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A new uncertain dominance and its properties in the framework of uncertainty theory

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Abstract
Theoretical analysis and empirical study results all show that there are situations in reality where observed data are not random variables. Thus, decision-making criteria based on probability theory are not suitable for people to make decisions. This paper proposes an uncertain dominance based on uncertainty theory to offer an alternative decision-making criterion for such situations. The paper first defines a new criterion of first- and second-order uncertain dominance, then proves some necessary conditions of them based on uncertainty theory. Some sufficient and necessary conditions of the first- and second-order uncertain dominance are given when uncertain variables are all normal or linear uncertain variables. In addition, the paper proves the link between the uncertain dominance criterion and the expected utility criterion and shows that the first-order uncertain dominance is suitable for all people to make decisions and the second-order uncertain dominance is suitable for risk-averse people to make decisions.

Keywords Uncertainty theory · Decision-making · Dominance · Uncertain dominance

1 Introduction

In order to choose among alternative choices, decision-making rules are needed. When choice results are random numbers, the expected value, the quantile, and the stochastic dominance are the three main decision-making criteria. With the expected value criterion, the choice with the maximum expected value should

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be selected, and with the quantile, the choice with the maximum quantile value at the predetermined confidence level should be selected. Rather than comparing one characteristic value of the two choices, the stochastic dominance criterion compares the two choices pointwise based on probability distributions. In addition, the stochastic dominance criterion is proved to be consistent with the expected utility criterion but only employs partial information on the preferences. Thus, it is theoretically appealing and the research on it, especially on second-order stochastic dominance, keeps hot.

These criteria all assume that choice results are random variables and they all make rankings based on probability theory. However, numerous empirical studies have revealed that in real life there are many situations where the distributions obtained from the observed data are not probability distributions and the imprecise parameters are not random variables. Thus, it is unsuitable to use the stochastic criteria to make decisions in these situations. For example, Ye and Liu (2022a) investigated the observed exchange rate data of US Dollar to Chinese Yuan from October 2019 to June 2021 and found that they are not from the same population and their residuals are not white noise in the sense of probability theory either. Therefore, the distribution got from these data cannot be a probability distribution. The investigations of Liu (2021); Ye and Yang (2021); Liu and Liu (2022), and Ye and Liu (2022b) also showed that none of the cumulative numbers of COVID-19 infections, Alibaba stock prices, or GDP values are random variables.

To model the imprecise quantity that is not random, in 2007, Liu proposed uncertain measure and developed uncertainty theory based on four axioms (Liu, 2007) and further refined it (Liu, 2009a). With the development of uncertainty theory, uncertain decision-making criteria in the framework of uncertainty theory have been proposed. The important ones include the uncertain expected value criterion (Liu, 2009b), the uncertain mean-risk criteria (Huang, 2010, 2012a, b), the expected uncertain utility criterion (Yao & Ji, 2014) and the utility criterion of the mean and variance of the uncertain choices (Huang & Jiang, 2021). As an alternative uncertain decision-making criterion, uncertain dominance was first studied and defined by Zuo and Ji (2009) via uncertainty distributions, and later redefined by Yao and Ji (2014) and Chen and Park (2017) via expected utility functions. Inspired by the advantage of stochastic dominance in handling stochastic decision making problems, we try to develop an uncertain dominance criterion as an alternative uncertain decision making rule. We propose a revised definition of uncertain dominance based on inverse uncertainty distributions and further discuss the properties of the rule. The uncertain dominance criterion can be applied to solve various uncertain decision-making problems such as uncertain portfolio optimization, project selection, supply chain management, optimal saving and consumption, etc.

Our main contributions are as follows. First, as an alternative uncertain decision making criterion, we propose a revised definition of first- and second-order uncertain dominance based on inverse uncertainty distributions. The difference and importance of our proposal is shown by comparing our definitions with those of Zuo and Ji (2009). Second, we discuss and offer the new properties of the proposed uncertain dominance criterion. Third, we prove the link between the proposed uncertain dominance criterion and the expected utility criterion and show that the first-order uncertain dominance is suitable for all people to make decisions and the second-order uncertain dominance is suitable for risk-averse people to make decisions.

The article is organized as follows. We define first- and second-order uncertain dominance, compare them with those by Zuo and Ji (2009) and discuss some necessary and sufficient conditions of the first- and second-order uncertain dominance in Sects. 2 and 3, respectively. Then we prove the link between the uncertain dominance criterion and the expected utility criterion in Sect. 4. Finally, we conclude in Sect. 5.

2 First-order uncertain dominance

Definition 1 Let ξ and η denote two uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. We say ξ dominates η by first-order uncertain dominance if $\Phi^{-1}(\alpha) \geq \Psi^{-1}(\alpha)$ for all $\alpha \in (0, 1)$, denoted by $\xi \succeq_1 \eta$.

Example 1 Let ξ and η represent two uncertain choice results and ξ a zigzag uncertain variable $\xi \sim \mathcal{Z}(3, 5, 8)$ and η a linear uncertain variable $\eta \sim \mathcal{L}(0, 2)$. According to Definition 1, we can get $\xi \succeq_1 \eta$. Please see Fig. 1.

Theorem 1 *Let ξ and η be two regular uncertain variables. Then $E[\xi] \geq E[\eta]$ if $\xi \succeq_1 \eta$.*

Proof Let Φ and Ψ denote the uncertainty distributions of ξ and η, respectively. Since $\xi \succeq_1 \eta$, we know from Definition 1 that $\Phi^{-1}(\alpha) \geq \Psi^{-1}(\alpha)$, $\forall \alpha \in (0, 1)$. According to the theorem of the expected value of the uncertain variable, for a regular uncertain variable ξ with inverse uncertainty distribution Φ^{-1}, its expected value can be obtained via

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) \mathrm{d}\alpha.$$

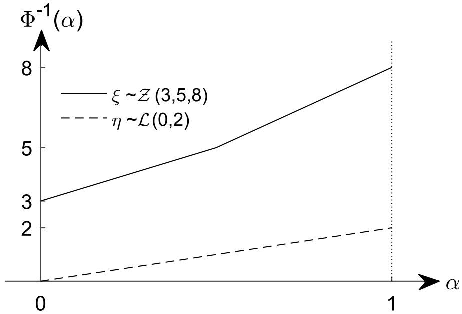

Fig. 1 Inverse uncertainty distributions of $\xi \sim \mathcal{Z}(3, 5, 8)$ and $\eta \sim \mathcal{L}(0, 2)$

Therefore, it follows from $\boldsymbol{\Phi}^{-1}(\alpha) \geq \boldsymbol{\Psi}^{-1}(\alpha)$, $\forall \alpha \in (0,1)$ that

$$E[\xi] = \int_0^1 \boldsymbol{\Phi}^{-1}(\alpha)\mathrm{d}\alpha \geq \int_0^1 \boldsymbol{\Psi}^{-1}(\alpha)\mathrm{d}\alpha = E[\eta].$$

□

Example 2 Let ξ and η be two normal uncertain variables $\xi \sim \mathcal{N}(e_1, \sigma_1)$ and $\eta \sim \mathcal{N}(e_2, \sigma_2)$, respectively. Then $\xi \succeq_1 \eta$ if and only if $e_1 \geq e_2$ and $\sigma_1 = \sigma_2$.

Example 3 Let ξ and η be two linear uncertain variables $\xi \sim \mathcal{L}(a_1, b_1)$ and $\eta \sim \mathcal{L}(a_2, b_2)$, respectively. Then $\xi \succeq_1 \eta$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Theorem 2 *Let ξ and η be two regular uncertain variables, and f a strictly increasing function. Then $f(\xi) \succeq_1 f(\eta)$ if $\xi \succeq_1 \eta$.*

Proof Let $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ denote the uncertainty distributions of ξ and η, respectively. If $\xi \succeq_1 \eta$, we know from Definition 1 that $\boldsymbol{\Phi}^{-1}(\alpha) \geq \boldsymbol{\Psi}^{-1}(\alpha)$ for all $\alpha \in (0,1)$. Then we get $f(\boldsymbol{\Phi}^{-1}(\alpha)) \geq f(\boldsymbol{\Psi}^{-1}(\alpha))$ for all $\alpha \in (0,1)$ because f is a strictly increasing function. According to the operational law of the uncertain variable, when f is a strictly increasing function, the inverse uncertainty distributions of $f(\xi)$ and $f(\eta)$ are just $f(\boldsymbol{\Phi}^{-1}(\alpha))$ and $f(\boldsymbol{\Psi}^{-1}(\alpha))$, respectively. Therefore, we have $f(\xi) \succeq_1 f(\eta)$. □

Theorem 3 *Let ξ and η be two regular uncertain variables, and f a strictly decreasing function. Then $f(\eta) \succeq_1 f(\xi)$ if $\xi \succeq_1 \eta$.*

Proof Let $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ denote the uncertainty distributions of ξ and η, respectively. If $\xi \succeq_1 \eta$, we know from Definition 1 that $\boldsymbol{\Phi}^{-1}(\alpha) \geq \boldsymbol{\Psi}^{-1}(\alpha)$ for all $\alpha \in (0,1)$. Then we get $f(\boldsymbol{\Psi}^{-1}(\alpha)) \geq f(\boldsymbol{\Phi}^{-1}(\alpha))$ for all $\alpha \in (0,1)$ because f is a strictly decreasing function. According to the operational law of the uncertain variable, when f is a strictly decreasing function, the inverse uncertainty distributions of $f(\xi)$ and $f(\eta)$ are $f(\boldsymbol{\Phi}^{-1}(1-\alpha))$ and $f(\boldsymbol{\Psi}^{-1}(1-\alpha))$, respectively. It is clear that $f(\boldsymbol{\Psi}^{-1}(1-\alpha)) \geq f(\boldsymbol{\Phi}^{-1}(1-\alpha))$ for all $\alpha \in (0,1)$ if $f(\boldsymbol{\Psi}^{-1}(\alpha)) \geq f(\boldsymbol{\Phi}^{-1}(\alpha))$ for all $\alpha \in (0,1)$. Therefore, we have $f(\eta) \succeq_1 f(\xi)$. □

3 Second-order uncertain dominance

First-order uncertain dominance is a too strong requirement. In real life, it is hard that one uncertain choice dominates another by first-order uncertain dominance. If in some ranges the distribution of one uncertain choice is below the distribution of another uncertain choice while in some other ranges the distribution of one uncertain choice is up to the distribution of another uncertain choice, which choice should we choose? This section answers the question.

Definition 2 Let ξ and η denote two uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. We say ξ dominates η by second-order uncertain dominance if

$$\int_0^\beta \Phi^{-1}(\alpha)\mathrm{d}\alpha \geq \int_0^\beta \Psi^{-1}(\alpha)\mathrm{d}\alpha, \quad \forall \beta \in (0,1),$$

denoted by $\xi \succeq_2 \eta$.

Example 4 Let ξ be a zigzag uncertain variable $\xi \sim \mathcal{Z}(1, 1.5, 5)$ and η a linear uncertain variable $\eta \sim \mathcal{L}(0,4)$, which represent two choice results. According to Definition 2, we can get that $\xi \succeq_2 \eta$. Please see Fig. 2.

Theorem 4 *Let ξ and η denote two uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then $\xi \succeq_2 \eta$ if $\xi \succeq_1 \eta$.*

Proof It follows from Definitions 1 and 2 directly. □

Example 5 Suppose $\xi_1 \sim \mathcal{L}(1,3), \xi_2 \sim \mathcal{L}(1,4), \eta_1 \sim (0.5, 2.5), \eta_2 \sim (1.5, 4.5)$ are four linear uncertain variables. Then according to the operational law of the uncertain variables, we know that the inverse uncertainty distribution Φ^{-1} of $\xi_1\xi_2$ is

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \cdot \Phi_2^{-1}(\alpha) = 6\alpha^2 + 5\alpha + 1, \ \alpha \in (0,1)$$

where $\Phi_1^{-1}(\alpha)$ and $\Phi_2^{-1}(\alpha)$ are inverse uncertainty distributions of ξ_1 and ξ_2, respectively, and the inverse uncertainty distribution Ψ^{-1} of $\eta_1\eta_2$ is

$$\Psi^{-1}(\alpha) = \Psi_1^{-1}(\alpha) \cdot \Psi_2^{-1}(\alpha) = 6\alpha^2 + 4.5\alpha + 0.75, \ \alpha \in (0,1)$$

where $\Psi_1^{-1}(\alpha)$ and $\Psi_2^{-1}(\alpha)$ are inverse uncertainty distributions of η_1 and η_2, respectively. Since $\Phi^{-1}(\alpha) = 6\alpha^2 + 5\alpha + 1 > 6\alpha^2 + 4.5\alpha + 0.75 = \Psi^{-1}(\alpha), \ \forall \alpha \in (0,1)$, we know

$$\xi_1\xi_2 \succeq_1 \eta_1\eta_2 \quad \text{and} \quad \xi_1\xi_2 \succeq_2 \eta_1\eta_2.$$

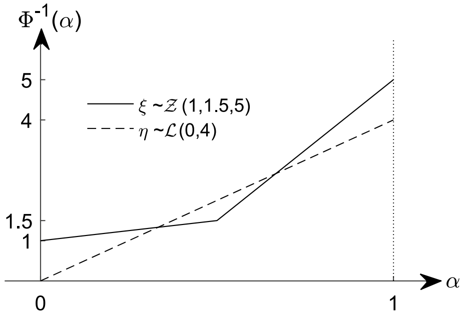

Fig. 2 Inverse uncertainty distributions of $\xi \sim \mathcal{Z}(1, 1.5, 5)$ and $\eta \sim \mathcal{L}(0,4)$

In the paper (Zuo & Ji, 2009), first- and second-order uncertain dominance were defined based on uncertainty distributions as follows. Let ξ and η denote two uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then ξ dominates η by first-order uncertain dominance if

$$\Phi(t) \le \Psi(t), \quad \forall t \in \Re.$$

And ξ dominates η by second-order uncertain dominance if

$$\int_{-\infty}^{x} \Phi(t)\mathrm{d}t \le \int_{-\infty}^{x} \Psi(t)\mathrm{d}t, \quad \forall x \in \Re.$$

Our definitions of first- and second-order uncertain dominance have several advantages over the definitions based on uncertainty distributions (Zuo & Ji, 2009). First, inverse uncertainty distributions rather than uncertainty distributions of the uncertain choice results are produced that we need to compare in real life. In real life, it is rare that we only compare two individual uncertain variables. Instead, we usually need to compare the alternative uncertain choice results produced by several uncertain variables, as illustrated in Example 5. According to the operational law of the uncertain variables, it is the inverse uncertainty distributions rather than uncertainty distributions that are produced. In Example 5, if ξ_1 and η_1 represent the uncertainty distributions of the uncertain prices of two new products and ξ_2 and η_2 denote the uncertainty distributions of the uncertain selling quantity of the two products, we can easily produce the inverse uncertainty distributions of the uncertain gross profits of the two new products according to the operational law of the uncertain variables. Yet it is usually difficult to get the uncertainty distributions of the uncertain gross profits of the products. Second, we know from the theorem of the expected value of the uncertain variable that $E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha$ and $E[\eta] = \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha$. Therefore, letting $\beta \to 1$, we have

$$\lim_{\beta\to 1} \int_0^{\beta} \Phi^{-1}(\alpha)\mathrm{d}\alpha = E[\xi], \quad \lim_{\beta\to 1} \int_0^{\beta} \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\eta],$$

which implies that with our Definition 2, for any $\beta \in [0, 1]$ the value of each $\int_0^{\beta} \Phi^{-1}(\alpha)\mathrm{d}\alpha$ and each $\int_0^{\beta} \Psi^{-1}(\alpha)\mathrm{d}\alpha$ converges to a constant. Therefore, we can design an algorithm to make comparison. However, for two uncertain variables ξ and η whose uncertainty distributions are Φ and Ψ, respectively, if $\xi \succeq_2 \eta$ is defined based on uncertainty distributions, it is seen that

$$\lim_{x\to\infty} \int_{-\infty}^{x} \Phi(t)\mathrm{d}t = \infty, \quad \lim_{x\to\infty} \int_{-\infty}^{x} \Psi(t)\mathrm{d}t = \infty,$$

which implies that neither $\int_{-\infty}^{x} \Phi(t)\mathrm{d}t$ nor $\int_{-\infty}^{x} \Psi(t)\mathrm{d}t$ converges to a definite value as $x \to \infty$. Thus, it is impossible to design an algorithm to make comparison because no terminal condition can be determined. Third, in our Definitions 1 and 2, the formulas all have economic meanings. For example, in Definition 2, the formulas $\int_0^{\beta} \Phi^{-1}(\alpha)\mathrm{d}\alpha$ and $\int_0^{\beta} \Psi^{-1}(\alpha)\mathrm{d}\alpha$ represent the weighted sum of the values that are equal to or smaller than $\Phi^{-1}(\beta)$ and $\Psi^{-1}(\beta)$, respectively, while in the definition of second-order uncertain dominance based on uncertainty distributions, $\int_{-\infty}^{x} \Phi(t)\mathrm{d}t$ and $\int_{-\infty}^{x} \Psi(t)\mathrm{d}t$ have no economic meaning.

Theorem 5 *Let ξ and η be two regular uncertain variables. Then $E[\xi] \geq E[\eta]$ if $\xi \succeq_2 \eta$.*

Proof According to Definition 2, if $\xi \succeq_2 \eta$, we have

$$\int_0^\beta \Phi^{-1}(\alpha)\mathrm{d}\alpha \geq \int_0^\beta \Psi^{-1}(\alpha)\mathrm{d}\alpha, \quad \forall \beta \in (0,1).$$

Since Φ and Ψ are regular, letting $\beta \to 1$, we have

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha \geq \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\eta].$$

□

Example 6 Let ξ and η be two normal uncertain variables $\xi \sim \mathcal{N}(e_1, \sigma_1)$ and $\eta \sim \mathcal{N}(e_2, \sigma_2)$, respectively. Then $\xi \succeq_2 \eta$ if and only if $e_1 \geq e_2$ and $\sigma_1 \leq \sigma_2$.

Example 7 Let ξ and η be two linear uncertain variables $\xi \sim \mathcal{L}(a_1, b_1)$ and $\eta \sim \mathcal{L}(a_2, b_2)$, respectively. Then $\xi \succeq_2 \eta$ if and only if $E[\xi] \geq E[\eta]$ and $a_1 \geq a_2$.

Example 8 Let ξ and η be two linear uncertain variables $\xi \sim \mathcal{L}(a_1, b_1)$ and $\eta \sim \mathcal{L}(a_2, b_2)$, respectively. Then $\xi \succeq_2 \eta$ if $E[\xi] \geq E[\eta]$ and $V[\xi] \leq V[\eta]$.

Please be aware that for two linear uncertain variables $\xi \sim \mathcal{L}(a_1, b_1)$ and $\eta \sim \mathcal{L}(a_2, b_2)$, it is possible that $E[\xi] \geq E[\eta]$, $V[\xi] \leq V[\eta]$ does not hold if $\xi \succeq_2 \eta$.

Example 9 Let ξ be a linear uncertain variable $\xi \sim \mathcal{L}(0, 4)$ and η another linear uncertain variable $\eta \sim \mathcal{L}(0, 2)$. It is easy to see $\xi \succeq_1 \eta$ and $\xi \succeq_2 \eta$. But we calculate that $V[\xi] = 1.33 > V[\eta] = 0.33$. Please see Fig. 3.

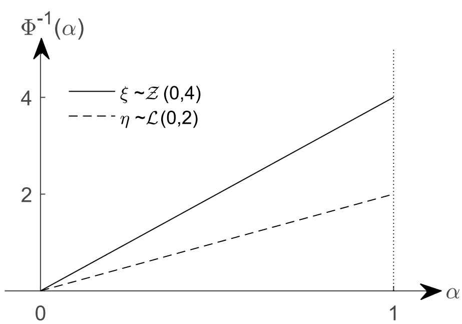

Fig. 3 Inverse uncertainty distributions of $\xi \sim \mathcal{L}(0, 4)$ and $\eta \sim \mathcal{L}(0, 2)$.

Theorem 6 *Let ξ and η be two regular uncertain variables whose uncertainty distributions are differentiable and f an increasing function with $f' > 0$ and $f'' < 0$. Then $f(\xi) \succeq_2 f(\eta)$ if $\xi \succeq_2 \eta$.*

Proof The tangent line $g(x)$ to the function $f(x)$ at any point $(x_0, f(x_0))$ is

$$g(x) = f(x_0) + f'(x_0)(x - x_0).$$

Since $f' > 0$ and $f'' < 0$, we know that $g(x) \geq f(x),\ \forall x \in \mathfrak{R}$, i.e.,

$$f(x_0) + f'(x_0)(x - x_0) \geq f(x),\ \forall x \in \mathfrak{R}.$$

Let Φ^{-1} and Ψ^{-1} denote the inverse uncertainty distributions of the regular uncertain variables ξ and η, respectively. Letting $x_0 = \Phi^{-1}(\alpha)$ and $x = \Psi^{-1}(\alpha)$, we get

$$f(\Phi^{-1}(\alpha)) - f(\Psi^{-1}(\alpha)) \geq f'(\Phi^{-1}(\alpha))(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)),\ \alpha \in (0, 1).$$

Therefore, we have that $\forall \beta \in (0, 1)$,

$$\int_0^\beta \Big(f(\Phi^{-1}(\alpha)) - f(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \geq \int_0^\beta f'(\Phi^{-1}(\alpha))\Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha.$$

Since $\Phi^{-1}(\alpha)$ is differentiable, integrating by parts yields

$$\begin{aligned}
&\int_0^\beta \Big(f(\Phi^{-1}(\alpha)) - f(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\
&\geq \int_0^\beta f'(\Phi^{-1}(\alpha))\mathrm{d}\int_0^\alpha (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau \\
&= f'(\Phi^{-1}(\alpha))\int_0^\alpha (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau\Big|_0^\beta \\
&\quad - \int_0^\beta f''(\Phi^{-1}(\alpha))(\Phi^{-1})'(\alpha)\int_0^\alpha (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau\mathrm{d}\alpha \\
&= f'(\Phi^{-1}(\beta))\int_0^\beta (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau \\
&\quad - \int_0^\beta f''(\Phi^{-1}(\alpha))(\Phi^{-1})'(\alpha)\int_0^\alpha (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau\mathrm{d}\alpha.
\end{aligned}$$

Since $\Phi^{-1}(\alpha)$ is increasing and differentiable, we have $(\Phi^{-1})'(\alpha) > 0$. Since $\xi \succeq_2 \eta$, we know from Definition 2 that $\int_0^\beta (\Phi^{-1}(\tau) - \Psi^{-1}(\tau))\mathrm{d}\tau \geq 0,\ \forall \beta \in (0, 1)$. Together with $f' > 0$ and $f'' < 0$ we get

$$\int_0^\beta \Big(f(\Phi^{-1}(\alpha)) - f(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \geq 0,\ \forall \beta \in (0, 1).$$

Therefore, $f(\xi) \succeq_2 f(\eta)$. □

Please be aware that in Theorem 6, $f'' < 0$ is required. Otherwise, $f(\xi) \succeq_2 f(\eta)$ may not hold.

Example 10 Let $\xi \sim \mathcal{L}(1, 3)$ and $\eta \sim \mathcal{L}(0, 4)$ be two linear uncertain variables and $f = t^2, t \geq 0$. Let Φ and Ψ denote the uncertainty distributions of ξ and η, respectively. Then we know $\Phi^{-1}(\alpha) = 1 + 2\alpha$ and $\Psi^{-1}(\alpha) = 4\alpha$ where $\alpha \in (0, 1)$. According to Definition 2, we know $\xi \succeq_2 \eta$. Let Φ_1 and Ψ_1 denote the uncertainty distributions of $f(\xi)$ and $f(\eta)$, respectively. Since f is increasing, according to the operational law of the uncertain variables, we get the inverse uncertainty distributions of $f(\xi)$ and $f(\eta)$ as follows.

$$\begin{aligned}\Phi_1^{-1}(\alpha) &= f(\Phi^{-1}(\alpha)) = (\Phi^{-1}(\alpha))^2 = (1 + 2\alpha)^2 \quad \text{and} \\ \Psi_1^{-1}(\alpha) &= f(\Psi^{-1}(\alpha)) = (\Psi^{-1}(\alpha))^2 = 16\alpha^2,\end{aligned}$$

where $\alpha \in (0, 1)$. Let $\beta = 0.9$. We can calculate that

$$\int_0^{0.9} \Phi_1^{-1}(\alpha)\mathrm{d}\alpha = 3.492 < 3.888 = \int_0^{0.9} \Psi_1^{-1}(\alpha)\mathrm{d}\alpha.$$

Thus, $f(\xi) \succeq_2 f(\eta)$ does not hold.

4 Uncertain dominance and expected utility

Utility is used to measure a person's satisfaction with a choice result. The bigger the utility function value is, the more satisfactory the person feels with the choice result. Different persons' utility values on the same choice result can be different, but it is believed that all people prefer more payment to less. In this paper, utility function is supposed to be a strictly increasing function whose derivative is positive. A person without any restriction on utility function is called an unsatisfied person. It is seen that all people are unsatisfied. In addition, it is widely accepted that most decision-makers are risk-averse. In this paper, we say a person is risk averse if his or her utility function U has $U'' < 0$. When the result of a choice is random, we have a random utility value. In the past, the links between stochastic dominance and expected utility have been provided. If the result of a person's choice is uncertain, the person's utility on the choice result is uncertain. Then, are there similar links between uncertain dominance and expected utility? This section answers the question.

Theorem 7 *Let ξ and η be two regular uncertain variables. Then $\xi \succeq_1 \eta$ if and only if $E[U(\xi)] \geq E[U(\eta)]$ for any utility function U.*

Proof (1) Suppose $\xi \succeq_1 \eta$. For any utility function U, since $U' > 0$, according to Theorem 2, we have $U(\xi) \succeq_1 U(\eta)$. It follows from Theorem 1 that $E[U(\xi)] \geq E[U(\eta)]$.

(2) Suppose $E[U(\xi)] \geq E[U(\eta)]$ for any utility function U. Let Φ and Ψ denote the uncertainty distributions of ξ and η, respectively. Assume there exists one value α_1

such that $\Phi^{-1}(\alpha_1) < \Psi^{-1}(\alpha_1)$. Since regular uncertainty distribution is continuous, we have

$$\Phi^{-1}(\alpha) < \Psi^{-1}(\alpha) \quad \text{for} \quad \alpha_1 \le \alpha \le \alpha_1 + \varepsilon,$$

where ε is any small enough number.

Choose the utility function below

$$U_0(t) = \begin{cases} t_1, \ t < t_1 \\ t, \ \ t_1 \le t \le t_2 \\ t_2, \ t > t_2 \end{cases}$$

where $t_1 = \Phi^{-1}(\alpha_1)$ and $t_2 = \Psi^{-1}(\alpha_1 + \varepsilon)$. It is seen that $U_0' \ge 0$ except at the values $t = \Phi^{-1}(\alpha_1)$ and $t = \Psi^{-1}(\alpha_1 + \varepsilon)$. It is clear that we can find a utility function U, i.e., $U' > 0$. Since $U_0' \ge 0$, according to the operational law of the uncertain variables, the inverse uncertainty distributions of $U_0(\xi)$ and $U_0(\eta)$ are $U_0(\Phi^{-1}(\alpha))$ and $U_0(\Psi^{-1}(\alpha))$, respectively. Then according to the theorem of the expected value of the uncertain variable, we have

$$\begin{aligned} & E[U_0(\xi)] - E[U_0(\eta)] \\ & \quad = \int_0^1 \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\ & \quad = \int_0^{\alpha_1} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha + \int_{\alpha_1}^{\alpha_1+\varepsilon} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\ & \quad\quad + \int_{\alpha_1+\varepsilon}^{1} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\ & \quad < \int_0^{\alpha_1} (t_1 - t_1)\mathrm{d}\alpha + \int_{\alpha_1}^{\alpha_1+\varepsilon} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha + \int_{\alpha_1+\varepsilon}^{1} (t_2 - t_2)\mathrm{d}\alpha \\ & \quad < 0, \end{aligned}$$

which is in contradiction to the statement that $E[U(\xi)] \ge E[U(\eta)]$ for any utility function U. Thus, the theorem is proved. □

Theorem 8 *Let ξ and η be two regular uncertain variables. Then $\xi \succeq_2 \eta$ if and only if $E[U(\xi)] \ge E[U(\eta)]$ for any utility function U with $U'' < 0$.*

Proof (1) Suppose $\xi \succeq_2 \eta$. Since $U' > 0$ and $U'' < 0$, according to Theorem 6, we have $U(\xi) \succeq_2 U(\eta)$. Then, it follows from Theorem 5 that $E[U(\xi)] \ge E[U(\eta)]$.

(2) Suppose $E[U(\xi)] \ge E[U(\eta)]$ for any utility function U with $U'' < 0$. Assume there exists one value β_0 such that $\int_0^{\beta_0}(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha))\mathrm{d}\alpha < 0$.

Choose the utility function below

$$U_0(t) = \begin{cases} t, \ \ t \le t_0 \\ t_0, \ t > t_0 \end{cases}$$

where $t_0 = \Psi^{-1}(\beta_0)$.

It is seen that $U_0' \geq 0$ and $U_0'' = 0$ except at $t = \Psi^{-1}(\beta_0)$. It is clear that we can find a utility function U such that $U'' < 0$. Then

$$\begin{aligned} E[U_0(\xi)] - E[U_0(\eta)] &= \int_0^1 \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\ &= \int_0^{\beta_0} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \\ &\quad + \int_{\beta_0}^1 \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha \end{aligned}$$

When $\Phi^{-1}(\beta_0) = \Psi^{-1}(\beta_0)$, then

$$\begin{aligned} E[U_0(\xi)] - E[U_0(\eta)] &= \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha + \int_{\beta_0}^1 (t_0 - t_0)\mathrm{d}\alpha \\ &= \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha < 0. \end{aligned}$$

When $\Phi^{-1}(\beta_0) < \Psi^{-1}(\beta_0)$,

$$\begin{aligned} \int_0^{\beta_0} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha &= \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha, \\ \int_{\beta_0}^1 \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha &< \int_{\beta_0}^1 (t_0 - t_0)\mathrm{d}\alpha. \end{aligned}$$

Then

$$\begin{aligned} E[U_0(\xi)] - E[U_0(\eta)] &< \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha + \int_{\beta_0}^1 (t_0 - t_0)\mathrm{d}\alpha \\ &= \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha < 0. \end{aligned}$$

When $\Phi^{-1}(\beta_0) > \Psi^{-1}(\beta_0)$,

$$\begin{aligned} \int_0^{\beta_0} \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha &< \int_0^{\beta_0} \Big(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha, \\ \int_{\beta_0}^1 \Big(U_0(\Phi^{-1}(\alpha)) - U_0(\Psi^{-1}(\alpha))\Big)\mathrm{d}\alpha &= \int_{\beta_0}^1 (t_0 - t_0)\mathrm{d}\alpha. \end{aligned}$$

Then

$$E[U_0(\xi)] - E[U_0(\eta)] < \int_0^{\beta_0} \left(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha) \right) \mathrm{d}\alpha + \int_{\beta_0}^{1} (t_0 - t_0) \mathrm{d}\alpha$$
$$= \int_0^{\beta_0} \left(\Phi^{-1}(\alpha) - \Psi^{-1}(\alpha) \right) \mathrm{d}\alpha < 0.$$

So $E[U_0(\xi)] < E[U_0(\eta)]$, which is in contradiction to the statement that $E[U(\xi)] \geq E[U(\eta)]$ for any utility function U with $U'' < 0$. Thus, the theorem is proved. □

5 Conclusion

This paper proposed a new uncertain dominance as an alternative decision-making criterion in the framework of uncertainty theory. It defined a new first- and second-order uncertain dominance based on inverse uncertainty distributions and proved some important necessary conditions of them. Some sufficient and necessary conditions of the first- and second-order uncertain dominance were given when uncertain variables are all normal or linear uncertain variables. Furthermore, the paper proved the link between the uncertain dominance and the expected uncertain utility criterion, which shows that the first-order uncertain dominance is suitable for all people to make decisions and the second-order uncertain dominance is suitable for risk-averse people to make decisions.

Research on uncertain dominance just starts up. There is a lot of work to do. Future work can be done in three directions. First direction concerns with theoretical research on uncertain dominance. Second direction is about the application of the uncertain dominance criterion in solving various decision-making problems such as portfolio optimization, project selection, supply chain management, optimal saving and consumption, etc. The third one is the research on the comparison and consistency of the uncertain dominance criterion and other uncertain decision-making criteria.

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