consensus measures to evaluate the consensus of all DMs, and defined close measurements to measure the consistency between DMs' personal preferences and group preferences. As we can see, the existing work ignored the ordinal consistency of ILFPR. The ordinal consistency of ILFPR is related to the ranking of alternatives. However, as one of the important characteristics of PR, ordinal consistency is often ignored. Therefore, it is necessary and meaningful to study the ordinal consistency of preference relation.

When transitivity is introduced into GDM with ILFPR, the following gaps are still existing. Meng et al. (2020) defined a threshold of group consistency index and proposed iterative learning strategy models. However, virtual terms, which make problems complex, may occur after consistency adjustment derived from Meng et al., (2019b). Virtual terms may be difficult to persuade the DMs to change their opinion in this optimal result as their new preference. In addition, little research considered the transitivity of intervals as an important part. Although some basic consistencies have been achieved in literature (Meng et al., 2016, 2019a, 2019b, 2020), ILFPR is modified without ordinal consistency optimization. This paper establishes a two-stage integer model to optimize it. In GDM process, the weights of DMs are related to the aggregation of individual preferences. For GDM with ILFPRs, the weights of DMs are obtained by optimization models constructed based on acceptable consensus degree (Meng et al., 2016, 2019a, 2020) or are given by subjective weighting method (Meng et al., 2019b). These ignored the decision-making interests and decision-making psychology of DMs. The priority weights are used to rank the alternatives. The existing method for deriving priority vector of ILFPR determined the priority vector by using the row arithmetic average value of ILFPR. They did not consider the mutual influence between different alternatives.

The motivations of this paper mainly come from the above shortcomings. The main contributions of this paper are as follows:

- (1) A two-stage optimization model is established to derive an ILFPR with ordinal consistency and acceptably interval additive consistency. The two-stage optimization model contains two integer optimization models and can ensure the modified linguistic term(s) are not virtual linguistic term(s).
- (2) A weight-determination method is presented based on the cooperative game. The advantage of the cooperative game is that it considers the decision-making interests and decision-making psychology of DMs.
- (3) A data envelopment analysis (DEA) cross-efficiency model is established to obtain the final priority vector of LPR derived from ILFPR. Based on the final priority vector, the expectation of ranking vector ILFPR is obtained by integrating stochastic preference analysis and Monte Carlo stochastic simulation method.

The rest of this paper is organized as follows. Section 2 reviews the relevant concepts about linguistic term set (LTS), LPR, and ILFPR. Section 3 defines the ordinal consistency of ILFPR and develops a two-stage optimization model for dealing with an ILFPR without ordinal consistency. Based on cooperative game, Sect. 4 develops a weight-determination model to calculate the weights of DMs. In Sect. 5, an input-oriented DEA model is presented to obtain the priority vector of ILFPR, and the expected ranking vector and its credibility are calculated by analyzing the randomly extracted judgment space based on stochastic preference analysis and Monte Carlo stochastic simulation method. Section 6 presents a numerical example to illustrate the application of our proposed method. In Sect. 7, we analyze different results of the numerical example by different existing methods. Finally, some conclusions are summarized in Sect. 8.

2 Preliminaries

In this section, some concepts of LTS, LPR, and ILFPR are reviewed.

2.1 LTS

Definition 2.1 Herrera et al. (1996) Let $S = \{s_\alpha | \alpha = 0, 1, 2, \dots, 2\tau\}$ be considered as a subscript symmetric LTS, where τ is a positive integer. It satisfies the following two properties:

- (1) Ordered property: if $s_\alpha, s_\beta \in S$ and $\alpha > \beta$, then $s_\alpha > s_\beta$, $\max(s_\alpha, s_\beta) = s_\alpha$.
- (2) Negation operator: $\text{neg}(s_\alpha) = s_{2\tau-\alpha}$.

In order to expand the range of discrete LTS S , Xu (2004) introduced a continuous LTS $S' = \{s_\alpha | \alpha \in [0, 2T]\}$. $T(T > \tau)$ represents a sufficiently large positive integer. If $s_\alpha \in S$, s_α is regarded as a simple linguistic term; otherwise, s_α is viewed as a virtual linguistic term. For S' , the subscript function $I(s_\alpha) = \alpha$ is mainly adopted.

2.2 LPR

Let $X = \{x_1, x_2, \dots, x_n\} (n \geq 2)$ be the set of alternatives. According to S and X , LPR can be used to express DMs' qualitative preferences. It is defined as follow:

Definition 2.2 Xu (2005) Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be a LPR, which satisfies $\tilde{a}_{ij} \in S$, $\tilde{a}_{ij} \oplus \tilde{a}_{ji} = s_{2\tau}$, $\tilde{a}_{ii} = s_\tau$, $\forall i, j = 1, 2, \dots, n$.

Definition 2.3 Meng et al. (2019a) Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be as before. Then $\tilde{A}$ is a consistent LPR, if it satisfies:

$$\tilde{a}_{ij} = \tilde{a}_{ik} \oplus \tilde{a}_{kj} - s_\tau, i, j, k = 1, 2, \dots, n. \quad (1)$$

2.3 ILFPR

Definition 2.4 Garcia et al. (2012) Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be an ILFPR, where $\bar{a}_{ij} = [a_{ij}^L, a_{ij}^U]$ is interval linguistic variable expressing the degree of preference from alternative x_i to x_j . The $\bar{A}$ satisfies:

Definition 2.5 Meng et al. (2020) Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be an ILFPR. The interval additive consistency index $ICI(\bar{A})$ of $\bar{A}$ is defined as follows:

$$ICI(\bar{A}) = \frac{1}{\tau n(n-1)(n-2)} \sum_{k=1, k \neq i, j}^n \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\left| \min \left\{ \left(I(a_{ij}^L) - I(a_{ik}^L) - I(a_{kj}^L) + \tau \right), 0 \right\} \right| + \left| \min \left\{ \left(I(a_{ij}^U) - I(a_{ik}^U) - I(a_{kj}^U) + \tau \right), 0 \right\} \right| \right) \quad (2)$$

Let $\overline{ICI} \in [0, 1]$ be an interval additive consistency index. If $ICI(\bar{A}) \leq \overline{ICI}$, then ILFPR $\bar{A}$ is acceptably interval additive consistency. Otherwise, $\bar{A}$ is unacceptably interval additive consistency. Particularly, when $ICI(\bar{A}) = 0$, $\bar{A}$ is completely interval additive consistency.

3 Ordinal consistency analysis of ILFPR

In this section, the ordinal consistency of ILFPR is defined and a two-stage integer optimization model is established to derive an ILFPR with ordinal and acceptably interval additive consistency from one without the two properties.

3.1 The ordinal consistency of ILFPR

Inspired by Xu et al. (2014), the score function and accuracy function are defined to compare interval linguistic variables.

Definition 3.1 Let $\bar{s} = [s_\alpha, s_\beta]$ be an interval linguistic variable. Then the score function F for $\bar{s}$ is defined as:

$$F(\bar{s}) = I(s_\alpha) + I(s_\beta) - 2\tau. \quad (3)$$

The accuracy function H for $\bar{s}$ is defined as:

$$H(\bar{s}) = I(s_\alpha) + \tau - I(s_\beta). \quad (4)$$

The comparison of two interval linguistic variables is presented according to Definition 3.1.

Definition 3.2 Let $\bar{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$, $\bar{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two interval linguistic variables.

- (1) If $F(\bar{s}_1) > F(\bar{s}_2)$, then $\bar{s}_1 > \bar{s}_2$.
- (2) If $F(\bar{s}_1) = F(\bar{s}_2)$, then:

- (a) $H(\bar{s}_1) = H(\bar{s}_2)$, then $\bar{s}_1 = \bar{s}_2$;
- (b) $H(\bar{s}_1) > H(\bar{s}_2)$, then $\bar{s}_1 > \bar{s}_2$; (c) $H(\bar{s}_1) < H(\bar{s}_2)$, then $\bar{s}_1 < \bar{s}_2$.

The following Example 1 is conducted to show the feasibility and effectiveness of our proposed comparison method.

Example 1. Let $\bar{s}_1 = [s_3, s_5]$, $\bar{s}_2 = [s_4, s_4]$, $\bar{s}_3 = [s_3, s_4]$, $\bar{s}_4 = [s_2, s_3]$ be four interval linguistic variables. Based on Definition 3.2, we obtain $F(\bar{s}_1) = F(\bar{s}_2) = -2 > F(\bar{s}_3) = -3 > F(\bar{s}_4) = -5$, $H(\bar{s}_2) = 5 > H(\bar{s}_1) = 3$. Then, we have $\bar{s}_2 > \bar{s}_1 > \bar{s}_3 > \bar{s}_4$.

In Meng et al. (2016), the likelihood measure was presented to measure the probability that one interval linguistic variable is larger than others. It is used to compare the above interval linguistic variables, we obtain the ranking results as $\bar{s}_2 > \bar{s}_1 > \bar{s}_3 > \bar{s}_4$, which is same as the ranking result obtained by our proposed method.

In Garcia et al. (2012), the dominance degree was developed by summing the upper and lower bounds of interval linguistic variable for comparing interval linguistic variables. Now, we utilize it to compare the four interval linguistic variables. Then, the ranking results is $\bar{s}_1 = \bar{s}_2 > \bar{s}_3 > \bar{s}_4$. It is different from the ranking result obtained by our proposed method.

Compared with likelihood measure, our proposed method can directly compare interval linguistic variables and simplify calculation process. Most notably, $\bar{s}_1 = [s_3, s_5]$ and $\bar{s}_2 = [s_4, s_4]$, they are not same. It means that the dominance degree has shortcoming. Therefore, our proposed comparison method is more effective and reasonable.

Inspired by Xu et al. (2014), the ordinal consistency of ILFPR is formally defined as follows.

Definition 3.3 Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be an ILFPR, where $\bar{a}_{ij} = [a_{ij}^L, a_{ij}^U]$, $i, j = 1, 2, \dots, n$. For all $i, j, k = 1, 2, \dots, n$:

- (1) if $\bar{a}_{ik} > [s_\tau, s_\tau]$ and $\bar{a}_{kj} \geq [s_\tau, s_\tau]$, then we have $\bar{a}_{ij} > [s_\tau, s_\tau]$.
- (2) if $\bar{a}_{ik} \geq [s_\tau, s_\tau]$ and $\bar{a}_{kj} > [s_\tau, s_\tau]$, then we have $\bar{a}_{ij} > [s_\tau, s_\tau]$.
- (3) if $\bar{a}_{ik} = [s_\tau, s_\tau]$ and $\bar{a}_{kj} = [s_\tau, s_\tau]$, then we have $\bar{a}_{ij} = [s_\tau, s_\tau]$.

Then $\bar{A}$ has ordinal consistency.

Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be as before. According to Definition 3.3, one of the following three cases must hold: (a) $\bar{a}_{ij} > [s_\tau, s_\tau]$; (b) $\bar{a}_{ij} = [s_\tau, s_\tau]$; (c) $\bar{a}_{ij} < [s_\tau, s_\tau]$. It can be seen that when comparing the two intervals, we will consider the score value of the intervals firstly, which means the priority of score value is higher accuracy degree.

In order to describe the relationship between $\bar{a}_{ij}$ and $[s_\tau, s_\tau]$. The corresponding four cases are given as follows:

- (1) If $F(\bar{a}_{ij}) > 0$, then $p_{ij} = 1$ and $q_{ij} = 0$, it can prove that $\bar{a}_{ij} > [s_\tau, s_\tau]$;
- (2) If $F(\bar{a}_{ij}) = 0$ and $H(\bar{a}_{ij}) > 0$, then $p_{ij} = 0, q_{ij} = 1, c_{ij} = 1$ and $d_{ij} = 0$, it can prove that $\bar{a}_{ij} > [s_\tau, s_\tau]$;
- (3) If $F(\bar{a}_{ij}) = 0$ and $H(\bar{a}_{ij}) = 0$, then $p_{ij} = 0, q_{ij} = 1, c_{ij} = 0$ and $d_{ij} = 1$, it can prove that $\bar{a}_{ij} = [s_\tau, s_\tau]$;
- (4) If $F(\bar{a}_{ij}) < 0$, then $p_{ij} = 0$ and $q_{ij} = 0$, it can prove that $\bar{a}_{ij} < [s_\tau, s_\tau]$.
where p_{ij} and q_{ij} are denoted as 0–1 variables to represent three cases of $F(\bar{a}_{ij})$,
 c_{ij} and d_{ij} are denoted as 0–1 variables to represent three cases of $H(\bar{a}_{ij})$.

For the sake of clarity, the relationships are shown in Table 1, where $i, j, k = 1, 2, \dots, n, i < k < j$. The above four cases are called conditions of ordinal consistency.

3.2 A two-stage optimization model for deriving ILFPR with ordinal consistency

Ordinal consistency is the minimum requirement to ensure that there is no contradictory judgment in a given ILFPR. To check whether the original ILFPR is ordinal consistency, the requirement of transitivity is added based on the conditions of transitivity. If one of these conditions is violated, the ILFPR is not ordinal consistency. For this situation, a two-stage integer optimization model is developed to derive an ILFPR with ordinal consistency.

For $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be an ILFPR without ordinal consistency, two important missions are considered to derive an ILFPR $\bar{A}^* = (\bar{a}_{ij}^*)_{n \times n}$ with ordinal consistency and acceptably interval additive consistency. One is that the score function is used to make the ILFPR has ordinal consistency. In this process, the score matrix $M = (m_{ij})_{n \times n}$ will be obtained based on the original $\bar{A} = (\bar{a}_{ij})_{n \times n}$, where $I(m_{ij}) = I(\bar{a}_{ij}^L) + I(\bar{a}_{ij}^U)$. It is regarded as a medium to ensure that the amount of information in each interval remains unchanged, and lays the foundation for the next stage to build a model for calculating the upper and lower bounds of the interval under the premise of the same amount of information. The other is that the score value derived from the one mission remains unchanged during the process and each element of $\bar{A}$ is adjusted to make the accuracy degree of each element of the modified ILFPR $\bar{A}^* = (\bar{a}_{ij}^*)_{n \times n}$ meets the ordinal consistency. Based on

Table 1 The relationships in four cases

Case	$\bar{a}_{ij}$	p_{ij}	q_{ij}	c_{ij}	d_{ij}
(a)	$> [s_\tau, s_\tau]$	1	0	0 or 1	0 or 1
(b)	$> [s_\tau, s_\tau]$	0	1	1	0
(c)	$= [s_\tau, s_\tau]$	0	1	0	1
(d)	$< [s_\tau, s_\tau]$	0	0	0 or 1	0 or 1

the above ideas and inspired by Wu and Tu (2021), a two-stage optimization model is obtained as the following. One stage is to achieve the first mission, the other is to realize the second mission.

The first stage model (5) is as below:

$$\min J_1 = \sum_{i=1}^n \sum_{j=1}^n \left| I(m_{ij}^*) - I(m_{ij}) \right|$$

$$s.t. \begin{cases} I(m_{ij}^*) - 2\tau + N(1 - p_{ij} + q_{ij}) > 0, I(m_{ij}^*) - 2\tau \leq Np_{ij}, \\ I(m_{ij}^*) - 2\tau + N(1 + p_{ij} - q_{ij}) \geq 0, I(m_{ij}^*) - 2\tau - N(1 + p_{ij} - q_{ij}) \leq 0, \\ N(p_{ij} + q_{ij}) + 2\tau - I(m_{ij}^*) > 0, \frac{1}{2\tau} I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0, \\ I(m_{ij}^*) - 2\tau + N(2 - p_{ik} - p_{kj}) > 0, I(m_{ij}^*) - 2\tau + N(2 - q_{ik} - q_{kj}) \geq 0, \\ I(m_{ij}^*) - 2\tau - N(2 - q_{ik} - q_{kj}) \leq 0, I(m_{ij}^*) - 2\tau + N(2 - p_{ik} - q_{kj}) > 0, \\ I(m_{ij}^*) - 2\tau + N(2 - q_{ik} - p_{kj}) > 0, p_{ij} + q_{ij} \leq 1, \\ \frac{1}{\tau n(n-1)(n-2)} \sum_{i < k < j} |I(m_{ij}^*) + I(m_{kj}^*) - I(m_{ik}^*) - 2\tau| \leq \overline{CI}, \\ p_{ij}, q_{ij} \in \{0, 1\}, \quad i, j, k = 1, 2, \dots, n, \end{cases} \quad (5)$$

where $I(m_{ij}) = I(\bar{a}_{ij}^L) + I(\bar{a}_{ij}^U)$, N is represented as a large integer number. The first to sixth constraints are to determine the relationship with $I(m_{ij}^*)$, p_{ij} and q_{ij} , and the seventh to twelfth constraints are to express the transitivity of score matrix. The thirteenth constraint condition is to ensure the acceptable consistency of the score matrix, which $\overline{CI}$ is denoted as the threshold of additive consistency index.

Completely additive consistency is a special case of acceptable consistency. $CI(\bar{A}^*) = 0$ means that $I(m_{ik}^*) + I(m_{kj}^*) - I(m_{ij}^*) - 2\tau = 0, \forall i < k < j$. It turns out that to obtain a consistent $\bar{A}^*$. It is obvious that model (5) has feasible solutions. For example, a matrix $\bar{A}^* = (\bar{a}_{ij}^*)_{n \times n}$ where each $I(m_{ij}^*) = I(\bar{a}_{ij}^{L*}) + I(\bar{a}_{ij}^{U*}) = 2\tau$ is a feasible solution. Since the objective function is bounded, we have the following property according to Weierstrass' Theorem (Borwein & Lewis, 2010).

Property 3.1. *Model (5) has at least one optimal solution.*

After solving model (5), the score matrix $M = (m_{ij}^*)_{n \times n}$ of the revised ILFPR $\bar{A}^*$ is obtained. Then, to calculate its upper and lower bounds, the second stage optimization denoted by the model (6) is established by comparing the accuracy under the condition of score determination. The first to sixth constraints are to determine the relationship with $I(\bar{a}_{ij}^{L*})$, $I(\bar{a}_{ij}^{U*})$, c_{ij} and d_{ij} , and the seventh to

twelfth constraints are to express the transitivity of accuracy value, The thirteenth constraint condition is to ensure the acceptable consistency of the intervals.

$$\min J_2 = \sum_{i=1}^n \sum_{j=1}^n \left| I(\bar{a}_{ij}^{L*}) - I(\bar{a}_{ij}^L) \right| + \sum_{i=1}^n \sum_{j=1}^n \left| I(\bar{a}_{ij}^{U*}) - I(\bar{a}_{ij}^U) \right|$$

$$s.t. \begin{cases} I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(1 - c_{ij} + d_{ij}) > 0, \\ I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) \leq Nc_{ij}, I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(1 + c_{ij} - d_{ij}) \geq 0, \\ I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) - N(1 + c_{ij} - d_{ij}) \leq 0, N(c_{ij} + d_{ij}) - (I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*})) > 0, \\ \frac{1}{2\tau} (I(\bar{a}_{ij}^{L*}) + 2\tau - I(\bar{a}_{ij}^{U*})) - 0.5(c_{ij} + d_{ij}) \geq 0, I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(2 - c_{ik} - c_{kj}) > 0, \\ I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(2 - d_{ik} - d_{kj}) \geq 0, I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) - N(2 - d_{ik} - d_{kj}) \leq 0, \\ I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(2 - c_{ik} - d_{kj}) > 0, I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(2 - d_{ik} - c_{kj}) > 0, \\ |ICI(\bar{A}^*)| \leq \overline{ICI}, I(\bar{a}_{ij}^{L*}) - I(\bar{a}_{ij}^{U*}) \leq 0, c_{ij} + d_{ij} \leq 1, c_{ij}, d_{ij} \in \{0, 1\}, i, j, k = 1, 2, \dots, n. \end{cases} \quad (6)$$

where $I(m_{ij}^*) - (I(\bar{a}_{ij}^{L*}) + I(\bar{a}_{ij}^{U*})) = 0$.

Complete consistency is a special case of acceptable consistency. $ICI(\bar{A}^*) = 0$ means that $I(\bar{a}_{ij}^L) - I(\bar{a}_{ik}^L) - I(\bar{a}_{kj}^L) + \tau = I(\bar{a}_{ij}^U) - I(\bar{a}_{ik}^U) - I(\bar{a}_{kj}^U) + \tau = 0, \forall i < k < j$. It turns out that to obtain a consistent $\bar{A}^*$. It is obvious that model (5) has feasible solutions. For example, a matrix $\bar{A}^* = (\bar{a}_{ij}^*)_{n \times n}$ where each $I(\bar{a}_{ij}^{L*}) = I(\bar{a}_{ij}^{U*}) = \tau$ is a feasible solution. Since the objective function is bounded, we have the following property according to Weierstrass' Theorem (Borwein & Lewis, 2010).

Property 3.2. Model (6) has at least one optimal solution.

Remark 1. When there are multiple optimal solutions, the DMs can provide more refined suggestions about which changes need to be changed. For instance, if there are two optimal solutions for which the preference $\bar{a}_{ij}^*$ should be modified, then $\bar{a}_{ij}^*$ has the priority to be first changed since more than once it appeared in the optimal solutions. The above concept is similar to the “core” concept used in the advantage-based rough set method (Greco et al., 2001). The “core” concept in the multi-optimization solution allows DMs to create recommendations in a new way, which is worthy of further study.

Proposition 3.1 The four cases in Table 1 can be described in the model (5) and (6): suppose M is a big integer number and $\bar{A}^* = (\bar{a}_{ij}^*)_{n \times n}$ is the modified ILFPR.

Proof In the following, it will show that each of the four cases in Table 1 can be obtained by (5) and (6).

Case (a) Prove that $p_{ij} = 1, q_{ij} = 0 \Rightarrow \bar{a}_{ij} > [s_\tau, s_\tau]$. Assume it is hold that $p_{ij} = 1$ and $q_{ij} = 0$. $I(m_{ij}^*) - 2\tau + N(1 - p_{ij} + q_{ij}) > 0$ leads to $I(m_{ij}^*) - 2\tau > 0$

and $\frac{1}{2\tau}I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0$ leads to $I(m_{ij}^*) - 2\tau \geq 0$. Other constraints have no requirement for $I(m_{ij}^*)$. It follows that $F(\bar{a}_{ij}^*) = I(m_{ij}^*) - 2\tau > 0$. Therefore, if $p_{ij} = 1$ and $q_{ij} = 0$, then $\bar{a}_{ij}^* > [s_\tau, s_\tau]$ according to Definition 3.2. To sum up, it shows that the series of models can describe Case (a).

Case (b) Prove that $p_{ij} = 0, q_{ij} = 1, c_{ij} = 1, d_{ij} = 0 \Rightarrow \bar{a}_{ij} > [s_\tau, s_\tau]$. Suppose we have $p_{ij} = 0$ and $q_{ij} = 1$. $I(m_{ij}^*) - 2\tau \leq Np_{ij}I(m_{ij}^*) - 2\tau + N(1 + p_{ij} - q_{ij}) \geq 0$, $I(m_{ij}^*) - 2\tau - N(1 + p_{ij} - q_{ij}) \leq 0$, $\frac{1}{2\tau}I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0$ lead to $I(m_{ij}^*) - 2\tau \leq 0, I(m_{ij}^*) - 2\tau \geq 0, I(m_{ij}^*) - 2\tau \leq 0, I(m_{ij}^*) - 2\tau \geq 0$ respectively. It follows that $I(m_{ij}^*) - 2\tau = 0$ and other constraints have no requirement for $I(m_{ij}^*)$. Therefore, if $p_{ij} = 0$ and $q_{ij} = 1$, then $F(\bar{a}_{ij}^*) = I(m_{ij}^*) - 2\tau = 0$. Then suppose we have $c_{ij} = 1$ and $d_{ij} = 0$. $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(1 - c_{ij} + d_{ij}) > 0$ leads to $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) > 0$ and $\frac{1}{2\tau}(I(\bar{a}_{ij}^{L*}) + 2\tau - I(\bar{a}_{ij}^{U*})) - 0.5(c_{ij} + d_{ij}) \geq 0$ leads to $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) \geq 0$. Other constraints have no requirement for $I(\bar{a}_{ij}^{L*})$ or $I(\bar{a}_{ij}^{U*})$. It follows that $H(\bar{a}_{ij}^*) = I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) > 0$. Therefore, if $p_{ij} = 0, q_{ij} = 1, c_{ij} = 1, d_{ij} = 0$, then $\bar{a}_{ij}^* > [s_\tau, s_\tau]$ according to Definition 3.2. To sum up, it shows that the series of models can describe Case (b).

Case (c) Prove that $p_{ij} = 0, q_{ij} = 1, c_{ij} = 0, d_{ij} = 1 \Rightarrow \bar{a}_{ij} = [s_\tau, s_\tau]$. Suppose we have $p_{ij} = 0$ and $q_{ij} = 1$. $I(m_{ij}^*) - 2\tau \leq Np_{ij}I(m_{ij}^*) - 2\tau + N(1 + p_{ij} - q_{ij}) \geq 0$, $I(m_{ij}^*) - 2\tau - N(1 + p_{ij} - q_{ij}) \leq 0$, $\frac{1}{2\tau}I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0$ lead to $I(m_{ij}^*) - 2\tau \leq 0, I(m_{ij}^*) - 2\tau \geq 0, I(m_{ij}^*) - 2\tau \leq 0, I(m_{ij}^*) - 2\tau \geq 0$ respectively. It follows that $I(m_{ij}^*) - 2\tau = 0$ and other constraints have no requirement for $I(m_{ij}^*)$. Therefore, if $p_{ij} = 0$ and $q_{ij} = 1$, then $F(\bar{a}_{ij}^*) = I(m_{ij}^*) - 2\tau = 0$. Then we suppose that $c_{ij} = 0$ and $d_{ij} = 1$. $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) \leq Nc_{ij}$, $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) + N(1 + c_{ij} - d_{ij}) \geq 0$, $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) - N(1 + c_{ij} - d_{ij}) \leq 0$ and $(I(\bar{a}_{ij}^{L*}) + 2\tau - I(\bar{a}_{ij}^{U*})) - N(c_{ij} + d_{ij}) \geq 0$ lead to $I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) = 0$. It follows that $H(\bar{a}_{ij}^*) = I(\bar{a}_{ij}^{L*}) + \tau - I(\bar{a}_{ij}^{U*}) = 0$ and other constraints have no requirement for $I(\bar{a}_{ij}^{L*})$ or $I(\bar{a}_{ij}^{U*})$. Therefore, if $p_{ij} = 0, q_{ij} = 1, c_{ij} = 0, d_{ij} = 1$, then $\bar{a}_{ij} = [s_\tau, s_\tau]$ according to Definition 3.2. To sum up, it shows that the series of models can describe Case (c).

Case (d) Prove that $p_{ij} = 0, q_{ij} = 0 \Rightarrow \bar{a}_{ij} < [s_\tau, s_\tau]$. When $p_{ij} = 0$ and $q_{ij} = 0$, $I(m_{ij}^*) - 2\tau \leq Np_{ij}$, $N(p_{ij} + q_{ij}) + 2\tau - I(m_{ij}^*) > 0$ and $\frac{1}{2\tau}I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0$

lead to $I(m_{ij}^*) - 2\tau < 0$. It follows that $H(\bar{a}_{ij}^*) = I(m_{ij}^*) - 2\tau < 0$. Therefore, if $p_{ij} = 0$ and $q_{ij} = 0$, then $\bar{a}_{ij} < [s_\tau, s_\tau]$. To sum up, it shows that the series of inequalities can describe Case (d).

By solving model (6), the upper and lower bounds of interval linguistic variable can be obtained. Hence, we can obtain the revised ILFPR $\bar{A}^*$ by solving the two-stage optimization model.

Example 2. Let $\bar{A}_1$ be an ILFPR shown as follows:

$$\bar{A}_1 = \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_6, s_7] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_7, s_7] & [s_2, s_4] \\ [s_3, s_4] & [s_3, s_3] & [s_5, s_5] & [s_5, s_6] \\ [s_6, s_7] & [s_6, s_8] & [s_4, s_5] & [s_5, s_5] \end{bmatrix}$$

According to Definition 3.3, $\bar{a}_{23} = [s_7, s_7] > [s_5, s_5]$, $\bar{a}_{34} = [s_5, s_6] > [s_5, s_5]$, but $\bar{a}_{24} = [s_2, s_4] < [s_5, s_5]$. According to Eq. (2), $ICI(\bar{A}_1) = 0.1917 > \overline{ICI} = 0.1$. It means that $\bar{A}_1$ doesn't satisfy ordinal consistency and acceptable interval additive consistency. By solving the models (5) and (6), we can obtain $\bar{A}_1^*$:

$$\bar{A}_1^* = \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_6, s_7] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_{5.6367}, s_7] & [s_2, s_4] \\ [s_3, s_4] & [s_3, s_{4.3633}] & [s_5, s_5] & [s_4, s_6] \\ [s_6, s_7] & [s_6, s_8] & [s_4, s_6] & [s_5, s_5] \end{bmatrix}.$$

It can be seen that the revised linguistic terms $s_{5.6367}$ and $s_{4.3633}$ don't belong to $\widehat{S}$, namely, are virtual terms. In actual decision-making problems, it may be hard to persuade DMs to adopt these optimization results as their new preferences. Inspired by Wu et al. (2020), we hope to add some constraint conditions to ensure the optimal adjusted linguistic terms belong to $\widehat{S}$, i.e., $I(a_{ij}^{L*}), I(a_{ij}^{U*}) \in \{0, 1, \dots, 2\tau\}$ to overcome this drawback.

Based on the model (5) and model (6), some constraint conditions are introduced, respectively. Add integer constraints to the model, therefore, model (7) and model (8) can be developed as follows:

$$\begin{aligned}
\min J_3 &= \sum_{i=1}^n \sum_{j=1}^n |I(m_{ij}^*) - I(m_{ij})| \\
s.t. &\begin{cases} I(m_{ij}^*) - 2\tau + N(1 - p_{ij} + q_{ij}) > 0, \\ I(m_{ij}^*) - 2\tau \leq Np_{ij}, I(m_{ij}^*) - 2\tau + N(1 + p_{ij} - q_{ij}) \geq 0, \\ I(m_{ij}^*) - 2\tau - N(1 + p_{ij} - q_{ij}) \leq 0, N(p_{ij} + q_{ij}) + 2\tau - I(m_{ij}^*) > 0, \\ \frac{1}{2\tau} I(m_{ij}^*) - (p_{ij} + q_{ij}) \geq 0, I(m_{ij}^*) - 2\tau + N(2 - p_{ik} - p_{kj}) > 0, \\ I(m_{ij}^*) - 2\tau + N(2 - q_{ik} - q_{kj}) \geq 0, I(m_{ij}^*) - 2\tau - N(2 - q_{ik} - q_{kj}) \leq 0, \\ I(m_{ij}^*) - 2\tau + N(2 - p_{ik} - q_{kj}) > 0, I(m_{ij}^*) - 2\tau + N(2 - q_{ik} - p_{kj}) > 0, \\ \frac{1}{\tau n(n-1)(n-2)} \sum_{i < k < j} |I(m_{ik}^*) + I(m_{kj}^*) - I(m_{ij}^*) - 2\tau| \leq \overline{CI}, \\ m_{ij}^* \in \{s_0, s_1, \dots, s_{4\tau}\}, p_{ij} + q_{ij} \leq 1, p_{ij}, q_{ij} \in \{0, 1\}, i, j, k = 1, 2, \dots, n. \end{cases}
\end{aligned} \tag{7}$$

$$\begin{aligned}
\min J_4 &= \sum_{i=1}^n \sum_{j=1}^n \left| I(\overline{a}_{ij}^{L*}) - I(\overline{a}_{ij}^L) \right| + \sum_{i=1}^n \sum_{j=1}^n \left| I(\overline{a}_{ij}^{U*}) - I(\overline{a}_{ij}^U) \right| \\
s.t. &\begin{cases} I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(1 - c_{ij} + d_{ij}) > 0, \\ I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) \leq Nc_{ij}, I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(1 + c_{ij} - d_{ij}) \geq 0, \\ I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) - N(1 + c_{ij} - d_{ij}) \leq 0, N(c_{ij} + d_{ij}) - (I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*})) > 0, \\ \frac{1}{2\tau} (I(\overline{a}_{ij}^{L*}) + 2\tau - I(\overline{a}_{ij}^{U*})) - 0.5(c_{ij} + d_{ij}) \geq 0, I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(2 - c_{ik} - c_{kj}) > 0, \\ I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(2 - d_{ik} - d_{kj}) \geq 0, I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) - N(2 - d_{ik} - d_{kj}) \leq 0, \\ I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(2 - c_{ik} - d_{kj}) > 0, I(\overline{a}_{ij}^{L*}) + \tau - I(\overline{a}_{ij}^{U*}) + N(2 - d_{ik} - c_{kj}) > 0, \\ ICI(\overline{A}^*) \leq \overline{ICI}, \overline{a}_{ij}^{L*}, \overline{a}_{ij}^{U*} \in \{s_0, s_1, \dots, s_{2\tau}\}, I(\overline{a}_{ij}^{L*}) - I(\overline{a}_{ij}^{U*}) \leq 0, \\ c_{ij} + d_{ij} \leq 1, c_{ij}, d_{ij} \in \{0, 1\}, i, j, k = 1, 2, \dots, n, \end{cases}
\end{aligned} \tag{8}$$

By solving models (7) and (8), we can obtain the revised ILFPR $\overline{A}^*$, which satisfies ordinal consistency and acceptable interval additive consistency.

Example 3. Let $\overline{A}_1$ be as before. By solving the model (7) and (8), we can obtain.

$$\bar{A}_1^* = \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_6, s_7] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_7, s_7] & [s_2, s_4] \\ [s_3, s_4] & [s_3, s_3] & [s_5, s_5] & [s_2, s_6] \\ [s_6, s_7] & [s_6, s_8] & [s_4, s_8] & [s_5, s_5] \end{bmatrix}.$$

4 Derive the weights of DMs in GDM with ILFPRs via cooperative games

In GDM, different group members may represent different decision-making interests and have different decision-making psychology, they can be seen as different alliances in a cooperative game. It can be used to calculate the importance of each DM. The Shapley value (Kalai & Samet, 1987) provides an objective method to determine the importance of each player's contribution in cooperative game. Inspired by Tao et al. (2015), this section presents a weight-determination method based on cooperative games to derive DMs' weights in GDM with ILFPRs.

4.1 Description of GDM with ILFPRs

Let $X = \{x_1, x_2, \dots, x_n\} (n \geq 2)$ be the set of alternatives and all to be evaluated by DMs, denoted as $D = \{d_1, d_2, \dots, d_K\} (K \geq 2)$. The d_k provides judgement by using an ILFPR $\bar{A}^{(k)} (i, j \in \{1, 2, \dots, n\}, k \in \{1, 2, \dots, K\})$ and $\bar{A}^{(k)*} = (\bar{a}_{ij}^{(k)*})_{n \times n}$ is the revised ILFPR of $\bar{A}^{(k)}$. Given that $W = (w_1, w_2, \dots, w_K)^T$ is a weighting vector of DMs and it satisfies $\sum_{k=1}^K w_k = 1$ and $w_k \geq 0$. The comprehensive ILFPR denoted as $\bar{A}^{(c)} = (\bar{a}_{ij}^{(c)})_{n \times n} = \left([\bar{a}_{ij}^{L(c)}, \bar{a}_{ij}^{U(c)}] \right)_{n \times n}$ can be computed by

$$\bar{A}^{(c)} = \sum_{k=1}^K w_k \bar{A}^{(k)*} = \left(w_k \bar{a}_{ij}^{(k)*} \right)_{n \times n} = \left(\left[\sum_{k=1}^K w_k \bar{a}_{ij}^{L(k)*}, \sum_{k=1}^K w_k \bar{a}_{ij}^{U(k)*} \right] \right)_{n \times n} \quad (9)$$

4.2 The weight-determination method based on cooperative games

For each $x_i \in X$, Liao et al. (2018) defined the interval linguistic distance measure based on the symmetric LTS, and the interval linguistic variables in our paper are defined based on the asymmetric LTS. Inspired by Liao et al. (2018), we define the distance measure of two interval linguistic variables as follows.

Definition 4.1. Let $W = (w_1, w_2, \dots, w_K)^T$ and $\bar{A}^{(c)} = (\bar{a}_{ij}^{(c)})_{n \times n}$ be as before. The deviation function is defined by:

$$dev_{ik} = \sum_{j=1}^n d(\bar{a}_{ij}^{(k)*}, \bar{a}_{ij}^{(c)}) \quad (10)$$

where $d(\bar{a}_{ij}^{(k)*}, \bar{a}_{ij}^{(c)})$ is defined by

$$d(\bar{a}_{ij}^{(c)}, a_{ij}^{(k)*}) = \sqrt{\frac{1}{8\tau^2} \left(\left(I(\bar{a}_{ij}^{L(c)}) - I(a_{ij}^{L(k)*}) \right)^2 + \left(I(\bar{a}_{ij}^{U(c)}) - I(a_{ij}^{U(k)*}) \right)^2 \right)} \quad (11)$$

It is expected in GDM that DMs' judgments become consistent as much as possible. However, the deviation is inevitable existing in the decision-making process. Therefore, a cooperative game method is established to distribute DMs' contributions and balance the deviation of all DMs'.

According to Eqs. (10–11), the following concepts of GDM with ILFPRs are defined.

Definition 4.2. $Dev = (dev_{ik})_{n \times K}$ is called the absolute deviation matrix of GDM with ILFPRs.

Definition 4.3. For a GDM with ILFPRs, $E = Dev^T \cdot Dev = (e_{ki})_{K \times K}$ is the group decision-making error information matrix.

Proposition 4.1. According to Eq. (11) $dev_{ik} = dev_{ki}$.

Proof. The conclusion that $dev_{ik} = dev_{ki}$ will be proved as.

$$\sum_{j=1}^n d(\bar{a}_{ij}^{(c)}, \bar{a}_{ij}^{(k)*}) = \sum_{j=1}^n d(\bar{a}_{ji}^{(c)}, \bar{a}_{ji}^{(k)*}). \quad (12)$$

Because $A = (a_{ij})_{n \times n}$ and $\bar{A}^{(k)*} = (\bar{a}_{ij}^{(k)*})_{n \times n}$ are complementary, it follows that

$$d(\bar{a}_{ij}^{(c)}, a_{ij}^{(k)*}) = \sqrt{\frac{1}{8\tau^2} \left(\left(I(\bar{a}_{ij}^{L(c)}) - I(a_{ij}^{L(k)*}) \right)^2 + \left(I(\bar{a}_{ij}^{U(c)}) - I(a_{ij}^{U(k)*}) \right)^2 \right)} = d(\bar{a}_{ji}^{(c)}, \bar{a}_{ji}^{(k)*})$$

which proves the Proposition 4.1.

Proposition 4.2. The group decision-making error information matrix E is symmetric.

The element on the original diagonal of E can be provided as decision error of the corresponding DM, the cooperative game method is adopted through the reciprocal matrix. In order to derive the weighting vector of DMs, the following weight-determination method is defined by Algorithm 1, which is provided as follows:

Algorithm 1

Step 1 Input ILFPRs $\bar{A}^{(k)}$ ($k = 1, 2, \dots, K$). Calculating an initial weighting vector of DMs, denoted as $L = (l_1, l_2, \dots, l_K)^T$, according to.

$$l_k = \left(\sum_{k=1}^K e_{kk}^{-1} \right)^{-1} \cdot e_{kk}^{-1}, \quad k = 1, 2, \dots, K, \quad (13)$$

where e_{kk} is the k -th diagonal element of the group decision-making error information matrix E .

Step 2 There are 2^K coalitions. Then, the characteristic functions of coalitions are composed of DMs. Based on the initial weighting vector, the cooperative result of the characteristic function of a coalition $s \in 2^K$ is defined by

$$v(s) = -\Lambda_s^T E_s \Lambda_s \quad (14)$$

where $\Lambda_s = \{\lambda_p, p \in s\}$ and $E_s = (e_{tk})_{n \times n}$, $t, k \in s$ ($|s|$ is the cardinality of s).

Step 3 Determining the contribution index of each DM in the group coalition. By applying the Shapley value method, the contribution index $\varphi_k(v)$ is defined as follows:

$$\varphi_k(v) = \frac{1}{K!} \sum_{s \subset 2^K} [v(s) - v(s - \{k\})] \quad (15)$$

and further transformed according to Borkotokey and Mesiar (2014):

$$\varphi_k(v) = \sum_{k \in s} \frac{(K - |s|)! \cdot (|s| - 1)!}{K!} [v(s) - v(s - \{k\})] \quad (16)$$

where s represents the subset in $\{1, 2, \dots, K\}$ that contains $\{k\}$.

Step 4 Calculating the updated weighting vector using contribution indices.

$$w_k = \frac{1}{\varphi_k(v)} \bigg/ \sum_{k=1}^K \varphi_k(v), \quad k = 1, 2, \dots, K. \quad (17)$$

Step 5 Determining whether the novel weighting vector meets the stop condition: If $W_{t+1}^T E_{t+1} W_{t+1} - W_t^T E_t W_t \leq \delta$ where δ is a predetermination threshold, then the final weighting vector is derived. Otherwise, the calculation returns to Step 1 and $W_t = W_{t+1}$.

Step 6 Output the weighting vector W of DMs.

After the calculation of the above steps, the final weighting vector is obtained. Meanwhile, the process of Algorithm 1 is shown in Fig. 1. To clearly explain Algorithm 1, Example 4 is provided as follows.

Example 4 Let $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4$ be four ILFPRs given by four DMs:

$$\begin{aligned}\bar{A}_1 &= \begin{bmatrix} [s_5, s_5] & [s_3, s_4] & [s_6, s_7] \\ [s_6, s_7] & [s_5, s_5] & [s_2, s_3] \\ [s_3, s_4] & [s_7, s_8] & [s_5, s_5] \end{bmatrix}, & \bar{A}_2 &= \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_1, s_2] \\ [s_6, s_7] & [s_8, s_9] & [s_5, s_5] \end{bmatrix} \\ \bar{A}_3 &= \begin{bmatrix} [s_5, s_5] & [s_7, s_9] & [s_4, s_5] \\ [s_1, s_3] & [s_5, s_5] & [s_2, s_6] \\ [s_5, s_6] & [s_4, s_8] & [s_5, s_5] \end{bmatrix}, & \bar{A}_4 &= \begin{bmatrix} [s_5, s_5] & [s_7, s_9] & [s_4, s_6] \\ [s_1, s_3] & [s_5, s_5] & [s_2, s_3] \\ [s_4, s_6] & [s_7, s_8] & [s_5, s_5] \end{bmatrix}.\end{aligned}$$

Based on Algorithm 1, the initial weighting vector is that $W_0 = (0.4064, 0.2434, 0.0578, 0.2923)^T$. Algorithm 1 reaches the termination condition after 24 iterations and the related results are listed in Table 2.

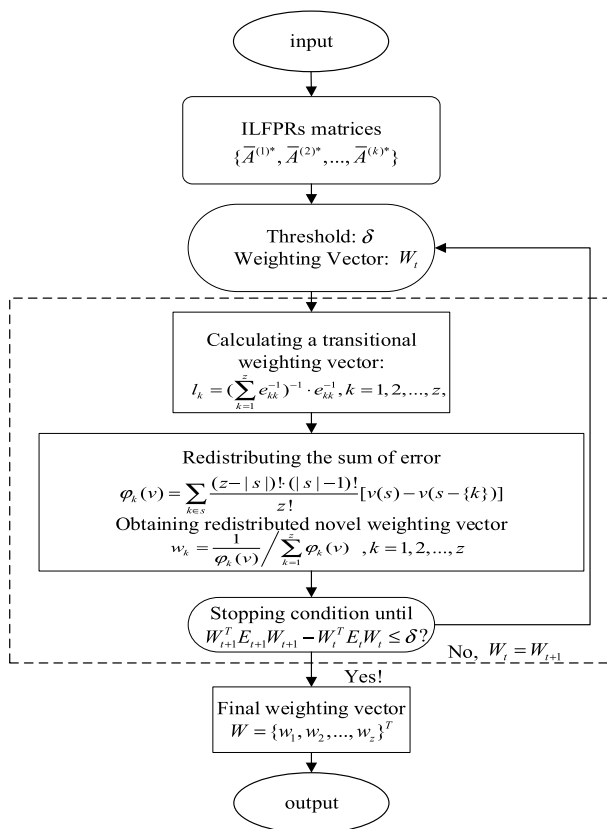


Fig. 1 The weight-determination method based on cooperative games

Theorem 4.1. Let E' be the decision error information matrix corresponding to W , then $W'^T E W' \geq W^T E' W$. The sequence of the sum of decision errors $W_t^T E_t W_t$ is convergent.

Proof. Owen (1995) has proved the following conclusion that $\sum_{k=1}^K \varphi_k(v) = v(O)$, Eq. (16) is being seen as a valid payment plan for $v(O)$, which satisfying: $v(\emptyset) = 0$, $v(O) \geq \sum_{k=1}^K v\{k\}$, $\hat{v}(O) = \sum_{k=1}^K \hat{v}_k(v)$ and $\hat{T}(v) = (\hat{v}_1(v), \dots, \hat{v}_K(v))$, where O is the largest coalition. The contribution distribution of the coalition is reset by utilizing the obtained weighting vector W , and it can be concluded that $\hat{v}_k(v) \geq v_k(v)$, $k = 1, 2, \dots, K$. Otherwise, the redistribution would be ended. Thus, it follows that:

$$\hat{v}(O) = \sum_{k=1}^K \hat{v}_k(v) \geq \sum_{k=1}^K v_k(v) = v(O). \quad (18)$$

□

According to $v(O) = -\Lambda^T E \Lambda$, $\hat{v}(O) = -\Lambda^T E' \Lambda$ and Eq. (18), we have

$$-W^T E' W \geq -\Lambda^T E \Lambda. \quad (19)$$

Under the premise of inputting an initial weighting vector W' , Eq. (17) determines a novel weighting vector W , which can decrease the sum of decision error. Assuming that $W' = W_0$, $E = E_0$ and $W = W_1$, $E' = E_1$, repeating the weighting algorithm mentioned above until $W_{t+1}^T E_{t+1} W_{t+1} - W_t^T E_t W_t \leq \delta$ and a series of weighting vectors, denoted as $W_2, W_3, \dots$, can be obtained. Correspondingly, the sums of decision errors are denoted as $E_2, E_3, \dots$ and, respectively, shown as

$$W_0^T E_0 W_0 \geq W_1^T E_1 W_1 \geq W_2^T E_2 W_2 \geq \dots, \quad (20)$$

In addition, we have $W_t^T E_t W_t \geq 0$. According to the monotone bounded principle, Algorithm 1 is convergent. It proves Theorem 4.1.

Table 2 The related results generated by Algorithm 1

Times	Error	The weights of DMs			
		w_1	w_2	w_3	w_4
0		0.4064	0.2434	0.0578	0.2923
1	0.05558665	0.23247124	0.19544231	0.34897377	0.22311268
2	0.02862190	0.35019475	0.24951711	0.22880026	0.17148789
3	0.00354893	0.27423959	0.21065520	0.30197061	0.21313460
4	0.00750106	0.32374011	0.23655531	0.25577997	0.18392461
...	...	...	...	...	...
23	0.00000104	0.30436015	0.22616558	0.27420140	0.19527287
24	0.00000067	0.30436935	0.22617054	0.27419273	0.19526737

5 A new novel GDM method with DEA crossing-efficiency and stochastic preference analysis

DEA is a non-parametric programming technique that has been successfully applied to evaluate the relative efficiency of a set of decision-making units (DMUs). It has been widely studied and applied (Liu et al., 2019a, 2019b). DEA is a data-oriented method, which is used to identify the efficiency production frontier and evaluate the relative efficiency of the decision-making unit with a certain amount of output generated by multiple production factor inputs. It can avoid the interference of subjective factors, and makes the evaluation results recognized by DMs. The DEA cross-efficiency model can consider comprehensive evaluation efficiency to avoid the generation of multiple optimal self-evaluation efficiencies. In this section, the DEA method and stochastic preference analysis will be used to obtain the final decision-making results, employing a novel benevolent DEA cross-efficiency model to GDM with ILFPRs.

5.1 Ranking vector of LPR based on DEA cross-efficiency

In order to facilitate the construction of the DEA cross-efficiency model, a derived function is proposed on the premise of keeping the information of ILFPR unchanged.

Definition 5.1. Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be a LPR on LTS $S' = \{s_\alpha | \alpha \in [0, 2\tau]\}$. Then.

$$\phi(\tilde{a}_{ij}) = I\left(\frac{1}{2\tau}\tilde{a}_{ij}\right), \quad i, j = 1, 2, \dots, n, \quad (21)$$

then ϕ is called the derived function of LPR, where $0 \leq \phi(\tilde{a}_{ij}) \leq 1$. $R = (\phi(\tilde{a}_{ij}))_{n \times n}$ is the derived matrix of $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$.

Theorem 5.1. If LPR $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ satisfies additive consistency, then its derived matrix $R = (\phi(\tilde{a}_{ij}))_{n \times n}$ also satisfies additive consistency.

Proof. Since the LPR $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ satisfies the additive consistency, it has

$$\phi(\tilde{a}_{ii}) = I\left(\frac{1}{2\tau}\tilde{a}_{ii}\right) = 0.5, \quad \phi(\tilde{a}_{ij}) + \phi(\tilde{a}_{ji}) = I\left(\frac{1}{2\tau}(\tilde{a}_{ij} + \tilde{a}_{ji})\right) = 1.$$

It is further concluded that:

$$\phi(\tilde{a}_{ik}) + \phi(\tilde{a}_{kj}) = I\left(\frac{1}{2\tau}(\tilde{a}_{ij} + s_\tau)\right) = \phi(\tilde{a}_{ij}) + 0.5, \quad i, j, k = 1, 2, \dots, n.$$

It means that the derived matrix $R = (\phi(\tilde{a}_{ij}))_{n \times n}$ is a FPR with additive consistency. $\square$

Let $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ be the ranking vector of $\tilde{A}$ and it satisfies $\varpi_i \geq 0$, $\sum_{i=1}^n \varpi_i = 1$. If $\tilde{A}$ has completely additive consistency, then $\phi(\tilde{a}_{ij}) = 0.5(\varpi_i - \varpi_j + 1)$ (Xu & Chen, 2008). For the additive consistent LPR $\tilde{A}$, DEA can be used to calculate its ranking vector. Let s_τ be the dummy input variable for all DMUs. Therefore, a DEA model established to calculate the efficiency value of alternative $x_j (j = 1, 2, \dots, n)$ is as follows:

$$\begin{aligned} & \max \gamma_d \\ & s.t. \begin{cases} \sum_{i=1}^n \xi_i \phi(\tilde{a}_{ik}) \geq \gamma_d \phi(\tilde{a}_{dk}), k = 1, 2, \dots, n, \\ \phi(s_\tau) \sum_{i=1}^n \xi_i \leq \phi(s_\tau), \\ \gamma_d \text{ free}, \xi_i \geq 0, \\ i = 1, 2, \dots, n. \end{cases} \end{aligned} \quad (22)$$

where ξ_i implies the contribution ratio in alternative x_i to build the composite unit, γ_d represents the efficiency value of alternative x_d . The ranking vector of $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ can be derived (Liu et al., 2019b):

$$\varpi_i = \frac{n+1}{n} - \frac{1}{n} \max_{1 \leq i \leq n} \{\gamma_i^*\} \sum_{i=1}^n \frac{1}{\gamma_i^*} + \left(\max_{1 \leq i \leq n} \{\gamma_i^*\} - \gamma_i^* \right) \frac{1}{\gamma_i^*} \quad (23)$$

where γ_i^* is the optimal solution of model (22), the efficiency score of x_i is $1/\gamma_i^*$.

When the LPR is not completely consistent, there could be two or more alternatives whose efficiency scores are equal to 1, and some priority weights derived from Eq. (23) are equivalent. To resolve this problem, we consider combining self-evaluation efficiency and peer evaluation efficiency of each alternative. Thus, a DEA cross-efficiency model is developed to obtain the ranking vector of LPR.

Firstly, an input-oriented DEA model is developed as follows:

$$\begin{aligned} & \max \theta_{dd} = \sum_{j=1}^N u_{dj} \phi(\tilde{a}_{dj}) \\ & s.t. \begin{cases} \sum_{j=1}^n u_{dj} \phi(\tilde{a}_{dj}) - \phi(s_\tau) \cdot v_d \leq 0, i = 1, 2, \dots, n \\ v_d \cdot \phi(s_\tau) = 1, v_d \geq 0, \\ u_{dj} \geq 0, j, r = 1, 2, \dots, n. \end{cases} \end{aligned} \quad (24)$$

In model (24), v_d and u_{dj} represent the weights of inputs and outputs, respectively. θ_{dd} is the self-evaluated efficiency of x_d .

Then, taking the self-evaluation efficiency θ_{dd}^* obtained by the model (24) as the ideal point, the benevolent DEA cross-efficiency model is adopted on the basis of the minimum deviation. Under the condition of keeping the self-evaluation efficiency of DMU_d constant, this model makes the evaluation value of other DMUs evaluated by DMU_d and the self-evaluation efficiency of other DMUs are as close as possible, which is more reasonable.

Let $\delta_{id} = \sum_{j=1}^n \omega_{dj} \phi(\tilde{a}_{ij}) - \theta_{ii}^* v_d \phi(s_\tau)$, $\delta_{id}^+ = \begin{cases} \delta_{id}, \delta_{id} \geq 0 \\ 0, \delta_{id} < 0 \end{cases}$, $\delta_{id}^- = \begin{cases} 0, \delta_{id} \geq 0 \\ -\delta_{id}, \delta_{id} < 0 \end{cases}$, then $|\delta_{id}| = \delta_{id}^+ + \delta_{id}^-$, $\delta_{id} = \delta_{id}^+ - \delta_{id}^-$. The benevolent DEA cross-efficiency model is as follows (Wang & Chin, 2010):

$$\min \sum_{i=1}^n (\delta_{id}^+ + \delta_{id}^-) \quad (25)$$

$$s.t. \begin{cases} \sum_{j=1}^n \omega_{dj} \left(\sum_{i=1}^n \phi(\tilde{a}_{ij}) \right) + v_d \left(\sum_{i=1}^n \phi(s_\tau) \right) = n \\ \sum_{r=1}^n \omega_{dj} \phi(\tilde{a}_{dj}) - \theta_{dd}^* v_d \phi(s_\tau) = 0 \\ \sum_{j=1}^n \omega_{dj} \phi(\tilde{a}_{ij}) - \theta_{ii}^* v_d \phi(s_\tau) - \delta_{id}^+ + \delta_{id}^- = 0, \\ \omega_{dj}, v_d, \delta_{id}^+, \delta_{id}^- \geq 0, d, i, j = 1, 2, \dots, n \end{cases}$$

where $\phi(s_\tau)$ denotes the dummy input of each DMU. v_d and ω_{dj} ($j = 1, 2, \dots, n$) are the weights of inputs and outputs respectively. δ_{id}^+ and δ_{id}^- are the deviation variable.

According to the optimal weights ($\omega_{d1}^*, \omega_{d2}^*, \dots, \omega_{dn}^*, v_d^*$) of the model (25), we can calculate the cross-efficiency of alternative x_i :

$$\theta_{di} = \frac{\sum_{j=1}^n \omega_{dj}^* \cdot \phi(\tilde{a}_{ij})}{v_d^* \phi(s_\tau)}, \quad d, i = 1, 2, \dots, n. \quad (26)$$

Furthermore, combined with the self-evaluation value and other evaluation values of each alternative, the final cross-efficiency value of x_i can be obtained:

$$\theta_i = \frac{1}{n} \sum_{d=1}^N \theta_{di}, \quad i = 1, 2, \dots, n. \quad (27)$$

Therefore, based on the final cross-efficiency value, we can calculate the final ranking vector of the LPR as follows:

$$\varpi_i = \frac{n+1}{n} - \frac{1}{n} \max_{1 \leq i \leq n} \left\{ \frac{1}{\theta_i} \right\} \sum_{i=1}^n \theta_i + \left(\max_{1 \leq i \leq n} \left\{ \frac{1}{\theta_i} \right\} - \frac{1}{\theta_i} \right) \theta_i, \quad i = 1, 2, \dots, n. \quad (28)$$

5.2 A GDM method with ILFPRs based on stochastic preference analysis

Based on the DEA cross-efficiency model, a method to calculate the ranking vector with LPR is proposed. According to all the possible values of the random variable, Monte Carlo simulation is used for statistical analysis to make GDM of ILFPRs.

From Sect. 4, the weights of experts have been derived as $W = (w_1, w_2, \dots, w_z)^T$. According to Eq. (9), the comprehensive ILFPR $\bar{A}^{(c)} = (\bar{a}_{ij}^{(c)})_{n \times n}$ is obtained, where $\bar{a}_{ij}^{(c)} = \sum_{k=1}^z w_k \bar{a}_{ij}^{(k)*} = [\bar{a}_{ij}^{L(c)}, \bar{a}_{ij}^{U(c)}]$, $\bar{a}_{ij}^{L(c)}$ and $\bar{a}_{ij}^{U(c)}$ are the lower and the upper bounds respectively. Assume that the random variable sets $\tilde{A} = \{\tilde{a}_{ij} | i, j = 1, 2, \dots, n; i \leq j\}$, where $\tilde{a}_{ij}$ is uniformly distributed over $[\bar{a}_{ij}^{L(c)}, \bar{a}_{ij}^{U(c)}]$, and $\tilde{a}_{ji} = \text{neg}(\tilde{a}_{ij})$. $f(\tilde{a}_{ij})$ is regarded as probability density function of $\tilde{a}_{ij}$. For $\tilde{A}$, the ranking vector $\varpi(\tilde{A}) = (\varpi_1(\tilde{A}), \varpi_2(\tilde{A}), \dots, \varpi_n(\tilde{A}))^T$ can be calculated based on the DEA cross-efficiency model. Then, the rank of the alternative x_i is as follows:

$$\text{rank}_i(\varpi(\tilde{A})) = 1 + \sum_{k=1}^n o(\varpi_k(\tilde{A}) > \varpi_i(\tilde{A})) = r,$$

where $o(\text{true}) = 1$, $o(\text{false}) = 0$. The group preference relation space $P_i^r(\varpi(\tilde{A}))$ with alternative x_i that ranks the r -th is as follows:

$$P_i^r(\varpi(\tilde{A})) = \{\tilde{A} \in \bar{A}^{(c)} | \text{rank}_i(\varpi(\tilde{A})) = r\} \quad (29)$$

In addition, by integrating the random vector $\tilde{A}$ in the whole preference space $\bar{A}^{(c)}$, the expected ranking vector of $\bar{A}^{(c)}$ can be obtained:

$$E(\varpi(\tilde{A})) = \int_{\bar{A}^{(c)}} f(\tilde{A}) \varpi(\tilde{A}) d\tilde{A} \quad (30)$$

According to the expected ranking vector, we can know that the expected ranking result of the alternative x_i is as follows:

$$\text{rank}_i^e = 1 + \sum_{k=1}^n o(E(\varpi(\tilde{A}))_k > E(\varpi(\tilde{A}))_i) \quad (31)$$

Furthermore, by integrating the subspace where the expected ranking result is established, the credibility of the expected ranking of alternative x_i in the whole preference space of GDM is obtained as follows:

$$T_i^e = \int_{\bar{A}^{(c)} : \text{rank}_i(\varpi(\tilde{A})) = \text{rank}_i^e} f(\tilde{A}) d\tilde{A} \quad (32)$$

In the interval linguistic fuzzy preference space, Monte Carlo simulation is used to obtain the expected ranking vector $E(\varpi(\tilde{A}))$, credibility T_i^e and the probability of other ranking cases.

Then, a novel decision-making method for GDM with ILFPRs is developed, which is concluded by Algorithm 2.

Algorithm 2

- Step 1.** The experts $D = \{d_1, d_2, \dots, d_z\}$ express their ILFPR $\bar{A}^{(k)} = (\bar{a}_{ij}^{(k)})_{n \times n}$ ($i, j = 1, 2, \dots, n; k = 1, 2, \dots, z$) on the alternative set $X = \{x_1, x_2, \dots, x_n\}$, respectively.
- Step 2.** Establishing a threshold $\overline{ICI}$ and compute the ICI of $\bar{A}^{(k)} = (\bar{a}_{ij}^{(k)})_{n \times n}$ according to Eq. (2). If $ICI(\bar{A}^{(k)}) \leq \overline{ICI}$, then transform into $\bar{A}^{(k)*} = (\bar{a}_{ij}^{(k)*})_{n \times n}$ and skip to Step 4; otherwise, go to the next step.
- Step 3.** Transforming the $\bar{A}^{(k)} = (\bar{a}_{ij}^{(k)})_{n \times n}$, contradict $ICI(\bar{A}^{(k)}) \leq \overline{ICI}$, into $\bar{A}^{(k)*} = (\bar{a}_{ij}^{(k)*})_{n \times n}$ that satisfy both ordinal consistency and acceptable additive consistency according to model (7) and (8).
- Step 4.** Applying Algorithm1 to compute the weights $W = (w_1, w_2, \dots, w_z)^T$ of DMs.
- Step 5.** Building the comprehensive ILFPR $\bar{A}^{(c)} = (\bar{a}_{ij}^{(c)})_{n \times n}$ according to Eq. (9).
- Step 6.** Obtaining the LPR $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, which generated from $\bar{A}^{(c)} = (\bar{a}_{ij}^{(c)})_{n \times n}$, where $\tilde{a}_{ij}$ obeys randomly generated from the uniform distribution of $[\bar{a}_{ij}^{L(c)}, \bar{a}_{ij}^{U(c)}]$ and $\tilde{a}_{ji} = 2\tau - \tilde{a}_{ij}$.
- Step 7.** Calculating the cross-efficiency $\theta_1, \theta_2, \dots, \theta_n$ by using models (24) and (25) and Eqs. (26–27) if there are equivalent self-efficiency for different DMUs.
- Step 8.** Calculating the ranking vector $\varpi_1, \varpi_2, \dots, \varpi_n$ corresponding to the alternative.
- Step 9.** Repeating the Step 6–8 N times and execute stochastic simulation to get the outcome of $E(\varpi(\tilde{A}))$, $rank_i^e$ and T_i^e .

6 Numerical example

In this section, the proposed GDM method is applied to an example of cold chain logistics selection in response to public emergencies.

The occurrence of public emergencies may cause heavy casualties and property losses and endanger social security. However, public emergencies cannot be predicted and controlled in advance. In the face of public emergencies, relevant departments should take timely and effective measures to minimize losses and maintain national security and social stability. When the epidemic virus breaks out, the control of cold chain food is very important. With the innovation of transportation mode, the new logistics industry often undertakes the distribution of cold chain food, which is called cold chain logistics. For example, the outbreak of coronavirus-19 (covid-19) is a public health emergency, which has a huge impact on China and even the world. When the first outbreak of covid-19 led to the closure of Wuhan, in order to meet the daily needs of residents, a large number of cold chain food, including primary agricultural

products (vegetables, fruits), processed food (frozen food, cooked food) and special drugs, need to be transported to Wuhan as soon as possible. Cold chain logistics can provide a strong guarantee for the distribution, transportation, and distribution of these cold chain materials. There are some excellent logistics platform enterprises in China, including rookie, Shunfeng, Jingdong, etc. These enterprises have a relatively perfect cold chain logistics distribution mechanism. They can provide cold chain supplies in time. Therefore, in order to minimize the economic and social losses caused by public health emergencies and maximize timeliness, it is particularly important to choose an excellent cold chain logistics platform company.

Supposing there are four high-quality logistics platform enterprises in possession of the cold chain logistics companies $\{x_1, x_2, x_3, x_4\}$. To select optimal alternative(s), four DMs $\{d_1, d_2, d_3, d_4\}$ are invited to give their evaluations based on LTS:

$$\widehat{S} = \left\{ \begin{array}{l} s_0 = \text{absolutely low}, \quad s_1 = \text{extremely low}, \quad s_2 = \text{very low}, \quad s_3 = \text{low}, \quad s_4 = \text{a little low}, \\ s_5 = \text{medium}, \quad s_6 = \text{a little high}, \quad s_7 = \text{high}, \quad s_8 = \text{very high}, \quad s_9 = \text{extremely high}, \\ s_{10} = \text{absolutely high} \end{array} \right\}.$$

The evaluations provided by DMs are respectively presented in the form of four ILFPRs $\overline{A}^{(k)} = (\overline{a}_{ij}^{(k)})_{4 \times 4} (k = 1, 2, 3, 4)$ as follows:

$$\begin{aligned} \overline{A}^{(1)} &= \begin{bmatrix} [s_5, s_5] & [s_6, s_7] & [s_6, s_7] & [s_3, s_4] \\ [s_3, s_4] & [s_5, s_5] & [s_5, s_6] & [s_2, s_3] \\ [s_3, s_4] & [s_4, s_5] & [s_5, s_5] & [s_5, s_6] \\ [s_6, s_7] & [s_7, s_8] & [s_4, s_5] & [s_5, s_5] \end{bmatrix}, & \overline{A}^{(2)} &= \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_6, s_7] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_7, s_7] & [s_2, s_4] \\ [s_3, s_4] & [s_3, s_3] & [s_5, s_5] & [s_5, s_6] \\ [s_6, s_7] & [s_6, s_8] & [s_4, s_5] & [s_5, s_5] \end{bmatrix} \\ \overline{A}^{(3)} &= \begin{bmatrix} [s_5, s_5] & [s_5, s_7] & [s_7, s_9] & [s_3, s_5] \\ [s_3, s_5] & [s_5, s_5] & [s_5, s_7] & [s_2, s_4] \\ [s_1, s_3] & [s_3, s_5] & [s_5, s_5] & [s_5, s_6] \\ [s_5, s_7] & [s_6, s_8] & [s_4, s_5] & [s_5, s_5] \end{bmatrix}, & \overline{A}^{(4)} &= \begin{bmatrix} [s_5, s_5] & [s_6, s_7] & [s_7, s_8] & [s_3, s_4] \\ [s_3, s_4] & [s_5, s_5] & [s_6, s_7] & [s_2, s_3] \\ [s_2, s_3] & [s_3, s_4] & [s_5, s_5] & [s_7, s_8] \\ [s_6, s_7] & [s_7, s_8] & [s_2, s_3] & [s_5, s_5] \end{bmatrix} \end{aligned}$$

Step 1 Let $\overline{ICI} = 0.1$. By Eq. (2), the interval additive consistency indices of $\overline{A}^{(k)} (k = 1, 2, 3, 4)$ are calculated as: $ICI(\overline{A}^{(1)}) = 0.15$, $ICI(\overline{A}^{(2)}) = 0.1917$, $ICI(\overline{A}^{(3)}) = 0.1583$, $ICI(\overline{A}^{(4)}) = 0.3$. Obviously, all are unacceptable interval additive consistency. Meanwhile, according to Definition 3.3, it has $\overline{a}_{23}^{(2)} = [s_7, s_7] > [s_5, s_5]$ and $\overline{a}_{34}^{(2)} = [s_5, s_6] > [s_5, s_5]$, which means that x_2 is better than x_3 , x_3 is better than x_4 by the second DM. Then x_2 is supposed to be better than x_4 which contradicts with $\overline{a}_{24}^{(2)} = [s_2, s_4] < [s_5, s_5]$ in ILFPR $\overline{A}^{(2)}$. Since there are matrices with unacceptable interval additive consistency and non-transitivity, they need to be revised in Step 2.

Step 2 Based on models (7) and (8), the ILFPR $\overline{A}^{(k)*} = (\overline{a}_{ij}^{(k)*})_{4 \times 4}$ with ordinal consistency and interval additive consistency are obtained as follows:

Table 3 Expected ranking vector under different simulation times

Simulation times	Expected ranking vectors			
	$E(v)_1$	$E(v)_2$	$E(v)_3$	$E(v)_4$
50	0.3713	0.158	0.1183	0.3524
100	0.367	0.154	0.1278	0.3512
150	0.3704	0.1568	0.122	0.3508
200	0.3685	0.1541	0.13	0.3475
250	0.3699	0.1557	0.1283	0.3461
300	0.3714	0.1561	0.1269	0.3456
350	0.3703	0.1566	0.1267	0.3465
400	0.3700	0.1562	0.1263	0.3475
450	0.3702	0.1559	0.1272	0.3467
500	0.3707	0.155	0.1273	0.347

$$\begin{aligned} \bar{A}^{(1)*} &= \begin{bmatrix} [s_5, s_5] & [s_6, s_7] & [s_6, s_7] & [s_3, s_4] \\ [s_3, s_4] & [s_5, s_5] & [s_5, s_5] & [s_2, s_3] \\ [s_3, s_4] & [s_5, s_5] & [s_5, s_5] & [s_4, s_5] \\ [s_6, s_7] & [s_7, s_8] & [s_5, s_6] & [s_5, s_5] \end{bmatrix}, & \bar{A}^{(2)*} &= \begin{bmatrix} [s_5, s_5] & [s_5, s_6] & [s_6, s_7] & [s_3, s_4] \\ [s_4, s_5] & [s_5, s_5] & [s_7, s_7] & [s_2, s_4] \\ [s_3, s_4] & [s_3, s_3] & [s_5, s_5] & [s_2, s_6] \\ [s_6, s_7] & [s_6, s_8] & [s_4, s_8] & [s_5, s_5] \end{bmatrix} \\ \bar{A}^{(3)*} &= \begin{bmatrix} [s_5, s_5] & [s_5, s_7] & [s_7, s_9] & [s_3, s_5] \\ [s_3, s_5] & [s_5, s_5] & [s_5, s_7] & [s_2, s_4] \\ [s_1, s_3] & [s_3, s_5] & [s_5, s_5] & [s_2, s_6] \\ [s_5, s_7] & [s_6, s_8] & [s_4, s_8] & [s_5, s_5] \end{bmatrix}, & \bar{A}^{(4)*} &= \begin{bmatrix} [s_5, s_5] & [s_6, s_7] & [s_7, s_8] & [s_4, s_4] \\ [s_3, s_4] & [s_5, s_5] & [s_6, s_7] & [s_2, s_3] \\ [s_2, s_3] & [s_3, s_4] & [s_5, s_5] & [s_2, s_6] \\ [s_6, s_6] & [s_7, s_8] & [s_4, s_8] & [s_5, s_5] \end{bmatrix}. \end{aligned}$$

Step 3 Calculating the weight of each DM by using Algorithm 1. An initial weighting vector is that $W_0 = (0.4064, 0.2434, 0.0578, 0.2923)^T$. According to Eqs. (10–11), the dev_0 and e_0 are obtained. Due to the space limitation, they are omitted. By using Algorithm 1, the final weighting vector is $W = (0.2953, 0.2502, 0.2538, 0.2008)^T$.

Step 4 According to Eq. (9), the revised ILFPR $\bar{A}^{(k)*}$ and weighting vector W , the comprehensive ILFPR $A^{(c)}$ is computed. Due to the space limitation, it is omitted.

Step 5 In this paper, the LPRs are generated randomly based on the uniform distribution in ILFPR. In order to establish the DEA cross-efficiency model, preprocess the derived function ϕ by Eq. (21).

Step 6 Based on model (24), model (25), Eqs. (26), and (27), the cross-efficiency $\theta = \{\theta_{11}, \theta_{22}, \theta_{33}, \theta_{44}\}$ is calculated. Then, calculate each ranking vector $v_i (i = 1, 2, 3, 4)$ corresponding to the alternative x_i .

Step 7 Stochastic preference analysis is used to obtain the final expected ranking vector and its confidence level is given. The results are shown in Table 3.

It can be seen that when the number of simulations exceeds 350, the expected value of the ranking vector tends to be stable, and the variation range is less than 0.002. When 500 simulation times are selected, the expected ranking vectors of alternatives and their confidence level are counted. According to the ranking statistics of 500 times of random simulation, the frequency distribution statistics diagram of ranking is made, as shown in Fig. 2. It can be determined that the best alternative in cold chain logistics capability is x_1 , and the confidence degree of its ranking first is 70.20%. Based on the expected ranking vector, the ranking result of cold chain logistics confidence level among five logistics enterprises is $x_1 > x_4 > x_2 > x_3$.

7 Comparative analysis

To demonstrate the advantages of our proposed methods, some existing priority vector deriving methods (Chen & Zhou, 2011; Liu et al., 2019b) are applied to the case study. Then some comparisons between the integrated GDM method and other existing methods are presented.

7.1 Comparison between our proposed method and Liu et al. (2019b)

In Liu et al. (2019b), they proposed an approach for GDM based on Monte Carlo stochastic simulation method and DEA model. According to Definition 5.1, the interval fuzzy preference relations are obtained. By using their method, we can obtain the priority vector $E = (0.2666, 0.2255, 0.2195, 0.2884)^T$. Then, the ranking order of the alternatives is $x_4 > x_1 > x_2 > x_3$. This result is slightly different from the results provided by our proposed method.

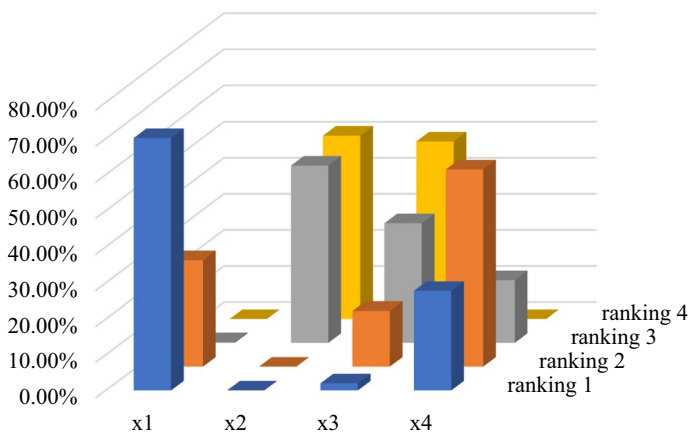


Fig. 2 Frequency distribution of ranking statistics of four companies

In a comparison between our proposed method and the method proposed by Liu et al. (2019b), we observe the following:

- Liu et al. (2019b) proposed a GDM method that involved additive consistency based on LPR while our method involves interval additive consistency and ordinal consistency on this basis. The former ignores the influence of interval additive consistency and order consistency in decision-making process. Ordinal consistency is the lowest condition to ensure that DMs will not give some contradictory preferences. The ranking result based on ordinal inconsistent ILFPR is unreliable. Based on the interval additivity consistency index, our proposed method contains a two-stage optimization model, which ensures the existence of transitivity to obtain acceptably interval additive consistent ILFPR. Therefore, our proposed method can provide more reliable decision-making results.
- Liu et al. (2019b) proposed a GDM method that introduced a weight model based on minimum deviation while our proposed method applies the cooperative game method to measure the weights of DMs. The former ignores the decision-making interests and decision-making psychology of DMs.

7.2 Comparison between our proposed method and Chen and Zhou (2011)

First, ILFPRs are transformed into interval fuzzy preference relations by using subscript function and Eq. (21). Second, we use the GDM method proposed by Chen and Zhou (2011) to solve the numerical example. Then, we can obtain the priority vector $E = (0.2564, 0.2284, 0.2343, 0.2806)^T$. Finally, the ranking order of the alternatives is $x_4 > x_1 > x_2 > x_3$. This result is slightly different from the results provided by our proposed method.

In a comparison between our proposed GDM method and the existing method (Chen & Zhou, 2011), we observe the following:

- Chen and Zhou (2011) ignored the ordering problem and the acceptable additive consistency problem of the interval fuzzy preference relations, which leads to inconsistent ranking results. Our proposed model not only defines the ordinal consistency based on the original information of ILFPR, developing a two-stage optimization model to derive the acceptably interval additive consistent ILFPR with ordinal consistency.
- Chen and Zhou (2011) applied the consistency indices to obtain the DMs' weights. However, it ignored the decision-making interests and decision-making psychology of DMs. Our proposed method uses the cooperative game method to calculate the weights of DMs. Therefore, our proposed method is more reliable and reasonable.

8 Conclusions

This paper mainly concentrates on the ordinal consistency of ILFPR and the method for GDM with ILFPRs. This paper first defines the ordinal consistency of ILFPR. For ILFPR without ordinal consistency, a two-stage optimization model is established to obtain revised ILFPR with ordinal consistency and acceptable interval additive consistency. Then, a weight-determination method for solving DMs' weights in GDM with ILFPRs is proposed based on cooperative games. Furthermore, each alternative is regarded as a DMU of DEA model, a cross-efficiency DEA model for LFP is constructed. For ILFPR, it can be viewed as composed of several LPRs. Hence, LPR can be extracted according to the uniform distribution. Finally, we use Monte Carlo stochastic simulation method to analyze the whole group preference space and obtain the expected ranking vector and its credibility of each alternative. The method can address the lack of ordinal consistency, calculate each ranking vector, and has strong applicability and high confidence level.

The main advantages of this article are: (1) using a two-stage optimization model based on interval additive consistency and ordinal consistency to modify unreasonable ILFPRs; (2) using cooperative game method based on the decision-making interests and decision-making psychology to measure the weights of DMs; (3) developing a DEA cross-efficiency model based on Monte Carlo simulation to obtain the expected ranking vector of alternatives in GDM. It can consider the mutual influence between different alternatives, avoid that the optimal solution of DEA is not unique, and fully use the whole decision-making information in ILFPR.

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